Sharp estimates for the Green function, 3G inequalities, and nonlinear Schrödinger problems in uniform cones

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Abstract

We find and prove sharp estimates for the Green function and 3G inequalities in uniform cones. These estimates are applied to give equivalent conditions for measures to satisfy the generalized Cranston-McConnell inequality, and to show the existence of infinitely many continuous positive solutions to certain nonlinear Schrödinger problems.

Keywords: Green function, Martin kernel, 3G inequality, Cranston-McConnell inequality, Kato class, nonlinear Schrödinger equation
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1 Introduction

We work in the Euclidean space $\mathbb{R}^n$, where $n \geq 3$. By G_Ω, we denote the Green function for a domain Ω, that is, for each $y \in \Omega$, the function $G_\Omega(\cdot, y)$ is the distributional solution to $-\Delta f = \delta_y$ in Ω and $f = 0$ at all Dirichlet regular boundary points of Ω (in addition, it is bounded near irregular points). We write $\delta_\Omega(x)$ for the distance from $x \in \Omega$ to the Euclidean boundary $\partial\Omega$ of Ω. By the symbol A, we denote an absolute positive constant whose value is unimportant and may change from line to line. If necessary, we use $A_0, A_1, \cdots$ to specify them. For two positive functions f_1 and f_2, we write $f_1 \approx f_2$ if there exists a constant $A \geq 1$ such that $A^{-1} f_1 \leq f_2 \leq A f_1$. The constant A will be called the constant of comparison.

The first purpose of the present paper is to show 3G inequalities in a cone by deriving a sharp global estimate for the Green function. Bogdan [7] and Hansen [11] proved in a bounded Lipschitz domain Ω that if we fix $x_0 \in \Omega$ and let $g(x) = \min\{1, G_\Omega(x, x_0)\}$, then

$$G_\Omega(x,y) \approx \frac{g(x)g(y)}{g(b)^2}|x-y|^{2-n} \quad \text{for } x, y \in \Omega \text{ and } b \in \mathcal{B}_0(x,y), \tag{1.1}$$

where $\mathcal{B}_0(x,y)$ is, roughly speaking, the set of points b in Ω that lie between x and y and satisfy $\delta_\Omega(b) \approx \max\{\delta_\Omega(x), \delta_\Omega(y), |x-y|\}$. See [7, p. 328] or Section 3 for the precise definition. Estimates of such a kind will play important roles when we treat the Green function. In fact, the following 3G inequality can be shown from (1.1). There exists a constant A such that

$$\frac{G_\Omega(x,y)G_\Omega(y,z)}{G_\Omega(x,z)} \leq A(|x-y|^{2-n} + |y-z|^{2-n}) \quad \text{for } x, y, z \in \Omega. \tag{1.2}$$

Before the estimate (1.1), the 3G inequality was proved by Cranston, Fabes and Zhao [8] to study the conditional gauge theory for the Schrödinger operator. Recently, Aikawa and Lundh [4] extended (1.2) to a bounded uniformly John domain, and gave some counterexample to (1.2). See also [12]. The constants appearing in (1.1) and (1.2) depend on the diameter of a domain, and it seems that there is no results such as (1.1) and (1.2) in "unbounded" domains with no explicit expressions of the Green functions. We shall find and establish a sharp global estimate for the Green function and 3G inequalities in particular unbounded domains, cones. We note that unbounded domains do not have (1.1) in general. For instance, considering the half space $\Omega = \{(x_1, \cdots, x_n) : x_n > 0\}$ and $x_0 = (0, \cdots, 0, 1)$, we see that $G_\Omega(rx_0, x_0) \approx r^{1-n}$ and $g(rx_0)g(x_0)g(b_r)^{-2}|rx_0 - x_0|^{2-n} \approx g(rx_0)r^{2-n} \approx r^{3-2n}$ for $r > 0$ sufficiently large and $b_r \in \mathcal{B}(rx_0, x_0)$. Therefore it is interesting to find a sharp global estimate for the Green function in a cone. Indeed, we will establish (1.1) using the Martin kernel at infinity instead of g. Our results will be stated in Section 3. As one of applications of a 3G inequality, we shall give equivalent conditions for measures ν to satisfy the generalized Cranston-McConnell inequality:

$$\int_\Omega G_\Omega(x,y)u(y)d\nu(y) \leq Au(x)$$

for all $x \in \Omega$ and all positive superharmonic functions u in Ω. We will see that if this inequality holds only for the Martin kernel at infinity, then one holds for all positive superharmonic functions.

The second purpose is to show the existence of infinitely many continuous solutions to the following nonlinear Schrödinger problem in a cone Ω:

$$\begin{cases} \Delta u - \mu u = f(\cdot, u) & \text{in } \Omega \text{ (in the sense of distributions)}, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial_r\Omega, \end{cases} \tag{1.3}$$

where μ and f are respectively a signed measure on Ω and a Borel measurable function in $\Omega \times (0, +\infty)$ with suitable properties stated in Section 5, and $\partial_r\Omega$ is the set of all Dirichlet regular points of $\partial\Omega$. Zhang and Zhao [18] studied (1.3) with $\mu = 0$ in a bounded Lipschitz domain containing the origin and showed, using the 3G inequality (1.2), the existence of singular solutions with the growth $|\cdot|^{2-n}$ near the origin. The existence of bounded solutions in an unbounded domain with a compact Lipschitz boundary was investigated in [19]. Bachar, Mâagli and Zribi [6] studied (1.3) with $\mu = 0$ in the half space and showed the existence of solutions with the growth x_n near infinity. Their discussion was based on the explicit expression of the Green function. Thus our purpose is to extend their result to cones by applying our sharp estimates for the Green function. In particular, we shall show the existence of solutions with the same growth as the Martin kernel at infinity. Note that a solution to (1.3) with the same "decay" at infinity as one to the linear equation $\Delta v - \mu v = 0$ was studied in [15].

The following notations will be used in this paper. By $B(x,r)$ and $S(x,r)$, we denote the open ball and the sphere of center x and radius r, respectively. When x is the origin, we write $B(r) = B(x,r)$ and $S(r) = S(x,r)$ for simplicity. A cone we consider is an unbounded domain of the form

$$\Gamma = \left\{ x \in \mathbb{R}^n \setminus \{0\} : \frac{x}{|x|} \in \omega \right\},$$

where ω is some relatively open subset of $S(1)$. In particular, we will consider uniform cones. See Section 2.

The plan of this paper is as follows. In Section 2, we shall collect definitions of a uniform cone and the Martin kernel, and give elementary and useful properties. We also state our key tools: the Carleson estimate and the boundary Harnack principle. In Section 3, we shall establish a sharp global estimate for the Green function in a uniform cone, and show new and classical 3G inequalities. Also, other inequalities that used in subsequent sections will be proved. In Section 4, we shall give a characterization of measures that enjoy the generalized Cranston-McConnell inequality, as an application of the new 3G inequality. We also introduce a certain class of measures which is bigger than the classical Kato class, and give some properties. In Section 5, we investigate (1.3) in a uniform cone.

2 Preliminaries

2.1 Uniform cones

We first state the definition of a uniform cone. A cone Γ in $\mathbb{R}^n$ is said to be a *uniform cone* if there exists a constant $A_0 \geq 1$ such that each pair of points x and y in $\Gamma \cap B(1)$ can be connected by a rectifiable curve γ in Γ for which

$$\begin{aligned} &\ell(\gamma) \leq A_0 |x-y|, \\ &\min\{\ell(\gamma(x,z)), \ell(\gamma(z,y))\} \leq A_0 \delta_\Gamma(z) \quad \text{for all } z \in \gamma, \end{aligned} \tag{2.1}$$

where $\ell(\gamma(x,z))$ denotes the length of the subarc $\gamma(x,z)$ of γ from x to z, and $\delta_\Gamma(z)$ stands for the distance from z to $\partial\Gamma$. We note that a uniform cone is a uniform domain in the sense of [10].

