Global estimates for non-symmetric Green type functions with applications to the p-Laplace equation [*]

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Abstract

This paper presents global estimates for non-symmetric Green type functions, which are applicable to singular functions for the p-Laplace equation.

Keywords: global estimate, quasi-symmetry, p-Green function, p-Martin kernel, p-harmonic measure
Mathematics Subject Classifications (2000): 31C45, 35J60

1 Introduction

In the study of the potential theory and its related fields, the Green function plays an important role. However it can not be represented explicitly except for the case of balls and the half-space, because its behavior depends on the shape of a given domain. The local behavior of the Green function near a singularity is independent of the shape of a domain and were investigated for general operators and domains. Grüter and Widman [9] obtained a local estimate of the Green function for uniformly elliptic operators which are not necessarily symmetric. Serrin [14, 15] and Kichenassamy and Véron [13] studied the local behavior of positive solutions of the p-Laplace equation near an isolated singularity. In contrast to a local estimate, a global estimate is effected by the shape of a domain. In a bounded $C^{1,1}$-domain Ω in $\mathbb{R}^n$ ($n \geq 3$), Zhao [16] established a global estimate of the Green function for the (classical) Laplacian. Indeed, in this case, the Green function $G(x, y)$ is estimated by using explicit functions:

$$G(x, y) \approx \min\left\{1, \frac{\delta_\Omega(x)\delta_\Omega(y)}{|x-y|^2}\right\} |x-y|^{2-n} \quad \text{for } x, y \in \Omega,$$

[*]This work was partially supported by Grant-in-Aid for Young Scientists (B) (No. 19740062), Japan Society for the Promotion of Science.

where $\delta_\Omega(x)$ is the distance from x to the boundary $\partial\Omega$ and the symbol $\approx$ means that the functions in the both sides are comparable. See [7] for two dimensional result. The estimate of this kind for the Green function for the fractional Laplacian was obtained by Chen and Song [6]. In non-smooth domains, the decay rate of the Green function is different at each boundary point, and for this reason it is impossible to estimate the Green function in terms of the explicit functions. To overcome this difficulty, Bogdan [5] introduced a supplementary set $\mathcal{B}(x,y)$ and established the following estimate in a bounded Lipschitz domain Ω:

$$G(x,y) \approx \frac{g(x)g(y)}{g(b)^2}|x-y|^{2-n} \quad \text{for } x,y\in\Omega \text{ and } b\in\mathcal{B}(x,y),$$

where $g(x)=\min\{1,G(x,x_0)\}$ with $x_0\in\Omega$ being fixed and the definition of $\mathcal{B}(x,y)$ is given in the sequel. Recently, Hansen [10] obtained this estimate in a bounded uniform domain. Actually, he discussed for a general function G satisfying properties stated below and the quasi-symmetry $G(x,y)\approx G(y,x)$ for $x,y\in\Omega$. In [12], the author established a global estimate for the Green function in a uniform cone. In this case, g is replaced by the Martin kernel at infinity.

The purpose of this paper is to present a global estimate for non-symmetric Green functions (of course, not quasi-symmetric). By the symbol A, we denote an absolute positive constant whose value is unimportant and may change from line to line. For two positive functions f and g, we write $f\approx g$ if there exists a constant A such that $f/A\le g\le Af$. The constant A will be called the constant of comparison. By $B(x,r)$, we denote the open ball of center x and radius r. First, we discuss on a pair of a bounded domain Ω in $\mathbb{R}^n$ ($n\ge 2$) and a function $G:\Omega\times\Omega\to(0,+\infty]$ with the following properties (I)–(IV) and either (V1) or (V2): Let $r_0>0$, $M>1$, $A_0>1$, $A_1>1$ and $\lambda>0$ be fixed constants.

(I) *Interior corkscrew condition*: For each $\xi\in\partial\Omega$ and $0<r<r_0$, there is $z\in\Omega\cap B(\xi,r)$ such that $\delta_\Omega(z)\ge r/M$.

(II) *Comparison principle*: Let D be an open set in Ω and let $y_1,y_2\in\Omega\setminus D$. Suppose that $f=G(\cdot,y_2)$ or $f=1$. If $G(\cdot,y_1)\le Af$ on $\partial D\cap\Omega$, then $G(\cdot,y_1)\le Af$ on D.

(III) *Harnack's inequality*: Let $k\in\mathbb{N}$. There is a constant $A=A(k)$ such that for each $y\in\Omega$, we have

$$G(x_1,y)\le AG(x_2,y)$$

whenever $x_1,x_2\in\Omega\setminus B(y,\delta_\Omega(y)/8)$ satisfy $|x_1-x_2|\le k\min\{\delta_\Omega(x_1),\delta_\Omega(x_2)\}$.

(IV) *Boundary Harnack principle*: For each $\xi\in\partial\Omega$ and $0<r<r_0$, we have

$$\frac{G(x_1,y_1)}{G(x_1,y_2)}\le A_1\frac{G(x_2,y_1)}{G(x_2,y_2)},$$

whenever $x_1,x_2\in\Omega\cap B(\xi,r)$ and $y_1,y_2\in\Omega\setminus B(\xi,A_0r)$.

(V1) *Singularity of logarithmic order*: For any fixed $y \in \Omega$,

$$G(x,y) \approx \log \frac{\delta_\Omega(y)}{|x-y|} \quad \text{for } x \in B(y, \delta_\Omega(y)/2).$$

(V2) *Singularity of finite order*: For any fixed $y \in \Omega$,

$$G(x,y) \approx |x-y|^{-\lambda} \quad \text{for } x \in B(y, \delta_\Omega(y)/2).$$

Here the constants of comparison in (V1) and (V2) are independent of x, y.

Remark 1.1. We do not impose the quasi-symmetry of G: $G(x,y) \approx G(y,x)$ for $x, y \in \Omega$.

The setup is like stated above. In particular, we fix $x_0 \in \Omega$ and put

$$g(x) = \min\{1, G(x, x_0)\} \quad \text{and} \quad g^*(x) = \min\{1, G(x_0, x)\}.$$

Let $\kappa \geq 1$. For $x, y \in \overline{\Omega}$, we define

$$\mathcal{B}(x,y) = \left\{ b \in \Omega : \frac{1}{\kappa} \max\{|x-b|, |b-y|\} \leq |x-y| \leq \kappa \delta_\Omega(b) \right\}.$$

This simple definition is different from one defined by Bogdan [5], but they are the same essentially. Observe from (I) and the boundedness of Ω that if κ is sufficiently large, then $\mathcal{B}(x,y)$ is nonempty for any pair x, y. The main result is as follows.

Theorem 1.2. *Suppose that a bounded domain Ω and a function $G : \Omega \times \Omega \to (0, +\infty]$ satisfy* (I)–(IV). *Then the following statements hold:*

(i) *If G satisfies* (V1)*, then we have for $x, y \in \Omega$ and $b \in \mathcal{B}(x,y)$,*

$$G(x,y) \approx \frac{g(x) g^*(y)}{g(b) g^*(b)} \left(1 + \log^+ \frac{\min\{\delta_\Omega(x), \delta_\Omega(y)\}}{|x-y|} \right), \tag{1.1}$$

where $\log^+ f = \max\{0, \log f\}$ and the constant of comparison depends only on Ω and the constants appearing in our setting.

(ii) *If G satisfies* (V2)*, then we have for $x, y \in \Omega$ and $b \in \mathcal{B}(x,y)$,*

$$G(x,y) \approx \frac{g(x) g^*(y)}{g(b) g^*(b)} |x-y|^{-\lambda}, \tag{1.2}$$

where the constant of comparison depends only on Ω and the constants appearing in our setting.

