Sharpness of the Korányi approach region

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Abstract

We prove a Littlewood type theorem which shows the sharpness of the Korányi approach region for the boundary behavior of Poisson-Szegö integrals on the unit ball of $\mathbb{C}^n$. Our result is stronger than Hakim and Sibony [3].

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1 Introduction

Let $\mathbb{C}^n$ be the n-dimensional complex space with inner product $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w}_j$, where $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$, and the associated norm $|z| = \sqrt{\langle z, z\rangle}$. We denote by B the unit ball of $\mathbb{C}^n$ and by S its boundary. Let σ be the normalized surface measure on S. For an integrable function f on S, the Poisson-Szegö integral of f is defined by

$$\mathcal{P}[f](z) = \int_S \frac{(1-|z|^2)^n}{|1-\langle z, \zeta\rangle|^{2n}} f(\zeta)\, d\sigma(\zeta) \qquad \text{for } z \in B.$$

In [4], Korányi investigated the boundary behavior of Poisson-Szegö integrals. For $\alpha > 1$ and $\xi \in S$, the Korányi approach region at ξ is given by

$$\mathcal{A}_\alpha(\xi) = \left\{ z \in B : |1 - \langle z, \xi\rangle| < \frac{\alpha}{2}(1-|z|^2) \right\}.$$

Theorem A. *Let $\alpha > 1$. If f is an integrable function on S, then the Poisson-Szegö integral $\mathcal{P}[f](z)$ has the limit $f(\xi)$ as $z \to \xi$ within $\mathcal{A}_\alpha(\xi)$ at almost every point ξ of S.*

When $n = 1$, this theorem is well-known as Fatou's theorem. In this case, $\mathcal{A}_\alpha(\xi)$ is a non-tangential approach region at ξ. The best possibility of this approach region was firstly proved by Littlewood [5] in the following sense: Let C_0 be a tangential curve in the unit disc D which ends at $z = 1$, and let C_θ be the curve C_0 rotated about the origin through an angle θ, so that C_θ touches the unit circle internally at $e^{i\theta}$. Then there exists a bounded harmonic function on D which admits no limits as $z \to e^{i\theta}$ along C_θ for almost every θ, $0 \le \theta \le 2\pi$. Aikawa [1] improved this result by showing that there exists a bounded harmonic function on D which admits no limit as $z \to e^{i\theta}$ along C_θ for *every* θ.

In [6], Nagel and Stein proved that the Poisson integral on the upper half space of $\mathbb{R}^{n+1}$ has the boundary limit at almost every point of $\mathbb{R}^n$ within a certain approach region which is not contained in any non-tangential approach regions. Sueiro [8] extended Nagel and Stein's result to $\mathbb{C}^n$ and proved that the Poisson-Szegö integral has the boundary limit at almost every point of S within a certain approach region which is not contained in any Korányi approach regions.

The purpose of the present paper is to prove a Littlewood type theorem in higher dimensions. Let γ be a curve in B which ends at $e_1 = (1, 0, \cdots, 0)$ and satisfies

$$\lim_{\substack{z \to e_1 \\ z \in \gamma}} \frac{|1 - \langle z, e_1 \rangle|}{1 - |z|^2} = \infty. \tag{1.1}$$

This means that, for each $\alpha > 1$, points of γ near e_1 lie outside $\mathcal{A}_\alpha(e_1)$. Let $\mathcal{U}$ denote the group of unitary transformations of $\mathbb{C}^n$. We write $U\gamma$ for the image of γ through $U \in \mathcal{U}$. Since U preserves inner products, $U\gamma$ touches S internally at Ue_1 and lies outside $\mathcal{A}_\alpha(Ue_1)$ near Ue_1 for every $\alpha > 1$.

Our main result is as follows.