Lemma 2.1. *If Γ is a uniform cone, then each pair of points x and y in Γ can be connected by a rectifiable curve γ in Γ satisfying* (2.1) *with the same constant A_0.*

Proof. Let $x, y \in \Gamma$ and let $r > \max\{|x|, |y|\}$. Then $x/r, y/r \in \Gamma \cap B(1)$. By assumption, there is a curve γ in Γ connecting x/r to y/r such that (2.1) holds for x/r

and y/r in place of x and y. Let $\gamma_r = \{rz : z \in \gamma\}$. Then γ_r is a curve in Γ connecting x to y. It also follows that $\ell(\gamma_r) = r\ell(\gamma) \le A_0|x-y|$ and that for $w \in \gamma_r$,

$$\begin{aligned}\min\{\ell(\gamma_r(x,w)), \ell(\gamma_r(w,y))\} &= r\min\{\ell(\gamma(x/r,w/r)), \ell(\gamma(w/r,y/r))\}\\ &\le rA_0\delta_\Gamma(w/r) = A_0\delta_\Gamma(w).\end{aligned}$$

Thus the lemma is proved. □

2.2 Quasi-hyperbolic metric and Harnack inequality

We state the Harnack inequality involving the quasi-hyperbolic metric in a uniform cone. The quasi-hyperbolic metric on Γ is defined by

$$k_\Gamma(x,y) = \inf_\gamma \int_\gamma \frac{ds(z)}{\delta_\Gamma(z)},$$

where the infimum is taken over all rectifiable curves γ in Γ connecting x to y, and ds stands for the line element on γ. Note from [10] that a uniform cone is characterized in terms of the quasi-hyperbolic metric:

$$k_\Gamma(x,y) \le A\log\left[\left(\frac{|x-y|}{\delta_\Gamma(x)}+1\right)\left(\frac{|x-y|}{\delta_\Gamma(y)}+1\right)\right] + A \quad \text{for } x,y \in \Gamma. \tag{2.2}$$

We also note from [3, Lemma 7.2] that if $z \in \Gamma$, then

$$k_{\Gamma\setminus\{z\}}(x,y) \le 3k_\Gamma(x,y) + \pi \quad \text{for } x,y \in \Gamma \setminus B(z, 2^{-1}\delta_\Gamma(z)).$$

A finite sequence of balls $\{B(x_j, 2^{-1}\delta_\Gamma(x_j))\}_{j=1}^M$ in Γ is called a Harnack chain of length M joining x and y if $x_1 = x$, $x_M = y$, and $x_{j+1} \in B(x_j, 2^{-1}\delta_\Gamma(x_j))$ for $j = 1, \cdots, M-1$. We observe that the shortest length of the Harnack chain joining x and y is comparable to $k_\Gamma(x,y)+1$. Therefore there exists a constant $A \ge 1$ depending only on Γ such that

$$\exp(-A(k_\Gamma(x,y)+1)) \le \frac{h(x)}{h(y)} \le \exp(A(k_\Gamma(x,y)+1)) \quad \text{for } x,y \in \Gamma, \tag{2.3}$$

whenever h is a positive harmonic function in Γ. As a consequence, we can obtain the following lemma.

Lemma 2.2. *Let Γ be a uniform cone, and let h be a positive harmonic function in Γ. If $x, y \in \Gamma$ satisfy $|x-y| \le A_3 \min\{\delta_\Gamma(x), \delta_\Gamma(y)\}$ for some $A_3 > 0$, then*

$$h(x) \approx h(y) \quad \textit{and} \quad G_\Gamma(x,y) \approx |x-y|^{2-n},$$

where the constants of comparisons depend only on A_3 and Γ.

2.3 Carleson estimate and Boundary Harnack principle

We next state the Carleson estimate and the boundary Harnack principle. We say that a property holds quasi-everywhere if it holds apart from a polar set. The following lemma is found in [2, Theorem 1 and Remark 2].

Lemma 2.3. *Let Γ be a uniform cone. Then there exist positive constants r_1 and $A_1 < 1$ depending only on Γ with the following properties: Let $\xi \in \partial\Gamma$ and $0 < r \le r_1$. If h_1 and h_2 are positive bounded harmonic functions in $\Gamma \cap B(\xi, r)$ vanishing quasi-everywhere on $\partial\Gamma \cap B(\xi, r)$, then*

$$\frac{h_1(x)}{h_2(x)} \approx \frac{h_1(x')}{h_2(x')} \quad \text{for } x, x' \in \Gamma \cap \overline{B(\xi, A_1 r)},$$

where the constant of comparison depends only on Γ. Moreover, if z is an arbitrary point in $\Gamma \cap S(\xi, A_1 r)$ such that $\delta_\Gamma(z) \ge A_2 r$ for some $A_2 > 0$, then

$$h_1(x) \le A h_1(z) \quad \text{for } x \in \Gamma \cap \overline{B(\xi, A_1 r)},$$

where the constant A depends only on A_2 and Γ.

Remark 2.4. In arguments below, a constant A_2 in Lemma 2.3 will be implicitly taken as $2^{-1}A_0^{-1}A_1$. The existence of a point $z \in \Gamma \cap S(\xi, A_1 r)$ with $\delta_\Gamma(z) \ge 2^{-1}A_0^{-1}A_1 r$ can be shown as follows. Let $x \in \Gamma \cap S(\xi, 2^{-1}A_1 r)$ and $y \in \Gamma \cap S(\xi, 2A_1 r)$. By Lemma 2.1, there exists a curve γ in Γ connecting x to y with the properties in (2.1). Then a point $z \in \gamma \cap S(\xi, A_1 r)$ satisfies $\delta_\Gamma(z) \ge 2^{-1}A_0^{-1}A_1 r$.

2.4 Martin kernels

We finally state the definition of the Martin kernels. Let Ω be a unbounded domain in $\mathbb{R}^n$. Recall that G_Ω is the Green function for Ω. We fix $x_0 \in \Omega$ (the reference point). Let $\xi \in \partial\Omega \cup \{\infty\}$, and let $\{y_j\}$ be a sequence in Ω converging to ξ. Then we see that some subsequence of $\{G_\Omega(\cdot, y_j)/G_\Omega(x_0, y_j)\}_j$ converges to a positive harmonic function in Ω. All limit functions obtained in this way are called *Martin kernels at ξ*. When we consider a cone Γ, the reference point x_0 is taken in $\Gamma \cap S(1)$.

Lemma 2.5. *If Γ is a uniform cone, then for each $\xi \in \partial\Gamma \cup \{\infty\}$, there exists a unique (minimal) Martin kernel $K_\Gamma(\cdot, \xi)$ at ξ. Moreover, there exist a non-negative constant α and a positive bounded continuous function θ on $\Gamma \cap S(1)$ such that*

$$K_\Gamma(x, 0) = |x|^{2-n-\alpha}\theta(x/|x|) \quad \text{and} \quad K_\Gamma(x, \infty) = |x|^{\alpha}\theta(x/|x|). \tag{2.4}$$

Proof. The first assertion for $\xi \in \partial\Gamma$ is found in [2, Theorem 3]. By the Kelvin transform, we also observe that there is a unique (minimal) Martin kernel at ∞. The representation (2.4) can be obtained in a similar way as in [13, p. 472]. ☐

It is noteworthy that if Γ is a uniform cone, then for $r > 0$,

$$G_\Gamma(x, y) = r^{2-n}G_\Gamma(x/r, y/r) \quad \text{and} \quad K_\Gamma(x, \infty) = r^{\alpha}K_\Gamma(x/r, \infty). \tag{2.5}$$

We also see from (2.2) and (2.3) that there exist positive constants A and $\beta \geq 1$ such that

$$\theta(z) \geq A\delta_\Gamma(z)^\beta \quad \text{for } z \in \Gamma \cap S(1). \tag{2.6}$$

Note that if $\Gamma \cap S(1)$ has a $C^{1,1}$-boundary, then we can take $\beta = 1$.

3 Sharp estimates for the Green function and 3G inequalities

Throughout this section, we suppose that Γ is a uniform cone in $\mathbb{R}^n$ with $n \geq 3$. To obtain (1.1) in a bounded Lipschitz domain Ω, Bogdan [7] defined $\mathcal{B}_0(x,y)$ as the set of all points b in Ω such that $B(b, \kappa_0 r) \subset \Omega \cap B(x, 3r) \cap B(y, 3r)$ if $r := \max\{\delta_\Omega(x), \delta_\Omega(y), |x-y|\} \leq r_2$, and $\mathcal{B}_0(x,y) = S(x_0, r_2)$ if $r > r_2$, where κ_0 and r_2 are some fixed positive constants. We know that $\mathcal{B}_0(x,y)$ plays a good role essentially when $\delta_\Omega(x)$ and $\delta_\Omega(y)$ are much smaller than $|x-y|$. From this view, we adopt the following somewhat simpler definition. Let $\kappa \geq 1$. For $x, y \in \Gamma$, we define

$$\mathcal{B}(x,y) = \left\{ b \in \Gamma : \max\{|x-b|, |b-y|\} \leq \kappa|x-y| \text{ and } \delta_\Gamma(b) \geq \frac{|x-y|}{\kappa} \right\}.$$

Although this definition does not possess the relation between $\delta_\Gamma(b)$ and $\max\{\delta_\Gamma(x), \delta_\Gamma(y)\}$, we have the following.

Proposition 3.1. *Let $x, y \in \Gamma$, and let A_0 be the constant in* (2.1)*. The following statements hold.*

(i) *If $\kappa \geq 2A_0$, then $\mathcal{B}(x,y)$ is non-empty, and $\mathcal{B}(x,y) = \mathcal{B}(y,x)$.*

(ii) *If $b \in \mathcal{B}(x,y)$, then $\delta_\Gamma(b) \geq (2\kappa^2)^{-1} \max\{\delta_\Gamma(x), \delta_\Gamma(y)\}$ and $|b| \geq (2\kappa^2)^{-1} \max\{|x|, |y|\}$.*

(iii) *If $r > 0$, then $\mathcal{B}(rx, ry) = \{b \in \Gamma : b/r \in \mathcal{B}(x,y)\}$.*

Proof. (i) Let $x, y \in \Gamma$. By Lemma 2.1, there exists a curve γ in Γ connecting x to y such that $\ell(\gamma) \leq A_0|x-y|$ and $\min\{|x-z|, |z-y|\} \leq A_0\delta_\Gamma(z)$ for all $z \in \gamma$. Let b be a point in γ such that $|x-b| = |b-y|$. Then

$$\max\{|x-b|, |b-y|\} \leq \ell(\gamma) \leq A_0|x-y|,$$
$$\delta_\Gamma(b) \geq A_0^{-1}|x-b| \geq (2A_0)^{-1}|x-y|.$$

Hence $\mathcal{B}(x,y)$ is non-empty whenever $\kappa \geq 2A_0$. The symmetry of $\mathcal{B}(x,y)$ is clear from the definition.