Immediately, we have the following.

Corollary 1.3. *The assumption is the same as Theorem 1.2. If $g \approx g^*$ on Ω, then G is quasi-symmetric: $G(x,y) \approx G(y,x)$ for $x, y \in \Omega$.*

Other applications of Theorem 1.2 to the p-Laplace equation will be stated in the next section.

2 Some applications and known properties of p-harmonic functions

In this section, we give some applications of Theorem 1.2. Let $1 < p < \infty$ and let Ω be a bounded domain in $\mathbb{R}^n$ ($n \geq 2$). A function $u \in W^{1,p}_{loc}(\Omega)$ is said to be *p-harmonic* on Ω if it is continuous on Ω and satisfies the p-Laplace equation

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in } \Omega \tag{2.1}$$

in the weak sense, that is,

$$\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla \phi\rangle dx = 0 \quad \text{for all } \phi \in C^1_0(\Omega).$$

A lower semicontinuous function $u : \Omega \to (-\infty, +\infty]$, where $u \not\equiv +\infty$, is called *p-superharmonic* on Ω if for each open set D with $\overline{D} \subset \Omega$ and each function h p-harmonic on D and continuous on $\overline{D}$, the inequality $u \geq h$ on ∂D implies $u \geq h$ on D. If $-u$ is p-superharmonic on Ω, then u is called *p-subharmonic* on Ω.

Let ν_n be the volume of the unit ball in $\mathbb{R}^n$, and let $c_{n,p} = (p-1)/(n-p)(n\nu_n)^{1/(p-1)}$ and $c_n = (n\nu_n)^{-1/(n-1)}$. Denote

$$\mu_y(x) = \begin{cases} c_{n,p}|x-y|^{(p-n)/(p-1)} & \text{if } 1 < p < n, \\ c_n \log \dfrac{1}{|x-y|} & \text{if } p = n. \end{cases}$$

This is the fundamental solution for Δ_p. Let δ_y be the Dirac measure at y. Kichenassamy and Véron [13] proved that if $1 < p \leq n$ and Ω is a smooth domain, then for each $y \in \Omega$ there exists a unique weak solution $u \in C^{1,\alpha}_{loc}(\Omega \setminus \{y\})$ of

$$\begin{cases} -\Delta_p u = \delta_y & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

such that $|\nabla u|^{p-1} \in L^1_{loc}(\Omega)$, $\nabla u \in L^p(\Omega \setminus B(y,r))$ for each $r > 0$, and $u/\mu_y \in L^\infty(\Omega)$. This solution is often called a *p-Green function* with pole at y and denoted by $G_p(\cdot, y)$. Note that the quasi-symmetry of G_p is unknown when $p \neq 2$.

Now, let $\{y_j\}$ be a sequence in Ω converging to $\xi \in \partial\Omega$. Then there is a subsequence $\{y_{j_k}\}$ such that the ratio $G_p(\cdot, y_{j_k})/G_p(x_0, y_{j_k})$ converges locally uniformly to a positive p-harmonic function on Ω. Such a limit function is called a *p-Martin kernel* with pole at ξ and written as $K_p(\cdot, \xi)$. Recently, Aikawa, Kilpeläinen, Shanmugalingam and Zhong [2] and Bidaut-Véron, Borghol and Véron [4] proved independently a boundary Harnack principle for p-harmonic functions in smooth domains and obtained, as a consequence, that there exist constants $\alpha > 1$ and $A > 1$ such that

$$\frac{1}{A}\delta_\Omega(x) \leq K_p(x,\xi) \leq A\frac{\delta_\Omega(x)}{|x-\xi|^\alpha} \quad \text{for } x \in \Omega.$$

We shall present some improvement.

Next, we recall the definition of p-harmonic measure. Let E be a subset of $\partial\Omega$ and let $\mathcal{X}_E$ denote the characteristic function of E. Define

$$\omega_p(x, E, \Omega) = \inf u(x) \quad \text{for } x \in \Omega,$$

where the infimum is taken over all nonnegative p-superharmonic functions u on Ω satisfying

$$\liminf_{x \to y} u(x) \geq \mathcal{X}_E(y) \quad \text{for all } y \in \partial\Omega.$$

Then $\omega_p(\cdot, E, \Omega)$ is p-harmonic on Ω and $0 \leq \omega_p(\cdot, E, \Omega) \leq 1$. See [11, Chapter 11] for more informations. We say $\omega_p(x_0, E, \Omega)$ a *p-harmonic measure* of E with respect to Ω for convenience, although this is not a measure except for the case $p = 2$.

2.1 Known properties

The following comparison principle and Harnack inequality are found in [11].

Lemma 2.1. *Suppose that u and v are respectively p-superharmonic and p-subharmonic functions on Ω satisfying*

$$\limsup_{\Omega \ni x \to \xi} v(x) \leq \liminf_{\Omega \ni x \to \xi} u(x) \quad \textit{for each } \xi \in \partial\Omega,$$

where both sides are not simultaneously $+\infty$ or $-\infty$. Then $v \leq u$ on Ω.

Lemma 2.2. *There exists a constant A depending only on p and n such that*

$$\sup_{B(x,r)} u \leq A \inf_{B(x,r)} u$$

for a nonnegative p-harmonic function u on $B(x, 2r)$.

The quasihyperbolic metric on Ω is defined by

$$k_\Omega(x, y) = \int_\gamma \frac{ds(z)}{\delta_\Omega(z)} \quad \text{for } x, y \in \Omega,$$

where the infimum is taken over all rectifiable curves γ in Ω connecting x and y and ds is the line element on γ. Observe that if Ω is a bounded $C^{1,1}$-domain, then there exists a constant A depending only on Ω such that

$$k_\Omega(x, y) \leq A \log^+ \frac{|x - y|}{\min\{\delta_\Omega(x), \delta_\Omega(y)\}} + A \quad \text{for } x, y \in \Omega. \tag{2.3}$$

Actually, this holds for uniform domains (see [8]). Also, if $z \in \Omega$ is fixed, then

$$k_{\Omega \setminus \{z\}}(x, y) \leq 3k_\Omega(x, y) + \pi \quad \text{for } x, y \in \Omega \setminus B(z, \delta_\Omega(z)/2). \tag{2.4}$$

See [1, Lemma 7.2]. A finite sequence of balls $\{B(x_j, \delta_\Omega(x_j)/2)\}_{j=1}^N$ is called a *Harnack chain* joining x and y in Ω if $x_1 = x$, $x_N = y$ and $x_{j+1} \in B(x_j, \delta_\Omega(x_j)/2)$ for $j = 1, \ldots, N-1$. The number N is called the *length* of the Harnack chain. Observe that the shortest length of the Harnack chain joining x and y in Ω is comparable to $k_\Omega(x, y) + 1$. Thus the following Harnack inequality is valid.

Lemma 2.3. *There exists a constant $A > 1$ depending only on p and n such that*

$$\exp(-A(k_\Omega(x,y)+1)) \le \frac{u(x)}{u(y)} \le \exp(A(k_\Omega(x,y)+1)) \quad \textit{for } x,y \in \Omega,$$

whenever u is a positive p-harmonic function on Ω.

This, together with (2.3) and (2.4), gives the following.

