Limits at infinity of superharmonic functions and solutions of semilinear elliptic equations of Matukuma type *

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Abstract

In an unbounded domain Ω in $\mathbb{R}^n$ $(n \geq 2)$ with a compact boundary or $\Omega = \mathbb{R}^n$, we investigate the existence of limits at infinity of positive superharmonic functions u on Ω satisfying a nonlinear inequality like as

$$-\Delta u(x) \leq \frac{c}{1+|x|^2} u(x)^p \quad \text{for } x \in \Omega,$$

where Δ is the Laplacian and $c > 0$ and $p > 0$ are constants. The result is applicable to positive solutions of semilinear elliptic equations of Matukuma type.

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1 Introduction

This paper is motivated by the following semilinear elliptic equation proposed by Matukuma in 1930 to study a gravitational potential u of a globular cluster of stars:

$$-\Delta u = \frac{u^p}{1+|x|^2} \quad \text{in } \mathbb{R}^3,$$

where Δ is the Laplacian and $p > 0$ is a constant. The equations of this kind have been studied widely by many mathematicians. Kenig and Ni [9] proved the existence of positive bounded solutions u of $-\Delta u = Vu^p$ in $\mathbb{R}^n$ $(n \geq 3)$, where V is a measurable function satisfying $|V(x)| \leq A(1+|x|^2)^{-1-\varepsilon}$ for some constants $A > 0$ and $\varepsilon > 0$. See also the reference therein. Using techniques of the probabilistic potential theory, Zhao [12] generalized their result and proved that if Ω is an unbounded domain in $\mathbb{R}^n$

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$(n \geq 3)$ with a compact Lipschitz boundary and V is a Green-tight function on Ω, then there are positive bounded solutions $u \in C(\overline{\Omega})$ of

$$\begin{cases} -\Delta u = V u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

satisfying

$$\lim_{x \to \infty} u(x) = \alpha \tag{1.2}$$

for a given small constant $\alpha > 0$. See also [3]. The corresponding result in two dimensions was obtained by Ufuktepe and Zhao [11]. They actually showed the existence of positive solutions u of (1.1) satisfying

$$\lim_{x \to \infty} \frac{u(x)}{\log |x|} = \alpha \tag{1.3}$$

for a given small constant $\alpha > 0$. In an unbounded cone, the existence of positive solutions of (1.1) which are comparable to the Martin kernel at infinity was studied in [7].

In this paper, we are interested in the following question:

Question. *Let Ω be an unbounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with a compact boundary or $\Omega = \mathbb{R}^n$ and let V be a nonnegative measurable function on Ω with suitable conditions. Does every positive solution u of $-\Delta u = V u^p$ in Ω satisfy* (1.2) *or* (1.3) *for some $\alpha \geq 0$?*

Remark 1.1. When $n \geq 3$ and V is a negative function with suitable properties, there is a positive solution u of $-\Delta u = V u^p$ in $\mathbb{R}^n$ such that $u(x) \to +\infty$ as $|x| \to +\infty$. See [4, 6, 10] and references therein. Thus the above question is significant in the case that V is nonnegative.

More generally, we shall discuss the above question for the class of positive superharmonic functions satisfying a certain nonlinear inequality, which includes all positive solutions of semilinear elliptic equations of Matukuma type. Let Ω be a domain in $\mathbb{R}^n$ $(n \geq 2)$. A lower semicontinuous function $u : \Omega \to (-\infty, +\infty]$, where $u \not\equiv +\infty$, is called *superharmonic* on Ω if it satisfies the mean value inequality

$$u(x) \geq \frac{1}{\nu_n r^n} \int_{B(x,r)} u(y) dy \quad \text{whenever } \overline{B(x,r)} \subset \Omega.$$

Here $B(x, r)$ denotes the open ball of center x and radius r, and ν_n is the volume of the unit ball in $\mathbb{R}^n$. It is well known that if u is a superharmonic function on Ω, then there exists a unique (Radon) measure μ_u on Ω such that

$$\int_\Omega \phi(x) d\mu_u(x) = -\int_\Omega u(x) \Delta\phi(x) dx \quad \text{for all } \phi \in C_0^\infty(\Omega),$$

where $C_0^\infty(\Omega)$ is the collection of all infinitely differentiable functions vanishing outside a compact set in Ω (cf. [2, Section 4.3]). The measure μ_u is called the *Riesz measure* associated with u.

