Estimates for the products of the Green function and the Martin kernel

Kentaro Hirata

Department of Mathematics, Hokkaido University,
Sapporo 060-0810, Japan *
e-mail: hirata@math.sci.hokudai.ac.jp

Abstract

Let Ω be a proper subdomain of $\mathbb{R}^n$, $n \geq 2$, and let $x_0 \in \Omega$ be fixed. By G_Ω and K_Ω we denote the Green function and the Martin kernel for Ω, respectively. Under a certain assumption on Ω near a boundary point ξ, we show that the product $G_\Omega(x, x_0)K_\Omega(x, \xi)$ is comparable to $|x - \xi|^{2-n}$ for x in a nontangential cone with vertex at ξ. We also give an estimate for the product $K_\Omega(x, \xi)K_\Omega(x, \eta)$ in a uniform domain, where η is another boundary point.

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1 Introduction

The purpose of this paper is to show a relationship between the boundary decay of the Green function and the boundary growth of the Martin kernel. This is motivated by the results [9, 10, 11, 12, 15] concerned with the boundary decay of the Green function for a Lipschitz domain and the result [18] concerned with the boundary growth of the Martin kernel near singularity. Now, we denote a point in $\mathbb{R}^n$ by $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

Theorem A. *Let $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ be a Lipschitz function such that $\phi(0') = 0$, and let $\Phi = \{(x', x_n) : x_n > \phi(x')\}$. Denote by $G_\Phi(\cdot, e)$ and $K_\Phi(\cdot, o)$ the Green function for Φ with pole at $e = (0', 1)$ and the Martin kernel of Φ with pole at $o = (0', 0)$, respectively. Define*

$$I^+ = \int_{\{|x'|<1\}} \frac{\max\{\phi(x'), 0\}}{|x'|^n} dx', \quad I^- = \int_{\{|x'|<1\}} \frac{\max\{-\phi(x'), 0\}}{|x'|^n} dx'.$$

Then the following statements hold.

*Current address: Faculty of Education and Human Studies, Akita University, Akita 010-8502, Japan e-mail: hirata@math.akita-u.ac.jp

(i) *If* $I^+ < +\infty$ *and* $I^- = +\infty$*, then*

$$\lim_{t\to 0+} \frac{G_\Phi(te,e)}{t} = +\infty \quad \textit{and} \quad \lim_{t\to 0+} \frac{K_\Phi(te,o)}{t^{1-n}} = 0.$$

(ii) *If* $I^+ = +\infty$ *and* $I^- < +\infty$*, then*

$$\lim_{t\to 0+} \frac{G_\Phi(te,e)}{t} = 0 \quad \textit{and} \quad \lim_{t\to 0+} \frac{K_\Phi(te,o)}{t^{1-n}} = +\infty.$$

(iii) *If* $I^+ < +\infty$ *and* $I^- < +\infty$*, then* $\lim_{t\to 0+} G_\Phi(te,e)/t$ *and* $\lim_{t\to 0+} K_\Phi(te,o)/t^{1-n}$ *exist, and each of them is positive and finite.*

The proof of Theorem A was based on the convergence of I^+, I^- and the minimal fine topology. The following question is natural: is the product $G_\Phi(te,e)K_\Phi(te,o)$ comparable to t^{2-n} for $0 < t < 1/2$? We shall show such an estimate in more general domains. Let Ω be a proper subdomain of $\mathbb{R}^n$, $n \geq 2$, and let $\delta_\Omega(x)$ stand for the distance from x to the boundary $\partial\Omega$. By $B(x,r)$ and $S(x,r)$, we denote the open ball and the sphere of center x and radius r, respectively.

Definition 1.1. We say that $\xi \in \partial\Omega$ satisfies *a local carrot condition* (abbreviated to LCC) if there exist constants $\kappa \geq 2$, $r_\xi > 0$ and $A_\xi \geq 1$ with the following property: for each positive $r \leq r_\xi$, there is a point $y_r \in \Omega \cap S(\xi,r)$ with $\delta_\Omega(y_r) \geq r/A_\xi$ such that each $x \in \Omega \cap B(\xi, r/\kappa)$ can be connected to y_r by a curve γ in $\Omega \cap B(\xi, \kappa r)$ for which

$$\ell(\gamma(x,z)) \leq A_\xi \delta_\Omega(z) \quad \text{for all } z \in \gamma, \tag{1.1}$$

where $\ell(\gamma(x,z))$ denotes the length of the subarc $\gamma(x,z)$ of γ from x to z.

Remark 1.2. In the study of minimal Martin boundary points of a John domain, Aikawa, Lundh and the author introduced the notion "a system of local reference points" by using the quasi-hyperbolic metric instead of the stronger condition (1.1). See [4, Definition 2.1]. For the above question, we do not need to assume a global condition on Ω, so we adopt (1.1) and the terminology "a local carrot condition".

Let $x_0 \in \Omega$ be fixed and $\alpha > 1$. A nontangential cone at $\xi \in \partial\Omega$ is denoted by

$$\Gamma_\alpha(\xi) = \{x \in \Omega \cap B(\xi, \delta_\Omega(x_0)/2) : |x - \xi| \leq \alpha\delta_\Omega(x)\}.$$

Note that $\Gamma_\alpha(\xi) \cap B(\xi, r)$ is nonempty for each $r > 0$ whenever (1.1) holds and $\alpha \geq A_\xi$. By the symbol A, we denote an absolute positive constant whose value is unimportant and may change from line to line. For two positive functions f_1 and f_2, we write $f_1 \approx f_2$ if there exists a constant $A \geq 1$ such that $f_1/A \leq f_2 \leq Af_1$. The constant A will be called the constant of comparison. The LCC at ξ implies that ξ has a unique Martin kernel (see Lemma 2.5). By $G_\Omega(\cdot, x_0)$ and $K_\Omega(\cdot, \xi)$, we denote the Green function for Ω with pole at x_0 and the Martin kernel of Ω at ξ, respectively.