Theorem. *Let γ be a curve in B which ends at e_1 and satisfies* (1.1). *Then there exists a bounded function f on S of which Poisson-Szegö integral $\mathcal{P}[f](z)$ admits no limits as $|z| \to 1$ along $U\gamma$ for every $U \in \mathcal{U}$, that is,*

$$\liminf_{\substack{|z| \to 1 \\ z \in U\gamma}} \mathcal{P}[f](z) \neq \limsup_{\substack{|z| \to 1 \\ z \in U\gamma}} \mathcal{P}[f](z) \qquad \textit{for every } U \in \mathcal{U}.$$

Remark 1. Since $\mathcal{U}$ acts transitively on S, for each $\xi \in S$ there is $U_\xi \in \mathcal{U}$ such that $\xi = U_\xi e_1$. Therefore, Theorem implies that there exists a bounded Poisson-Szegö integral which admits no limits as $z \to \xi$ along $U_\xi \gamma$ at *every* point ξ of S. Moreover, we can make f satisfy

$$\liminf_{\substack{|z| \to 1 \\ z \in U\gamma}} \mathcal{P}[f](z) = \inf_{\zeta \in S} f(\zeta) \quad \text{and} \quad \limsup_{\substack{|z| \to 1 \\ z \in U\gamma}} \mathcal{P}[f](z) = \sup_{\zeta \in S} f(\zeta)$$

for every $U \in \mathcal{U}$.

Remark 2. By Sueiro's result, the limit in (1.1) can not be replaced by the upper limit.

As a related topic in higher dimensions, there is the following result due to Hakim and Sibony [3].

Theorem B. *Suppose $n > 1$. Let $\alpha > 1$ and $h : (0, 1] \to [\alpha, \infty)$ be a decreasing function such that*

$$\lim_{x \to 0+} h(x) = \infty,$$

and let

$$D_{\alpha,h}(\xi) = \left\{ z \in B : \begin{array}{l} |1 - \langle z, \xi \rangle| \le \alpha(1 - |\langle z, \xi \rangle|) \ \textit{and} \\ |1 - \langle z, \xi \rangle| \le h(|1 - \langle z, \xi \rangle|)(1 - |z|) \end{array} \right\}.$$

Then there exists a bounded holomorphic function on B which admits no limits as $z \to \xi$ within $D_{\alpha,h}(\xi)$ at almost every point ξ of S.

We notice that the approach region $D_{\alpha,h}(\xi)$ is wider than any Korányi approach regions in the complex tangential directions, but is the same in the special real direction. Our theorem is stronger than Theorem B in the following points:

- It improves no convergence "almost everywhere" to "everywhere".
- It establishes that a tangential approach in the special real direction can not be allowed in Theorem A.
- The existence of a bounded Poisson-Szegö integral which fails to have a boundary limit is ensured even if we replace $D_{\alpha,h}(e_1)$ by much smaller curve γ satisfying (1.1).

Also, our method is different from Hakim and Sibony's. Theorem B is proved by constructing a higher dimensional Blaschke product. However, we will prove Theorem in Section 3 by constructing a bounded function on S and using lower and upper estimates of Poisson-Szegö integrals in Section 2. In the proofs we adapt ideas from [1, 2]. Whereas the polar and the euclidean coordinates were used to construct a bounded function on the unit circle and on $\mathbb{R}^n$ in [1, 2], they are not applicable in our case. This is an important difference between [1, 2] and our case.

Throughout the paper we use the symbols $A_0, A_1, A_2, \cdots$ to denote absolute positive constants depending only on the dimension n.

2 Estimates of Poisson-Szegö integrals

In this section we give lower and upper estimates for Poisson-Szegö integrals. To this end, we start with introducing a non-isotropic ball in S. We observe that the function $d(z, w) = |1 - \langle z, w\rangle|^{1/2}$ satisfies the triangle inequality on $B \cup S$, and defines a metric on S. See [7, Lemma 7.3]. For $\xi \in S$ and $r > 0$, we write $Q(\xi, r) = \{\zeta \in S : d(\zeta, \xi) < r\}$, the non-isotropic ball of center ξ and radius r. Note that, to emphasize the metric d, we use the slightly different definition from Stoll's book. We observe that $\sigma(Q(U\xi, r)) = \sigma(Q(\xi, r))$ for any unitary transformations U and that

$$\lim_{r\to 0} \frac{\sigma(Q(\xi, r))}{r^{2n}} = \frac{2^n}{4\sqrt{\pi}} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2}+1)} \tag{2.1}$$

See [7, p. 84]. Moreover, there is a constant $A_0 > 1$ depending only on the dimension n such that

$$A_0^{-1} r^{2n} \le \sigma(Q(\xi, r)) \le A_0 r^{2n} \tag{2.2}$$

for $\xi \in S$ and $0 \le r \le \operatorname{diam} S = \sqrt{2}$. Here $\operatorname{diam} F = \sup\{d(\eta, \zeta) : \eta, \zeta \in F\}$ for $F \subset S$.