Throughout the paper, we suppose that Ω is an unbounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with a compact boundary or $\Omega = \mathbb{R}^n$, and study positive superharmonic functions u on Ω whose Riesz measure is absolutely continuous with respect to Lebesgue measure. The Radon-Nikodým derivative is denoted by f_u. Note that f_u is nonnegative and that $f_u = -\Delta u$ if $u \in C^2(\Omega)$. Our results are as follows.

Theorem 1.2. *Let $n \geq 3$. Suppose that*

$$0 \leq p < \frac{n}{n-2}.$$

If u is a positive superharmonic function on Ω satisfying

$$f_u(x) \leq \frac{c}{|x|^2} u(x)^p \quad \text{for almost every } x \in \Omega \setminus B(0,R) \tag{1.4}$$

with some constants $c > 0$ and $R > 0$, then u has a finite limit at infinity.

In contrast, the following theorem shows that the above question is negative when p is greater than $n/(n-2)$.

Theorem 1.3. *Let $n \geq 3$ and $c > 0$. If*

$$p > \frac{n}{n-2},$$

then for each $\beta > 0$, there exists a positive function $u \in C^2(\mathbb{R}^n)$ satisfying

$$0 \leq -\Delta u \leq \frac{c}{1+|x|^2} u^p \quad \text{in } \mathbb{R}^n$$

such that

$$\limsup_{x \to \infty} \frac{u(x)}{|x|^\beta} = +\infty.$$

The two dimensional result corresponding to Theorem 1.2 is stated as follows.

Theorem 1.4. *Let $n = 2$ and let $p \geq 0$ be arbitrary constant. If u is a positive superharmonic function on Ω satisfying*

$$f_u(x) \leq \frac{c}{|x|^2(\log|x|)^p} u(x)^p \quad \text{for almost every } x \in \Omega \setminus B(0,R) \tag{1.5}$$

with some constants $c > 0$ and $R > 1$, then $u(x)/\log|x|$ has a finite limit at infinity.

2 Proofs of Theorems 1.2 and 1.4

We begin with some notation and terminology. The symbol A stands for an absolute positive constant whose value is unimportant and may change from line to line. For simplicity, we write $B(r) = B(0,r)$ and $D = \mathbb{R}^n \setminus \overline{B(1)}$. Let $G_D(x,y)$ and $K_D(x,y)$

denote the Green function for D and the Martin kernel of D, respectively. Observe that for $x \in D$ and $y \in \partial B(1)$,

$$K_D(x,y) = a_y \frac{|x|^2 - 1}{|x-y|^n}, \tag{2.1}$$

$$K_D(x,\infty) = \begin{cases} a_2 \log |x| & (n=2), \\ a_n(1-|x|^{2-n}) & (n \geq 3). \end{cases} \tag{2.2}$$

The reference point is taken at $x_0 = (2, 0, \cdots, 0)$, so that $a_y = |x_0 - y|^n/3$, $a_2 = (\log 2)^{-1}$ and $a_n = (1 - 2^{2-n})^{-1}$. In the proof below, we will use some facts concerning the minimal fine topology. Let E be a subset of D. By $\widehat{R}^E_{K_D(\cdot,\infty)}$, we denote the lower semicontinuous regularization of the reduced function defined by

$$R^E_{K_D(\cdot,\infty)}(x) = \inf_u u(x),$$

where the infimum is taken over all nonnegative superharmonic functions u on D satisfying $K_D(\cdot,\infty) \leq u$ on E. In general, $\widehat{R}^E_{K_D(\cdot,\infty)} \leq K_D(\cdot,\infty)$. A set E is called *minimally thin* at infinity (with respect to D) if

$$\widehat{R}^E_{K_D(\cdot,\infty)}(x) < K_D(x,\infty) \quad \text{for some } x \in D.$$

We say that a function u on D has *minimal fine limit* ℓ at infinity if there exists a subset E of D, which is minimally thin at infinity, such that

$$\lim_{D \setminus E \ni x \to \infty} u(x) = \ell.$$

The following lemma is a special case of the Fatou-Naïm-Doob theorem (cf. [2, Theorem 9.4.6]).