Theorem 1.3. *Let* Ω *be a proper subdomain of* $\mathbb{R}^n$, $n \geq 3$*, and suppose that* $\xi \in \partial\Omega$ *satisfies the LCC. Then*

$$G_\Omega(x,x_0)K_\Omega(x,\xi) \approx |x-\xi|^{2-n} \quad \textit{for } x \in \Gamma_\alpha(\xi), \tag{1.2}$$

where the constant of comparison depends only on α, ξ *and* Ω.

Remark 1.4. In Section 4, we give a bounded domain such that (1.2) fails to hold, which is also a simple counterexample to the 3G inequality.

We say that Ω is *a uniform domain* if there exists a constant $A_0 \geq 1$ such that each pair of points $x, y \in \overline{\Omega}$ can be connected by a curve γ with $\gamma \setminus \{x, y\} \subset \Omega$ for which

$$\begin{aligned} &\ell(\gamma) \leq A_0 |x - y|, \\ &\min\{\ell(\gamma(x, z)), \ell(\gamma(z, y))\} \leq A_0 \delta_\Omega(z) \quad \text{for all } z \in \gamma. \end{aligned} \tag{1.3}$$

If Ω is a uniform domain, then all boundary points satisfy the LCC. Moreover, the constant of comparison in (1.2) can be taken independently of $\xi \in \partial\Omega$.

Corollary 1.5. *Let Ω be a uniform domain in $\mathbb{R}^n$, $n \geq 3$. Then*

$$G_\Omega(x, x_0) K_\Omega(x, \xi) \approx |x - \xi|^{2-n} \quad \textit{for } \xi \in \partial\Omega \textit{ and } x \in \Gamma_\alpha(\xi),$$

where the constant of comparison depends only on α and Ω.

Only the upper bound in Corollary 1.5 follows from the following 3G inequality. Let Ω be a bounded uniform domain in $\mathbb{R}^n$, $n \geq 3$. Then there exists a constant A depending only on Ω such that

$$\frac{G_\Omega(x, y) G_\Omega(x, z)}{G_\Omega(y, z)} \leq A \left(|x - y|^{2-n} + |x - z|^{2-n} \right) \quad \text{for } x, y, z \in \Omega. \tag{1.4}$$

See Cranston-Fabes-Zhao [13] for Lipschitz domains and Aikawa-Lundh [5] for uniformly John domains, and also Bogdan [8] and Hansen [17] in which a certain global estimate for the Green function was obtained. If we let $z = x_0$ and let $y \to \xi \in \partial\Omega$, then for $x \in \Omega \cap B(\xi, \delta_\Omega(x_0)/2)$,

$$K_\Omega(x, \xi) G_\Omega(x, x_0) \leq A(|x - \xi|^{2-n} + |x - x_0|^{2-n}) \leq A|x - \xi|^{2-n}.$$

Corollary 1.5 asserts that the product $G_\Omega(\cdot, x_0) K_\Omega(\cdot, \xi)$ is bounded from below by the function $|\cdot - \xi|^{2-n}$ as well.

The 3G inequality in two dimensions was proved by Bass-Burdzy [7]: for any bounded domains Ω in $\mathbb{R}^2$, there exists a constant A depending only on Ω such that

$$\frac{G_\Omega(x, y) G_\Omega(x, z)}{G_\Omega(y, z)} \leq A \left(1 + \log^+ \frac{1}{|x - y|} + \log^+ \frac{1}{|x - z|} \right) \quad \text{for } x, y, z \in \Omega.$$

If Ω is a bounded uniform domain in $\mathbb{R}^2$, then the same reasoning as above gives that for $x \in \Omega$ close to $\xi \in \partial\Omega$,

$$K_\Omega(x, \xi) G_\Omega(x, x_0) \leq A \log \frac{1}{|x - \xi|}.$$

When ξ is an isolated boundary point (i.e. $B(\xi, \varepsilon) \setminus \{\xi\} \subset \Omega$ for some $\varepsilon > 0$), this is sharp. Indeed, letting $\delta = \min\{1, \varepsilon, |x_0 - \xi|\}/2$, we obtain by the Harnack inequality that for $x \in B(\xi, \delta) \setminus \{\xi\}$,

$$K_\Omega(x, \xi) = \frac{G_{\Omega \cup \{\xi\}}(x, \xi)}{G_{\Omega \cup \{\xi\}}(x_0, \xi)} \geq \frac{G_{B(\xi, 2\delta)}(x, \xi)}{A G_\Omega(x_0, x)} \geq \frac{2\delta}{A G_\Omega(x, x_0)} \log \frac{1}{|x - \xi|}.$$

However, if Ω is the unit disc of $\mathbb{R}^2$, then $K_\Omega(r\xi, \xi)G_\Omega(r\xi, o) \approx 1$ for $\xi \in \partial\Omega$ and $1/2 < r < 1$. To obtain comparison estimate (1.2) for $n = 2$, we need some exterior condition. Let us define the Green capacity of a compact set E in an open set U by

$$\mathrm{Cap}_U(E) = \mu(U),$$

where μ is the associated Riesz measure of the regularized reduced function $\widehat{R}_1^E$ on U. We say that $\xi \in \partial\Omega$ satisfies *a capacity density condition* (abbreviated to CDC) if there exist constants $r'_\xi > 0$ and $A'_\xi > 0$ such that

$$\inf_{0<r<r'_\xi} \mathrm{Cap}_{B(\xi,2r)}(\overline{B(\xi, r)} \setminus \Omega) \geq A'_\xi.$$

Theorem 1.6. *Let Ω be a proper subdomain of $\mathbb{R}^2$, and suppose that $\xi \in \partial\Omega$ satisfies the LCC and the CDC. Then*

$$G_\Omega(x, x_0)K_\Omega(x, \xi) \approx 1 \quad \textit{for } x \in \Gamma_\alpha(\xi),$$

where the constant of comparison depends only on α, ξ and Ω.

A uniform domain Ω is said to be *NTA* if there are constants $r_0 > 0$ and $A > 1$ such that for each $\xi \in \partial\Omega$ and $0 < r < r_0$, there is a ball $B(z, r/A)$ contained in $B(\xi, r) \setminus \Omega$. Observe that all boundary points of an NTA domain satisfy the CDC, and the constants r'_ξ and A'_ξ can be taken uniformly for $\xi \in \partial\Omega$.