Lemma 2.4. *Suppose that Ω is a bounded $C^{1,1}$-domain. Let $z \in \Omega$ and $k \in \mathbb{N}$. Then there exists a constant A depending only on k, p and Ω such that for a positive p-harmonic function u on $\Omega \setminus \{z\}$, we have*

$$u(x) \le Au(y),$$

whenever $x, y \in \Omega \setminus B(z, \delta_\Omega(z)/2)$ satisfy $|x-y| \le k \min\{\delta_\Omega(x), \delta_\Omega(y)\}$.

Recently, the following boundary Harnack principle was established in [2, 4]. See also [3].

Lemma 2.5. *Suppose that Ω is a bounded $C^{1,1}$-domain. Then there exists a constant A depending only on p and Ω such that for each $\xi \in \partial\Omega$ and $0 < r < r_0$, we have*

$$\frac{u(x)}{v(x)} \le A\frac{u(y)}{v(y)} \quad \textit{for } x,y \in \Omega \cap B(\xi, r),$$

whenever u and v are positive p-harmonic functions on $\Omega \cap B(\xi, 2r)$ vanishing continuously on $\partial\Omega \cap B(\xi, 2r)$. Moreover,

$$G_p(x, x_0) \approx \delta_\Omega(x) \quad \textit{whenever } \delta_\Omega(x) < r_0.$$

Lemma 2.6. *Suppose that Ω is a bounded domain satisfying the exterior corkscrew condition. Then for $y \in \Omega$ and $x \in B(y, \delta_\Omega(y)/2)$,*

$$G_p(x,y) \approx \begin{cases} |x-y|^{(p-n)/(p-1)} & \textit{if } 1 < p < n, \\ \log \dfrac{\delta_\Omega(y)}{|x-y|} & \textit{if } p = n, \end{cases} \tag{2.5}$$

where the constant of comparison depends only on p and Ω.

Proof. In this proof, we denote by $G_{p,\Omega}(x,y)$ the p-Green function for Ω. Observe that for $x \in B(y,r)$,

$$G_{p,B(y,r)}(x,y) = \begin{cases} c_{n,p}\left(|x-y|^{(p-n)/(p-1)} - r^{(p-n)/(p-1)}\right) & \text{if } 1 < p < n, \\ c_n \log \dfrac{r}{|x-y|} & \text{if } p = n. \end{cases}$$

Therefore

$$G_{p,\Omega}(x,y) \ge G_{p,B(y,\delta_\Omega(y))}(x,y) \ge \begin{cases} \dfrac{1}{A}|x-y|^{(p-n)/(p-1)} & \text{if } 1 < p < n, \\ c_n \log \dfrac{\delta_\Omega(y)}{|x-y|} & \text{if } p = n. \end{cases}$$

Also, if $1 < p < n$, then

$$G_{p,\Omega}(x,y) \le c_{n,p}|x-y|^{(p-n)/(p-1)}.$$

Hence (2.5) holds in the case $1 < p < n$.

Consider the case $p = n$. If $\delta_\Omega(y) \ge r_0$, then

$$G_{p,\Omega}(x,y) \le G_{p,B(y,\operatorname{diam}\Omega)}(x,y) \le A \log \frac{\delta_\Omega(y)}{|x-y|}.$$

If $\delta_\Omega(y) < r_0$, then the exterior corkscrew condition implies that there is a point $w \in B(y, 2\delta_\Omega(y))$ such that $B(w, \delta_\Omega(y)/A) \subset \mathbb{R}^n \setminus \overline{\Omega}$. Therefore

$$G_{n,\Omega}(x,y) \le G_{n,\mathbb{R}^n \setminus \overline{B(w,\delta_\Omega(y)/A)}}(x,y) \le A \log \frac{\delta_\Omega(y)}{|x-y|},$$

where the last inequality follows from the formula of $G_{n,\mathbb{R}^n\setminus\overline{B(w,r)}}(x,y)$ obtained in [2] (see (5.2) in this paper). Thus the lemma is proved. $\square$

2.2 Applications of Theorem 1.2

It is well known that if Ω is a bounded $C^{1,1}$-domain, then there exists $r_0 > 0$ such that for each $\xi \in \partial\Omega$, there are two balls $B(z_\xi, r_0) \subset \Omega$ and $B(w_\xi, r_0) \subset \mathbb{R}^n \setminus \overline{\Omega}$ satisfying $\partial B(z_\xi, r_0) \cap \partial B(w_\xi, r_0) = \{\xi\}$. In the sequel, we fix r_0, z_ξ and w_ξ. By $\overline{z_\xi \xi}$, we denote the (open) line segment between z_ξ and ξ. Lemmas 2.1, 2.4 and 2.5 imply that $G = G_p$ satisfies (II)–(IV). Also, we observe from Lemma 2.6 that $G = G_p$ satisfies (V1) if $p = n$; (V2) with $\lambda = (n-p)/(p-1)$ if $1 < p < n$. Applying Theorem 1.2, we shall prove the following equivalence.

Theorem 2.7. *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$ ($n \ge 2$) and $x_0 \in \Omega$ be fixed. Let $1 < p \le n$. The following statements are equivalent:*

(a) $\omega_p(x_0, \partial\Omega \cap B(\xi, r), \Omega) \approx r^{(n-1)/(p-1)}$ *whenever $\xi \in \partial\Omega$ and $0 < r < r_0$,*

(b) $G_p(x,y) \approx G_p(y,x)$ *for any pair $x, y \in \Omega$,*

(c) $G_p(x_0, x) \approx \delta_\Omega(x)$ *whenever $\delta_\Omega(x) < r_0$,*

(d) *for each $\xi \in \partial\Omega$, there is a positive p-harmonic function u on Ω vanishing continuously on $\partial\Omega \setminus \{\xi\}$ such that $u(x_0) = 1$ and $u(x) \approx \delta_\Omega(x)^{(1-n)/(p-1)}$ for $x \in \overline{z_\xi \xi}$,*

(e) *for each $\xi \in \partial\Omega$,*

$$K_p(x,\xi) \approx \frac{\delta_\Omega(x)}{|x-\xi|^{(n+p-2)/(p-1)}} \quad \textit{for } x \in \Omega.$$

When $p = n$, we can prove (d) in Theorem 2.7 and obtain the following estimates.

Corollary 2.8. *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$ ($n \geq 2$), and let $p = n$. Then the following estimates hold:*

(i) $\omega_n(x_0, \partial\Omega \cap B(\xi, r), \Omega) \approx r$ *whenever* $\xi \in \partial\Omega$ *and* $0 < r < r_0$,

(ii) $G_n(x, y) \approx \log\left(1 + \dfrac{\delta_\Omega(x)\delta_\Omega(y)}{|x-y|^2}\right)$ *for* $x, y \in \Omega$,

(iii) $K_n(x, \xi) \approx \dfrac{\delta_\Omega(x)}{|x-\xi|^2}$ *for* $x \in \Omega$ *and* $\xi \in \partial\Omega$,

where the constants of comparison depend only on n and Ω.

For the case $p < n$, we could not show any of (a)–(e) in Theorem 2.7. However, using Theorem 1.2, we shall prove the following one-side estimate.

Proposition 2.9. *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$ ($n \geq 2$). There exists a constant $A \geq 1$ depending only on p and Ω such that if $1 < p \leq 2$, then*

$$K_p(x, \xi) \geq \frac{1}{A} \frac{\delta_\Omega(x)}{|x-\xi|^{(n+p-2)/(p-1)}} \quad \textit{for } x \in \Omega;$$

if $2 \leq p < n$, then

$$K_p(x, \xi) \leq A \frac{\delta_\Omega(x)}{|x-\xi|^{(n+p-2)/(p-1)}} \quad \textit{for } x \in \Omega.$$

The proofs of the results in this section will be given in Sections 4 and 5.