(ii) We first show $\delta_\Gamma(b) \geq (2\kappa^2)^{-1} \max\{\delta_\Gamma(x), \delta_\Gamma(y)\}$. By symmetry, it suffices to prove $\delta_\Gamma(b) \geq (2\kappa^2)^{-1}\delta_\Gamma(x)$. Suppose to the contrary that there is $b \in \mathcal{B}(x,y)$ such that $\delta_\Gamma(b) < (2\kappa^2)^{-1}\delta_\Gamma(x)$. Then $|x-y| \leq \kappa\delta_\Gamma(b) \leq (2\kappa)^{-1}\delta_\Gamma(x)$, and so $|x-b| \leq \kappa|x-y| \leq 2^{-1}\delta_\Gamma(x)$. Hence

$$\delta_\Gamma(b) \geq \delta_\Gamma(x) - |x-b| \geq 2^{-1}\delta_\Gamma(x) \geq (2\kappa^2)^{-1}\delta_\Gamma(x).$$

This is a contradiction. We next show $|b| \geq (2\kappa^2)^{-1} \max\{|x|, |y|\}$. It is enough to prove $|b| \geq (2\kappa^2)^{-1}|x|$. Suppose to the contrary that there is $b \in \mathcal{B}(x, y)$ such that $|b| < (2\kappa^2)^{-1}|x|$. Then $|x-y| \leq (2\kappa)^{-1}|x|$ since $\delta_\Gamma(b) \leq |b|$, and so $|x-b| \leq 2^{-1}|x|$. Hence

$$|b| \geq |x| - |x-b| \geq 2^{-1}|x| \geq (2\kappa)^{-1}|x|.$$

This is a contradiction.

(iii) Since $\delta_\Gamma(rb) = r\delta_\Gamma(b)$ and $|rx - ry| = r|x-y|$, we can obtain (iii) immediately. ☐

Our sharp global estimate for the Green function in Γ is as follows.

Theorem 3.2. *For $x, y \in \Gamma$ and $b \in \mathcal{B}(x, y)$,*

$$G_\Gamma(x, y) \approx \frac{K_\Gamma(x, \infty)K_\Gamma(y, \infty)}{K_\Gamma(b, \infty)^2}|x-y|^{2-n}, \tag{3.1}$$

where the constant of comparison depends only on κ and Γ.

Proof. We first show (3.1) for $x, y \in \Gamma \cap B(1)$ and $b \in \mathcal{B}(x, y)$. We may assume by symmetry that $\delta_\Gamma(x) \leq \delta_\Gamma(y)$. Let $A_4 = A_1^{-1} \max\{5, 2r_1^{-1}\}$, where $0 < A_1 < 1$ and $r_1 > 0$ are the constants in Lemma 2.3. We consider two cases: $|x-y| \leq A_4\delta_\Gamma(x)$ and $|x-y| > A_4\delta_\Gamma(x)$.

Case 1 : $|x-y| \leq A_4\delta_\Gamma(x)$. Since $|x-b| \leq \kappa|x-y| \leq \kappa^2\delta_\Gamma(b)$ by the definition of $\mathcal{B}(x, y)$, it follows from Lemma 2.2 that

$$K_\Gamma(b, \infty) \approx K_\Gamma(x, \infty) \approx K_\Gamma(y, \infty) \quad \text{and} \quad G_\Gamma(x, y) \approx |x-y|^{2-n}.$$

Hence we obtain (3.1) in this case.

Case 2 : $|x-y| > A_4\delta_\Gamma(x)$. Note from our choice of A_4 that

$$A_1^{-1}A_4^{-1}|x-y| \leq 2A_1^{-1}A_4^{-1} \leq r_1.$$

Let $\xi \in \partial\Gamma$ be a point such that $\delta_\Gamma(x) = |x-\xi|$. Then

$$|y-\xi| \geq |x-y| - |x-\xi| \geq (1 - A_4^{-1})|x-y| \geq A_1^{-1}A_4^{-1}|x-y|.$$

We take $x_1 \in \Gamma \cap S(\xi, A_4^{-1}|x-y|)$ with $\delta_\Gamma(x_1) \approx |x-y|$ (cf. Remark 2.4). By Lemma 2.3, we have

$$\frac{G_\Gamma(x, y)}{K_\Gamma(x, \infty)} \approx \frac{G_\Gamma(x_1, y)}{K_\Gamma(x_1, \infty)}. \tag{3.2}$$

We take y_1 as follows. If $\delta_\Gamma(y) \geq A_4^{-1}|x-y|$, then we let $y_1 = y$. If $\delta_\Gamma(y) < A_4^{-1}|x-y|$, then, letting $\eta \in \partial\Gamma$ be a point such that $\delta_\Gamma(y) = |y-\eta|$, we take $y_1 \in \Gamma \cap S(\eta, A_4^{-1}|x-y|)$ with $\delta_\Gamma(y_1) \approx |x-y|$. Note in the latter case that

$$|x_1 - \eta| \geq |x-y| - |x-x_1| - |y-\eta| \geq (1 - 3A_4^{-1})|x-y| \geq A_1^{-1}A_4^{-1}|x-y|.$$

We have by Lemma 2.3

$$\frac{G_\Gamma(x_1, y)}{K_\Gamma(y, \infty)} \approx \frac{G_\Gamma(x_1, y_1)}{K_\Gamma(y_1, \infty)}. \tag{3.3}$$

This is true for any pairs of y and y_1. Since $|x_1-y_1| \approx |x-y| \le A\min\{\delta_\Gamma(x_1), \delta_\Gamma(y_1)\}$, it follows from Case 1 that for $b_1 \in \mathcal{B}(x_1, y_1)$,

$$G_\Gamma(x_1, y_1) \approx \frac{K_\Gamma(x_1, \infty)K_\Gamma(y_1, \infty)}{K_\Gamma(b_1, \infty)^2}|x-y|^{2-n}. \tag{3.4}$$

Note that $\delta_\Gamma(b_1) \ge \kappa^{-1}|x_1-y_1| \approx |x-y|$ and

$$|b-b_1| \le |b-x| + |x-x_1| + |x_1-b_1| \le (\kappa + 2A_4^{-1})|x-y| + \kappa|x_1-y_1| \approx |x-y|,$$

and so $|b-b_1| \le A\min\{\delta_\Gamma(b), \delta_\Gamma(b_1)\}$. Therefore Lemma 2.2 yields that

$$K_\Gamma(b_1, \infty) \approx K_\Gamma(b, \infty). \tag{3.5}$$

Combining (3.2), (3.3), (3.4) and (3.5), we obtain (3.1) in this case. Thus (3.1) holds for $x, y \in \Gamma \cap B(1)$ and $b \in \mathcal{B}(x, y)$.

Finally, to establish (3.1) for all $x, y \in \Gamma$ and $b \in \mathcal{B}(x, y)$, we let $r > \max\{|x|, |y|\}$. Then $x/r, y/r \in \Gamma \cap B(1)$ and $b/r \in \mathcal{B}(x/r, y/r)$ by Proposition 3.1. Therefore we have from the above observation

$$G_\Gamma(x/r, y/r) \approx \frac{K_\Gamma(x/r, \infty)K_\Gamma(y/r, \infty)}{K_\Gamma(b/r, \infty)^2}\left|\frac{x}{r} - \frac{y}{r}\right|^{2-n}.$$

Hence (3.1) follows from (2.5). Thus the proof is complete. $\square$

In what follows, we take $\kappa = 2A_0$ but we continue to use the symbol κ. If x and y are separated enough, then the Green function is comparable to the product of the Martin kernels at the origin and at infinity.

Corollary 3.3. *For $x, y \in \Gamma$ with $2|y| \le |x|$,*

$$G_\Gamma(x, y) \approx |x|^{2-n-2\alpha}K_\Gamma(x, \infty)K_\Gamma(y, \infty) = K_\Gamma(x, 0)K_\Gamma(y, \infty),$$

where the constant of comparison depends only on Γ.

Proof. Let $x, y \in \Gamma$ satisfy $2|y| \le |x|$, and let $b \in \mathcal{B}(x, y)$. Then

$$(2\kappa)^{-1}|x| \le \kappa^{-1}|x-y| \le \delta_\Gamma(b) \le |b| \le |b-x| + |x| \le \kappa|x-y| + |x| \le 3\kappa|x|,$$

and so $\delta_\Gamma(b/|b|) = |b|^{-1}\delta_\Gamma(b) \ge (6\kappa^2)^{-1}$. It follows from Lemma 2.5 that $K_\Gamma(b, \infty) \approx |b|^\alpha \approx |x|^\alpha$. Hence we obtain from Theorem 3.2 and (2.4) that

$$G_\Gamma(x, y) \approx K_\Gamma(x, \infty)K_\Gamma(y, \infty)|x|^{2-n-2\alpha} = K_\Gamma(x, 0)K_\Gamma(y, \infty).$$

Thus the corollary follows. $\square$

As an important application of Theorem 3.2, we obtain the following 3G inequality.