Let $T > 0$ and $\xi \in S$. For an integrable function g on S, we define the truncated maximal function at ξ by

$$\mathcal{M}_T[g](\xi) = \sup_{r\ge T} r^{-2n} \int_{Q(\xi,r)} |g(\zeta)|\, d\sigma(\zeta).$$

By the argument in [7, Theorem 7.8], we obtain the following estimate for the Poisson-Szegö integral. For completeness we give the proof.

Lemma 1. *There exists a positive constant A_1 depending only on the dimension n such that if g is an integrable function on S and $C > 0$, then*

$$|\mathcal{P}[g](t\xi)| \leq A_1\left((1-t)^{-n}\int_{Q(\xi, C\sqrt{1-t})}|g(\zeta)|\,d\sigma(\zeta) + C^{-2n}\mathcal{M}_{C\sqrt{1-t}}[g](\xi)\right)$$

for $\xi \in S$ and $0 < t < 1$.

Proof. Let $\xi \in S$ and $0 < t < 1$ be fixed, and let

$$\begin{aligned} &V_0 = Q(\xi, C\sqrt{1-t}),\\ &V_j = Q(\xi, 2^jC\sqrt{1-t}) \setminus Q(\xi, 2^{j-1}C\sqrt{1-t}) \qquad (j = 1, \cdots, N), \end{aligned}$$

where N is the smallest integer such that $2^NC\sqrt{1-t} > \sqrt{2}$. Then

$$|\mathcal{P}[g](t\xi)| \leq \sum_{j=0}^{N}\int_{V_j}\frac{(1-t^2)^n}{|1-\langle t\xi, \zeta\rangle|^{2n}}|g(\zeta)|\,d\sigma(\zeta).$$

Since $|1 - \langle t\xi, \zeta\rangle| \geq 1 - t$ for $\zeta \in S$, it follows that

$$\int_{V_0}\frac{(1-t^2)^n}{|1-\langle t\xi, \zeta\rangle|^{2n}}|g(\zeta)|\,d\sigma(\zeta) \leq \frac{2^n}{(1-t)^n}\int_{Q(\xi, C\sqrt{1-t})}|g(\zeta)|\,d\sigma(\zeta).$$

Let $j = 1, \cdots, N$. By the triangle inequality, we have for $\zeta \in V_j$,

$$2^{j-1}C\sqrt{1-t} \leq d(\xi, \zeta) \leq d(\xi, t\xi) + d(t\xi, \zeta) \leq 2d(t\xi, \zeta) = 2|1 - \langle t\xi, \zeta\rangle|^{1/2}.$$

Hence it follows that

$$\begin{aligned} \int_{V_j}\frac{(1-t^2)^n}{|1-\langle t\xi, \zeta\rangle|^{2n}}|g(\zeta)|\,d\sigma(\zeta) &\leq \frac{2^{9n}}{2^{4nj}C^{4n}(1-t)^n}\int_{Q(\xi, 2^jC\sqrt{1-t})}|g(\zeta)|\,d\sigma(\zeta)\\ &\leq \frac{2^{9n}}{2^{2nj}C^{2n}}\mathcal{M}_{C\sqrt{1-t}}[g](\xi). \end{aligned}$$

Noting that $\sum_{j=1}^{N} 2^{-2nj} < 1$, we obtain the lemma with $A_1 = 2^{9n}$. □

As a consequence of Lemma 1, we obtain the following upper and lower estimates.

Lemma 2. *The following statements hold.*

(i) *If g is an integrable function on S, then*

$$|\mathcal{P}[g](t\xi)| \leq A_2\mathcal{M}_{\sqrt{1-t}}[g](\xi) \qquad \textit{for } \xi \in S \textit{ and } 0 < t < 1,$$

where A_2 is a positive constant depending only on the dimension n.