Corollary 1.7. *Let Ω be an NTA domain in $\mathbb{R}^2$. Then*

$$G_\Omega(x, x_0)K_\Omega(x, \xi) \approx 1 \quad \textit{for } \xi \in \partial\Omega \textit{ and } x \in \Gamma_\alpha(\xi),$$

where the constant of comparison depends only on α and Ω.

Remark 1.8. Since the Green function and the Martin kernel are conformal invariant (cf. [14, Section 6.3]), it is easy to see that if Ω is a Jordan domain in $\mathbb{R}^2$ and $\xi \in \partial\Omega$, then $G_\Omega(x, x_0)K_\Omega(x, \xi) \approx 1$ for $x \in \psi^{-1}(\{(r, 0) : 1/2 < r < 1\})$, where ψ is a conformal mapping from Ω onto the unit disc such that $\psi(x_0) = (0, 0)$ and $\psi(\xi) = (1, 0)$. In view of this, the LCC is not essential when $n = 2$. However $\partial\Omega$ does not need to be a Jordan curve and may have infinitely many components.

Without the assumptions on I^+, I^- in Theorem A, we can obtain the following relationships as a consequence of Corollaries 1.5 and 1.7.

Corollary 1.9. *Let Φ be as in Theorem A and let $\alpha > 0$. Then the following hold:*

(i) $\displaystyle\liminf_{t\to 0} \frac{G_\Phi(te, e)}{t^\alpha} = 0$ *if and only if* $\displaystyle\limsup_{t\to 0} \frac{K_\Phi(te, o)}{t^{2-n-\alpha}} = +\infty$.

(ii) $\displaystyle\limsup_{t\to 0} \frac{G_\Phi(te, e)}{t^\alpha} = +\infty$ *if and only if* $\displaystyle\liminf_{t\to 0} \frac{K_\Phi(te, o)}{t^{2-n-\alpha}} = 0$.

Next, we give an estimate for the product of two Martin kernels with different singularities in a uniform domain. Let $\xi, \eta \in \partial\Omega$ and let γ be a curve connecting ξ and η such that $\gamma \setminus \{\xi, \eta\} \subset \Omega$ and (1.3) holds. We denote by $z_{\xi,\eta}$ the middle point of γ so that $\ell(\gamma(\xi, z_{\xi,\eta})) = \ell(\gamma(z_{\xi,\eta}, \eta)) = \ell(\gamma)/2$, and define

$$g(\xi, \eta) = \max\left\{1, \frac{|\xi - \eta|^{2-n}}{G_\Omega(z_{\xi,\eta}, x_0)^2}\right\}.$$

Theorem 1.10. *Let Ω be a bounded uniform domain in $\mathbb{R}^n$, $n \geq 2$, and let $\xi, \eta \in \partial\Omega$ be distinct. Suppose that γ is a curve connecting ξ and η such that $\gamma \setminus \{\xi, \eta\} \subset \Omega$ and* (1.3) *holds. Then the following statements hold.*

(i) *If $n \geq 3$, then*

$$K_\Omega(x, \xi) K_\Omega(x, \eta) \approx g(\xi, \eta) \left(|x - \xi|^{2-n} + |x - \eta|^{2-n}\right) \quad \textit{for } x \in \gamma, \tag{1.5}$$

where the constant of comparison depends only on Ω.

(ii) *If $n = 2$ and Ω is a bounded NTA domain, then* (1.5) *holds.*

Corollary 1.11. *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 2$, and let $\xi, \eta \in \partial\Omega$ be distinct. Suppose that γ is a curve connecting ξ and η such that $\gamma \setminus \{\xi, \eta\} \subset \Omega$ and* (1.3) *holds. Then*

$$K_\Omega(x, \xi) K_\Omega(x, \eta) \approx \frac{1}{|\xi - \eta|^n} \left(|x - \xi|^{2-n} + |x - \eta|^{2-n}\right) \quad \textit{for } x \in \gamma,$$

where the constant of comparison depends only on Ω.

2 Preparatory material

Throughout this section, we suppose that Ω is a proper subdomain of $\mathbb{R}^n$, $n \geq 2$. The quasi-hyperbolic metric on Ω is defined by

$$k_\Omega(x, y) = \inf_\gamma \int_\gamma \frac{ds(z)}{\delta_\Omega(z)},$$

where the infimum is taken over all rectifiable curves γ in Ω connecting x and y, and ds stands for the line element on γ. We say that $\{B(x_j, \delta_\Omega(x_j)/2)\}_{j=1}^N$ is a Harnack chain joining x and y in Ω if $x_1 = x$, $x_N = y$ and $x_{j+1} \in B(x_j, \delta_\Omega(x_j)/2)$ for $j = 1, \ldots, N-1$. The number N is called the length of the Harnack chain. Observe that the shortest length of the Harnack chain joining x and y in Ω is comparable to $k_\Omega(x, y) + 1$. The following Harnack inequality is valid.

Lemma 2.1. *There exists a constant $A > 1$ depending only on the dimension n such that*

$$\exp(-A(k_\Omega(x, y) + 1)) \leq \frac{h(x)}{h(y)} \leq \exp(A(k_\Omega(x, y) + 1)) \quad \textit{for } x, y \in \Omega,$$

whenever h is a positive harmonic function on Ω.

To apply Lemma 2.1 to the Green function, we need the following lemma (cf. [4, Lemma 7.2]).

Lemma 2.2. *Let $z \in \Omega$. Then*

$$k_{\Omega\setminus\{z\}}(x,y) \le 3k_\Omega(x,y) + \pi \quad \textit{for } x,y \in \Omega \setminus B(z, \delta_\Omega(z)/2).$$

Lemma 2.3. *Suppose that $\xi \in \partial\Omega$ satisfies the LCC. Then there exists a constant A depending only on A_ξ such that if $0 < r < r_\xi$, then*

$$k_{\Omega\cap B(\xi,\kappa^3 r)}(x, y_r) \le A \log \frac{r}{\delta_\Omega(x)} + A \quad \textit{for } x \in \Omega \cap B(\xi, r/\kappa),$$

where $y_r \in \Omega \cap S(\xi, r)$ is as in Definition 1.1.