3 Proof of Theorem 1.2

In this section, we suppose that a bounded domain Ω satisfies (I) and a function $G : \Omega \times \Omega \to (0, +\infty]$ satisfies (II)–(IV) and either (V1) or (V2). We need to prepare several lemmas. By $S(x, r)$ we denote the sphere of center x and radius r. The constants A appearing in this section may depend on the constants in our setting (I)–(IV) and (V1) or (V2).

Lemma 3.1. *There is a constant A such that for each $\xi \in \partial\Omega$ and $0 < r < r_0$, we have*

$$\frac{G(x_1, y_1)}{G(x_1, y_2)} \leq A \frac{G(x_2, y_1)}{G(x_2, y_2)}$$

whenever $x_1, x_2 \in \Omega \setminus B(\xi, 3r)$ and $y_1, y_2 \in \Omega \cap B(\xi, r)$.

Proof. Let $y_1, y_2 \in \Omega \cap B(\xi, r)$ be fixed and take $w \in \Omega \cap S(\xi, 3r)$ with $\delta_\Omega(w) \approx r$. We claim that

$$G(x, y_1) \approx \frac{G(w, y_1)}{G(w, y_2)} G(x, y_2) \quad \text{for } x \in \Omega \cap S(\xi, 3r). \tag{3.1}$$

Let $x \in \Omega \cap S(\xi, 3r)$. If $\delta_\Omega(x) < r/A_0$, then we take $z \in \Omega \cap S(\eta, r/A_0)$ with $\delta_\Omega(z) \approx r$, where $\eta \in \partial\Omega$ is a point such that $|\eta - x| = \delta_\Omega(x)$. Then $B(\eta, r) \cap B(\xi, r) = \emptyset$. We apply (IV) to obtain

$$\frac{G(x, y_1)}{G(x, y_2)} \approx \frac{G(z, y_1)}{G(z, y_2)}.$$

Since $|z - w| \le A \min\{\delta_\Omega(z), \delta_\Omega(w)\}$, it follows from (III) that

$$G(z, y_1) \approx G(w, y_1) \quad \text{and} \quad G(z, y_2) \approx G(w, y_2).$$

Hence (3.1) holds for such x. If $\delta_\Omega(x) \ge r/A_0$, then (III) implies that $G(x, y_j) \approx G(w, y_j)$ for $j = 1, 2$, and so (3.1) holds. The lemma follows from (3.1) and (II) with $D = \Omega \setminus B(\xi, 3r)$. □

Lemma 3.2. *Let $x, y \in \Omega$ satisfy $|x - y| \le k \min\{\delta_\Omega(x), \delta_\Omega(y)\}$. Then the following statements hold:*

(i) *If G satisfies* (V1)*, then*

$$G(x, y) \approx 1 + \log^+ \frac{\min\{\delta_\Omega(x), \delta_\Omega(y)\}}{|x - y|} \approx G(y, x),$$

where the constant of comparison depends on k.

(ii) *If G satisfies* (V2)*, then*

$$G(x, y) \approx |x - y|^{-\lambda} \approx G(y, x),$$

where the constant of comparison depends on k.

Proof. Without loss of generality, we may assume that $\delta_\Omega(x) \le \delta_\Omega(y)$. If $|x - y| < \delta_\Omega(y)/4$, then $y \in B(x, \delta_\Omega(x)/3)$. Therefore

$$G(x, y) \approx \begin{cases} \log \dfrac{\delta_\Omega(y)}{|x - y|} \approx \log \dfrac{\delta_\Omega(x)}{|y - x|} \approx G(y, x) & \text{if } G \text{ satisfies (V1)}, \\ |x - y|^{-\lambda} = |y - x|^{-\lambda} \approx G(y, x) & \text{if } G \text{ satisfies (V2)}. \end{cases}$$

Suppose $|x - y| \ge \delta_\Omega(y)/4$. Let $z \in S(x, \delta_\Omega(x)/8)$ and $w \in S(y, \delta_\Omega(y)/8)$. Observe that $|x - w| \le A \min\{\delta_\Omega(x), \delta_\Omega(w)\}$ and $|z - y| \le A \min\{\delta_\Omega(z), \delta_\Omega(y)\}$. By (III), we have

$$G(x, y) \approx G(w, y) \approx \begin{cases} 1 & \text{if } G \text{ satisfies (V1)}, \\ \delta_\Omega(y)^{-\lambda} & \text{if } G \text{ satisfies (V2)}, \end{cases}$$

and

$$G(y, x) \approx G(z, x) \approx \begin{cases} 1 & \text{if } G \text{ satisfies (V1)}, \\ \delta_\Omega(x)^{-\lambda} & \text{if } G \text{ satisfies (V2)}. \end{cases}$$

Since $\delta_\Omega(x)/4 \le \delta_\Omega(y)/4 \le |x - y| \le k\delta_\Omega(x) \le k\delta_\Omega(y)$, we obtain the lemma. □

Lemma 3.3. *The following statements hold:*

(i) *Let A_2 be a positive constant. If $\delta_\Omega(x) \ge A_2$, then*

$$g(x) \approx 1 \quad \textit{and} \quad g^*(x) \approx 1,$$

where the constants of comparison depend on A_2.

(ii) *If $x \in \Omega \setminus B(x_0, \delta_\Omega(x_0)/2)$, then $g(x) \approx G(x, x_0)$.*

(iii) *If $x, y \in \Omega$ satisfy $|x - y| \le k \min\{\delta_\Omega(x), \delta_\Omega(y)\}$, then*

$$g(x) \approx g(y) \quad \textit{and} \quad g^*(x) \approx g^*(y),$$

where the constants of comparison depend on k.

Proof. We give the proof in only the case that G satisfies (V1), because the another case can be proved similarly.

(i) Since $|x - x_0| \le A \min\{\delta_\Omega(x), \delta_\Omega(x_0)\}$, Lemma 3.2 yields that $g(x) \approx 1 \approx g^*(x)$.

(ii) Since $G(\cdot, x_0) \approx 1$ on $S(x_0, \delta_\Omega(x_0)/2)$, it follows from (II) that

$$G(\cdot, x_0) \le A \quad \text{on } \Omega \setminus B(x_0, \delta_\Omega(x_0)/2).$$

Hence $g \approx G(\cdot, x_0)$ there.

(iii) We first show $g(x) \approx g(y)$. If $x \in B(x_0, \delta_\Omega(x_0)/2)$, then

$$\delta_\Omega(y) \ge \frac{\delta_\Omega(x)}{k+1} \ge \frac{\delta_\Omega(x_0)}{2(k+1)}.$$

Hence $g(x) \approx 1 \approx g(y)$ by (i). If $y \in B(x_0, \delta_\Omega(x_0)/2)$, then $g(x) \approx 1 \approx g(y)$ by the same reasoning. If $x, y \notin B(x_0, \delta_\Omega(x_0)/2)$, then $G(x, x_0) \approx G(y, x_0)$ by (III). Hence $g(x) \approx g(y)$.