(ii) *Let $\xi \in S$, $0 < r < 1$ and $C > 0$. If g is a measurable function on S such that $g = 1$ on $Q(\xi, C\sqrt{1-r})$ and $|g| \le 1$ on S, then*

$$\mathcal{P}[g](t\xi) \ge 1 - \frac{A_3}{C^{2n}} \qquad \textit{for } r \le t < 1,$$

where A_3 is a positive constant depending only on the dimension n.

Proof. Putting $C = 1$ in Lemma 1, we obtain (i) with $A_2 = 2A_1$. Let us show (ii). We put $h = (1-g)/2$. Then $h = 0$ on $Q(\xi, C\sqrt{1-r})$ and $|h| \le 1$ on S. Applying Lemma 1 to h, we obtain from (2.2) that for $r \le t < 1$,

$$\mathcal{P}[h](t\xi) \le \frac{A_1}{C^{2n}} \mathcal{M}_{C\sqrt{1-t}}[h](\xi) \le \frac{A_1}{C^{2n}} \sup_{\rho \ge C\sqrt{1-t}} \frac{\sigma(Q(\xi,\rho))}{\rho^{2n}} \le \frac{A_0 A_1}{C^{2n}}.$$

Since $\mathcal{P}[g] = 1 - 2\mathcal{P}[h]$, we obtain (ii) with $A_3 = 2A_0A_1$. □

3 Proof of Theorem

Let π be the radial projection to S defined by $\pi(z) = z/|z|$ for $z \ne 0$. We note that (1.1) implies

$$\lim_{\substack{z \to e_1 \\ z \in \gamma}} \frac{d(z, e_1)}{d(z, \pi(z))} = \infty, \tag{3.1}$$

since $1 - |z|^2 \ge 1 - |z| = d(z, \pi(z))^2$ for $z \in B \setminus \{0\}$. Recall that

$$\operatorname{diam} F = \sup_{\eta, \zeta \in F} d(\eta, \zeta) \qquad \text{for } F \subset S.$$

Lemma 3. *Let γ be the curve as in Theorem. Then there exist sequences of positive numbers $\{a_j\}_{j=1}^\infty$, $\{b_j\}_{j=1}^\infty$ and subcurves $\{\gamma_j\}_{j=1}^\infty$ of γ with the following properties:*

(i) $0 < a_j < b_j < a_{j+1} < b_{j+1} < 1$ *and* $\lim_{j\to\infty} a_j = 1$*;*

(ii) $a_j \le |z| \le b_j$ *for* $z \in \gamma_j$*;*

(iii) $\operatorname{diam} \pi(\gamma_j) \le \sqrt{1 - b_{j-1}}$ *if* $j \ge 2$*;*

(iv) $\displaystyle\lim_{j\to\infty} \frac{\operatorname{diam} \pi(\gamma_j)}{\sqrt{1-a_j}} = \infty.$

Proof. Let $\alpha_j > 1$ be such that $\alpha_j \to \infty$ as $j \to \infty$. We shall choose $\{a_j\}$, $\{b_j\}$ and $\{\gamma_j\}$, inductively. By (3.1), we can find a_1 with $\inf_{z\in\gamma} |z| < a_1 < 1$ and

$$d(z, e_1) \ge \alpha_1 d(z, \pi(z)) \qquad \text{for } z \in \gamma \cap \{|z| \ge a_1\}.$$

Let γ' be the connected component of $\gamma \cap \{|z| \ge a_1\}$ which ends at e_1. Since there is $z_0 \in \gamma' \cap \{|z| = a_1\}$, we have from the triangle inequality that

$$\begin{aligned} \operatorname{diam} \pi(\gamma') &\ge d(\pi(z_0), e_1) \\ &\ge d(z_0, e_1) - d(z_0, \pi(z_0)) \\ &\ge (\alpha_1 - 1) d(z_0, \pi(z_0)) \\ &= (\alpha_1 - 1)\sqrt{1 - a_1}. \end{aligned}$$