Proof. This follows from (1.1). □

Lemma 2.4. *Suppose that $\xi \in \partial\Omega$ satisfies the LCC. Let $0 < r < r_\xi$. If $z, w \in \Omega \setminus B(\xi, \kappa^3 r)$, then*

$$\frac{G_\Omega(x,z)}{G_\Omega(x,w)} \approx \frac{G_\Omega(y,z)}{G_\Omega(y,w)} \quad \textit{for } x,y \in \Omega \cap B(\xi, r/\kappa^3),$$

where the constant of comparison depends only on r_ξ, A_ξ and Ω.

Proof. This can be proved by the similar way as in [4], so we just skech the proof. Note from Lemma 2.3 that ξ has a system of local reference points y_r of order 1 (see [4, Definition 2.1] for its definition). The existence of a curve with (1.1) shows that there is $\tau > 0$ such that $\int_{\Omega\cap B(\xi,r)} (r/\delta_\Omega(x))^\tau dx \le Ar^n$ for $0 < r < r_\xi$ (see [4, Lemma 4.1]). As in [4, Lemma 5.1], we can obtain the following Carleson estimate: for $x \in \Omega \cap S(\xi, r/\kappa^2)$ and $z \in \Omega \setminus B(\xi, \kappa^3 r)$,

$$G_\Omega(x,z) \le AG_\Omega(y_r, z). \tag{2.1}$$

Let $\omega(x, E, U)$ denote the harmonic measure of a Borel set E for an open set U evaluated at x. Then the similar argument to [4, Lemma 6.1] gives that for $x \in \Omega \cap B(\xi, r/\kappa^3)$ and $w \in \Omega \setminus B(\xi, \kappa^3 r)$,

$$\omega(x, \Omega \cap S(\xi, r/\kappa^2), \Omega \cap B(\xi, r/\kappa^2)) \le A\frac{G_\Omega(x,w)}{G_\Omega(y_r,w)}. \tag{2.2}$$

Therefore the maximum principle, together with (2.1) and (2.2), yields that for $x \in \Omega \cap B(\xi, r/\kappa^3)$ and $z, w \in \Omega \setminus B(\xi, \kappa^3 r)$,

$$G_\Omega(x,z) \le A\frac{G_\Omega(y_r,z)}{G_\Omega(y_r,w)} G_\Omega(x,w).$$

Changing the roles of z and w, we obtain the opposite inequality. Thus the lemma follows. □

Let $\xi \in \partial\Omega$ and let $\{y_j\}$ be a sequence in Ω converging to ξ. Observe that there is a subsequence $\{y_{j_k}\}$ such that $\{G_\Omega(\cdot, y_{j_k})/G_\Omega(x_0, y_{j_k})\}$ converges to a positive harmonic function on Ω. We call such a limit function *the Martin kernel of* Ω *(with pole) at* ξ. A positive harmonic function h is said to be *minimal* if every positive harmonic function less than or equal to h coincides with a constant multiple of h.

Lemma 2.5. *Suppose that $\xi \in \partial\Omega$ satisfies the LCC. Then ξ has a unique Martin kernel and it is minimal.*

Proof. This follows from Lemma 2.4 and the Martin representation theorem. □

3 Proofs of Theorems 1.3 and 1.6

Proof of Theorem 1.3. Suppose that $\xi \in \partial\Omega$ satisfies the LCC and put

$$A_1 = \max\left\{\kappa^3, \frac{\delta_\Omega(x_0)}{r_\xi}\right\}.$$

We may assume without loss of generality that $r_\xi \le \delta_\Omega(x_0)/2$. Let $x \in \Gamma_\alpha(\xi)$ and let $r = |x-\xi|/(\kappa^3 A_1)$. Then $\kappa^3 r < r_\xi$, since $|x-\xi| < \delta_\Omega(x_0) \le A_1 r_\xi$. Also, we have $|x-\xi| \ge \kappa^6 r$ and $|x_0-\xi| \ge \delta_\Omega(x_0) \ge |x-\xi| \ge \kappa^6 r$. Let $y_r \in \Omega \cap S(\xi, r)$ be such that $\delta_\Omega(y_r) \ge r/A_\xi$. Then Lemma 2.4 gives

$$\frac{G_\Omega(x,y)}{G_\Omega(x_0,y)} \approx \frac{G_\Omega(x,y_r)}{G_\Omega(x_0,y_r)} \quad \text{for } y \in \Omega \cap B(\xi, r).$$

Letting $y \to \xi$, we obtain

$$K_\Omega(x,\xi) \approx \frac{G_\Omega(x,y_r)}{G_\Omega(x_0,y_r)}. \tag{3.1}$$

We claim

$$G_\Omega(x_0, y_r) \approx G_\Omega(x_0, x). \tag{3.2}$$

To show this, we consider two cases.

Case 1: $\rho := \kappa|x-\xi| < r_\xi$**.** The LCC and Lemma 2.3 show that there is $y_\rho \in \Omega \cap S(\xi, \rho)$ with $\delta_\Omega(y_\rho) \ge \rho/A_\xi$ such that

$$k_\Omega(z, y_\rho) \le A \log \frac{\rho}{\delta_\Omega(z)} + A \quad \text{for } z \in \Omega \cap \overline{B(\xi, \rho/\kappa)}.$$

Observe that $x, y_r \in \Omega \cap \overline{B(\xi, \rho/\kappa)}$, $\delta_\Omega(x) \ge |x-\xi|/\alpha = \rho/(\alpha\kappa)$ and $\delta_\Omega(y_r) \ge \rho/(A_\xi A_1 \kappa^4)$. Therefore

$$k_\Omega(x, y_\rho) \le A \quad \text{and} \quad k_\Omega(y_r, y_\rho) \le A.$$

Since $x, y_r, y_\rho \in \Omega \setminus B(x_0, \delta_\Omega(x_0)/2)$, it follows from Lemmas 2.1 and 2.2 that

$$G_\Omega(x_0, y_r) \approx G_\Omega(x_0, y_\rho) \approx G_\Omega(x_0, x).$$

Thus (3.2) holds in this case.