We next show $g^*(x) \approx g^*(y)$. Without loss of generality, we may assume that $\delta_\Omega(x) \le \delta_\Omega(y)$ and $\delta_\Omega(x_0) \ge 3r_0$. If $\delta_\Omega(y) \ge r_0/(k+1)$, then

$$\delta_\Omega(x) \ge \frac{\delta_\Omega(y)}{k+1} \ge \frac{r_0}{(k+1)^2}.$$

Hence $g^*(x) \approx 1 \approx g^*(y)$ by (i). Suppose $\delta_\Omega(y) < r_0/(k+1)$. Take $\xi \in \partial\Omega$ with $|x - \xi| = \delta_\Omega(x)$ and let $\rho = (k+1)\delta_\Omega(x)$. Then $|y - \xi| \le \rho < r_0$. By Lemma 3.1, we have for $x_1 \in \Omega \cap S(\xi, 3\rho)$ with $\delta_\Omega(x_1) \approx \rho$,

$$G(x_0, x) \approx \frac{G(x_1, x)}{G(x_1, y)} G(x_0, y).$$

Since $G(x_1, x) \approx G(x, x_1) \approx G(y, x_1) \approx G(x_1, y)$ by Lemma 3.2 and (III), we obtain $g^*(x) \approx g^*(y)$. $\square$

We now give the proof of Theorem 1.2.

Proof of Theorem 1.2(i). Let A_0 and r_0 be as in (IV). Let

$$A_3 = \max\left\{5A_0, 3\kappa, \frac{3\operatorname{diam}\Omega}{r_0}\right\},$$

and let A_4 be a constant such that $A_4 > A_3$. We may assume that $\delta_\Omega(x_0) \ge A_0 r_0$. To prove (1.1), we split the proof into several cases:

Case 1: $|x-y| \le A_4 \min\{\delta_\Omega(x), \delta_\Omega(y)\}$,
Case 2: $|x-y| \ge A_3 \min\{\delta_\Omega(x), \delta_\Omega(y)\}$;
Subcase 2.1: $\delta_\Omega(x) \le \delta_\Omega(y)$ and **Subcase 2.2**: $\delta_\Omega(y) < \delta_\Omega(x)$.

Case 1. By the definition of $\mathcal{B}(x,y)$,

$$\max\{|x-b|, |y-b|\} \le \kappa|x-y| \le \kappa^2 A_4 \min\{\delta_\Omega(x), \delta_\Omega(y), \delta_\Omega(b)\}.$$

Therefore $g(x) \approx g(b)$ and $g^*(y) \approx g^*(b)$ by Lemma 3.3. This and Lemma 3.2 give (1.1).
Case 2. Since $1 + \log^+(\min\{\delta_\Omega(x), \delta_\Omega(y)\}/|x-y|) = 1$, it is enough to show

$$G(x,y) \approx \frac{g(x)g^*(y)}{g(b)g^*(b)}. \tag{3.2}$$

Let $r = |x-y|/A_3$. Then $r \le r_0/3$ and $\delta_\Omega(b) \ge r$.
Subcase 2.1. Note that $\delta_\Omega(x) \le r$. Take $\xi \in \partial\Omega$ with $|x-\xi| = \delta_\Omega(x)$. Then $|x_0 - \xi| \ge A_0 r$ and $|y-\xi| \ge |x-y| - |x-\xi| \ge A_0 r$. Let $x_1 \in \Omega \cap S(\xi, r/2)$ be such that $\delta_\Omega(x_1) \approx r$. Observe that $|b - x_1| \le Ar \le A\min\{\delta_\Omega(b), \delta_\Omega(x_1)\}$, and so

$$g(b) \approx g(x_1) \approx G(x_1, x_0)$$

by Lemma 3.3. It follows from (IV) that

$$G(x,y) \approx \frac{G(x_1,y)}{G(x_1,x_0)} G(x,x_0) \approx \frac{g(x)}{g(b)} G(x_1,y). \tag{3.3}$$

We still distinguish two more cases.
Subcase 2.1.1: $\delta_\Omega(y) \ge r$**.** Observe that $\delta_\Omega(x_1) \le r \approx |x_1 - y| \le A\min\{\delta_\Omega(x_1), \delta_\Omega(y)\}$. By Lemma 3.2,

$$G(x_1, y) \approx 1.$$

Also, $|y-b| \le Ar \le A\min\{\delta_\Omega(y), \delta_\Omega(b)\}$ and Lemma 3.3 imply

$$g^*(y) \approx g^*(b).$$

Hence (3.2) follows from (3.3).
Subcase 2.1.2: $\delta_\Omega(y) < r$**.** Observe that

$$|x_1 - y| \ge |x-y| - |x - x_1| \ge r \ge \delta_\Omega(y),$$
$$|b-y| \ge \delta_\Omega(b) - \delta_\Omega(y) \ge \frac{|x-y|}{\kappa} - \delta_\Omega(y) \ge r \ge \delta_\Omega(y),$$

and so $x_1, b \notin B(y, \delta_\Omega(y))$. Since $|b - x_1| \le Ar \le A\min\{\delta_\Omega(b), \delta_\Omega(x_1)\}$, it follows from (III) that $G(x_1, y) \approx G(b, y)$. This and (3.3) give

$$G(x,y) \approx \frac{g(x)}{g(b)} G(b,y). \tag{3.4}$$

Take $y_1 \in \Omega \cap S(\eta, r)$ with $\delta_\Omega(y_1) \approx r$, where $\eta \in \partial\Omega$ is a point such that $|y - \eta| = \delta_\Omega(y)$. Observe that $|b - \eta| \ge \delta_\Omega(b) \ge |x - y|/\kappa \ge 3r$. Therefore Lemma 3.1 gives

$$G(b,y) \approx \frac{G(b,y_1)}{G(x_0,y_1)} G(x_0,y) \approx \frac{g^*(y)}{g^*(y_1)} G(b,y_1). \tag{3.5}$$

Since $\delta_\Omega(y_1) \le r \approx |y_1 - b| \le A\min\{\delta_\Omega(y_1), \delta_\Omega(b)\}$, we have from Lemmas 3.2 and 3.3

$$G(b,y_1) \approx 1 \quad \text{and} \quad g^*(y_1) \approx g^*(b).$$

These, together with (3.4) and (3.5), yield (3.2).

Subcase 2.2. Note that $\delta_\Omega(y) \le r \le r_0/3$. We still distinguish two more cases.

Subcase 2.2.1: $\delta_\Omega(x) \ge r$**.** Take $y_2 \in \Omega \cap S(\eta, r)$ with $\delta_\Omega(y_2) \approx r$, where $\eta \in \partial\Omega$ is a point such that $|y - \eta| = \delta_\Omega(y)$. Observe that

$$|x - \eta| \ge |x - y| - |y - \eta| \ge 3r,$$

and so $x, x_0 \notin B(\eta, 3r)$. Lemma 3.1 gives

$$G(x,y) \approx \frac{G(x,y_2)}{G(x_0,y_2)} G(x_0,y) \approx \frac{g^*(y)}{g^*(y_2)} G(x,y_2). \tag{3.6}$$

Since $\delta_\Omega(y_2) \le r \approx |x - y_2| \le A\min\{\delta_\Omega(x), \delta_\Omega(y_2)\}$, it follows from Lemma 3.2 that

$$G(x,y_2) \approx 1.$$

Also, $\max\{|x - b|, |y_2 - b|\} \le Ar \le A\min\{\delta_\Omega(x), \delta_\Omega(y_2), \delta_\Omega(b)\}$ and Lemma 3.3 imply

$$g(x) \approx g(b) \quad \text{and} \quad g^*(y_2) \approx g^*(b).$$

Thus (3.2) follows.