Let γ'' be a subcurve of γ' connecting a point in $\{|z| = a_1\}$ and a point near e_1 such that
$$\operatorname{diam} \pi(\gamma'') \geq \frac{1}{2} \operatorname{diam} \pi(\gamma').$$
We take b_1 so that $\sup_{z \in \gamma''} |z| < b_1 < 1$, and let γ_1 be the connected component of $\gamma \cap \{a_1 \leq |z| \leq b_1\}$ containing γ''. Then
$$\operatorname{diam} \pi(\gamma_1) \geq \operatorname{diam} \pi(\gamma'') \geq \frac{\alpha_1 - 1}{2} \sqrt{1 - a_1}.$$
We next choose a_2, b_2 and γ_2 as follows. Let a_2 be such that $b_1 < a_2 < 1$ and
$$\frac{1}{4} \sqrt{1 - b_1} \geq d(z, e_1) \geq \alpha_2 d(z, \pi(z)) \qquad \text{for } z \in \gamma \cap \{|z| \geq a_2\}. \tag{3.2}$$
By repeating the above procedure, we can find b_2 and γ_2 with $a_2 < b_2 < 1$ and $a_2 \leq |z| \leq b_2$ for $z \in \gamma_2$, and
$$\operatorname{diam} \pi(\gamma_2) \geq \frac{\alpha_2 - 1}{2} \sqrt{1 - a_2}.$$
It also follows from (3.2) and $\alpha_2 > 1$ that
$$d(\pi(z), e_1) \leq d(z, e_1) + d(z, \pi(z)) \leq \frac{1}{2} \sqrt{1 - b_1} \qquad \text{for } z \in \gamma_2,$$
and so $\operatorname{diam} \pi(\gamma_2) \leq \sqrt{1 - b_1}$. Hence γ_2 satisfies (iii). Continuing this procedure, we obtain the required sequences. $\square$

In the rest of this section, we suppose that $\{a_j\}$, $\{b_j\}$ and $\{\gamma_j\}$ are as in Lemma 3, and put
$$\ell_j = \frac{\operatorname{diam} \pi(\gamma_j)}{4}, \quad c_j = \left(\frac{\operatorname{diam} \pi(\gamma_j)}{\sqrt{1 - a_j}} \right)^{1/2} \quad \text{and} \quad \rho_j = c_j \sqrt{1 - a_j}$$
to simplify the notation. Note from Lemma 3 that
$$\lim_{j \to \infty} \ell_j = 0, \quad \lim_{j \to \infty} \frac{\rho_j}{\ell_j} = 0 \quad \text{and} \quad \lim_{j \to \infty} c_j = \infty. \tag{3.3}$$
Therefore, in the argument below, we may assume that $\rho_j < \ell_j$ for every $j \in \mathbb{N}$.

For each $j \in \mathbb{N}$, let us choose finitely many points $\{\eta_j^\nu\}_\nu$ in S such that

(P1) $S = \bigcup_\nu Q(\eta_j^\nu, \ell_j)$,

(P2) $\{Q(\eta_j^\nu, \ell_j/2)\}_\nu$ are mutually disjoint.

This is possible. In fact, we first take an arbitrary $\eta_j^1 \in S$, and take $\eta_j^\mu \in S \setminus \bigcup_{\nu=1}^{\mu-1} Q(\eta_j^\nu, \ell_j)$ inductively as long as $S \setminus \bigcup_{\nu=1}^{\mu-1} Q(\eta_j^\nu, \ell_j) \neq \emptyset$. Since S is compact, we can get finitely many points $\{\eta_j^\nu\}_\nu$ satisfying (P1). It also fulfills that $d(\eta_j^\nu, \eta_j^\mu) \geq \ell_j$ if

$\nu \neq \mu$ by the definition of the non-isotropic ball. Hence (P2) follows from the triangle inequality.