Theorem 3.4 (New 3G inequality)**.** *There exists a constant A depending only on Γ such that for $x, y, z \in \Gamma$,*

$$\frac{G_\Gamma(x, y)G_\Gamma(y, z)}{G_\Gamma(x, z)} \le A\left(\frac{K_\Gamma(y, \infty)}{K_\Gamma(x, \infty)}G_\Gamma(x, y) + \frac{K_\Gamma(y, \infty)}{K_\Gamma(z, \infty)}G_\Gamma(y, z)\right). \tag{3.6}$$

Proof. We may assume by (2.5) and symmetry that $x, y, z \in \Gamma \cap B(1)$ and $|x-y| \le |y-z|$. It is enough to show that

$$\frac{G_\Gamma(y,z)}{G_\Gamma(x,z)} \le A \frac{K_\Gamma(y,\infty)}{K_\Gamma(x,\infty)}. \tag{3.7}$$

Let $b_{x,z} \in \mathcal{B}(x,z)$ and $b_{y,z} \in \mathcal{B}(y,z)$. Since $|x-z| \le |x-y| + |y-z| \le 2|y-z|$, it follows from Theorem 3.2 that

$$\frac{G_\Gamma(y,z)}{G_\Gamma(x,z)} \le A \left(\frac{K_\Gamma(b_{x,z},\infty)}{K_\Gamma(b_{y,z},\infty)} \right)^2 \frac{K_\Gamma(y,\infty)}{K_\Gamma(x,\infty)}.$$

We claim that

$$\frac{K_\Gamma(b_{x,z},\infty)}{K_\Gamma(b_{y,z},\infty)} \le A. \tag{3.8}$$

To show this, we let $A_5 = 3\kappa A_1^{-1}$ and $r = A_5 \max\{|y-z|, \delta_\Gamma(z)\}$. We consider two cases: $r \ge r_1$ and $r < r_1$. Here $0 < A_1 < 1$ and $r_1 > 0$ are the constants in Lemma 2.3.

Case 1 : $r \ge r_1$. Since $|b_{x,z}| \le 1 + 2\kappa$, it follows from Lemma 2.5 that $K_\Gamma(b_{x,z},\infty) \le A$. Also, we have $K_\Gamma(b_{y,z},\infty) \ge A > 0$ because of $\delta_\Gamma(b_{y,z}) \ge (2\kappa^2)^{-1} A_5^{-1} r_1$ by Proposition 3.1. Hence (3.8) holds in this case.

Case 2 : $r < r_1$. Let $\zeta \in \partial\Gamma$ be a point such that $\delta_\Gamma(z) = |z-\zeta|$. Then $|b_{x,z} - \zeta| \le |b_{x,z} - z| + |z-\zeta| \le (2\kappa+1)A_5^{-1} r \le A_1 r$. We take $w \in \Gamma \cap S(\zeta, A_1 r)$ with $\delta_\Gamma(w) \approx r$. Note that

$$\begin{aligned} |w - b_{y,z}| &\le |w-\zeta| + |\zeta - z| + |z - b_{y,z}| \le (A_1 + (1+\kappa)A_5^{-1})r \\ &\le A \min\{\delta_\Gamma(w), \delta_\Gamma(b_{y,z})\}, \end{aligned}$$

since $\delta_\Gamma(b_{y,z}) \ge (2\kappa^2)^{-1} A_5^{-1} r$ by Proposition 3.1. Hence Lemmas 2.3 and 2.2 yield that

$$K_\Gamma(b_{x,z},\infty) \le A K_\Gamma(w,\infty) \approx K_\Gamma(b_{y,z},\infty).$$

Hence (3.8) holds. Thus the theorem is proved. □

Theorem 3.5. *There exists a constant A depending only on Γ such that for $x, y \in \Gamma$,*

$$\frac{G_\Gamma(x,y) K_\Gamma(y,\infty)}{K_\Gamma(x,\infty)} \le A |x-y|^{2-n}. \tag{3.9}$$

Proof. We may assume by (2.5) that $x, y \in \Gamma \cap B(1)$. Let $b \in \mathcal{B}(x,y)$. Then we have by Theorem 3.2

$$\frac{G_\Gamma(x,y) K_\Gamma(y,\infty)}{K_\Gamma(x,\infty)} \approx \left(\frac{K_\Gamma(y,\infty)}{K_\Gamma(b,\infty)} \right)^2 |x-y|^{2-n}.$$

To obtain (3.9), it suffices to show $K_\Gamma(y,\infty)/K_\Gamma(b,\infty) \le A$. This can be proved in a similar way as in the proof of Theorem 3.4. For the reader's convenience, we give a proof. Let $r = A_5 \max\{|x-y|, \delta_\Gamma(y)\}$, where $A_5 = 3\kappa A_1^{-1}$. If $r \ge r_1$, then

$\delta_\Gamma(b) \geq (2\kappa^2)^{-1}A_5^{-1}r_1$ by Proposition 3.1, and therefore $K_\Gamma(y,\infty) \leq AK_\Gamma(b,\infty)$. Suppose in the sequel that $r < r_1$. Let $\xi \in \partial\Gamma$ be a point such that $\delta_\Gamma(b) = |b-\xi|$, and take $w \in \Gamma \cap S(\xi, A_1 r)$ with $\delta_\Gamma(w) \approx r$. Then

$$|y-\xi| \leq |y-b| + \delta_\Gamma(b) \leq 2|y-b| + \delta_\Gamma(y) \leq (2\kappa+1)A_5^{-1}r \leq A_1 r$$

and

$$\begin{aligned}|b-w| &\leq |b-\xi| + |\xi - w| \leq |b-y| + \delta_\Gamma(y) + |\xi - w| \\ &\leq ((\kappa+1)A_5^{-1} + A_1)r \leq A\min\{\delta_\Gamma(b), \delta_\Gamma(w)\}.\end{aligned}$$

Hence we obtain from Lemmas 2.3 and 2.2

$$K_\Gamma(y,\infty) \leq AK_\Gamma(w,\infty) \approx K_\Gamma(b,\infty).$$

Thus the theorem is proved. ☐

Remark 3.6. From the proof of Theorem 3.5, it follows in general that

$$\max\{K_\Gamma(x,\infty), K_\Gamma(y,\infty)\} \leq AK_\Gamma(b,\infty) \quad \text{for } x,y \in \Gamma \text{ and } b \in \mathcal{B}(x,y). \tag{3.10}$$

The following 3G inequality has been studied widely in many bounded domains, but it may be unknown in cones.

Corollary 3.7 (Classical 3G inequality). *There exists a constant A depending only on Γ such that for $x, y, z \in \Gamma$,*

$$\frac{G_\Gamma(x,y)G_\Gamma(y,z)}{G_\Gamma(x,z)} \leq A(|x-y|^{2-n} + |y-z|^{2-n}).$$

Proof. This follows immediately from Theorems 3.4 and 3.5. ☐

The following two lemmas will be used in the subsequent sections, so we prove them here.

Lemma 3.8. *There exists a constant A depending only on Γ such that for $x, y \in \Gamma$,*

$$K_\Gamma(x,\infty)K_\Gamma(y,\infty) \leq A\max\{|x|,|y|\}^{n-2+2\alpha}G_\Gamma(x,y),$$

where $\alpha \geq 0$ is the constant in Lemma 2.5.

Proof. Let $b \in \mathcal{B}(x,y)$ and $R = \max\{|x|,|y|\}$. Since $|b| \leq |b-x| + |x| \leq \kappa|x-y| + |x| \leq (2\kappa+1)R$, we have by Lemma 2.5

$$\frac{|x-y|^{2-n}}{K_\Gamma(b,\infty)^2} \geq A\frac{R^{2-n}}{R^{2\alpha}}.$$

Thus the lemma follows from Theorem 3.2. ☐

Lemma 3.9. *Let $r > 0$ and $R > 0$. Then there exists a constant A depending only on r, R and Γ such that for $x, y \in \Gamma \cap B(R)$ with $|x-y| \geq r$,*

$$G_\Gamma(x,y) \leq AK_\Gamma(x,\infty)K_\Gamma(y,\infty).$$

Proof. Let $\alpha \geq 0$ and θ be as in Lemma 2.5, and let $\beta \geq 1$ be as in (2.6). Then we have

$$K_\Gamma(b,\infty) = |b|^\alpha \theta(b/|b|) \geq A|b|^{\alpha-\beta}\delta_\Gamma(b)^\beta \quad \text{for } b \in \mathcal{B}(x,y). \tag{3.11}$$

Since $|b| \leq (2\kappa+1)R$ and $\delta_\Gamma(b) \geq \kappa^{-1}|x-y| \geq \kappa^{-1}r$, it follows from Theorem 3.2 that

$$G_\Gamma(x,y) \leq A(r,R,\Gamma)K_\Gamma(x,\infty)K_\Gamma(y,\infty),$$

where $A(r,R,\Gamma)$ is a positive constant depending only on r, R and Γ. Thus the lemma is proved. $\square$

Remark 3.10. If $\alpha \geq 1 = \beta$, then $K_\Gamma(b,\infty) \geq A\max\{|x|,|y|,|x-y|\}^{\alpha-1}|x-y|$ by (3.11) and Proposition 3.1, and therefore we have

$$G_\Gamma(x,y) \leq A\frac{K_\Gamma(x,\infty)K_\Gamma(y,\infty)}{\max\{|x|,|y|,|x-y|\}^{2(\alpha-1)}|x-y|^n} \quad \text{for } x,y \in \Gamma,$$

where the constant A depends only on Γ. If $\alpha < 1 = \beta$, then $|b| \leq (2\kappa+1)\max\{|x|,|y|\}$, and so $K_\Gamma(b,\infty) \geq A\max\{|x|,|y|\}^{\alpha-1}|x-y|$. Therefore we have

$$G_\Gamma(x,y) \leq A\frac{K_\Gamma(x,\infty)K_\Gamma(y,\infty)}{\max\{|x|,|y|\}^{2(\alpha-1)}|x-y|^n} \quad \text{for } x,y \in \Gamma,$$

where the constant A depends only on Γ.

