Comparison of the Contested Garment Principle and the Proportional Distribution Principle: A Simulation

Toshitaka Fukiharu
(Faculty of Economics, Hiroshima University)

March, 2008
October, 2010 (Revised)

Introduction

Suppose that a person dies with property left and the property is valued \$*P*. Furthermore, he (or she) owes to n persons with the total debt, valued \$*D*. When $P<D$, there remains the problem of how the property should be divided between the creditors. Let d_i be the value of debt to the creditor, i (i=1,...,n), so that $D=\sum_{i=1}^{n} d_i$. One of the ways to divide and distribute \$*P* to the creditors is the Proportional Distribution principle: PD principle. According to the PD principle, each creditor, i, receives Pd_i/D (i=1,...,n). The origin of PD principle dates back to the Greek philosopher, Aristotle. This principle is not the only distribution principle. The Contested Garment principle: CG principle, is also known. The example of the distribution of \$*P* to $d=\{d_1,...,d_n\}$ by CG principle was provided in *Mishnah* of the *Talmud*, the collection of Jewish dictums. The extension of this example to the general principle was attempted by Aumann and Maschler [1985]. The aim of this paper is to examine the relation between the PD principle and the CG principle. Especially, constructing the social utility function from the individual utility functions, the simulation approach examines which principle is better.

I. The Contested Garment Principle: CG principle, and the General Formula

The *Talmud* was the collection of Jewish dictums, edited in the 6th century in Babylonia and in the 5th century in Jerusalem. In *Mishnah* of the *Talmud*, an example of the distribution of the property left to the three creditors is provided. Suppose that a person, John, dies with property left and the property is valued \$100. Furthermore, he owes to 3 persons with the total debt, valued \$600. John owes the 1st creditor \$100, the 2nd creditor, \$200, and the 3rd creditor, \$300. *Mishnah* advises the 3 creditors to receive \$100/3 each. When John's property is \$200, *Mishnah* advises the 1st creditor to receive \$50, the 2nd to receive \$75, and the 3rd, \$75. When John's property is \$300, *Mishnah* advises the 1st creditor to receive \$50, the 2nd to receive \$100, and the 3rd, \$150. This example with 3 cases is summarized as in the following table. They are named Case 1 to Case 3.

Table 1

Property / Debt	100	200	300	Case
100	100 / 3	100 / 3	100 / 3	1
200	50	75	75	2
300	50	100	150	3

According to the PD principle, the distribution of the John's property to the 3 creditors is made as in what follows.

Table 2

Property / Debt	100	200	300	Case
100	100 / 6	200 / 6	300 / 6	1 A
200	200 / 6	400 / 6	100	2 A
300	50	100	150	3 A

Aumann and Maschler [1985] showed that the CG principle is a solution of cooperative game. The algorithm for the CG principle is provided in Suzuki [1994]. The CG algorithm, *f*[*x*], is formulated by *Mathematica* as in what follows, where $x=\{P, d_1, d_2, ..., d_n\}$, and $P<D=\sum_{i=1}^{n} d_i$. Thus, when *x* is given arbitrarily, *f*[*x*] computes the distribution of *P* to *n* creditors according to the CG principle.

```
In[1]:= f[x_] := Module[{s}, s = Sum[x[[i]], {i, 2, Length[x]}];
          Which[x[[1]] ≤ (x[[2]] / 2) * (Length[x] - 1),
           Table[x[[1]] / (Length[x] - 1), {i, 1, Length[x] - 1}],
           (x[[2]] / 2) * (Length[x] - 1) < x[[1]] < s - (x[[2]] / 2) * (Length[x] - 1),
           g[1, x], s - (x[[2]] / 2) * (Length[x] - 1) <= x[[1]],
           Table[x[[i]] - (s - x[[1]]) / (Length[x] - 1), {i, 2, Length[x]}]]]

In[2]:= g[k_, x_] := Module[{y, s}, (y = Flatten[
             {x[[1]] - Sum[x[[i]] / 2, {i, 2, k + 1}], Table[x[[i]], {i, k + 2, Length[x]}]}]);
          s = Sum[y[[i]], {i, 2, Length[y]}]; Which[y[[1]] ≤ (y[[2]] / 2) * (Length[y] - 1),
           Flatten[{Table[x[[i]] / 2, {i, 2, k + 1}],
             Table[y[[1]] / (Length[y] - 1), {i, 2, Length[y]}]}],
           (y[[2]] / 2) * (Length[y] - 1) < y[[1]] < s - (y[[2]] / 2) * (Length[y] - 1), g[k + 1, x],
           s - (y[[2]] / 2) * (Length[y] - 1) ≤ y[[1]], Flatten[{Table[x[[i]] / 2, {i, 2, k + 1}],
             Table[y[[i]] - (s - y[[1]]) / (Length[y] - 1), {i, 2, Length[y]}]}]]]
```

First, we ascertain that *f*[*x*] computes the distribution of *P* to *n* creditors in Table 1

```
In[3]:= b0[1] = {100, 100, 200, 300}; b0[2] = {200, 100, 200, 300}; b0[3] = {300, 100, 200, 300};
        {{1, a0[1] = f[b0[1]]}, {2, a0[2] = f[b0[2]]}, {3, a0[3] = f[b0[3]]}}
```

Out[3]= $\left\{\left\{1, \left\{\frac{100}{3}, \frac{100}{3}, \frac{100}{3}\right\}\right\}, \{2, \{50, 75, 75\}\}, \{3, \{50, 100, 150\}\}\right\}$

In Suzuki [1994, p.371], an extension of Table 1 to the 7 cases of 5 creditors is attempted as in the following Table 3. These cases are named Case 4 to Case 10.