Lemma 2.1. *If u is a nonnegative superharmonic function on D, then $u/K_D(\cdot,\infty)$ has a finite minimal fine limit ℓ at infinity. Moreover, if u is a Green potential, then $\ell = 0$.*

The following lemma is well known.

Lemma 2.2. *Let $0 < \varepsilon < 1$ and let $\{x_i\}$ be a sequence in D such that $x_i \to \infty$ ($i \to +\infty$). Then the set $\bigcup_i B(x_i, \varepsilon|x_i|)$ is not minimally thin at infinity.*

Proof. Consider the inverse and the Kelvin transform with respect to the unit sphere. Then the inverse of $B(x_i, \varepsilon|x_i|)$ is the ball of center $x_i^*/(1-\varepsilon^2)$ and radius $\varepsilon|x_i^*|/(1-\varepsilon^2)$, where x^* denotes the inverse of a point x. Since the minimal thinness is invariant under the inversion, this lemma follows from [1, Lemma 5]. $\square$

More detailed informations about minimal thinness and minimal fine limit are found in [2, Chapter 9]. After showing three propositions below, we shall present proofs of Theorems 1.2 and 1.4.

Proposition 2.3. *Suppose that v is a positive superharmonic function on Ω whose Riesz measure is absolutely continuous with respect to Lebesgue measure, say $d\mu_v(x) = f_v(x)dx$. Let $\{z_i\}$ be a sequence in Ω such that $z_i \to \infty$ ($i \to +\infty$). If there are constants $A > 0$ and $0 < \rho \le 1/2$ such that*

$$f_v(x) \le \frac{A}{|x|^2} \quad \text{for almost every } x \in \bigcup_i B(z_i, \rho|z_i|),$$

then the following statements hold:

(i) *If $n \ge 3$, then $v(z_i)$ has a finite limit as $i \to +\infty$.*

(ii) *If $n = 2$, then $v(z_i)/\log|z_i|$ has a finite limit as $i \to +\infty$.*

Here the value of the limit is independent of $\{z_i\}$.

Proof. Since Ω has a compact boundary or $\Omega = \mathbb{R}^n$, we may assume, without loss of generality, that $D \subset \Omega$. In view of (2.2), it is enough to show that

$$\lim_{i\to+\infty} \frac{v(z_i)}{K_D(z_i,\infty)} \text{ exists and its value is finite and independent of } \{z_i\}. \tag{2.3}$$

For $0 < \varepsilon < \rho$, let

$$V_1(x) = \int_{D\setminus B(x,\varepsilon|x|)} G_D(x,y) f_v(y)dy,$$
$$V_2(x) = \int_{D\cap B(x,\varepsilon|x|)} G_D(x,y) f_v(y)dy.$$

By the Riesz decomposition on D, we have

$$v(x) = h(x) + V_1(x) + V_2(x),$$

where h is a nonnegative harmonic function on D. Moreover, the Martin representation gives

$$h(x) = \alpha_v K_D(x,\infty) + \int_{\partial B(1)} K_D(x,y)d\nu(y),$$

where ν is a measure on $\partial B(1)$ and

$$\alpha_v = \inf_{x\in D} \frac{h(x)}{K_D(x,\infty)} = \inf_{x\in D} \frac{v(x)}{K_D(x,\infty)}.$$

Therefore it follows from (2.1) and (2.2) that

$$\lim_{x\to\infty} \frac{h(x)}{K_D(x,\infty)} = \alpha_v.$$

To show (2.3), it suffices to prove that

$$\lim_{i\to+\infty} \frac{V_1(z_i)}{K_D(z_i,\infty)} = 0, \tag{2.4}$$

and

$$\limsup_{i\to+\infty} \frac{V_2(z_i)}{K_D(z_i,\infty)} \le A\varepsilon^2. \tag{2.5}$$