Case 2: $\kappa|x-\xi| \ge r_\xi$**.** Since $r \ge r_\xi/(A_1\kappa^4)$, it follows from the Harnack inequality on the compact set $\Gamma_\alpha(\xi) \setminus B(\xi, r_\xi/(A_1\kappa^4))$ that $G_\Omega(x_0, y_r) \approx G_\Omega(x_0, x)$, where the constant of comparison depends only on ξ and Ω. Thus (3.2) follows.

We next claim

$$G_\Omega(x, y_r) \approx |x-\xi|^{2-n}. \tag{3.3}$$

Let $w \in S(y_r, \delta_\Omega(y_r)/2)$. Then the similar argument as above gives

$$G_\Omega(x, y_r) \approx G_\Omega(w, y_r) \approx |w-y_r|^{2-n}. \tag{3.4}$$

Since $|w-y_r| \approx r \approx |x-\xi|$, we obtain (3.3). Combining (3.1), (3.2) and (3.3), we complete the proof of Theorem 1.3. $\square$

Proof of Corollary 1.5. If Ω is a uniform domain, then κ, r_ξ and A_ξ can be taken uniformly for $\xi \in \Omega$. Therefore (5.1) gives (3.2) and (3.3) with the comparison constant depending only on α and Ω. $\square$

Proof of Theorem 1.6. The proofs of (3.1), (3.2) and the first estimate in (3.4) are independent of the dimension. It is enough to show that $G_\Omega(w, y_r) \approx 1$ for $w \in S(y_r, \delta_\Omega(y_r)/2)$. This will be shown in Proposition 3.2 below. $\square$

Lemma 3.1. *Let Ω be a proper subdomain of $\mathbb{R}^n$, $n \ge 2$, and let $z, w \in \Omega$ satisfy $|z-w| \le \delta_\Omega(z)/4$. Suppose that u is a subharmonic function on $B(z, \delta_\Omega(z)) \cup B(w, \delta_\Omega(w))$ such that $u \le M$. If $u \le (1-\theta)M$ on $B(z, \delta_\Omega(z)/8)$ for some $0 < \theta < 1$, then*

$$u \le \Big(1 - \Big(\frac{4}{17}\Big)^n \theta\Big) M \quad \textit{on } B(w, \delta_\Omega(w)/8).$$

Proof. Let $x \in B(w, \delta_\Omega(w)/8)$. Observe that

$$B(z, \delta_\Omega(z)/8) \subset B(x, 17\delta_\Omega(z)/32) \subset B(w, \delta_\Omega(w)).$$

Write $E_1 = B(x, 17\delta_\Omega(z)/32)$ and $E_2 = E_1 \setminus B(z, \delta_\Omega(z)/8)$. By the mean value inequality, we have

$$\begin{aligned} u(x) &\le \frac{1}{|E_1|} \int_{E_1} u(y)dy \le \frac{1}{|E_1|} \big((1-\theta)M|E_1 \setminus E_2| + M|E_2|\big) \\ &\le M\Big(1 - \Big(\frac{4}{17}\Big)^n \theta\Big), \end{aligned}$$

where $|E|$ denotes the volume of a set E. Thus the lemma follows. $\square$

Proposition 3.2. *Let Ω be a proper subdomain of $\mathbb{R}^2$ and suppose that $\xi \in \partial\Omega$ satisfies the LCC and the CDC. Then*

$$G_\Omega(x, y) \approx 1 \quad \textit{for } x \in \Gamma_\alpha(\xi) \textit{ and } y \in S(x, \delta_\Omega(x)/2),$$

where the constant of comparison depends only on α, ξ and Ω.

Proof. Clearly, $G_\Omega(x,y) \ge G_{B(x,\delta_\Omega(x))}(x,y) \approx 1$ for $y \in S(x,\delta_\Omega(x)/2)$. Let us show

$$G_\Omega(x,y) \le A \quad \text{for } x \in \Gamma_\alpha(\xi) \text{ and } y \in S(x,\delta_\Omega(x)/2). \tag{3.5}$$

The method is based on Aikawa [3, Proof of Lemma 2]. The CDC at ξ implies that

$$\mathrm{Cap}_{B(\xi,2r)}(\overline{B(\xi,r)} \setminus \Omega) \ge A \quad \text{whenever } 0 < r < \delta_\Omega(x_0), \tag{3.6}$$

where $A > 0$ depends only on r'_ξ, A'_ξ and $\delta_\Omega(x_0)$. Let $r = \delta_\Omega(x)/2$ and let $M = \sup_{S(x,r)} G_\Omega(x,\cdot)$. Then the maximum principle gives that for $z \in \Omega \cap B(\xi,r)$,

$$G_\Omega(x,z) \le M\omega(z, S(x,r), \Omega \setminus \overline{B(x,r)}) \le M\omega(z, S(\xi,r), B(\xi,r) \setminus E),$$

where $E = \overline{B(\xi,r/2)} \setminus \Omega$ and $\omega(z,F,U)$ is the harmonic measure of a set F for an open set U evaluated at z. By [1, Lemma 3] and (3.6), we have

$$\sup_{B(\xi,r/2)} \omega(\cdot, S(\xi,r), B(\xi,r) \setminus E) \le 1 - \frac{1}{A}\,\mathrm{Cap}_{B(\xi,r)}(E) \le 1 - \theta,$$

where $0 < \theta < 1$. Therefore

$$G_\Omega(x,z) \le M(1-\theta) \quad \text{for } z \in \Omega \cap B(\xi,r/2). \tag{3.7}$$