Table 3

Property / Debt	100	200	300	400	500	Case
200	40	40	40	40	40	4
250	50	50	50	50	50	5
510	50	100	120	120	120	6
750	50	100	150	200	250	7
1130	50	120	220	320	420	8
1250	50	150	250	350	450	9
1400	80	180	280	380	480	10

We ascertain that *f*[*x*] computes the distribution of *P* to *n* creditors in Table 3.

```
In[4]:= b0[4] = {200, 100, 200, 300, 400, 500}; b0[5] = {250, 100, 200, 300, 400, 500};
        b0[6] = {510, 100, 200, 300, 400, 500}; b0[7] = {750, 100, 200, 300, 400, 500};
        b0[8] = {1130, 100, 200, 300, 400, 500}; b0[9] = {1250, 100, 200, 300, 400, 500};
        b0[10] = {1400, 100, 200, 300, 400, 500}; {{4, a0[4] = f[b0[4]]},
         {5, a0[5] = f[b0[5]]}, {6, a0[6] = f[b0[6]]}, {7, a0[7] = f[b0[7]]},
         {8, a0[8] = f[b0[8]]}, {9, a0[9] = f[b0[9]]}, {10, a0[10] = f[b0[10]]}}

Out[4]= {{4, {40, 40, 40, 40, 40}}, {5, {50, 50, 50, 50, 50}}, {6, {50, 100, 120, 120, 120}},
         {7, {50, 100, 150, 200, 250}}, {8, {50, 120, 220, 320, 420}},
         {9, {50, 150, 250, 350, 450}}, {10, {80, 180, 280, 380, 480}}}
```

In Suzuki [1994, p.375], another extension of Table 1 to the 5 cases of 5 creditors is attempted as in the following Table 4. They are named Case 11 to Case 15.

Table 4

Property / Debt	10	50	100	500	1000	Case
200	5	25	50	60	60	11
500	5	25	50	210	210	12
830	5	25	50	250	500	13
1100	5	25	50	260	760	14
1480	5	25	50	450	950	15

We ascertain that *f*[*x*] computes the distribution of *P* to *n* creditors in Table 4.

```
In[5]:= b0[11] = {200, 10, 50, 100, 500, 1000}; b0[12] = {500, 10, 50, 100, 500, 1000};
        b0[13] = {830, 10, 50, 100, 500, 1000}; b0[14] = {1100, 10, 50, 100, 500, 1000};
        b0[15] = {1480, 10, 50, 100, 500, 1000}; {a0[11] = f[b0[11]],
         a0[12] = f[b0[12]], a0[13] = f[b0[13]], a0[14] = f[b0[14]], a0[15] = f[b0[15]]}

Out[5]= {{5, 25, 50, 60, 60}, {5, 25, 50, 210, 210},
         {5, 25, 50, 250, 500}, {5, 25, 50, 260, 760}, {5, 25, 50, 450, 950}}
```

2. Comparison of the CG Principle and the PD Principle: Risk Averter Case

In this section, the comparison is conducted between the CG principle and the PD principle. In a society where people are risk averters, we examine which principle is more desirable. When people are risk averters, they have utility function of income, where marginal utility decreases. For the purpose of the use in simulation, we assume in this section that every creditor has the same utility function of income, $U[y]$, where y is income:

$$U[y]=y^{1/2} \qquad (1\text{-}1)$$

There are n creditors for John's property left, P, where $P<D=\sum_{i=1}^{n} d_i$. Let r_i be the receipt for the creditor i.Whether the distribution principle is CD principle or PD principle, we have

$$r_i \le d_i \quad (i=1,...,n) \qquad (2)$$

For each creditor i, the *individual* dissatisfaction from this distribution is evaluated by

$$U[r_i]\text{-}U[d_i] \quad (i=1,...,n) \qquad (3)$$

One of the simplest *social* evaluation of the dissatisfaction in this distribution is the sum of each *individual* dissatisfaction:

$$\sum_{i=1}^{n} (U[r_i] - U[d_i]) \le 0 \qquad (4)$$

Given P, and $d=\{d_1,...,d_n\}$, when the CD principle is adopted, the *social* dissatisfaction is computed by

$$C^1[P, d]=\sum_{i=1}^{n} (U[f_i[P, d]] - U[d_i]) \le 0 \qquad (5)$$

where $f[P, d]=\{f_1[P, d], f_2[P, d], ... , f_n[P, d]\}$. Given P, and $d=\{d_1,...,d_n\}$, when the PD principle is adopted, the *social* dissatisfaction is computed by

$$C^2[P, d]=\sum_{i=1}^{n} (U[P d_i / D] - U[d_i]) \le 0 \qquad (6)$$

When $C^1[P, d]>C^2[P, d]$ holds, the CG principle is superior to the PD principle. In section 1, 15 cases were constructed as the distribution according to the CG principle. In what follows, for each case k, $\{k, C^1[P, d], C^2[P, d]\}$ are computed for k (k=1,...,15).

```
In[6]:= gA[x_] := x^(1/2);
        c1[i_] := Table[gA[(a0[i][[j]])] - gA[(b0[i][[j + 1]])], {j, 1, Length[a0[i]]}];
        b1[i_] := Table[(b0[i][[1]] / Sum[b0[i][[j]], {j, 2, Length[b0[i]]}]) * b0[i][[k]],
          {k, 2, Length[b0[i]]}];
        c2[i_] := Table[gA[b1[i][[j]]] - gA[b0[i][[j + 1]]], {j, 1, Length[b0[i]] - 1}];
        dataA = Table[{j, N[{Sum[c1[j][[i]], {i, 1, Length[c1[j]]}],
            Sum[c2[j][[i]], {i, 1, Length[c2[j]]}]}]}, {j, 1, 15}]

Out[6]= {{1, {-24.1421, -24.5356}}, {2, {-17.0711, -17.5242}}, {3, {-12.1441, -12.1441}},
         {4, {-52.2005, -53.2154}}, {5, {-48.468, -49.6026}}, {6, {-33.8889, -34.9463}},
         {7, {-24.5513, -24.5513}}, {8, {-12.583, -11.0691}}, {9, {-8.77193, -7.30345}},
         {10, {-3.32695, -2.8423}}, {11, {-44.4177, -48.4558}}, {12, {-30.9269, -33.485}},
         {13, {-21.7376, -21.7376}}, {14, {-16.2171, -13.8018}}, {15, {-7.87439, -4.13923}}}
```