First, we show (2.4). Observe from Lemma 2.1 that $(V_1+V_2)/K_D(\cdot,\infty)$ has minimal fine limit 0 at infinity. Since Lemma 2.2 implies that the set $\bigcup_i B(z_i,\varepsilon|z_i|/2)$ is not minimally thin at infinity, we find a sequence $x_i \in B(z_i,\varepsilon|z_i|/2)$ such that

$$\lim_{i\to+\infty} \frac{V_1(x_i)+V_2(x_i)}{K_D(x_i,\infty)} = 0.$$

If $|z_i| > 10$, then the Harnack inequality gives $V_1(z_i) \le A\{V_1(x_i)+V_2(x_i)\}$ and $K_D(x_i,\infty) \le AK_D(z_i,\infty)$. Therefore

$$\lim_{i\to+\infty} \frac{V_1(z_i)}{K_D(z_i,\infty)} = 0,$$

and so (2.4) holds.

Next, we show (2.5). Since

$$f_v(y) \le \frac{A}{|y|^2} \le \frac{A}{|z_i|^2} \quad \text{for a.e. } y \in B(z_i,\rho|z_i|),$$

it follows that for all i,

$$V_2(z_i) \le \frac{A}{|z_i|^2}\int_{B(z_i,\varepsilon|z_i|)} G_D(z_i,y)dy.$$

Using

$$G_D(x,y) \le \begin{cases} A\log\dfrac{|x|^2}{|x-y|} & (n=2) \\ |x-y|^{2-n} & (n\ge 3) \end{cases} \quad \text{for } y \in B(x,|x|/2),$$

we obtain

$$V_2(z_i) \le \begin{cases} A\varepsilon^2 & (n\ge 3), \\ A\varepsilon^2(1+\log|z_i| - \log\varepsilon) & (n=2). \end{cases}$$

Therefore (2.5) follows from (2.2). Thus Proposition 2.3 is proved. □

Also, we obtain the following proposition, using a technique in our previous paper [8].

Proposition 2.4. *Let $n \ge 3$. Suppose that*

$$0 < p < \frac{n}{n-2},$$

and that u is a positive superharmonic function on Ω satisfying (1.4). *Let $\{z_i\}$ be a sequence in Ω such that $z_i \to \infty$ ($i \to +\infty$). Then there eixst a constant A and $i_0, \ell \in \mathbb{N}$ such that*

$$u \le A \quad \textit{on} \ \bigcup_{i\ge i_0} B(z_i, 2^{-\ell-3}|z_i|).$$

Proof. Without loss of generality, we may assume that $R = 1$ and $D \subset \Omega$. As in the proof of Proposition 2.3, we have

$$u(x) = \alpha_u K_D(x, \infty) + \int_{\partial B(1)} K_D(x, y) d\mu(y) + \int_D G_D(x, y) f_u(y) dy, \tag{2.6}$$

where μ is a measure on $\partial B(1)$. Therefore, by (2.1) and (2.2),

$$u(x) \le A + \int_D G_D(x, y) f_u(y) dy \quad \text{for } x \in D \setminus B(2), \tag{2.7}$$

where A depends only on u. Let ℓ be a positive integer determined in the sequel. Since $u/K_D(\cdot, \infty)$ has minimal fine limit α_u at infinity and the set $\bigcup_i B(z_i, 2^{-\ell-3}|z_i|)$ is not minimally thin at infinity, we can find a sequence $w_i \in B(z_i, 2^{-\ell-3}|z_i|)$ with

$$u(w_i) \le A \frac{u(w_i)}{K_D(w_i, \infty)} \le A, \tag{2.8}$$

whenever i is sufficiently large.