We put

$$M_j = \bigcup_{\nu} \{\zeta \in S : d(\zeta, \eta_j^\nu) = \ell_j\}.$$

Then $\pi(U\gamma_j) \cap M_j \neq \emptyset$ for any unitary transformations U. In fact, there is ν such that $\pi(U\gamma_j) \cap Q(\eta_j^\nu, \ell_j) \neq \emptyset$ by (P1). Since $\operatorname{diam} \pi(U\gamma_j) = \operatorname{diam} \pi(\gamma_j) = 4\ell_j$ and $\operatorname{diam} Q(\eta_j^\nu, \ell_j) \leq 2\ell_j$, we have $\pi(U\gamma_j) \cap \{\zeta \in S : d(\zeta, \eta_j^\nu) = \ell_j\} \neq \emptyset$, and so $\pi(U\gamma_j) \cap M_j \neq \emptyset$. Let G_j be the subset of B given by

$$G_j = \{z \in B : a_j \leq |z| \leq b_j \text{ and } \pi(z) \in M_j\}.$$

Since $U\gamma_j \subset \{a_j \leq |z| \leq b_j\}$ by Lemma 3 (ii), it follows that $U\gamma_j \cap G_j \neq \emptyset$. We also put

$$E_j = \bigcup_{\nu} R_j^\nu,$$

where $R_j^\nu = \{\zeta \in S : \ell_j - \rho_j < d(\zeta, \eta_j^\nu) < \ell_j + \rho_j\}$ is the non-isotropic ring. Since the value $\sigma(R_j^\nu)$ is independent of η_j^ν by unitary invariance, we write κ_j for this value. We note that

$$\lim_{j \to \infty} \frac{\kappa_j}{\ell_j^{2n}} = 0. \tag{3.4}$$

In fact, we obtain from (2.1) and (3.3) that for $\eta \in S$,

$$\begin{aligned}
\frac{\kappa_j}{\ell_j^{2n}} &= \frac{\sigma(Q(\eta, \ell_j + \rho_j)) - \sigma(Q(\eta, \ell_j - \rho_j))}{\ell_j^{2n}} \\
&= \left(\frac{\ell_j + \rho_j}{\ell_j}\right)^{2n} \frac{\sigma(Q(\eta, \ell_j + \rho_j))}{(\ell_j + \rho_j)^{2n}} - \left(\frac{\ell_j - \rho_j}{\ell_j}\right)^{2n} \frac{\sigma(Q(\eta, \ell_j - \rho_j))}{(\ell_j - \rho_j)^{2n}} \\
&\longrightarrow 0 \qquad \text{as } j \to \infty.
\end{aligned}$$

Lemma 4. *Let $\{E_j\}$ be as above, and let χ_{E_j} denote the characteristic function of E_j. Then the following properties hold.*

(i) $\displaystyle \lim_{j \to \infty} \left(\sup_{|z| \leq b_{j-1}} \mathcal{P}[\chi_{E_j}](z) \right) = 0.$

(ii) $\displaystyle \lim_{j \to \infty} \sigma(E_j) = 0.$

Proof. Let $z \in B$ be such that $|z| \leq b_{j-1}$. By Lemma 2 (i), we have

$$\begin{aligned}
\mathcal{P}[\chi_{E_j}](z) &\leq A_2 \mathcal{M}_{\sqrt{1-|z|}}[\chi_{E_j}](\pi(z)) \\
&\leq A_2 \sup_{r \geq \sqrt{1-|z|}} r^{-2n} \sum_{\nu} \sigma(R_j^\nu \cap Q(\pi(z), r)) \\
&\leq A_2 \sup_{r \geq \sqrt{1-|z|}} r^{-2n} N_j(z, r) \kappa_j,
\end{aligned}$$

where $N_j(z,r)$ is the number of η_j^ν such that $R_j^\nu \cap Q(\pi(z),r) \neq \emptyset$. Since $\sqrt{1-|z|} \geq \operatorname{diam}\pi(\gamma_j)$ by Lemma 3 (iii), we observe from $\rho_j < \ell_j \leq r/4$ that if $R_j^\nu \cap Q(\pi(z),r) \neq \emptyset$, then $Q(\eta_j^\nu, \ell_j/2) \subset Q(\pi(z), 2r)$. Therefore it follows from (2.2) and (P2) that $N_j(z,r) \leq A_4(r/\ell_j)^{2n}$ with a positive constant A_4 depending only on the dimension n. Hence we obtain

$$\mathcal{P}[\chi_{E_j}](z) \leq A_2 A_4 \frac{\kappa_j}{\ell_j^{2n}},$$

so that (i) follows from (3.4).