Out of 15cases, 7 cases exhibit the ones in which the CG principle is superior.

```
In[7]:= Select[dataA, #[[2, 1]] > #[[2, 2]] &]

Out[7]= {{1, {-24.1421, -24.5356}}, {2, {-17.0711, -17.5242}},
         {4, {-52.2005, -53.2154}}, {5, {-48.468, -49.6026}}, {6, {-33.8889, -34.9463}},
         {11, {-44.4177, -48.4558}}, {12, {-30.9269, -33.485}}}
```

Out of 15cases, 3 cases exhibit the ones in which the CG principle is equivalent to the PD principle.

```
In[8]:= Select[dataA, #[[2, 1]] == #[[2, 2]] &]

Out[8]= {{3, {-12.1441, -12.1441}}, {7, {-24.5513, -24.5513}}, {13, {-21.7376, -21.7376}}}
```

Out of 15cases, 5 cases exhibit the ones in which the PD principle is superior.

```
In[9]:= Select[dataA, #[[2, 1]] < #[[2, 2]] &]

Out[9]= {{8, {-12.583, -11.0691}}, {9, {-8.77193, -7.30345}},
         {10, {-3.32695, -2.8423}}, {14, {-16.2171, -13.8018}}, {15, {-7.87439, -4.13923}}}
```

This simulation does not necessarily imply that the CG principle is superior to the PD principle when the creditors are risk-averters. In order to examine the superiority, let us conduct the following simulation. In this simulation, there are 2 creditors. 10000 pairs of d and P, where $d=\{d_1, d_2\}$, $d_1 \leq d_2 \leq 1000000$, $P \leq d_1 + d_2$, are selected randomly. From these pairs, we construct 10000 pairs of $\{C^1[P, d], C^2[P, d]\}$. Out of 10000 pairs, 4996 pairs exhibits the ones in which the CG principle is superior: *i.e.* $C^1[P, d] > C^2[P, d]$.

```
In[10]:= dataB1 = Table[a1 = 2; a2 = Sort[Table[Random[Integer, {1, 1000000}], {a1}]];
            a3 = Random[Integer, {1, Apply[Plus, a2]}]; b = Flatten[{a3, a2}]; a = f[b];
            k1 = Table[gA[(a[[j]])] - gA[(b[[j + 1]])], {j, 1, Length[a]}]; b1[x_] :=
             Table[(x[[1]] / Sum[x[[j]], {j, 2, Length[x]}]) * x[[k]], {k, 2, Length[x]}];
            k2 = Table[gA[b1[b][[j]]] - gA[b[[j + 1]]], {j, 1, Length[b] - 1}];
            N[{Apply[Plus, k1], Apply[Plus, k2]}], {10000}];

In[11]:= Length[Select[dataB1, #[[1]] > #[[2]] &]]

Out[11]= 5026
```

Out of 10000 pairs, nothing exhibit the ones in which the CG principle is equivalent to the PD principle: *i.e.* $C^1[P, d] = C^2[P, d]$.

```
In[12]:= Length[Select[dataB1, #[[1]] == #[[2]] &]]

Out[12]= 0
```

Out of 10000 pairs, 4996 pairs exhibit the ones in which the CG principle is inferior: *i.e.* $C^1[P, d] < C^2[P, d]$.

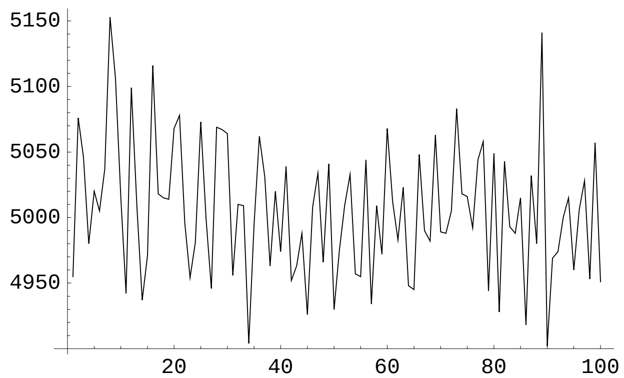
```
In[13]:= Length[Select[dataB1, #[[1]] < #[[2]] &]]

Out[13]= 4974
```