Fix $z \in \Omega \cap S(\xi,r/4)$ with $\delta_\Omega(z) \ge r/(4\alpha)$, and let $w \in S(x,3r/2)$. Then $\delta_\Omega(w) \ge r/2$ and $|z-w| \le Ar$. We observe, as in the proof of Theorem 1.3, that

$$k_{\Omega\setminus\{x\}}(z,w) \le 3k_\Omega(z,w) + \pi \le A,$$

where A depends only on α, ξ and Ω. Therefore z and w can be joined by $\{B(w_j, \delta_{\Omega\setminus\{x\}}(w_j)/4)\}_{j=1}^N$ such that $w_1 = z$, $w_N = w$ and $w_{j+1} \in B(w_j, \delta_{\Omega\setminus\{x\}}(w_j)/4)$ for $j = 1,\dots,N-1$, where N depends only on α, ξ and Ω. Note from (3.7) that $G_\Omega(x,\cdot) \le M(1-\theta)$ on $B(w_1, \delta_{\Omega\setminus\{x\}}(w_1)/8)$. Apply Lemma 3.1 repeatedly. Then

$$G_\Omega(x,w) \le M\Big(1 - \Big(\frac{4}{17}\Big)^{nN}\theta\Big) \quad \text{for } w \in S(x, \frac{3}{2}r). \tag{3.8}$$

Observe that for $y \in B(x,3r/2)$,

$$G_{B(x,3r/2)}(x,y) = G_\Omega(x,y) - R_{G_\Omega(x,\cdot)}^{\Omega\setminus\overline{B(x,3r/2)}}(y),$$

where $R^F_{G_\Omega(x,\cdot)}$ is the reduced function of $G_\Omega(x,\cdot)$ relative to a set F in Ω. By (3.8),

$$\sup_{S(x,r)} G_\Omega(x,\cdot) - M\Big(1 - \Big(\frac{4}{17}\Big)^{nN}\theta\Big) \le \sup_{S(x,r)} G_{B(x,3r/2)}(x,\cdot) = \log\frac{3}{2}.$$

Hence we obtain $M \le \log(3/2)\cdot(17/4)^{nN}/\theta$, and thus (3.5) holds. $\square$

4 Counterexample

In this section, we give an example of a domain on which (1.2) fails to hold. Let us denote a point $x \in \mathbb{R}^n$ by $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, and write $o = (0', 0)$.

Example 4.1. Suppose that $n \geq 3$. Let Ω be the inverse of Ω^* with respect to $S(o,1)$, where

$$\Omega^* = \{(x', x_n) : |x'| < 1/2,\ x_n > 0\} \setminus \overline{B(o,1)}.$$

Let $x_0 = (0', 1/2)$. Then

$$\limsup_{x \to o,\ x \in E} \frac{G_\Omega(x, x_0) K_\Omega(x, o)}{|x|^{2-n}} = +\infty, \tag{4.1}$$

where $E = \{(0', x_n) : 0 < x_n < 1/4\}$.

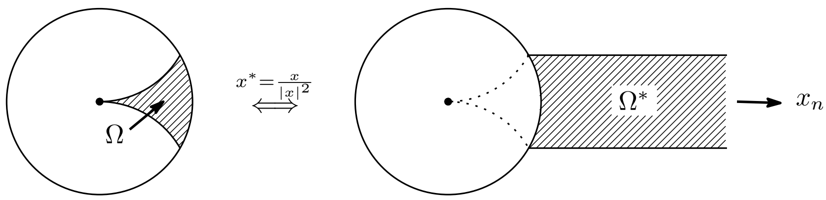

Figure 1: Ω and Ω^*.

Proof. Suppose to the contrary that there is a constant A such that

$$G_\Omega(x, x_0) K_\Omega(x, o) \leq A|x|^{2-n} \quad \text{for } x \in E.$$

Let $K_{\Omega^*}(\cdot, +\infty)$ denote the Martin kernel of Ω^* at $+\infty$, i.e. the limit function of $G_{\Omega^*}(\cdot, (y', y_n))/G_{\Omega^*}(x_0^*, (y', y_n))$ as $y_n \to +\infty$. Since

$$K_{\Omega^*}(x, +\infty) = \left(\frac{2}{|x|}\right)^{n-2} K_\Omega(x/|x|^2, o),$$
$$G_{\Omega^*}(x, x_0^*) = (2|x|)^{2-n} G_\Omega(x/|x|^2, x_0)$$

for $x \in \Omega^*$, it follows that for $x \in E^*$,

$$\begin{aligned} G_{\Omega^*}(x, x_0^*) K_{\Omega^*}(x, +\infty) &= |x|^{2(2-n)} G_\Omega(x/|x|^2, x_0) K_\Omega(x/|x|^2, o) \\ &\leq A|x|^{2-n}. \end{aligned} \tag{4.2}$$

Let $\omega = \{(x', x_n) : |x'| < 1/2,\ -\infty < x_n < +\infty\}$. Note that $\Omega^* \subset \omega$ and $\Omega^* \cap \{x_n > 1\} = \omega \cap \{x_n > 1\}$, and that the Martin kernels of ω at $+\infty$ and $-\infty$ are respectively of the form

$$K_\omega(x, +\infty) = e^{\tau x_n} f(x') \quad \text{and} \quad K_\omega(x, -\infty) = Ae^{-\tau x_n} f(x'), \tag{4.3}$$

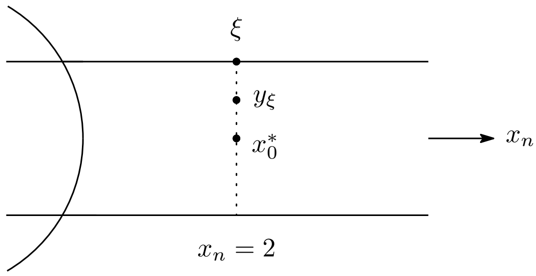

Figure 2: Positions of ξ and y_ξ.

