微分積分学 II レポート(No.2) ・解答

問題. 次の恒等式が成り立つような実定数 A, B, C を求めよ.

(1) $\dfrac{1}{x^2-1} = \dfrac{A}{x-1} + \dfrac{B}{x+1}. \quad \cdots (*)$

- $(*)$ の両辺に $x-1$ をかけて, $\dfrac{1}{x+1} = A + \dfrac{B}{x+1} \cdot (x-1).$

  よって $x=1$ を代入して, $A = \dfrac{1}{2}.$

- $(*)$ の両辺に $x+1$ をかけて, $\dfrac{1}{x-1} = \dfrac{A}{x-1} \cdot (x+1) + B.$

  よって $x=-1$ を代入して, $B = -\dfrac{1}{2}.$

(2) (1) と同様にして, $A = 2$, $B = -1$.

(3) $\dfrac{x^2+1}{x^2(x+1)} = \dfrac{A}{x+1} + \dfrac{B}{x} + \dfrac{C}{x^2}. \quad \cdots (*)$

- $(*)$ の両辺に $x+1$ をかけて, $\dfrac{x^2+1}{x^2} = A + \left(\dfrac{B}{x} + \dfrac{C}{x^2}\right)(x+1).$

  よって $x=-1$ を代入して, $A = 2$.

- $(*)$ の両辺に x^2 をかけて, $\dfrac{x^2+1}{x+1} = \dfrac{A}{x+1} \cdot x^2 + Bx + C.$

  よって $x=0$ を代入して, $C = 1$.

- $(*)$ に $x=1$ を代入して, $1 = \dfrac{A}{2} + B + C = B + 2$. よって $B = -1$.

(4) $\dfrac{x+1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}. \quad \cdots (*)$

- $(*)$ の x をかけて, $\dfrac{x+1}{x^2+1} = A + \dfrac{Bx+C}{x^2+1} \cdot x.$

  よって $x=0$ を代入して, $A = 1$.

- $(*)$ に $x=1$ を代入して, $1 = A + \dfrac{B+C}{2} = \dfrac{1}{2}B + \dfrac{1}{2}C + 1.$

  $(*)$ に $x=-1$ を代入して, $0 = -A + \dfrac{-B+C}{2} = -\dfrac{1}{2}B + \dfrac{1}{2}C - 1.$

  よってこの連立方程式を解いて, $B = -1$, $C = 1$.

問題. 次の不定積分を求めよ (ただし積分定数は C とする).

(1) $\displaystyle\int \frac{1}{(x+1)^2}dx = -\frac{1}{x+1} + C.$

(2) $\displaystyle\int \frac{x+1}{x^2+1}dx = \int \frac{x}{x^2+1}dx + \int \frac{1}{x^2+1}dx = \frac{1}{2}\log(x^2+1) + \arctan x + C.$

(3)
$$\int \frac{1}{x^2-1}dx = \frac{1}{2}\int\left(\frac{1}{x-1} - \frac{1}{x+1}\right)dx$$
$$= \frac{1}{2}\left(\log|x-1| - \log|x+1|\right) + C = \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C.$$

(4)
$$\int \frac{x}{x^2-3x+2}dx = \int\left(\frac{2}{x-2} - \frac{1}{x-1}\right)dx$$
$$= 2\log|x-2| - \log|x-1| + C = \log\frac{(x-2)^2}{|x-1|} + C.$$

(5)
$$\int \frac{x^2+1}{x^2(x+1)}dx = \int\left(\frac{2}{x+1} - \frac{1}{x} + \frac{1}{x^2}\right)dx$$
$$= 2\log|x+1| - \log|x| - \frac{1}{x} + C = \log\frac{(x+1)^2}{|x|} - \frac{1}{x} + C.$$