Continuing this procedure 100 times, we obtain the following dataA3, with 100 elements. Each element, $\{m_1, m_0, m_2\}$, is the result of each constructed 10000 pairs, and consists of 3 numbers. The 1st number, m_1, exhibits the number of pairs in which the CG principle is superior: *i.e.* $C^1[P, d] > C^2[P, d]$. The 2nd number, m_0, exhibits the number of pairs in which the CG principle is equivalent to the PD principle: *i.e.* $C^1[P, d] = C^2[P, d]$. Finally, the 3rd number, m_2, exhibits the number of pairs in which the CG principle is inferior: *i.e.* $C^1[P, d] < C^2[P, d]$.

```
In[14]:= dataA3 = {{4955, 0, 5045}, {5076, 0, 4924}, {5046, 0, 4954}, {4980, 0, 5020},
    {5020, 0, 4980}, {5005, 0, 4995}, {5037, 0, 4963}, {5153, 0, 4847}, {5106, 0, 4894},
    {5016, 0, 4984}, {4942, 0, 5058}, {5099, 0, 4901}, {5008, 0, 4992}, {4937, 1, 5062},
    {4971, 0, 5029}, {5116, 0, 4884}, {5018, 0, 4982}, {5015, 0, 4985}, {5014, 0, 4986},
    {5068, 0, 4932}, {5078, 0, 4922}, {4996, 0, 5004}, {4954, 0, 5046}, {4981, 0, 5019},
    {5073, 0, 4927}, {4999, 0, 5001}, {4946, 0, 5054}, {5069, 0, 4931}, {5067, 0, 4933},
    {5064, 0, 4936}, {4956, 0, 5044}, {5010, 0, 4990}, {5009, 0, 4991}, {4904, 0, 5096},
    {4995, 0, 5005}, {5062, 0, 4938}, {5031, 0, 4969}, {4963, 0, 5037}, {5020, 0, 4980},
    {4974, 0, 5026}, {5039, 0, 4961}, {4952, 0, 5048}, {4963, 0, 5037}, {4988, 0, 5012},
    {4926, 0, 5074}, {5008, 0, 4992}, {5034, 0, 4966}, {4966, 0, 5034}, {5041, 0, 4959},
    {4930, 0, 5070}, {4975, 0, 5025}, {5009, 0, 4991}, {5033, 0, 4967}, {4957, 0, 5043},
    {4955, 0, 5045}, {5044, 0, 4956}, {4934, 0, 5066}, {5009, 0, 4991}, {4972, 0, 5028},
    {5068, 0, 4932}, {5012, 0, 4988}, {4983, 0, 5017}, {5023, 0, 4977}, {4948, 0, 5052},
    {4945, 1, 5054}, {5048, 0, 4952}, {4990, 0, 5010}, {4982, 0, 5018}, {5063, 0, 4937},
    {4989, 0, 5011}, {4988, 0, 5012}, {5005, 0, 4995}, {5083, 0, 4917}, {5018, 0, 4982},
    {5016, 0, 4984}, {4992, 0, 5008}, {5044, 0, 4956}, {5058, 0, 4942}, {4944, 0, 5056},
    {5049, 0, 4951}, {4928, 0, 5072}, {5043, 0, 4957}, {4993, 0, 5007}, {4988, 0, 5012},
    {5015, 0, 4985}, {4918, 0, 5082}, {5032, 0, 4968}, {4980, 0, 5020},
    {5141, 0, 4859}, {4902, 0, 5098}, {4969, 0, 5031}, {4974, 0, 5026},
    {5000, 0, 5000}, {5015, 0, 4985}, {4960, 1, 5039}, {5006, 0, 4994},
    {5028, 0, 4972}, {4953, 0, 5047}, {5057, 0, 4943}, {4951, 1, 5048}};
```

The following figure shows the number of pairs in which the CG principle is superior: *i.e.* $C^1[P, d] > C^2[P, d]$, for each session. We may conclude that the average of the number of pairs, in which the CG principle is superior, is 5000.

```
In[15]:= dA3 = Table[dataA3[[i, 1]], {i, 1, Length[dataA3]}]; ListPlot[dA3, PlotJoined → True];
```



Next, let us conduct the following simulation. In this simulation, there are at most 100 creditors. 10000 pairs of d and P, where $d=\{d_1, ..., d_n\}$, $d_1 \leq ... \leq d_n \leq 1000000$, $P \leq d_1 + ... + d_n$, $n \leq 100$, are selected randomly. From these pairs, we construct 10000 pairs of $\{C^1[P, d], C^2[P, d]\}$. Out of 10000 pairs, 4949 pairs exhibit the ones in which the CG principle is superior: *i.e.* $C^1[P, d] > C^2[P, d]$, 111 pairs exhibit the ones in which the CG principle is equivalent to the PD principle: *i.e.* $C^1[P, d] = C^2[P, d]$, and finally 4940 pairs exhibit the ones in which the CG principle is inferior: *i.e.* $C^1[P, d] < C^2[P, d]$.

```
In[16]:= Timing[dataB3 = Table[a1 = Random[Integer, {1, 100}];
            a2 = Sort[Table[Random[Integer, {1, 1000000}], {a1}]];
            a3 = Random[Integer, {1, Apply[Plus, a2]}]; b = Flatten[{a3, a2}]; a = f[b];
            k1 = Table[gA[(a[[j]])] - gA[(b[[j + 1]])], {j, 1, Length[a]}]; b1[x_] :=
             Table[(x[[1]] / Sum[x[[j]], {j, 2, Length[x]}]) * x[[k]], {k, 2, Length[x]}];
            k2 = Table[gA[b1[b][[j]]] - gA[b[[j + 1]]], {j, 1, Length[b] - 1}];
            N[{Apply[Plus, k1], Apply[Plus, k2]}], {10000}];
          {Length[Select[dataB3, #[[1]] > #[[2]] &]],
           Length[Select[dataB3, #[[1]] == #[[2]] &]],
           Length[Select[dataB3, #[[1]] < #[[2]] &]]}]

General::spell1 :
 Possible spelling error: new symbol name "dataB3" is similar to existing symbol "dataA3". More…

Out[16]= {2249.22 Second, {4872, 104, 5024}}
```