where $\tau > 0$ and $A > 0$ are constants and f is a positive function on $\{x' \in \mathbb{R}^{n-1} : |x'| < 1/2\}$ vanishing continuously on $\{x' : |x'| = 1/2\}$. Let $\xi = (\xi', 2) \in \partial\omega$, and let y_ξ be the point in the line segment $\overline{\xi x_0^*}$ such that $|y_\xi - \xi| = 1/4$. The boundary Harnack principle gives

$$\frac{G_{\Omega^*}(y, x_0^*)}{K_\omega(y, -\infty)} \approx \frac{G_{\Omega^*}(y_\xi, x_0^*)}{K_\omega(y_\xi, -\infty)} \quad \text{for } y = (y', 2) \in \omega \cap B(\xi, 1/4),$$

where the constant of comparison is independent of y, y_ξ and ξ. Observe from the Harnack inequality that $G_{\Omega^*}(y, x_0^*) \geq A > 0$ and $K_\omega(y, -\infty) \approx K_\omega(x_0^*, -\infty) \approx 1$ for $y = (y', 2)$ with $\delta_\omega(y) \geq 1/4$. Therefore

$$K_\omega(y, -\infty) \leq A G_{\Omega^*}(y, x_0^*) \tag{4.4}$$

for $y = (y', 2) \in (\omega \cap B(\xi, 1/4)) \cup \{\delta_\omega(y) \geq 1/4\}$. The arbitrariness of $\xi = (\xi', 2) \in \partial\omega$ shows that (4.4) holds for all $y = (y', 2) \in \omega$, and so for all $y \in \{(y', y_n) \in \omega : y_n \geq 2\}$ by the maximum principle. It follows from (4.2) and (4.3) that for $x \in E^*$,

$$\frac{K_{\Omega^*}(x, +\infty)}{K_\omega(x, +\infty)} \approx K_\omega(x, -\infty) K_{\Omega^*}(x, +\infty) \leq A|x|^{2-n}.$$

As $x \in E^*$ and $x_n \to +\infty$, we have a contradiction, because

$$\limsup_{x_n \to +\infty} \frac{K_{\Omega^*}((0', x_n), +\infty)}{K_\omega((0', x_n), +\infty)} > 0 \tag{4.5}$$

(see Remark 4.2 below). Hence (4.1) holds. $\square$

Remark 4.2. We see from [6, Theorems 9.2.6 and 9.3.3] that

$$\limsup_{x_n \to +\infty} \frac{K_{\Omega^*}((x', x_n), +\infty)}{K_\omega((x', x_n), +\infty)} > 0.$$

As in the proof of Example 4.1, the boundary Harnack principle and the usual Harnack inequality give that for each $x_n \geq 2$,

$$\frac{K_{\Omega^*}((x', x_n), +\infty)}{K_\omega((x', x_n), +\infty)} \approx \frac{K_{\Omega^*}((0', x_n), +\infty)}{K_\omega((0', x_n), +\infty)} \quad \text{for } |x'| < 1/2.$$

Thus (4.5) follows.

Remark 4.3. Aikawa and Lundh [5] constructed a bounded domain in $\mathbb{R}^n$, $n \geq 3$, such that 3G inequality (1.4) fails to hold. A domain Ω in Example 4.1 is also one of conterexamples to (1.4). Indeed, as stated in the introduction, (1.4) implies that $G_\Omega(x, x_0)K_\Omega(x, o) \leq A|x|^{2-n}$ for $x \in \Omega$ close to o. But this contradicts (4.1).

5 Proof of Theorem 1.10

If Ω is a uniform domain, then the constants κ, r_ξ and A_ξ in (1.1) can be taken uniformly for $\xi \in \partial\Omega$. In this case, Lemma 2.4 is restated as follows: there is a constant $r_1 > 0$ depending only on Ω such that if $\xi \in \partial\Omega$ and $0 < r \leq r_1$, then

$$\frac{G_\Omega(x,z)}{G_\Omega(x,w)} \approx \frac{G_\Omega(y,z)}{G_\Omega(y,w)}$$

for $x, y \in \Omega \cap \overline{B(\xi, r)}$ and $z, w \in \Omega \setminus B(\xi, \kappa^6 r)$, where the constant of comparison depends only on Ω. This was indeed proved in [2] and is called *the uniform boundary Harnack principle* (abbreviated to UBHP). Recall that a uniform domain Ω is characterized in terms of the quasi-hyperbolic metric (cf. [16]):

$$k_\Omega(x,y) \leq A \log\left(\frac{|x-y|}{\min\{\delta_\Omega(x), \delta_\Omega(y)\}} + 1\right) + A \quad \text{for } x, y \in \Omega. \tag{5.1}$$

The following lemma is an elementary consequence of (5.1) and Lemma 2.1.

Lemma 5.1. *Let Ω be a uniform domain in $\mathbb{R}^n$, $n \geq 3$, or an NTA domain in $\mathbb{R}^2$. If $x, y \in \Omega$ satisfy $\delta_\Omega(y)/2 \leq |x-y| \leq A_2 \min\{\delta_\Omega(x), \delta_\Omega(y)\}$ for some constant A_2, then*

$$G_\Omega(x,y) \approx |x-y|^{2-n},$$

where the constant of comparison depends only on A_2 and Ω.

Proof of Theorem 1.10. We give a proof only when $n \geq 3$. We may assume without loss of generality that $\delta_\Omega(x_0) \geq (\kappa^6 + 2)A_0 r_1$, where A_0 is the constant in (1.3). Let $\xi, \eta \in \partial\Omega$ be distinct and let γ be a curve connecting ξ and η such that $\gamma \setminus \{\xi, \eta\} \subset \Omega$ and (1.3) holds. Put $r = |\xi - \eta|/(\kappa^6 + 2)$. We consider two cases.
Case 1: $r \leq r_1$**.** Let $x \in \gamma \cap \overline{B(\xi, r)}$. Then $x, x_0 \in \Omega \setminus B(\eta, \kappa^6 r)$. The UBHP gives

$$K_\Omega(x, \eta) \approx \frac{G_\Omega(x, w_\eta)}{G_\Omega(x_0, w_\eta)}, \tag{5.2}$$

where $w_\eta \in \gamma \cap S(\eta, r) \subset \Omega \setminus B(\xi, \kappa^6 r)$. We again apply the UBHP to obtain

$$\frac{G_\Omega(x, w_\eta)}{G_\Omega(x, x_0)} \approx \frac{G_\Omega(w_\xi, w_\eta)}{G_\Omega(w_\xi, x_0)}, \tag{5.3}$$

where $w_\xi \in \gamma \cap S(\xi, r)$. Note from (1.3) that $x \in \Gamma_{A_0}(\xi)$. Therefore (5.2), (5.3) and Corollary 1.5 give

$$K_\Omega(x, \eta) \approx \frac{G_\Omega(w_\xi, w_\eta)}{G_\Omega(w_\xi, x_0)G_\Omega(w_\eta, x_0)} \frac{|x-\xi|^{2-n}}{K_\Omega(x, \xi)}. \tag{5.4}$$