Fix a sufficiently large i and let $z \in B(z_i, 2^{-\ell-3}|z_i|)$ and $1 \le j \le \ell$. Since

$$|z - w_i| \le |z - z_i| + |z_i - w_i| \le 2^{-\ell-2}|z_i| \le 2^{-\ell-1}|z| \le 2^{-j-1}|z|,$$

it follows from the Harnack inequality, (2.6) and (2.8) that for $x \in B(z, 2^{-j-1}|z|)$,

$$\begin{aligned} \int_{D \setminus B(z, 2^{-j}|z|)} G_D(x, y) f_u(y) dy &\le A \int_{D \setminus B(z, 2^{-j}|z|)} G_D(w_i, y) f_u(y) dy \\ &\le A u(w_i) \le A. \end{aligned} \tag{2.9}$$

This and (2.7) yield that

$$u(x) \le A_0 + \int_{B(z, 2^{-j}|z|)} \frac{f_u(y)}{|x - y|^{n-2}} dy \quad \text{for } x \in B(z, 2^{-j-1}|z|), \tag{2.10}$$

where A_0 is a constant depending only on u. Also, since

$$G_D(w_i, y) \ge \frac{1}{A} |z|^{2-n} \quad \text{for } y \in B(z, 2^{-1}|z|),$$

we have by (2.6) and (2.8)

$$\begin{aligned} |z|^{2-n} \int_{B(z, 2^{-1}|z|)} f_u(y) dy &\le A \int_{B(z, 2^{-1}|z|)} G_D(w_i, y) f_u(y) dy \\ &\le A u(w_i) \le A. \end{aligned} \tag{2.11}$$

Let $r = |z|$ and let $\psi_z(\zeta) = r^2 f_u(z + r\zeta)$. Making the change of variables $x = z + r\eta$ and $y = z + r\zeta$, we have from (2.10) and (2.11) that

$$\int_{B(1/2)} \psi_z(\zeta) d\zeta \le A, \tag{2.12}$$

and

$$u(z+r\eta) \le \Psi_{z,j}(\eta) \quad \text{for } \eta \in B(2^{-j-1}), \tag{2.13}$$

where

$$\Psi_{z,j}(\eta) = A_0 + \int_{B(2^{-j})} \frac{\psi_z(\zeta)}{|\eta-\zeta|^{n-2}} d\zeta.$$

Suppose that $0 < p < n/(n-2)$. Let

$$\max\{1,p\} < q < \frac{n}{n-2} \quad \text{and} \quad \ell = \left[\frac{\log(q/(q-1))}{\log(q/p)}\right] + 1.$$

Put $s = q/p > 1$. We claim that for $\kappa \ge 1$ there exists a constant A depending only on κ, c, p, q, A_0 and n such that

$$\int_{B(2^{-j-1})} \psi_z(\zeta)^{\kappa s} d\zeta \le A + A\left(\int_{B(2^{-j})} \psi_z(\zeta)^{\kappa} d\zeta\right)^q. \tag{2.14}$$

Indeed, by the Jensen inequality for the probability measure

$$\frac{|\eta-\zeta|^{2-n} d\zeta}{\int_{B(2^{-j})} |\eta-\zeta|^{2-n} d\zeta} \quad \text{on } B(2^{-j}),$$

we have

$$\left(\int_{B(2^{-j})} \frac{\psi_z(\zeta)}{|\eta-\zeta|^{n-2}} d\zeta\right)^{\kappa} \le A \int_{B(2^{-j})} \frac{\psi_z(\zeta)^{\kappa}}{|\eta-\zeta|^{n-2}} d\zeta \quad \text{for } \eta \in B(1).$$

Using the inequality $(a+b)^t \le 2^t(a^t+b^t)$ for $a,b,t>0$, we have

$$\int_{B(2^{-j})} \Psi_{z,j}(\eta)^{\kappa q} d\eta \le A + A \int_{B(2^{-j})} \left(\int_{B(2^{-j})} \frac{\psi_z(\zeta)^{\kappa}}{|\eta-\zeta|^{n-2}} d\zeta\right)^q d\eta.$$

The Minkowski inequality and $q(n-2) < n$ imply that the integral of the right hand side is bounded by

$$\left(\int_{B(2^{-j})} \left(\int_{B(2^{-j})} \frac{d\eta}{|\eta-\zeta|^{q(n-2)}}\right)^{1/q} \psi_z(\zeta)^{\kappa} d\zeta\right)^q \le A\left(\int_{B(2^{-j})} \psi_z(\zeta)^{\kappa} d\zeta\right)^q.$$