The second session produces the similar data to the 1st session.

```
In[17]:= Timing[dataB3 = Table[a1 = Random[Integer, {1, 100}];
            a2 = Sort[Table[Random[Integer, {1, 1000000}], {a1}]];
            a3 = Random[Integer, {1, Apply[Plus, a2]}]; b = Flatten[{a3, a2}]; a = f[b];
            k1 = Table[gA[(a[[j]])] - gA[(b[[j + 1]])], {j, 1, Length[a]}]; b1[x_] :=
             Table[(x[[1]] / Sum[x[[j]], {j, 2, Length[x]}]) * x[[k]], {k, 2, Length[x]}];
            k2 = Table[gA[b1[b][[j]]] - gA[b[[j + 1]]], {j, 1, Length[b] - 1}];
            N[{Apply[Plus, k1], Apply[Plus, k2]}], {10000}];
          {Length[Select[dataB3, #[[1]] > #[[2]] &]],
           Length[Select[dataB3, #[[1]] == #[[2]] &]],
           Length[Select[dataB3, #[[1]] < #[[2]] &]]}]

Out[17]= {2280.02 Second, {4909, 93, 4998}}
```

Continuing this procedure 100 times, we obtain the similar result to the 2 creditors case. We may conclude that the average of the number of pairs, in which the CG principle is superior, is 5000.

3. Comparison of the CG Principle and the PD Principle: Risk Lover Case

In this section, the comparison is conducted between the CG principle and the PD principle in a society where people are risk lovers, examining which principle is more desirable. When people are risk lovers, they have utility function of income, where marginal utility increases. For the purpose of the use in simulation, we assume in this section that every creditor has the same utility function of income, $U[y]$, where y is income:

$$U[y]=y^2 \qquad (1\text{-}2)$$

There are n creditors for John's property left, P, where $P<D=\sum_{i=1}^{n} d_i$. Let r_i be the receipt for the creditor i. Whether the distribution principle is CD principle or PD principle, we have (2). For each creditor i, the *individual* dissatisfaction from this distribution is evaluated by (3). One of the simplest *social* evaluation of the dissatisfaction in this distribution is (4): the sum of each *individual* dissatisfaction. Given P, and $d=\{d_1,...,d_n\}$, when the CD principle is adopted, the *social* dissatisfaction is computed by (5). Given P, and $d=\{d_1,...,d_n\}$, when the PD principle is adopted, the *social* dissatisfaction is computed by (6). When $C^1[P, d]>C^2[P, d]$ holds, the CG principle is superior to the PD principle. In section 1, 15 cases were constructed as the distribution according to the CG principle. In what follows, for each case k, $\{k, C^1[P, d], C^2[P, d]\}$ are computed for k (k=1,...,15).

```
In[18]:= gB[x_] := x^2;
         c1[i_] := Table[gB[(a0[i][[j]])] - gB[(b0[i][[j + 1]])], {j, 1, Length[a0[i]]}];
         b1[i_] := Table[(b0[i][[1]] / Sum[b0[i][[j]], {j, 2, Length[b0[i]]}]) * b0[i][[k]],
            {k, 2, Length[b0[i]]}];
         c2[i_] := Table[gB[b1[i][[j]]] - gB[b0[i][[j + 1]]], {j, 1, Length[b0[i]] - 1}];
         dataA = Table[{j, N[{Sum[c1[j][[i]], {i, 1, Length[c1[j]]}],
              Sum[c2[j][[i]], {i, 1, Length[c2[j]]}]}]}, {j, 1, 15}]

Out[22]= {{1, {-136667., -136111.}}, {2, {-126250., -124444.}}, {3, {-105000., -105000.}},
          {4, {-542000., -540222.}}, {5, {-537500., -534722.}}, {6, {-494300., -486420.}},
          {7, {-412500., -412500.}}, {8, {-205900., -237869.}}, {9, {-137500., -168056.}},
          {10, {-58000., -70888.9}}, {11, {-1.25225×10^6, -1.24427×10^6}},
          {12, {-1.17125×10^6, -1.14805×10^6}}, {13, {-946950., -946950.}},
          {14, {-614250., -708185.}}, {15, {-154450., -258971.}}}
```

Out of 15cases, 5 cases exhibit the ones in which the CG principle is superior when the creditors are risk-lovers. As is expected, these cases are those which exhibit the ones in which the CG principle is inferior when the creditors are risk-averters.

```
In[23]:= Select[dataA, #[[2, 1]] > #[[2, 2]] &]

Out[23]= {{8, {-205900., -237869.}}, {9, {-137500., -168056.}},
          {10, {-58000., -70888.9}}, {14, {-614250., -708185.}}, {15, {-154450., -258971.}}}
```

Out of 15cases, 3 cases exhibit the ones in which the CG principle is equivalent to the PD principle.

```
In[24]:= Select[dataA, #[[2, 1]] == #[[2, 2]] &]

Out[24]= {{3, {-105000., -105000.}}, {7, {-412500., -412500.}}, {13, {-946950., -946950.}}}
```

Out of 15cases, 7 cases exhibit the ones in which the PD principle is superior when the creditors are risk-lovers. As is expected, these cases are those which exhibit the ones in which the CG principle is superior when the creditors are risk-averters.

```
In[25]:= Select[dataA, #[[2, 1]] < #[[2, 2]] &]

Out[25]= {{1, {-136667., -136111.}}, {2, {-126250., -124444.}},
          {4, {-542000., -540222.}}, {5, {-537500., -534722.}}, {6, {-494300., -486420.}},
          {11, {-1.25225×10^6, -1.24427×10^6}}, {12, {-1.17125×10^6, -1.14805×10^6}}}
```