Let $z_{\xi,\eta}$ be the middle point of γ. Observe from (1.3) that $\delta_\Omega(w_\xi)$, $\delta_\Omega(w_\eta)$, $\delta_\Omega(z_{\xi,\eta})$ are greater than r/A_0, and that $|w_\xi - z_{\xi,\eta}|$, $|w_\eta - z_{\xi,\eta}|$ are bounded by $\ell(\gamma) \le A_0|\xi-\eta| = A_0(\kappa^6+2)r$. Therefore $k_\Omega(w_\xi, z_{\xi,\eta}) \le A$ and $k_\Omega(w_\eta, z_{\xi,\eta}) \le A$ by (5.1). Since $w_\xi, w_\eta, z_{\xi,\eta} \in \Omega \setminus B(x_0, \delta_\Omega(x_0)/2)$, it follows from Lemmas 2.1 and 2.2 that

$$G_\Omega(w_\xi, x_0) \approx G_\Omega(z_{\xi,\eta}, x_0) \approx G_\Omega(w_\eta, x_0). \tag{5.5}$$

Also, we have by Lemma 5.1

$$G_\Omega(w_\xi, w_\eta) \approx |w_\xi - w_\eta|^{2-n} \approx r^{2-n} \approx |\xi-\eta|^{2-n}. \tag{5.6}$$

Combining (5.4), (5.5) and (5.6), we obtain

$$K_\Omega(x,\xi)K_\Omega(x,\eta) \approx \frac{|\xi-\eta|^{2-n}}{G_\Omega(z_{\xi,\eta},x_0)^2}|x-\xi|^{2-n} \tag{5.7}$$

whenever $x \in \gamma \cap \overline{B(\xi,r)}$. If $x \in \gamma(\xi, z_{\xi,\eta}) \setminus B(\xi,r)$, then $|x-w_\xi| \le Ar \le A\delta_\Omega(x)$ by (1.3). Therefore Lemma 2.1 and (5.1) give

$$K_\Omega(x,\xi)K_\Omega(x,\eta) \approx K_\Omega(w_\xi,\xi)K_\Omega(w_\xi,\eta).$$

Since $|x-\xi| \approx r = |w_\xi - \xi|$, it follows from (5.7) with $x = w_\xi$ that (5.7) holds for $x \in \gamma(\xi, z_{\xi,\eta})$. Observe that $|x-\xi|^{2-n} \approx |x-\xi|^{2-n} + |x-\eta|^{2-n}$ for $x \in \gamma(\xi, z_{\xi,\eta})$ and $|\xi-\eta|^{2-n}/G_\Omega(z_{\xi,\eta},x_0)^2 \ge A(\Omega) > 0$. Hence we obtain

$$K_\Omega(x,\xi)K_\Omega(x,\eta) \approx g(\xi,\eta)\left(|x-\xi|^{2-n} + |x-\eta|^{2-n}\right) \tag{5.8}$$

for $x \in \gamma(\xi, z_{\xi,\eta})$. Similarly, we can obtain (5.8) for $x \in \gamma(z_{\xi,\eta}, \eta)$.
Case 2: $r > r_1$. Let $x \in \gamma \cap \overline{B(\xi, r_1)}$ and let $w_0 \in \gamma \cap S(\xi, r_1)$. Then

$$K_\Omega(w_0,\eta) \approx 1 \quad \text{and} \quad G_\Omega(w_0,x_0) \approx 1,$$

where the constants of comparisons depend on r_1, $\delta_\Omega(x_0)$ and $\operatorname{diam}(\Omega)$. Note that $|\xi-\eta| = (\kappa^6+2)r \ge \kappa^6 r_1$. By the UBHP and Corollary 1.5,

$$K_\Omega(x,\eta) \approx \frac{K_\Omega(w_0,\eta)}{G_\Omega(w_0,x_0)}G_\Omega(x,x_0) \approx \frac{|x-\xi|^{2-n}}{K_\Omega(x,\xi)} \approx \frac{|x-\xi|^{2-n}+|x-\eta|^{2-n}}{K_\Omega(x,\xi)}.$$

If $x \in \gamma(\xi, z_{\xi,\eta}) \setminus B(\xi, r_1)$, then $\delta_\Omega(x) \ge r_1/A_0$ by (1.3), and so

$$K_\Omega(x,\xi) \approx 1 \approx K_\Omega(x,\eta) \quad \text{and} \quad |x-\xi| \approx 1 \approx |x-\eta|,$$

where the constants of comparisons depend on r_1/A_0, $\delta_\Omega(x_0)$ and $\operatorname{diam}(\Omega)$. Since $|\xi-\eta|^{2-n}/G_\Omega(z_{\xi,\eta},x_0)^2 \le A(\Omega)$, we obtain $K_\Omega(x,\xi)K_\Omega(x,\eta) \approx g(\xi,\eta)(|x-\xi|^{2-n}+|x-\eta|^{2-n})$ for $x \in \gamma(\xi, z_{\xi,\eta})$. Similarly, we obtain this for $x \in \gamma(z_{\xi,\eta},\eta)$. Thus the proof of Theorem 1.10 is complete. □

Proof of Corollary 1.11. Let γ be a curve connecting ξ and η such that $\gamma \setminus \{\xi,\eta\} \subset \Omega$ and (1.3) holds, and let $z_{\xi,\eta}$ be the middle point of γ. Then

$$\frac{1}{2A_0}|\xi-\eta| \le \frac{1}{A_0}\ell(\gamma(\xi,z_{\xi,\eta})) \le \delta_\Omega(z_{\xi,\eta}) \le \ell(\gamma(\xi,z_{\xi,\eta})) \le A_0|\xi-\eta|.$$

It is known that if Ω is a bounded $C^{1,1}$-domain, then $G_\Omega(z,x_0) \approx \delta_\Omega(z)$ for $z \in \Omega \setminus B(x_0, \delta_\Omega(x_0)/2)$. Hence Corollary 1.11 follows from Theorem 1.10. □

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