Therefore

$$\int_{B(2^{-j})} \Psi_{z,j}(\eta)^{\kappa q} d\eta \le A + A\left(\int_{B(2^{-j})} \psi_z(\zeta)^{\kappa} d\zeta\right)^q.$$

Since $r^2 = |z|^2 \le A|z+r\eta|^2$ for $\eta \in B(1/2)$, it follows from (1.4) and (2.13) that

$$\begin{aligned} \psi_z(\eta) = r^2 f_u(z+r\eta) &\le A u(z+r\eta)^p \\ &\le A\Psi_{z,j}(\eta)^p \quad \text{for a.e. } \eta \in B(2^{-j-1}). \end{aligned}$$

Hence

$$\int_{B(2^{-j-1})} \psi_z(\eta)^{\kappa q/p} d\eta \le A + A\left(\int_{B(2^{-j})} \psi_z(\zeta)^{\kappa} d\zeta\right)^q,$$

and so (2.14) holds.

Our choice of ℓ implies that $s^\ell \geq q/(q-1)$, equivalent to $s^\ell \leq (s^\ell - 1)q$. Therefore

$$\frac{s^\ell}{s^\ell - 1}(n-2) \leq q(n-2) < n.$$

By the Hölder inequality,

$$\Psi_{z,\ell+1}(0) \leq A + A\left(\int_{B(2^{-\ell-1})} \psi_z(\zeta)^{s^\ell} d\zeta\right)^{1/s^\ell}.$$

Using (2.14) ℓ times, we have

$$\begin{aligned}\int_{B(2^{-\ell-1})} \psi_z(\zeta)^{s^\ell} d\zeta &\leq A + A\left(\int_{B(2^{-\ell})} \psi_z(\zeta)^{s^{\ell-1}} d\zeta\right)^q \\ &\leq \cdots \\ &\leq A + A\left(\int_{B(1/2)} \psi_z(\zeta) d\zeta\right)^{q^\ell}.\end{aligned}$$

Hence we conclude from (2.13) and (2.12) that

$$u(z) \leq \Psi_{z,\ell+1}(0) \leq A.$$

This completes the proof of Proposition 2.4. □

The two dimensional analogue of Proposition 2.4 is stated as follows.

Proposition 2.5. *Let $n = 2$ and let $p > 0$ be arbitrary constant. Suppose that u is a positive superharmonic function on Ω satisfying*

$$f_u(x) \leq \frac{c}{|x|^2(\log|x|)^{p-1}} u(x)^p \quad \text{for almost every } x \in \Omega \setminus B(0, R) \tag{2.15}$$

with some constants $c > 0$ and $R > 1$. Let $\{z_i\}$ be a sequence in Ω such that $z_i \to \infty$ ($i \to +\infty$). Then there eixst a constant A and $i_0 \in \mathbb{N}$ such that

$$\frac{u(x)}{\log|x|} \leq A \quad \text{for } x \in \bigcup_{i \geq i_0} B(z_i, 2^{-5}|z_i|).$$

Proof. The proof is similar to that of Proposition 2.4, so we give only an outline. We may assume that $R = 1$ and $D \subset \Omega$. The Riesz decomposition (2.6) and (2.2) yield that

$$u(x) \leq A\log|x| + \int_D G_D(x,y) f_u(y) dy \quad \text{for } x \in D \setminus B(2).$$

Fix a sufficiently large i and let $z \in B(z_i, 2^{-5}|z_i|)$. As in the proof of Proposition 2.4, we can find $w \in B(z, 2^{-3}|z|)$ with

$$u(w) \leq A\log|w| \leq A\log|z|.$$

Therefore, if $j = 1, 2$, then the Harnack inequality gives that for $x \in B(z, 2^{-j-1}|z|)$,

$$\int_{D \setminus B(z, 2^{-j}|z|)} G_D(x,y) f_u(y) dy \le A \int_{D \setminus B(z, 2^{-j}|z|)} G_D(w,y) f_u(y) dy$$
$$\le A u(w) \le A \log |z|.$$