This simulation does not necessarily imply that the PD principle is superior to the CG principle when the creditors are risk-lovers. In order to examine the superiority, let us conduct the following simulation. In this simulation, there are 2 creditors. 10000 pairs of d and P, where $d=\{d_1, d_2\}$, $d_1 \leq d_2 \leq 1000000$, $P \leq d_1 + d_2$, are selected randomly. From these pairs, we construct 10000 pairs of $\{C^1[P, d], C^2[P, d]\}$. Out of 10000 pairs, 4957 pairs exhibits the ones in which the CG principle is superior: *i.e.* $C^1[P, d] > C^2[P, d]$.

```
In[26]:= dataB = Table[a1 = 2; a2 = Sort[Table[Random[Integer, {1, 1000000}], {a1}]];
            a3 = Random[Integer, {1, Apply[Plus, a2]}]; b = Flatten[{a3, a2}]; a = f[b];
            k1 = Table[gB[(a[[j]])] - gB[(b[[j + 1]])], {j, 1, Length[a]}]; b1[x_] :=
             Table[(x[[1]] / Sum[x[[j]], {j, 2, Length[x]}]) * x[[k]], {k, 2, Length[x]}];
            k2 = Table[gB[b1[b][[j]]] - gB[b[[j + 1]]], {j, 1, Length[b] - 1}];
            N[{Apply[Plus, k1], Apply[Plus, k2]}], {10000}];

     General::spell1 :
      Possible spelling error: new symbol name "dataB" is similar to existing symbol "dataA". More…

In[27]:= Length[Select[dataB, #[[1]] > #[[2]] &]]

Out[27]= 4940

In[28]:= Length[Select[dataB, #[[1]] == #[[2]] &]]

Out[28]= 0

In[29]:= Length[Select[dataB, #[[1]] < #[[2]] &]]

Out[29]= 5060
```

In exactly the sam way as in section 2, we conduct the following simulations. In the first simulation, we repeat the same session 100times. In the second simulation, there are at most 100 creditors. 10000 pairs of d and P, where $d=\{d_1, ..., d_n\}$, $d_1 \leq ... \leq d_n \leq 1000000$, $P \leq d_1 + ... + d_n$, $n \leq 100$, are selected randomly. From these pairs, we construct 10000 pairs of $\{C^1[P, d], C^2[P, d]\}$. Essentially the same results obtain for the two simulation as in section 2.

4. Comparison of the CG Principle and the PD Principle: Risk- Averter Case with Additional Assumption

A common assumption is adopted in all the examples in section 1. It is that

$$d_1 = \text{Min}[d_1, ... d_n] \leq P \qquad (7)$$

In this section, adding (7) as an assumption, we conduct the same simulation for the risk-averter case.

When n=2, first, selecting 1000 pair of random values from [1,1000000], $\{P, d_1, d_2\}$, such that $d_1 \leq d_2$, $P \leq d_1 + d_2$, we derive $\{m^1, m^0, m^2\}$, $m^1 + m^0 + m^2 = 1000$, in which, m^1 is the number of cases of "CG is better than PD": $0 \geq C^2[P, d] > C^1[P, d]$, m^0 is the number of cases of "CG is indifferent to PD": $0 \geq C^2[P, d] = C^1[P, d]$, and m^2 is the number of cases of "CG is worse than PD": $0 \geq C^1[P, d] > C^2[P, d]$. The following data is the result of 100 such simulations.

```
In[30]:= g[x_] := x^(1 / 2)
```

```
In[31]:= dataBB =
   Table[dataB = Table[a1 = 2; a2 = Sort[Table[Random[Integer, {1, 1000000}], {a1}]];
      a3 = Random[Integer, {Min[a2], Apply[Plus, a2]}]; b = Flatten[{a3, a2}];
      a = f[b]; k1 = Table[g[(a[[j]])] - g[(b[[j + 1]])], {j, 1, Length[a]}]; b1[x_] :=
       Table[(x[[1]] / Sum[x[[j]], {j, 2, Length[x]}]) * x[[k]], {k, 2, Length[x]}];
      k2 = Table[g[b1[b][[j]]] - g[b[[j + 1]]], {j, 1, Length[b] - 1}];
      N[{Apply[Plus, k1], Apply[Plus, k2]}], {1000}];
    {Length[Select[dataB, #[[1]] > #[[2]] &]],
     Length[Select[dataB, #[[1]] == #[[2]] &]],
     Length[Select[dataB, #[[1]] < #[[2]] &]]}, {100}]

General::spell1 :
 Possible spelling error: new symbol name "dataBB" is similar to existing symbol "dataB". More…

Out[31]= {{234, 0, 766}, {235, 0, 765}, {281, 0, 719}, {234, 0, 766}, {270, 0, 730},
  {237, 0, 763}, {259, 0, 741}, {245, 0, 755}, {249, 0, 751}, {228, 0, 772},
  {252, 0, 748}, {252, 0, 748}, {234, 0, 766}, {281, 0, 719}, {244, 0, 756},
  {255, 0, 745}, {228, 0, 772}, {256, 0, 744}, {237, 0, 763}, {268, 0, 732},
  {250, 0, 750}, {262, 0, 738}, {265, 0, 735}, {258, 0, 742}, {253, 0, 747},
  {276, 0, 724}, {254, 0, 746}, {260, 0, 740}, {254, 0, 746}, {259, 0, 741},
  {235, 0, 765}, {292, 0, 708}, {231, 0, 769}, {232, 0, 768}, {246, 0, 754},
  {250, 0, 750}, {276, 0, 724}, {245, 0, 755}, {240, 0, 760}, {272, 0, 728},
  {230, 0, 770}, {243, 0, 757}, {237, 0, 763}, {252, 0, 748}, {258, 0, 742},
  {247, 0, 753}, {224, 0, 776}, {235, 0, 765}, {256, 0, 744}, {235, 0, 765},
  {272, 0, 728}, {262, 0, 738}, {259, 0, 741}, {249, 0, 751}, {261, 0, 739},
  {232, 0, 768}, {233, 0, 767}, {256, 0, 744}, {232, 0, 768}, {253, 0, 747},
  {261, 0, 739}, {240, 0, 760}, {256, 0, 744}, {239, 0, 761}, {282, 0, 718},
  {256, 0, 744}, {272, 0, 728}, {247, 0, 753}, {233, 0, 767}, {235, 0, 765},
  {243, 0, 757}, {253, 0, 747}, {262, 0, 738}, {256, 0, 744}, {264, 0, 736},
  {252, 0, 748}, {230, 0, 770}, {250, 0, 750}, {235, 0, 765}, {257, 0, 743},
  {252, 0, 748}, {256, 0, 744}, {224, 0, 776}, {248, 0, 752}, {259, 0, 741},
  {245, 0, 755}, {256, 0, 744}, {253, 0, 747}, {274, 0, 726}, {234, 0, 766},
  {264, 0, 736}, {272, 0, 728}, {249, 0, 751}, {249, 0, 751}, {248, 0, 752},
  {253, 0, 747}, {233, 0, 767}, {261, 0, 739}, {216, 0, 784}, {246, 0, 754}}
```