Subcase 2.2.2: $\delta_\Omega(x) < r$**.** Let $x_2 \in \Omega \cap B(\xi, \delta_\Omega(y))$, where $\xi \in \partial\Omega$ is a point such that $|x - \xi| = \delta_\Omega(x)$. It follows from (IV) and Subcase 2.1 that

$$G(x,y) \approx \frac{G(x,x_0)}{G(x_2,x_0)} G(x_2,y) \approx \frac{g(x)}{g(x_2)} \frac{g(x_2) g^*(y)}{g(b_{x_2,y}) g^*(b_{x_2,y})}, \tag{3.7}$$

where $b_{x_2,y} \in \mathcal{B}(x_2, y)$. Since $|b_{x_2,y} - b| \le Ar \le A\min\{\delta_\Omega(b_{x_2,y}), \delta_\Omega(b)\}$, Lemma 3.3 gives

$$g(b_{x_2,y}) \approx g(b) \quad \text{and} \quad g^*(b_{x_2,y}) \approx g^*(b).$$

Hence (1.1) follows. The proof of Theorem 1.2(i) is complete. $\square$

Proof of Theorem 1.2(ii). The proof of (ii) is similar to (i), so we give only a sketch.
Case 1. The same reasoning as in (i) gives (1.1).
Subcase 2.1.1. By Lemma 3.2, $G(x_1, y) \approx |x_1 - y|^{-\lambda} \approx |x - y|^{-\lambda}$. Since $g^*(y) \approx g^*(b)$, we obtain (1.2) from (3.3).
Subcase 2.1.2. Lemmas 3.2 and 3.3 give $G(b, y_1) \approx |b - y_1|^{-\lambda} \approx |x - y|^{-\lambda}$ and $g^*(y_1) \approx g^*(b)$. Hence (3.4) and (3.5) yield (1.2).
Subcase 2.2.1. By Lemma 3.2, $G(x, y_2) \approx |x-y_2|^{-\lambda} \approx |x-y|^{-\lambda}$. Since $g(x) \approx g(b)$ and $g^*(y_2) \approx g^*(b)$, we obtain (1.2) from (3.6).
Subcase 2.2.2. Since $|x_2 - y| \approx r \approx |x - y|$, it follows from (IV) and Subcase 2.1 that

$$G(x, y) \approx \frac{g(x)}{g(x_2)} \frac{g(x_2)g^*(y)}{g(b_{x_2,y})g^*(b_{x_2,y})} |x - y|^{-\lambda}.$$

See (3.7). Since $g(b_{x_2,y}) \approx g(b)$ and $g^*(b_{x_2,y}) \approx g^*(b)$, we obtain (1.2). $\square$

4 Proof of Theorem 2.7

In this section, we suppose that Ω is a bounded $C^{1,1}$-domain in $\mathbb{R}^n$ ($n \geq 2$). Let

$$\gamma_p(x, y) = \begin{cases} |x - y|^{(p-n)/(p-1)} & \text{if } 1 < p < n, \\ 1 + \log^+ \dfrac{\min\{\delta_\Omega(x), \delta_\Omega(y)\}}{|x - y|} & \text{if } p = n. \end{cases}$$

Applying Theorem 1.2 and Lemma 3.3 to $G = G_p$, we have the following.

Lemma 4.1. *For $x, y \in \Omega$ and $b \in \mathcal{B}(x, y)$,*

$$G_p(x, y) \approx \frac{g(x)g^*(y)}{g(b)g^*(b)} \gamma_p(x, y), \tag{4.1}$$

where the constant of comparison depends only on p and Ω. Moreover, if $|x - y| \leq k \min\{\delta_\Omega(x), \delta_\Omega(y)\}$, then

$$G_p(x, y) \approx \gamma_p(x, y),$$

where the constant of comparison depends only on k, p and Ω.

Lemma 4.1 yields the following estimate for K_p.

Lemma 4.2. *For $x \in \Omega$, $\xi \in \partial\Omega$ and $b \in \mathcal{B}(x, \xi)$,*

$$K_p(x, \xi) \approx \frac{g(x)}{g(b)g^*(b)} |x - \xi|^{(p-n)/(p-1)},$$

where the constant of comparison depends only on p and Ω.

Proof. We find a sequence $\{y_j\}$ in Ω converging to ξ such that $\{G_p(\cdot, y_j)/G_p(x_0, y_j)\}$ converges to $K_p(\cdot, \xi)$ on Ω. Let $x \in \Omega$ be fixed. Without loss of generality, we may assume that all y_j satisfy $|y_j-\xi| \le |y_1-\xi| \le \delta_\Omega(x)/(2\kappa^2+2)$ and $4|y_j-\xi| \le |x-y_j|$. Let $b \in \mathcal{B}_\kappa(x, y_1)$ and $j \in \mathbb{N}$. Since

$$|x-y_j| \le |x-y_1| + |y_1-\xi| + |\xi-y_j| \le \frac{5}{4}|x-y_1| + \frac{1}{4}|x-y_j|,$$

we have

$$|x-y_j| \le 2|x-y_1| \le 2\kappa\delta_\Omega(b). \tag{4.2}$$

Also, since $|x-y_1| \le 2|x-y_j|$ by the same reasoning as above, we have

$$\begin{aligned} |x-b| &\le \kappa|x-y_1| \le 2\kappa|x-y_j|, \\ |b-y_j| &\le |b-x| + |x-y_j| \le (2\kappa+1)|x-y_j|. \end{aligned} \tag{4.3}$$

Hence $b \in \bigcap_j \mathcal{B}_{2\kappa+1}(x, y_j)$. Let us apply Lemma 4.1. Since $\delta_\Omega(b_j) \ge \delta_\Omega(x_0)/(\kappa^2+1)$ for $b_j \in \mathcal{B}(x_0, y_j)$, it follows from Lemma 3.3 that $g(b_j) \approx 1 \approx g^*(b_j)$, and so

$$\frac{G_p(x,y_j)}{G_p(x_0,y_j)} \approx \frac{g(x)}{g(b)g^*(b)} \frac{\gamma_p(x,y_j)}{\gamma_p(x_0,y_j)} \quad \text{for } x \in \Omega.$$

Letting $j \to \infty$, we obtain

$$K_p(x,\xi) \approx \frac{g(x)}{g(b)g^*(b)} |x-\xi|^{(p-n)/(p-1)}. \tag{4.4}$$

Also, letting $j \to \infty$ in (4.2) and (4.3), we have $b \in \mathcal{B}_{2\kappa+1}(x,\xi)$. From Lemma 3.3, we see that (4.4) holds for all $b \in \mathcal{B}_\kappa(x,\xi)$. This completes the proof. $\square$

Also, we need the following relationship between the p-harmonic measure and the p-Green function.

Lemma 4.3. *Let $\xi \in \partial\Omega$ and $0 < r < r_0$. Let $x \in \Omega$ satisfy $|x-\xi| = \delta_\Omega(x) = r$. Then*

$$\omega_p(y, \partial\Omega \cap B(\xi, r), \Omega) \approx r^{(n-p)/(p-1)} G_p(y,x) \quad \textit{for } y \in \Omega \setminus B(\xi, 2r),$$

where the constant of comparison depends only on p and Ω.