Taking $z=0$ in (i), we obtain

$$\sigma(E_j) = \mathcal{P}[\chi_{E_j}](0) \longrightarrow 0 \qquad \text{as } j \to \infty.$$

Thus (ii) follows. □

We now construct a bounded function f on S satisfying the property in Theorem.

Proof of Theorem. In view of Lemma 4, taking a subsequence of j if necessary, we may assume that

$$\mathcal{P}[\chi_{E_j}](z) \leq 2^{-j} \qquad \text{for } |z| \leq b_{j-1}, \tag{3.5}$$

and $\sigma(E_j) \leq 2^{-j}$. Then $\sigma(\bigcap_k \bigcup_{j=k}^\infty E_j) = 0$. Let

$$f_j(\zeta) = \begin{cases} (-1)^{I_j(\zeta)} & \text{if } \zeta \in \bigcup_{i=1}^j E_i, \\ 0 & \text{if } \zeta \notin \bigcup_{i=1}^j E_i, \end{cases}$$

where $I_j(\zeta)$ is the maximum integer i such that $\zeta \in E_i$ for $\zeta \in \bigcup_{i=1}^j E_i$. Then we observe that f_j converges almost everywhere on S to

$$f(\zeta) = \begin{cases} (-1)^{I(\zeta)} & \text{if } \zeta \in \bigcup_{j=1}^\infty E_j \setminus \bigcap_k \bigcup_{j=k}^\infty E_j, \\ 0 & \text{if } \zeta \notin \bigcup_{j=1}^\infty E_j \text{ or } \zeta \in \bigcap_k \bigcup_{j=k}^\infty E_j, \end{cases}$$

where $I(\zeta)$ is the maximum integer i such that $\zeta \in E_i$ for $\zeta \in \bigcup_{j=1}^\infty E_j \setminus \bigcap_k \bigcup_{j=k}^\infty E_j$. We also see that

(a) $f_j = (-1)^j$ on E_j and $|f_j| \leq 1$ on S,

(b) $|f_{j+1} - f_j| \leq 2\chi_{E_{j+1}}$,

(c) $\mathcal{P}[f_j]$ converges to $\mathcal{P}[f]$ on B.

Let U be a unitary transformation. Since $U\gamma$ intersects G_j for every j as stated in the paragraph defining G_j, we can take $z_j \in U\gamma \cap G_j$. Note that $a_j \leq |z_j| \leq b_j$ and $Q(\pi(z_j), c_j\sqrt{1-a_j}) \subset E_j$. If j is even, then it follows from Lemma 2 (ii), Lemma 3

(i) and (3.5) that

$$\begin{aligned}
\mathcal{P}[f](z_j) &= \mathcal{P}[f_j](z_j) + \sum_{k=j}^{\infty} \mathcal{P}[f_{k+1} - f_k](z_j) \\
&\geq \mathcal{P}[f_j](z_j) - \sum_{k=j}^{\infty} \mathcal{P}[|f_{k+1} - f_k|](z_j) \\
&\geq 1 - \frac{A_3}{c_j^{2n}} - 2\sum_{k=j}^{\infty} \mathcal{P}[\chi_{E_{k+1}}](z_j) \\
&\geq 1 - \frac{A_3}{c_j^{2n}} - 2\sum_{k=j}^{\infty} 2^{-k-1} \\
&= 1 - \frac{A_3}{c_j^{2n}} - 2^{1-j}.
\end{aligned}$$

Similarly, if j is odd, then

$$\mathcal{P}[f](z_j) \leq -1 + \frac{A_3}{c_j^{2n}} + 2^{1-j}.$$

Hence we obtain

$$\liminf_{\substack{|z|\to 1 \\ z \in U\gamma}} \mathcal{P}[f](z) = -1 < 1 = \limsup_{\substack{|z|\to 1 \\ z \in U\gamma}} \mathcal{P}[f](z)$$

by (3.3). Thus the theorem is proved. □

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