Mean of 1000 pairs of m^1s is approximately, AC[2], 250. In other words, the probability of "CG is better than PD", AC[2]/1000, is approximately 25%.

```
In[32]:= AB[2] = Table[dataBB[[i, 1]], {i, 1, Length[dataBB[2]]}]; AC[2] = N[Mean[AB[2]]]

Out[32]= 234.
```

When n=3, first, selecting 1000 pair of random values from [1,1000000], $\{P, d_1, d_2, d_3\}$, such that $d_1 \leq d_2, d_3$, $P \leq d_1+d_2+d_3$, we derive $\{m^1, m^0, m^2\}$, $m^1+m^0+m^2$=1000, in which, m^1 is the number of cases of "CG is better than PD": $0 \geq C^2[P, d] > C^1[P, d]$, m^0 is the number of cases of "CG is indifferent to PD": $0 \geq C^2[P, d] = C^1[P, d]$, and m^2 is the number of cases of "CG is worse than PD": $0 \geq C^1[P, d] > C^2[P, d]$. Mean of 1000 pairs of m^1s, AC[3], is approximately 400. In other words, the probability of "CG is better than PD", AC[3]/1000 is approximately 40%.

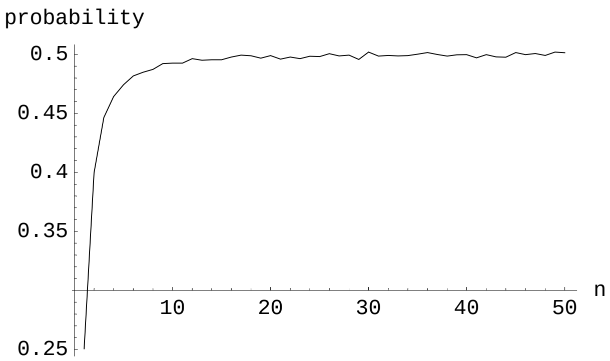
```
In[33]:= dataBB =
  Table[dataB = Table[a1 = 3; a2 = Sort[Table[Random[Integer, {1, 1000000}], {a1}]];
     a3 = Random[Integer, {Min[a2], Apply[Plus, a2]}]; b = Flatten[{a3, a2}];
     a = f[b]; k1 = Table[g[(a[[j]])] - g[(b[[j + 1]])], {j, 1, Length[a]}]; b1[x_] :=
      Table[(x[[1]] / Sum[x[[j]], {j, 2, Length[x]}]) * x[[k]], {k, 2, Length[x]}];
     k2 = Table[g[b1[b][[j]]] - g[b[[j + 1]]], {j, 1, Length[b] - 1}];
     N[{Apply[Plus, k1], Apply[Plus, k2]}], {1000}];
   {Length[Select[dataB, #[[1]] > #[[2]] &]],
    Length[Select[dataB, #[[1]] == #[[2]] &]],
    Length[Select[dataB, #[[1]] < #[[2]] &]]}, {100}]

Out[33]= {{422, 0, 578}, {388, 0, 612}, {422, 0, 578}, {407, 0, 593}, {412, 0, 588},
  {390, 0, 610}, {391, 0, 609}, {406, 0, 594}, {408, 0, 592}, {370, 0, 630},
  {412, 0, 588}, {391, 0, 609}, {396, 0, 604}, {393, 0, 607}, {428, 0, 572},
  {393, 0, 607}, {425, 0, 575}, {422, 0, 578}, {386, 0, 614}, {394, 0, 606},
  {389, 0, 611}, {407, 0, 593}, {423, 0, 577}, {404, 0, 596}, {405, 0, 595},
  {416, 0, 584}, {411, 0, 589}, {383, 0, 617}, {418, 0, 582}, {389, 0, 611},
  {390, 0, 610}, {406, 0, 594}, {425, 0, 575}, {410, 0, 590}, {367, 0, 633},
  {382, 0, 618}, {415, 0, 585}, {421, 0, 579}, {389, 0, 611}, {403, 0, 597},
  {422, 0, 578}, {410, 0, 590}, {403, 0, 597}, {389, 0, 611}, {405, 0, 595},
  {392, 0, 608}, {391, 0, 609}, {432, 0, 568}, {414, 0, 586}, {384, 0, 616},
  {398, 0, 602}, {433, 0, 567}, {383, 0, 617}, {402, 0, 598}, {395, 0, 605},
  {417, 0, 583}, {417, 0, 583}, {402, 0, 598}, {394, 0, 606}, {413, 0, 587},
  {425, 0, 575}, {405, 0, 595}, {417, 0, 583}, {384, 0, 616}, {403, 0, 597},
  {401, 0, 599}, {369, 0, 631}, {404, 0, 596}, {390, 0, 610}, {412, 0, 588},
  {363, 0, 637}, {399, 0, 601}, {393, 0, 607}, {405, 0, 595}, {403, 0, 597},
  {415, 0, 585}, {436, 0, 564}, {392, 0, 608}, {397, 0, 603}, {406, 0, 594},
  {412, 0, 588}, {430, 0, 570}, {407, 0, 593}, {402, 0, 598}, {407, 0, 593},
  {417, 0, 583}, {392, 0, 608}, {405, 0, 595}, {396, 0, 604}, {423, 0, 577},
  {393, 0, 607}, {386, 0, 614}, {411, 0, 589}, {391, 0, 609}, {409, 0, 591},
  {393, 0, 607}, {393, 0, 607}, {398, 0, 602}, {406, 0, 594}, {401, 0, 599}}

In[34]:= AB[3] = Table[dataBB[[i, 1]], {i, 1, Length[dataBB]}]; AC[3] = N[Mean[AB[3]]]

Out[34]= 402.96
```