Proof. Let $z \in \Omega \cap S(\xi, 2r)$ be a point such that $\delta_\Omega(z) \approx r$. By the similar way to the proof of Lemma 3.1, we have

$$\frac{G_p(y,x)}{G_p(z,x)} \approx \frac{\omega_p(y, \partial\Omega \cap B(\xi,r), \Omega)}{\omega_p(z, \partial\Omega \cap B(\xi,r), \Omega)} \quad \text{for } y \in \Omega \setminus B(\xi, 2r). \tag{4.5}$$

In view of Lemma 4.1, we have

$$G_p(z,x) \approx \begin{cases} 1 & \text{if } p = n, \\ |x-z|^{(p-n)/(p-1)} \approx r^{(p-n)/(p-1)} & \text{if } 1 < p < n. \end{cases} \tag{4.6}$$

Let $\Gamma = \{x \in B(w_\xi, r_0) : \angle x\xi w_\xi < \pi/4\}$, where w_ξ is a point in $\mathbb{R}^n \setminus \Omega$ stated before Theorem 2.7. Since $\Gamma \subset \mathbb{R}^n \setminus \Omega$, it follows from Lemma 2.1 that

$$\omega_p(x, \partial\Omega \cap B(\xi, r), \Omega) \geq \omega_p(x, S(\xi, r) \cap \Gamma, B(\xi, r)) \quad \text{for } x \in \Omega \cap B(\xi, r).$$

Note that the value

$$A_5 = \inf_{B(\xi, r/2)} \omega_p(\cdot, S(\xi, r) \cap \Gamma, B(\xi, r)) > 0$$

is invariant under dilation. Therefore $\omega_p(\cdot, \partial\Omega \cap B(\xi, r), \Omega) \geq A_5$ on $\Omega \cap B(\xi, r/2)$, and so $\omega_p(z, \partial\Omega \cap B(\xi, r), \Omega) \approx 1$ by Lemma 2.4. This, together with (4.5) and (4.6), yields that

$$\omega_p(y, \partial\Omega \cap B(\xi, r), \Omega) \approx r^{(n-p)/(p-1)} G_p(y, x) \quad \text{for } y \in \Omega \setminus B(\xi, 2r).$$

Thus the lemma is proved. □

Proof of Theorem 2.7. (e) ⇒ (d). Since $\delta_\Omega(x) = |x - \xi|$ for $x \in \overline{z_\xi \xi}$, this implication is clear.

(d) ⇒ (c). Let $x \in \Omega$ satisfy $\delta_\Omega(x) < r_0$ and let $\xi \in \partial\Omega$ be a point such that $|x - \xi| = \delta_\Omega(x)$. Then $x \in \overline{z_\xi \xi}$. Suppose that u is a positive p-harmonic function on Ω satisfying the properties in (d). Let $y \in \overline{z_\xi \xi}$ be a point such that $\delta_\Omega(y) = \delta_\Omega(x)/2$. Using Lemmas 2.5 and 4.1, we observe that

$$u(x) = \frac{u(x)}{u(x_0)} \approx \frac{G_p(x, y)}{G_p(x_0, y)} \approx \frac{\delta_\Omega(x)^{(p-n)/(p-1)}}{G_p(x_0, y)}.$$

Since $u(x) \approx \delta_\Omega(x)^{(1-n)/(p-1)}$, we have $G_p(x_0, y) \approx \delta_\Omega(x)$. Therefore Lemma 3.3 yields

$$G_p(x_0, x) \approx G_p(x_0, y) \approx \delta_\Omega(x).$$

(c) ⇒ (b). Let $x \in \Omega$. If $\delta_\Omega(x) \geq r_0$, then $|x - x_0| \leq A \min\{\delta_\Omega(x), \delta_\Omega(x_0)\}$. It follows from Lemma 4.1 that

$$G_p(x, x_0) \approx \gamma_p(x, x_0) = \gamma_p(x_0, x) \approx G_p(x_0, x).$$

If $\delta_\Omega(x) < r_0$, then $G_p(x, x_0) \approx \delta_\Omega(x)$, and so $G_p(x, x_0) \approx G_p(x_0, x)$ by assumption. Hence, in any case,

$$g \approx g^* \quad \text{on } \Omega.$$

This and Lemma 4.1 give that $G_p(x, y) \approx G_p(y, x)$ for all $x, y \in \Omega$.

(b) ⇒ (a). Let $x \in \overline{z_\xi \xi}$ be a point such that $\delta_\Omega(x) = r < r_0$. Since

$$G_p(x_0, x) \approx G_p(x, x_0) \approx \delta_\Omega(x) = r,$$

it follows from Lemma 4.3 with $y = x_0$ that

$$\omega_p(x_0, \partial\Omega \cap B(\xi, r), \Omega) \approx r^{(n-p)/(p-1)} G_p(x_0, x) \approx r^{(n-1)/(p-1)}.$$

(a) $\Rightarrow$ (c). Let $\delta_\Omega(x) < r_0$. Then Lemma 4.3 gives

$$G_p(x_0, x) \approx \delta_\Omega(x).$$

(c) $\Rightarrow$ (e). By Lemma 4.2, we have

$$K_p(x,\xi) \approx \frac{\delta_\Omega(x)}{\delta_\Omega(b)^2}|x-\xi|^{(p-n)/(p-1)} \quad \text{for } x \in \Omega \text{ and } b \in \mathcal{B}(x,\xi).$$

Since $\delta_\Omega(b) \approx |x-\xi|$, we obtain the required estimate. The proof of Theorem 2.7 is complete. □

5 Proofs of Corollary 2.8 and Proposition 2.9

In this section, we prove Corollary 2.8 by showing the existence of a positive p-harmonic function with the properties in Theorem 2.7(d). Let y^* denote the inverse of y with respect to $S(z,r)$:

$$y^* = z + \frac{r^2}{|y-z|^2}(y-z).$$

We observe that the n-Green functions for $B(z,r)$ and $\mathbb{R}^n \setminus \overline{B(z,r)}$ are given respectively by

$$G_{n,B(z,r)}(x,y) = \begin{cases} \dfrac{1}{(n\nu_n)^{1/(n-1)}} \log \dfrac{r}{|x-y|} & \text{if } y = z, \\ \dfrac{1}{(n\nu_n)^{1/(n-1)}} \log \dfrac{|y-z||x-y^*|}{r|x-y|} & \text{if } y \neq z, \end{cases} \tag{5.1}$$

and

$$G_{n,\mathbb{R}^n\setminus\overline{B(z,r)}}(x,y) = \frac{1}{(n\nu_n)^{1/(n-1)}} \log \frac{|y-z||x-y^*|}{r|x-y|}. \tag{5.2}$$

See [2] for example.

Lemma 5.1. *Let $\xi \in \partial\Omega$. Then there is an n-Martin kernel with pole at ξ such that*

$$K_n(x,\xi) \approx \frac{1}{\delta_\Omega(x)} \quad \textit{for } x \in \overline{z_\xi \xi},$$

where the constant of comparison depends only on n and Ω.