4 Generalized Cranston-McConnell inequality and extended Kato class

In [9], Cranston and McConnell proved that if Ω is a domain in $\mathbb{R}^2$ with the finite volume vol(Ω), then there exists a constant A such that

$$\sup_{x,\,h} \frac{1}{h(x)}\int_\Omega G_\Omega(x,y)h(y)dy \leq A\mathrm{vol}(\Omega),$$

where the supremum is taken over all $x \in \Omega$ and all positive harmonic functions h in Ω. In general, if Ω has infinite volume, then the left hand side of the above inequality diverges. We consider the following generalization. In the rest of this section, we suppose that Γ is a uniform cone in $\mathbb{R}^n$ with $n \geq 3$. By $\mathcal{U}_+(\Gamma)$, we denote the class of all positive superharmonic functions in Γ. We say that a measure ν on Γ enjoys the generalized Cranston-McConnell inequality if

$$\|\nu\|_{CM} := \sup_{x\in\Gamma,\, u\in\mathcal{U}_+(\Gamma)} \frac{1}{u(x)}\int_\Gamma G_\Gamma(x,y)u(y)d\nu(y) < +\infty. \tag{4.1}$$

See [1, 5, 14, 16, 17] for investigations and related topics to the generalized Cranston-McConnell inequality, and see reference therein. We give necessary and sufficient conditions for ν to satisfy the generalized Cranston-McConnell inequality. To simplify the notation, we write

$$H_\Gamma(x,y) = \frac{G_\Gamma(x,y)K_\Gamma(y,\infty)}{K_\Gamma(x,\infty)} \quad \text{for } x,y \in \Gamma,$$

and set

$$\|\nu\|_H = \sup_{x\in\Gamma} \int_\Gamma H_\Gamma(x,y)d\nu(y).$$

The following theorem means that ν enjoys the generalized Cranston-McConnell inequality if (4.1) holds only for $u = K_\Gamma(\cdot,\infty)$.

Theorem 4.1. *Let ν be a measure on Γ. The following statements are equivalent:*

(i) $\|\nu\|_{CM} < +\infty$;

(ii) $\|\nu\|_H < +\infty$;

(iii) $\|\nu\|_G := \sup_{x,z\in\Gamma} \int_\Gamma \frac{G_\Gamma(x,y)G_\Gamma(y,z)}{G_\Gamma(x,z)} d\nu(y) < +\infty.$

Moreover, $\|\nu\|_{CM} \approx \|\nu\|_H \approx \|\nu\|_G$, where the constants of comparisons depend only on Γ.

Proof. If (i) holds, then we obtain (ii) by taking $u = K_\Gamma(\cdot,\infty)$ in (4.1). If (ii) holds, then we have by Theorem 3.4

$$\int_\Gamma \frac{G_\Gamma(x,y)G_\Gamma(y,z)}{G_\Gamma(x,z)} d\nu(y) \le A \int_\Gamma \{H_\Gamma(x,y) + H_\Gamma(z,y)\} d\nu(y).$$

Hence (iii) follows. We finally show that (iii) implies (i). By assumption, we have

$$\int_\Gamma G_\Gamma(x,y)G_\Gamma(y,z)d\nu(y) \le \|\nu\|_G G_\Gamma(x,z) \quad \text{for } x,z \in \Gamma.$$

Let $G_\Gamma\mu$ be a Green potential in Γ of a measure μ. It then follows from Fubini's theorem that

$$\int_\Gamma G_\Gamma(x,y)G_\Gamma\mu(y)d\nu(y) \le \|\nu\|_G G_\Gamma\mu(x).$$

Since every positive superharmonic function can be approximated by an increasing sequence of Green potentials, the monotone convergence theorem yields (i). Thus the theorem is proved. □

The following is an immediate consequence of Theorems 3.5 and 4.1.

Corollary 4.2. *If ν is a measure on Γ satisfying*

$$\sup_{x\in\Gamma} \int_\Gamma |x-y|^{2-n} d\nu(y) < +\infty,$$

then ν enjoys the generalized Cranston-McConnell inequality.

As another consequence, we obtain the following.

Corollary 4.3. *Let ν be a measure on Γ such that $\|\nu\|_H < +\infty$. Then for each $R > 0$,*

$$\int_{\Gamma \cap B(R)} K_\Gamma(y,\infty)d\nu(y) < +\infty.$$

Proof. Let $R > 1$. Then Lemma 3.8 with $x = x_0$ and Theorem 4.1 yield that

$$\int_{\Gamma \cap B(R)} K_\Gamma(y,\infty)d\nu(y) \le A \int_\Gamma G_\Gamma(x_0,y)d\nu(y) \le A\|\nu\|_{CM} < +\infty,$$

and thus the corollary follows. □

We now introduce an extended Kato class.

Definition 4.4. We say that a measure ν on Γ belongs to the extended Kato class $\mathcal{K}(\Gamma)$ if ν fulfills the following properties:

$$\lim_{r\to 0}\left(\sup_{x\in\Gamma}\int_{\Gamma\cap B(x,r)} H_\Gamma(x,y)d\nu(y)\right) = 0, \tag{4.2}$$

$$\lim_{R\to+\infty}\left(\sup_{x\in\Gamma}\int_{\Gamma\setminus B(R)} H_\Gamma(x,y)d\nu(y)\right) = 0. \tag{4.3}$$

We also say that a Borel measurable function f in Γ belongs to the extended Kato class $\mathcal{K}(\Gamma)$ if the measure $d\nu = |f|dy$ belongs to $\mathcal{K}(\Gamma)$.

The classical Kato class $\mathcal{K}_0(\Gamma)$ is the set of all measures ν on Γ satisfying (4.2) and (4.3) with the Newtonian kernel $|x-y|^{2-n}$ instead of $H_\Gamma(x,y)$. By Theorem 3.5, we easily see $\mathcal{K}_0(\Gamma) \subset \mathcal{K}(\Gamma)$.

Proposition 4.5. *If ν is a measure on Γ such that $\nu \in \mathcal{K}(\Gamma)$, then $\|\nu\|_H < +\infty$. Moreover, for each $R > 0$,*

$$\int_{\Gamma\cap B(R)} K_\Gamma(y,\infty)^2 d\nu(y) < +\infty.$$

Proof. By the definition of $\mathcal{K}(\Gamma)$, there exist positive numbers r_2 and R_2 such that

$$\sup_{x\in\Gamma}\int_{\Gamma\cap B(x,r_2)} H_\Gamma(x,y)d\nu(y) \le 1 \quad \text{and} \quad \sup_{x\in\Gamma}\int_{\Gamma\setminus B(R_2)} H_\Gamma(x,y)d\nu(y) \le 1. \tag{4.4}$$

Since $\Gamma \cap B(R_2)$ is covered by a finite sequence of balls $\{B(x_j,r_2)\}_{j=1}^m$ with $x_j \in \Gamma \cap B(R_2)$, it follows from Lemma 3.8 and (4.4) that

$$\int_{\Gamma\cap B(R_2)} K_\Gamma(y,\infty)^2 d\nu(y) \le A\sum_{j=1}^m \int_{\Gamma\cap B(x_j,r_2)} H_\Gamma(x_j,y)d\nu(y) < +\infty.$$

Therefore we have by Corollary 3.3 and Lemma 3.9

$$\sup_{x\in\Gamma}\int_{\Gamma\cap B(R_2)\setminus B(x,r_2)} H_\Gamma(x,y)d\nu(y) < +\infty.$$