We continue this process for $n \le 50$. First, selecting 1000 pair of random values from [1,1000000], $\{P, d_1, d_2, ..., d_n\}$, such that $d_1 \le d_2, ..., d_n$, $P \le d_1 + ... + d_n$, we derive $\{m^1, m^0, m^2\}$, $m^1+m^0+m^2=1000$, in which, m^1 is the number of cases of "CG is better than PD": $0 \ge C^2[P, d] > C^1[P, d]$, m^0 is the number of cases of "CG is indifferent to PD": $0 \ge C^2[P, d] = C^1[P, d]$, and m^2 is the number of cases of "CG is worse than PD": $0 \ge C^1[P, d] > C^2[P, d]$. Let us define AC[$n$]=$m^1$. In this way, we collect, AC[n], {n=1, ...,51}. The following data, dataMEAN is the collection of AC[n]/1000. {n=1, ...,51}

```
In[135]:=
  dataMEAN = Table[AC[i] / 1000, {i, 2, 51}]

Out[135]=
  {0.2506, 0.39965, 0.44638, 0.46413, 0.47416, 0.4817, 0.48477, 0.48724, 0.492,
   0.49255, 0.49251, 0.49628, 0.49494, 0.49536, 0.49524, 0.49768, 0.49926,
   0.4987, 0.49672, 0.49888, 0.4959, 0.49772, 0.4963, 0.4982, 0.49815, 0.50058,
   0.49848, 0.49929, 0.4956, 0.50185, 0.49846, 0.49893, 0.4986, 0.49888,
   0.50003, 0.50155, 0.49982, 0.49844, 0.49958, 0.49968, 0.49702, 0.49962,
   0.4978, 0.49762, 0.50148, 0.49961, 0.50064, 0.49891, 0.50175, 0.50135}
```

In[136]:=

```
ListPlot[dataMEAN, PlotJoined → True,
  PlotRange → All, AxesLabel → {"n", "probability"}];
```



Conclusion

The aim of this paper is to compare the Proportional Distribution principle: PD principle and the Contested Garment principle: CG principle, from the viewpoint of desirability in terms of social utility function. The two principles can be utilized in dividing and distributing the property left, *P*, to the creditors, where d_i is the value of debt to the creditor, *i* (*i*=1,..., *n*) and $d=\{d_1,...,d_n\}$. In Section 1, the algorithm according to the CG principle, advocated in *Mishnah* of the *Talmud*, the collection of Jewish dictums, is programmed in terms of *Mathematica.*

In Section 2, we assume that all the creditors have ha same risk-averter type utility function of income. Defining the social utility function by the sum of utility difference between the receipt and the credit. Note that the social utility is always non-positive. If the social utility according to one of principles is greater than the other, then, the former principle is superior by definition. In this section, two simulations are conducted. In the 1st simulation, there are 2 creditors. 10000 pairs of *P* and *d* are selected randomly. From these pairs, we construct 10000 pairs of social utilities according to the 2 principles. We conclude that 50% exhibit that the CG principle is superior to the PD principle, while remaining 50% exhibit that the PD principle is superior to the CD principle. In the 2nd simulation, there are at most 100 creditors. Applying the same procedure we conclude that 50% exhibit that the CG principle is superior to the PD principle, while the remaining 50% exhibit that the PD principle is superior to the CD principle.

In Section 3, we assume that all the creditors have ha same risk-lover type utility function of income. Applying the same procedure as in Section 2 we conclude that 50% exhibit that the CG principle is superior to the PD principle, while the remaining 50% exhibit that the PD principle is superior to the CD principle. Thus, we conclude that there is no superiority of one principle to the other, rather they are equivalent from the social viewpoint.

In Section 4, an additional assumption is made: min $d \leq P$, in other words, the property left, *P*, is sufficient to satisfy at least one creditor. With this additional assumption, the same simulations are conducted as in Section 2: the risk-averter case. The conclusion of simulations in this section is as follows. When there are two creditors, the probability of "CD principle is superior" is far less than 50%, around 25%. As the number of creditors increases, the probability of "CD principle is superior" rises, however, probably less than 50%.

The conclusions in this paper may explain the history of the adoption of PD principle in many societies, including Japan, since the PD principle is much simpler in the computation of receipts for the creditors and the probability of "CD principle is socially superior" is probably less than 50% when the creditors are risk-averters.

References

1. Aumann, R. J. and M. Maschler (1985), "Game Theoretic Analysis of a Bankruptcy Problem from the Talmud", *Journal of Economic Theory*, 36,195-213.

2. Funaki, Y. (2000), "Bankruptcy Problem", in *Thinking in Game Theory*, M. Nakayama, S. Muto, Y. Funaki, eds. (in Japanese), Yuhikaku.

3. Suzuki, M. (1994), *The New Game Theory* (in Japanese), Keiso-Shobo.