Proof. Let $r > 0$ be small and let $\{y_j\}$ be a sequence in $\overline{z_\xi \xi}$ converging to ξ such that

$$\lim_{j\to\infty} \frac{G_n(x,y_j)}{G_n(x_0,y_j)} = K_n(x,\xi) \quad \text{and} \quad \lim_{j\to\infty} \frac{G_{n,B(z_\xi,r_0)}(x,y_j)}{G_{n,B(z_\xi,r_0)}(z_\xi,y_j)} = K_{n,B(z_\xi,r_0)}(x,\xi).$$

Let $j \in \mathbb{N}$ be sufficiently large so that $y_j \in B(\xi, r/2)$. Then, by (5.1) and (5.2), we have for $x \in \overline{z_\xi \xi} \setminus B(\xi,r)$,

$$G_{n,B(z_\xi,r_0)}(x,y_j) \le G_n(x,y_j) \le G_{n,\mathbb{R}^n\setminus\overline{B(w_\xi,r_0)}}(x,y_j) \le AG_{n,B(z_\xi,r_0)}(x,y_j).$$

Therefore, by Lemma 2.4,

$$\frac{G_n(x, y_j)}{G_n(x_0, y_j)} \approx \frac{G_n(x, y_j)}{G_n(z_\xi, y_j)} \approx \frac{G_{n,B(z_\xi, r_0)}(x, y_j)}{G_{n,B(z_\xi, r_0)}(z_\xi, y_j)} \quad \text{for } x \in \overline{z_\xi \xi} \setminus B(\xi, r).$$

Letting $j \to \infty$ and $r \to 0$, we obtain

$$K_n(x, \xi) \approx K_{n,B(z_\xi, r_0)}(x, \xi) = \frac{r_0^2 - |x - z_\xi|^2}{|x - \xi|^2} \quad \text{for } x \in \overline{z_\xi \xi}.$$

Here the last equality follows from (5.1) and direct computations (see [2]). Thus the lemma is proved. □

Proof of Corollary 2.8. Let $\xi \in \partial\Omega$ and let $K_n(\cdot, \xi)$ be an n-Martin kernel with pole at ξ obtained in Lemma 5.1. Observe that $K_n(\cdot, \xi)$ vanishes continuously on $\partial\Omega \setminus \{\xi\}$ and $K_n(x_0, \xi) = 1$. Hence (i) and (iii) follow from Theorem 2.7.

We show (ii). By Theorem 2.7, we have $g \approx g^*$. It follows from Lemma 4.1 that

$$G_n(x, y) \approx \frac{g(x)g(y)}{g(b)^2} \left(1 + \log^+ \frac{\min\{\delta_\Omega(x), \delta_\Omega(y)\}}{|x - y|}\right) \quad \text{for } x, y \in \Omega \text{ and } b \in \mathcal{B}(x, y).$$

Without loss of generality, we may assume that $\delta_\Omega(x) \le \delta_\Omega(y)$.
Case 1: $|x - y| \le \delta_\Omega(y)/4$**.** Observe that $(3/4)\delta_\Omega(y) \le \delta_\Omega(x)$, $|x - y| \le \delta_\Omega(x)/3$ and $g(b) \approx g(x) \approx g(y)$. Therefore

$$G_n(x, y) \approx 1 + \log \frac{\delta_\Omega(x)}{|x - y|} \approx \log\left(1 + \frac{\delta_\Omega(x)\delta_\Omega(y)}{|x - y|^2}\right).$$

Case 2: $\delta_\Omega(y)/4 < |x - y|$**.** Observe that $1 + \log^+(\delta_\Omega(x)/|x - y|) \approx 1$ and

$$\frac{g(x)g(y)}{g(b)^2} \approx \frac{\delta_\Omega(x)\delta_\Omega(y)}{\delta_\Omega(b)^2}.$$

Since $\delta_\Omega(b) \approx |x - y|$, we obtain

$$G_n(x, y) \approx \frac{\delta_\Omega(x)\delta_\Omega(y)}{|x - y|^2} \approx \log\left(1 + \frac{\delta_\Omega(x)\delta_\Omega(y)}{|x - y|^2}\right).$$

Thus (ii) is proved. □

Let $z \in \mathbb{R}^n$ and $r > 0$. For $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n \setminus \{z\}$, we define

$$g_{p,z,r}(x, y) = |x - y|^{(p-n)/(p-1)} - \left(\frac{|y - z||x - y^*|}{r}\right)^{(p-n)/(p-1)},$$

where y^* is the inverse of y with respect to $S(z, r)$. Note that $g_{p,z,r}(\cdot, y)$ is not annihilated by Δ_p on $\mathbb{R}^n \setminus \{y\}$ when $p \ne 2$.

Lemma 5.2. *Let $x \ne y$. If $1 < p \le 2$ and $x, y \in \mathbb{R}^n \setminus B(z, r)$, then $\Delta_p g_{p,z,r}(x, y) \le 0$. If $2 \le p < n$ and $x, y \in B(z, r)$, then $\Delta_p g_{p,z,r}(x, y) \ge 0$.*

Proof. Since the sign of Δ_p is invariant under dilation and translation, it suffices to show the lemma for $g = g_{p,0,1}(\cdot, y)$. By direct computations, we have

$$\begin{aligned}\Delta_p g &= |\nabla g|^{p-4} \left((p-2) \sum_{j,k} \frac{\partial g}{\partial x_j} \frac{\partial g}{\partial x_k} \frac{\partial^2 g}{\partial x_j \partial x_k} + |\nabla g|^2 \sum_j \frac{\partial^2 g}{\partial x_j^2} \right) \\ &= |\nabla g|^{p-4} \frac{(n-p)^3}{(p-1)^4} \frac{(p+n-2)(p-2)|y|^2}{(|y||x-y||x-y^*|)^{(n-p)/(p-1)+2}} \\ &\quad \times g_{p,0,1}(x,y) \left(1 - \left\langle \frac{x-y}{|x-y|}, \frac{x-y^*}{|x-y^*|} \right\rangle^2 \right).\end{aligned}$$

Note that $g_{p,0,1}(x,y) \geq 0$ if $x, y \in B(0,1)$ or $x, y \in \mathbb{R}^n \setminus B(0,1)$. Hence we can obtain the lemma. $\square$

Proof of Proposition 2.9. Let $\delta_\Omega(y) < r_0/2$ and let $\xi \in \partial\Omega$ be a point such that $|y - \xi| = \delta_\Omega(y)$.
Case 1: $1 < p \leq 2$**.** Observe that

$$g_{p,w_\xi,r_0}(\cdot, y) \approx \delta_\Omega(y)^{(p-n)/(p-1)} \approx G_p(\cdot, y) \quad \text{on } S(y, \delta_\Omega(y)/2).$$

In view of Lemma 5.2, we obtain from the Harnack inequality and Lemma 2.1 that

$$G_p(x_0, y) \approx G_p(z_\xi, y) \leq A g_{p,w_\xi,r_0}(z_\xi, y) \approx \delta_\Omega(y).$$

Hence $g^*(y) \leq A\delta_\Omega(y)$. This, together with $g(x) \approx \delta_\Omega(x)$ and Lemma 4.2, yields that

$$K_p(x,\xi) \geq \frac{1}{A} \frac{\delta_\Omega(x)}{\delta_\Omega(b)^2} |x-\xi|^{(p-n)/(p-1)} \quad \text{for } x \in \Omega \text{ and } b \in \mathcal{B}(x,\xi).$$

Since $\delta_\Omega(b) \approx |x - \xi|$, we obtain the required estimate.
Case 2: $2 \leq p < n$**.** The proof is similar to Case 1. Observe that

$$g_{p,z_\xi,r_0}(\cdot, y) \approx \delta_\Omega(y)^{(p-n)/(p-1)} \approx G_p(\cdot, y) \quad \text{on } S(y, \delta_\Omega(y)/2).$$

Therefore $\delta_\Omega(y) \approx g_{p,z_\xi,r_0}(z_\xi, y) \leq AG_p(z_\xi, y) \approx G_p(x_0, y)$, and so $g^*(y) \geq \delta_\Omega(y)/A$. Hence we obtain

$$K_p(x,\xi) \leq A \frac{\delta_\Omega(x)}{|x-\xi|^{2+(n-p)/(p-1)}} \quad \text{for } x \in \Omega.$$

Thus Proposition 2.9 is proved. $\square$

Acknowledgement

I would like to thank the referee and the associate editor for their helpful suggestions.

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