(6)
$$\int \frac{x+1}{x(x^2+1)}dx = \int\left(\frac{1}{x} - \frac{x}{x^2+1} + \frac{1}{x^2+1}\right)dx$$
$$= \log|x| - \frac{1}{2}\log(x^2+1) - \arctan x + C$$
$$= \log\frac{|x|}{\sqrt{x^2+1}} - \arctan x + C.$$

(7) $\displaystyle\int \tan^2 x dx = \int \left(\frac{1}{\cos^2 x} - 1\right) dx = \tan x - x + C.$

(8) $\displaystyle\int \frac{1}{\tan x} dx = \int \frac{\cos x}{\sin x} dx = \log|\sin x| + C.$

(9) $\tan \frac{x}{2} = t$ とおくと, $dx = \dfrac{2}{1+t^2} dt.$

$$\int \frac{1}{1+\cos x} dx = \int \frac{1}{1+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt = \int 1 dt = t + C = \tan \frac{x}{2} + C.$$

(10) $\tan \frac{x}{2} = t$ とおくと, $dx = \dfrac{2}{1+t^2} dt.$

$$\int \frac{1}{1+\sin x} dx = \int \frac{1}{1+\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2} dt$$

$$= \int \frac{2}{(1+t)^2} dt = -\frac{2}{1+t} + C = -\frac{2}{1+\tan \frac{x}{2}} + C.$$

(11) $\tan x = t$ とおくと, $dx = \dfrac{1}{1+t^2} dt.$

$$\int \frac{1}{\sin^2 x} dx = \int \frac{1}{\frac{t^2}{1+t^2}} \cdot \frac{1}{1+t^2} dt = \int \frac{1}{t^2} dt = -\frac{1}{t} + C = -\frac{1}{\tan x} + C.$$

(12) $\displaystyle\int \sqrt{2x-1}dx = \frac{2}{3}(2x-1)^{\frac{3}{2}} \cdot \frac{1}{2} + C = \frac{1}{3}(2x-1)^{\frac{3}{2}} + C.$

(13) $2x^2 - 1 = t$ とおくと, $4xdx = dt$.

$$\int x\sqrt{2x^2-1}dx = \frac{1}{4}\int \sqrt{t}dt = \frac{1}{4} \cdot \frac{2}{3}t^{\frac{3}{2}} + C = \frac{1}{6}(2x^2-1)^{\frac{3}{2}} + C.$$

(14) $\sqrt{2x-1} = t$ とおくと, $x = \dfrac{t^2+1}{2}$, $dx = tdt$.

$$\begin{aligned}\int \frac{x}{\sqrt{2x-1}}dx &= \int \frac{\frac{t^2+1}{2}}{t} \cdot tdt = \int \left(\frac{1}{2}t^2 + \frac{1}{2}\right)dt \\ &= \frac{1}{6}t^3 + \frac{1}{2}t + C = \frac{1}{6}\sqrt{(2x-1)^3} + \frac{1}{2}\sqrt{2x-1} + C.\end{aligned}$$

(15) $\sqrt{2x-1} = t$ とおくと, $x = \dfrac{t^2+1}{2}$, $dx = tdt$.

$$\begin{aligned}\int x\sqrt{2x-1}dx &= \int \left(\frac{t^2+1}{2} \cdot t\right) \cdot tdt = \int \left(\frac{1}{2}t^4 + \frac{1}{2}t^2\right)dt \\ &= \frac{1}{10}t^5 + \frac{1}{6}t^3 + C = \frac{1}{10}\sqrt{(2x-1)^5} + \frac{1}{6}\sqrt{(2x-1)^3} + C.\end{aligned}$$

(16) $\sqrt{2x-1} = t$ とおくと, $x = \dfrac{t^2+1}{2}$, $dx = tdt$.

$$\begin{aligned}\int \frac{1}{x\sqrt{2x-1}}dx &= \int \frac{1}{\frac{t^2+1}{2} \cdot t} \cdot tdt = \int \frac{2}{1+t^2}dt \\ &= 2\arctan t + C = 2\arctan\sqrt{2x-1} + C.\end{aligned}$$