This, together with (4.4), yields that $\|\nu\|_H < +\infty$. □

Lemma 4.6. *If ν is a measure on Γ such that $\nu \in \mathcal{K}(\Gamma)$, then for each $z_0 \in \overline{\Gamma}$,*

$$\lim_{r\to 0}\int_{\Gamma\cap B(z_0,r)} K_\Gamma(y,\infty)^2 d\nu(y) = 0.$$

Proof. Let $0 < r < 1$. Then there is $x \in \Gamma$ such that $B(z_0, r) \subset B(x, 2r)$. By Lemma 3.8, we have

$$\begin{aligned}\int_{\Gamma\cap B(z_0,r)} K_\Gamma(y,\infty)^2 d\nu(y) &\le \int_{\Gamma\cap B(x,2r)} K_\Gamma(y,\infty)^2 d\nu(y)\\ &\le A(z_0,\Gamma)\int_{\Gamma\cap B(x,2r)} H_\Gamma(x,y)d\nu(y).\end{aligned}$$

Hence the conclusion follows from (4.2). □

Proposition 4.7. *If ν is a measure on Γ such that $\nu \in \mathcal{K}(\Gamma)$, then for each $z_0 \in \overline{\Gamma}$,*

$$\lim_{r\to 0}\left(\sup_{x\in\Gamma}\int_{\Gamma\cap B(z_0,r)} H_\Gamma(x,y)d\nu(y)\right) = 0. \tag{4.5}$$

Proof. Let $\varepsilon > 0$. Since $\nu \in \mathcal{K}(\Gamma)$, there exist positive numbers r_3 and R_3 such that

$$\sup_{x\in\Gamma}\int_{\Gamma\cap B(x,r_3)} H_\Gamma(x,y)d\nu(y) \le \varepsilon \quad \text{and} \quad \sup_{x\in\Gamma}\int_{\Gamma\setminus B(R_3)} H_\Gamma(x,y)d\nu(y) \le \varepsilon.$$

Let $r > 0$ and $x \in \Gamma$. Then we have by Corollary 3.3 and Lemma 3.9

$$\begin{aligned}\int_{\Gamma\cap B(z_0,r)} H_\Gamma(x,y)d\nu(y) &\le 2\varepsilon + \int_{\Gamma\cap B(z_0,r)\cap B(R_3)\setminus B(x,r_3)} H_\Gamma(x,y)d\nu(y)\\ &\le 2\varepsilon + A\int_{\Gamma\cap B(z_0,r)} K_\Gamma(y,\infty)^2 d\nu(y).\end{aligned}$$

Hence (4.5) follows from Lemma 4.6. □

In the sequel, let $\alpha \ge 0$ and θ be as in Lemma 2.5. We give examples for the strictness of the inclusion $\mathcal{K}_0(\Gamma) \subset \mathcal{K}(\Gamma)$ when $\partial\Gamma \cap S(1)$ has a $C^{1,1}$-boundary. Note in this case that $\theta(z) \approx \delta_\Gamma(z)$ for $z \in \Gamma\cap S(1)$ and that if $\tau \ge 1$, then for $\xi \in \partial\Gamma\cap S(1)$ and $r > 0$,

$$\int_{\Gamma\cap B(\xi,r)} \delta_\Gamma(y)^{-\tau} dy = +\infty. \tag{4.6}$$

Since $\mathcal{K}_0(\Gamma) \subset L^1(\Gamma \cap B(2))$, we see in each example below that $V \in \mathcal{K}(\Gamma) \setminus \mathcal{K}_0(\Gamma)$ if $1 \le p < 2 < q$.

Example 4.8. Suppose that $\alpha \ge 1$ and $\theta(z) \approx \delta_\Gamma(z)$, and let

$$V(y) = (1 + |y|)^{p-q}\delta_\Gamma(y)^{-p}.$$

Then $V \in \mathcal{K}(\Gamma)$ if and only if $p < 2 < q$.

Proof. Note that $V(y) \approx (1+|y|)^{p-q}|y|^{p(\alpha-1)}K_\Gamma(y,\infty)^{-p}$. We first show the sufficiency, so we assume that $p < 2 < q$. Let $r > 0$ and $x \in \Gamma$. If $p < 0$, then $H_\Gamma(x,y)V(y) \le A|x-y|^{2-n}$. Therefore

$$\int_{\Gamma\cap B(x,r)} H_\Gamma(x,y)V(y)dy \le Ar^2,$$

and so V satisfies (4.2). We consider the case $p \ge 0$. It follows from Theorem 3.2 and (3.10) that for $b \in \mathcal{B}(x,y)$,

$$H_\Gamma(x,y)V(y) \le AK_\Gamma(b,\infty)^{-p}|x-y|^{2-n}(1+|y|)^{p-q}|y|^{p(\alpha-1)}.$$

Since $\alpha \ge 1$, it follows from Proposition 3.1 that $K_\Gamma(b,\infty) \approx |b|^{\alpha-1}\delta_\Gamma(b) \ge A|y|^{\alpha-1}|x-y|$. Therefore we have

$$\int_{\Gamma\cap B(x,r)} H_\Gamma(x,y)V(y)dy \le A\int_{\Gamma\cap B(x,r)} |x-y|^{2-n-p}dy \le Ar^{2-p}.$$

Hence V satisfies (4.2). We next show that V satisfies (4.3). Observe from Remark 3.10 and Lemma 2.5 that

$$H_\Gamma(x,y)V(y) \le A\frac{|y|^{2-q}}{|x-y|^n}.$$

Let $R > 0$. Then it follows from the above observation that

$$\begin{aligned}\sup_{x\in\Gamma}\int_{\Gamma\setminus B(R)} H_\Gamma(x,y)V(y)dy &\le A(r^{2-p}+I_1+I_2+I_3+I_4)\\ &\le A\left(r^{2-p}+R^{2-q}+\sup_{|x|\ge 2^{-1}R}|x|^{2-q}\log\frac{3|x|}{r}\right),\end{aligned}$$

where

$$\begin{aligned} I_1 &= \sup_{|x|\le 2^{-1}R}\int_{\Gamma\setminus(B(R)\cup B(x,r))}\frac{|y|^{2-q}}{|x-y|^n}dy,\\ I_2 &= \sup_{|x|\ge 2^{-1}R}\int_{\{|y|\le 2^{-1}|x|\}\setminus(B(R)\cup B(x,r))}\frac{|y|^{2-q}}{|x-y|^n}dy,\\ I_3 &= \sup_{|x|\ge 2^{-1}R}\int_{\{2^{-1}|x|\le|y|\le 2|x|\}\setminus(B(R)\cup B(x,r))}\frac{|y|^{2-q}}{|x-y|^n}dy,\\ I_4 &= \sup_{|x|\ge 2^{-1}R}\int_{\{2|x|\le|y|\}\setminus(B(R)\cup B(x,r))}\frac{|y|^{2-q}}{|x-y|^n}dy. \end{aligned} \tag{4.7}$$

Hence V satisfies (4.3), and thus $V \in \mathcal{K}(\Gamma)$. Conversely, we suppose that $V \in \mathcal{K}(\Gamma)$. Note that $H_\Gamma(x,y) \approx |x-y|^{2-n} \ge A\delta_\Gamma(x)^{2-n}$ for $y \in B(x, 2^{-1}\delta_\Gamma(x))$. Hence we have

$$H_\Gamma(x,y)V(y) \ge A\delta_\Gamma(x)^{2-n-p}(1+|x|)^{p-q} \quad \text{for } y \in B(x,2^{-1}\delta_\Gamma(x)).$$

Let $r > 0$, and let $x \in \Gamma \cap S(1)$ be a point such that $\delta_\Gamma(x) \le r$. Since $B(x, 2^{-1}\delta_\Gamma(x)) \subset \Gamma \cap B(x, r)$, it follows that

$$\int_{\Gamma \cap B(x,r)} H_\Gamma(x,y)V(y)dy \ge A\delta_\Gamma(x)^{2-p}.$$

Therefore it must be $p < 2$ to satisfy (4.2). Let $R > 0$. Similarly, if $x = \rho x_0$ with $\rho \ge 2R$, then $B(x, 2^{-1}\delta_\Gamma(x)) \subset \Gamma \setminus B(R)$, and so

$$\int_{\Gamma \setminus B(R)} H_\Gamma(x,y)V(y)dy \ge A\rho^{2-p}(1+\rho)^{p-q}.$$

Hence it must be $q > 2$ to satisfy (4.3). ☐

Example 4.9. Suppose that $0 \le \alpha < 1$ and $\theta(z) \approx \delta_\Gamma(z)$, and let

$$V(y) = (1+|y|)^{\alpha p - q}|y|^{p(1-\alpha)}\delta_\Gamma(y)^{-p}.$$

Then $V \in \mathcal{K}(\Gamma)$ if and only if $p < 2 < q$.

Proof. Note that $V(y) \approx (1+|y|)^{\alpha p - q}K_\Gamma(y,\infty)^{-p}$. We first assume that $p < 2 < q$. Let $0 < r < 1$ and $x \in \Gamma$. If $p < 0$, then $H_\Gamma(x,y)V(y) \le A|x-y|^{2-n}$. Therefore

$$\int_{\Gamma \cap B(x,r)} H_\Gamma(x,y)V(y)dy \le Ar^2,$$

and so V satisfies (4.2). We consider the case $p \ge 0$. If $y \in B(x,r)$, then $1+|y| \approx 1+|x|$, and so

$$K_\Gamma(b,\infty) \approx |b|^{\alpha-1}\delta_\Gamma(b) \ge A(1+|x|)^{\alpha-1}|x-y| \quad \text{for } b \in \mathcal{B}(x,y).$$

It follows from Theorem 3.2 and (3.10) that

$$\int_{\Gamma \cap B(x,r)} H_\Gamma(x,y)V(y)dy \le A(1+|x|)^{p-q}\int_{\Gamma \cap B(x,r)} |x-y|^{2-n-p}dy \le Ar^{2-p}.$$

Hence V satisfies (4.2). We next show that V satisfies (4.3). Observe from Remark 3.10 and Lemma 2.5 that

$$H_\Gamma(x,y)V(y) \le A\frac{|y|^{2\alpha-q}}{\max\{|x|,|y|\}^{2(\alpha-1)}|x-y|^n}.$$

Let $R > 0$. Then it follows from the above observation that

$$\begin{aligned}\sup_{x\in\Gamma}\int_{\Gamma\setminus B(R)} H_\Gamma(x,y)V(y)dy &\le A(r^{2-p} + I_1 + I_2' + I_3 + I_4)\\ &\le A\left(r^{2-p} + R^{2-q} + \sup_{|x|\ge 2^{-1}R}|x|^{2-q}\log\frac{3|x|}{r}\right),\end{aligned}$$

where I_1, I_3 and I_4 are as in (4.7), and

$$I_2' = \sup_{|x| \geq 2^{-1}R} \int_{\{|y| \leq 2^{-1}|x|\} \setminus (B(R) \cup B(x,r))} \frac{|y|^{2\alpha - q}}{|x|^{2(\alpha-1)}|x-y|^n} dy.$$

Hence V satisfies (4.3), and thus $V \in \mathcal{K}(\Gamma)$. Conversely, we suppose that $V \in \mathcal{K}(\Gamma)$. Observe that

$$H_\Gamma(x,y)V(y) \geq A\delta_\Gamma(x)^{2-n-p}(1+|x|)^{\alpha p - q}|x|^{p(1-\alpha)} \quad \text{for } y \in B(x, 2^{-1}\delta_\Gamma(x)).$$

Let $r > 0$ and let $x \in \Gamma \cap S(1)$ be a point such that $\delta_\Gamma(x) \leq r$. Then $B(x, 2^{-1}\delta_\Gamma(x)) \subset \Gamma \cap B(x,r)$, and so

$$\int_{\Gamma \cap B(x,r)} H_\Gamma(x,y)V(y)dy \geq A\delta_\Gamma(x)^{2-p}.$$

Therefore it must be $p < 2$ to satisfy (4.2). Let $R > 0$. Similarly, if $x = \rho x_0$ with $\rho \geq 2R$, then $B(x, 2^{-1}\delta_\Gamma(x)) \subset \Gamma \setminus B(R)$, and so

$$\int_{\Gamma \setminus B(R)} H_\Gamma(x,y)V(y)dy \geq A\rho^{2-\alpha p}(1+\rho)^{\alpha p - q}.$$

Therefore it must be $q > 2$ to satisfy (4.3). ☐

5 Nonlinear Schrödinger problem

In this section, we consider the following nonlinear Schrödinger problem:

$$\begin{cases} \Delta u - \mu u = f(\cdot, u) & \text{in } \Gamma \text{ (in the sense of distributions)}, \\ u > 0 & \text{in } \Gamma, \\ u = 0 & \text{on } \partial_r \Gamma, \end{cases} \tag{5.1}$$

where $\partial_r\Gamma$ denotes the set of all Dirichlet regular points of $\partial\Gamma$. We assume the following condition on μ and f. By $|\mu|$, we denote the total variational measure of a signed measure μ.

(P1) μ is a signed measure on Γ such that $|\mu| \in \mathcal{K}(\Gamma)$ and $\||\mu|\|_H < 2^{-1}$.

(P2) f is a Borel measurable function in $\Gamma \times (0, +\infty)$ such that $f(x, \cdot)$ is continuous in $(0, +\infty)$ for each $x \in \Gamma$.

(P3) $|f(x,t)| \leq t\psi(x,t)$ for $(x,t) \in \Gamma \times (0,+\infty)$, where ψ is some non-negative Borel measurable function in $\Gamma \times (0, +\infty)$ such that for each $x \in \Gamma$, $\psi(x, \cdot)$ is non-decreasing in $(0, +\infty)$ and $\lim_{t \to 0+} \psi(x,t) = 0$.

(P4) $\varphi(x) = \psi(x, K_\Gamma(x, \infty))$ belongs to $\mathcal{K}(\Gamma)$.

Theorem 5.1. *Let Γ be a uniform cone in $\mathbb{R}^n$ with $n \geq 3$. Assume that μ and f satisfy (P1)–(P4). Then the problem* (5.1) *has infinitely many continuous solutions. More precisely, there exists a positive constant λ_0 such that for each $\lambda \in (0, \lambda_0]$, the problem* (5.1) *has a continuous solution u satisfying that*

$$\frac{2(1-2|||\mu|||_H)}{3-2|||\mu|||_H}\lambda K_\Gamma(x,\infty) \leq u(x) \leq \frac{4}{3-2|||\mu|||_H}\lambda K_\Gamma(x,\infty) \quad \textit{for } x \in \Gamma, \tag{5.2}$$

and

$$\lim_{x\to\infty} \frac{u(x)}{K_\Gamma(x,\infty)} = \lambda. \tag{5.3}$$

The proof is based on the Schauder's fixed point argument. In the sequel, we suppose that Γ is a uniform cone in $\mathbb{R}^n$ with $n \geq 3$ and that μ and f satisfy (P1)–(P4). Let $C(\Gamma)$ denote the space of all bounded continuous functions in Γ endowed with the uniform norm $\|\cdot\|_\infty$, and let

$$C_0(\overline{\Gamma}\cup\{\infty\}) = \left\{w \in C(\overline{\Gamma}\cup\{\infty\}) : \lim_{x\to\infty} w(x) = 0\right\}.$$

We also write

$$C_1(\Gamma) = \{w \in C(\Gamma) : 0 < w(x) \leq 1 \text{ for } x \in \Gamma\}.$$

For $w \in C_1(\Gamma)$, we define

$$Fw(x) = \int_\Gamma H_\Gamma(x,y)w(y)d\mu(y) + \frac{1}{K_\Gamma(x,\infty)}\int_\Gamma G_\Gamma(x,y)f(y,w(y)K_\Gamma(y,\infty))dy.$$

In what follows, we write $d\nu(y) = d|\mu|(y) + \varphi(y)dy$ to simplify the notation. Note from (P1) and (P4) that $\nu \in \mathcal{K}(\Gamma)$.

Lemma 5.2. *The class $\{Fw : w \in C_1(\Gamma)\}$ is equicontinuous in $\overline{\Gamma}\cup\{\infty\}$. Moreover, $\{Fw : w \in C_1(\Gamma)\} \subset C_0(\overline{\Gamma}\cup\{\infty\})$.*

Proof. Let $z_0 \in \overline{\Gamma}$ and $\delta > 0$. Let $x_1, x_2 \in \Gamma \cap B(z_0, 2^{-1}\delta)$. It follows from (P3) and $w \leq 1$ that

$$\begin{aligned}
&|Fw(x_1) - Fw(x_2)| \\
&\quad \leq 2\sup_{x\in\Gamma}\int_{\Gamma\setminus B(\delta^{-1})} H_\Gamma(x,y)d\nu(y) + 2\sup_{x\in\Gamma}\int_{\Gamma\cap B(z_0,\delta)} H_\Gamma(x,y)d\nu(y) \\
&\qquad + \int_{\Gamma\cap B(\delta^{-1})\setminus B(z_0,\delta)} \left|\frac{G_\Gamma(x_1,y)}{K_\Gamma(x_1,\infty)} - \frac{G_\Gamma(x_2,y)}{K_\Gamma(x_2,\infty)}\right| K_\Gamma(y,\infty)d\nu(y).
\end{aligned}$$

By (4.3) and Proposition 4.7, the first two quantities of the right hand side are bounded by ε whenever δ is sufficiently small. If $z_0 \in \Gamma$, then $B(z_0,\delta) \subset \Gamma$ for sufficiently small δ, and so $G_\Gamma(\cdot,y)/K_\Gamma(\cdot,\infty)$ is continuous in $B(z_0, 2^{-1}\delta)$ whenever $y \in \Gamma \setminus B(z_0,\delta)$. If $z_0 \in \partial\Gamma$, then $G_\Gamma(\cdot,y)/K_\Gamma(\cdot,\infty)$ can be extended continuously to $\overline{\Gamma} \cap B(z_0,\delta_1)$ whenever $y \in \Gamma \setminus B(z_0,\delta)$ and $\delta_1 \in (0,\delta)$ is sufficiently small (cf. [2, Theorem 2]). In any cases, it follows from Corollary 4.3 and Lebesgue's convergence theorem that, as $|x_1 - x_2| \to 0$,

$$\int_{(\Gamma \cap B(\delta^{-1})) \setminus B(z_0,\delta)} \left| \frac{G_\Gamma(x_1,y)}{K_\Gamma(x_1,\infty)} - \frac{G_\Gamma(x_2,y)}{K_\Gamma(x_2,\infty)} \right| K_\Gamma(y,\infty)d\nu(y) \to 0.$$

Hence Fw is continuous in $\overline{\Gamma}$ uniformly for w. We next consider $z_0 = \infty$. Let $R > 0$. Then

$$|Fw(x)| \le \int_{\Gamma \cap B(R)} H_\Gamma(x,y)d\nu(y) + \sup_{z \in \Gamma} \int_{\Gamma \setminus B(R)} H_\Gamma(z,y)d\nu(y).$$

By (4.3), the second term of the right hand side is bounded by ε whenever R is sufficiently large. Note from Corollary 3.3 that

$$H_\Gamma(x,y) \le AR^{2-n-2\alpha} K_\Gamma(y,\infty)^2 \quad \text{for } x \in \Gamma \setminus B(2R) \text{ and } y \in \Gamma \cap B(R).$$

It follows from Proposition 4.5 and Lebesgue's convergence theorem that $|Fw(x)| \to 0$ uniformly for $w \in F$ as $x \to \infty$. Thus $\{Fw : w \in C_1(\Gamma)\}$ is equicontinuous in $\overline{\Gamma} \cup \{\infty\}$, and is contained in $C_0(\overline{\Gamma} \cup \{\infty\})$. ☐

Let $|||\mu|||_H < 2^{-1}$. For $\lambda > 0$, we define

$$\mathcal{W}_\lambda = \left\{ w \in C(\overline{\Gamma} \cup \{\infty\}) : \frac{2(1-2|||\mu|||_H)}{3-2|||\mu|||_H}\lambda \le w(x) \le \frac{4}{3-2|||\mu|||_H}\lambda \ \text{ for } x \in \Gamma \right\}.$$

Obviously, $\mathcal{W}_\lambda$ is a non-empty bounded closed convex set in $C(\overline{\Gamma} \cup \{\infty\})$. Note that if $\lambda \le 2^{-1}$, then $\mathcal{W}_\lambda \subset C_1(\Gamma)$. We define the operator T_λ on $\mathcal{W}_\lambda$ by

$$T_\lambda w(x) = \lambda - Fw(x) \quad \text{for } x \in \Gamma,$$

and write $T_\lambda(\mathcal{W}_\lambda) = \{T_\lambda w : w \in \mathcal{W}_\lambda\}$.

Lemma 5.3. *There exists a positive constant* $\lambda_0 \le 2^{-1}$ *such that if* $0 < \lambda \le \lambda_0$, *then* $T_\lambda(\mathcal{W}_\lambda) \subset \mathcal{W}_\lambda$. *Moreover,* $T_\lambda(\mathcal{W}_\lambda)$ *is relatively compact in* $C(\overline{\Gamma} \cup \{\infty\})$.

Proof. Let $0 < \lambda \le 2^{-1}$ and let $w \in \mathcal{W}_\lambda$. Since $Fw \in C(\overline{\Gamma} \cup \{\infty\})$ by Lemma 5.2, we have $T_\lambda w \in C(\overline{\Gamma} \cup \{\infty\})$. It suffices to show that there exists a positive constant $\lambda_0 \le 2^{-1}$ such that if $0 < \lambda \le \lambda_0$, then

$$\frac{2(1-2|||\mu|||_H)}{3-2|||\mu|||_H}\lambda \le T_\lambda w(x) \le \frac{4}{3-2|||\mu|||_H}\lambda \quad \text{for } x \in \Gamma. \tag{5.4}$$

Let $0 < \tau \le 1$ and define

$$\Psi_\tau(x) = \int_\Gamma H_\Gamma(x,y)\psi(y, \tau K_\Gamma(y,\infty))dy \quad \text{for } x \in \Gamma.$$

As in the proof of Lemma 5.2, we see that $\Psi_\tau \in C_0(\overline{\Gamma} \cup \{\infty\})$. Moreover, for each $x \in \Gamma$, it follows from (P3) that the function $\tau \mapsto \Psi_\tau(x)$ is non-decreasing and $\Psi_\tau(x) \to 0$ as $\tau \to 0$. Therefore we have by Dini's theorem

$$\lim_{\tau \to 0} \left(\sup_{x \in \Gamma} \Psi_\tau(x) \right) = 0.$$

We take $0 < \tau_0 \le 2^{-1}$ so that $\sup_{x\in\Gamma}\Psi_{\tau_0}(x) \le 4^{-1}(1-2\|\|\mu\|\|_H)$. Let $\lambda_0 = 4^{-1}(3-2\|\|\mu\|\|_H)\tau_0$ and let $0<\lambda\le\lambda_0$. Then $4(3-2\|\|\mu\|\|_H)^{-1}\lambda \le \tau_0$, and so it follows from (P3) that for $x\in\Gamma$,

$$|T_\lambda w(x)-\lambda| = |Fw(x)| \le \frac{4\lambda\|\|\mu\|\|_H}{3-2\|\|\mu\|\|_H} + \frac{4\lambda}{3-2\|\|\mu\|\|_H}\Psi_{\tau_0}(x) \le \frac{1+2\|\|\mu\|\|_H}{3-2\|\|\mu\|\|_H}\lambda.$$

Hence we obtain (5.4), and thus $T_\lambda(\mathcal{W}_\lambda)\subset\mathcal{W}_\lambda$. Since $T_\lambda(\mathcal{W}_\lambda)$ is uniformly bounded and is equicontinuous in $\overline{\Gamma}\cup\{\infty\}$ by Lemma 5.2, it follows from Ascoli-Arzelà's theorem that $T_\lambda(\mathcal{W}_\lambda)$ is relatively compact in $C(\overline{\Gamma}\cup\{\infty\})$. Thus the lemma is proved. □

Lemma 5.4. *Let* $0<\lambda\le\lambda_0$. *Then* T_λ *is continuous on* $\mathcal{W}_\lambda$.

Proof. Let $\{w_j\}$ be a sequence in $\mathcal{W}_\lambda$ converging to $w\in\mathcal{W}_\lambda$ with respect to $\|\cdot\|_\infty$. Then, for each $x\in\Gamma$, it follows from (P1)–(P4), Proposition 4.5 and Lebesgue's convergence theorem that, as $j\to+\infty$,

$$\begin{aligned}&\int_\Gamma H_\Gamma(x,y)|w_j(y)-w(y)|d|\mu|(y)\to 0,\\ &\frac{1}{K_\Gamma(x,\infty)}\int_\Gamma G_\Gamma(x,y)|f(y,w_j(y)K_\Gamma(y,\infty))-f(y,w(y)K_\Gamma(y,\infty))|dy\to 0.\end{aligned}$$

Hence $T_\lambda w_j$ converges pointwisely to $T_\lambda w$ in Γ as $j\to+\infty$. Since $T_\lambda(\mathcal{W}_\lambda)$ is relatively compact in $C(\overline{\Gamma}\cup\{\infty\})$, the pointwise convergence implies the uniform convergence. Therefore $\|T_\lambda w_j - T_\lambda w\|_\infty\to 0$ as $j\to+\infty$. Thus T_λ is continuous on $\mathcal{W}_\lambda$. □

Let us prove Theorem 5.1.

Proof of Theorem 5.1. Let $0<\lambda\le\lambda_0$, where λ_0 is the positive constant in Lemma 5.3. Note again that $\mathcal{W}_\lambda$ is a non-empty bounded closed convex set in $C(\overline{\Gamma}\cup\{\infty\})$. Since T_λ is compact mapping from $\mathcal{W}_\lambda$ into itself, it follows from Schauder's fixed point theorem that there exists $w\in\mathcal{W}_\lambda$ such that $w=T_\lambda w=\lambda-Fw$, that is, for $x\in\Gamma$,

$$w(x)=\lambda-\int_\Gamma H_\Gamma(x,y)w(y)d\mu(y)-\frac{1}{K_\Gamma(x,\infty)}\int_\Gamma G_\Gamma(x,y)f(y,w(y)K_\Gamma(y,\infty))dy.$$

Let $u(x)=w(x)K_\Gamma(x,\infty)$. Then $u>0$ in Γ, and $u=0$ on $\partial_r\Gamma$ since $K_\Gamma(\cdot,\infty)$ is so. Also, we have for $x\in\Gamma$,

$$\frac{2(1-2\|\|\mu\|\|_H)}{3-2\|\|\mu\|\|_H}\lambda K_\Gamma(x,\infty)\le u(x)\le\frac{4}{3-2\|\|\mu\|\|_H}\lambda K_\Gamma(x,\infty)$$

and

$$u(x)=\lambda K_\Gamma(x,\infty)-\int_\Gamma G_\Gamma(x,y)u(y)d\mu(y)-\int_\Gamma G_\Gamma(x,y)f(y,u(y))dy.$$

Since $K_\Gamma(\cdot, \infty)$ is harmonic in Γ, we see that u is a distributional solution to $\Delta u - \mu u = f(\cdot, u)$ in Γ. Finally, since $Fw \in C_0(\overline{\Gamma} \cup \{\infty\})$, it follows that

$$\lim_{x\to\infty} \frac{u(x)}{K_\Gamma(x,\infty)} = \lambda - \lim_{x\to\infty} Fw(x) = \lambda.$$

Thus the proof of Theorem 5.1 is complete. □

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