Note that for $x, y \in B(z, 2^{-1}|z|)$,

$$G_D(x,y) \le A \log \frac{5|z|^2}{|x-y|}.$$

Hence we obtain

$$u(x) \le A \log |z| + A \int_{B(z, 2^{-j}|z|)} f_u(y) \log \frac{5|z|^2}{|x-y|} dy \quad \text{for } x \in B(z, 2^{-j-1}|z|). \tag{2.16}$$

Also, since

$$G_D(w,y) \ge \frac{1}{A} \log |z| \quad \text{for } y \in B(z, 2^{-1}|z|),$$

we obtain

$$\log |z| \int_{B(z, 2^{-1}|z|)} f_u(y) dy \le A \int_{B(z, 2^{-1}|z|)} G_D(w,y) f_u(y) dy$$
$$\le A u(w) \le A \log |z|,$$

and so

$$\int_{B(z, 2^{-1}|z|)} f_u(y) dy \le A. \tag{2.17}$$

Let $r = |z|$ and let

$$\psi_z(\zeta) = \frac{r^2}{\log r} f_u(z + r\zeta).$$

Making the change of variables $x = z + r\eta$ and $y = z + r\zeta$, we have from (2.16) and (2.17) that

$$\int_{B(1/2)} \psi_z(\zeta) d\zeta \le \frac{A}{\log r} \le A, \tag{2.18}$$

and for $\eta \in B(2^{-j-1})$,

$$\begin{aligned} \frac{u(z + r\eta)}{\log r} &\le A + A \int_{B(2^{-j})} \psi_z(\zeta) \log \frac{5r}{|\eta - \zeta|} d\zeta \\ &\le A + A \int_{B(2^{-j})} \psi_z(\zeta) \log \frac{5}{|\eta - \zeta|} d\zeta =: \Psi_{z,j}(\eta). \end{aligned} \tag{2.19}$$

Here the second inequality follows by (2.18). Let $q > \max\{1, p\}$ and put $s = q/p > 1$. Using the Minkowski inequality, we have

$$\left(\int_{B(1/2)} \Psi_{z,1}(\eta)^q d\eta \right)^{1/q} \le A + A \int_{B(1/2)} \psi_z(\zeta) d\zeta.$$

Since

$$0 \le \psi_z(\eta) = \frac{r^2}{\log r} f_u(z + r\eta) \le \frac{A}{(\log r)^p} u(z + r\eta)^p$$
$$\le A\Psi_{z,1}(\eta)^p \quad \text{for a.e. } \eta \in B(1/4)$$

by (2.15) and (2.19), we have

$$\int_{B(1/4)} \psi_z(\eta)^s d\eta \le A + A\left(\int_{B(1/2)} \psi_z(\zeta) d\zeta\right)^q.$$

By (2.19) and the Hölder inequality,

$$\frac{u(z)}{\log r} \le \Psi_{z,2}(0) \le A + A\left(\int_{B(1/4)} \psi_z(\zeta)^s d\zeta\right)^{1/s}$$
$$\le A + A\left(\int_{B(1/2)} \psi_z(\zeta) d\zeta\right)^p.$$

Hence (2.18) yields $u(z)/\log|z| \le A$. This completes the proof. □

Now, Theorems 1.2 and 1.4 are proved immediately. Let $\{z_i\}$ be arbitrary sequence in Ω such that $z_i \to \infty$ ($i \to +\infty$). If $n \ge 3$, then we have by (1.4) and Proposition 2.4

$$f_u(x) \le \frac{c}{|x|^2} u(x)^p \le \frac{A}{|x|^2} \quad \text{for a.e. } x \in \bigcup_{i \ge i_0} B(z_i, 2^{-\ell-3}|z_i|).$$

If $n = 2$, then we have by (1.5) and Proposition 2.5

$$f_u(x) \le \frac{c}{|x|^2(\log|x|)^p} u(x)^p \le \frac{A}{|x|^2} \quad \text{for a.e. } x \in \bigcup_{i \ge i_0} B(z_i, 2^{-5}|z_i|).$$

Hence Theorems 1.2 and 1.4 follow from Proposition 2.3.

3 Proof of Theorem 1.3

In this section, we give a proof of Theorem 1.3. Suppose that $p > n/(n-2)$. Let $\beta > 0$ and let

$$\gamma = 1 - \beta(p-1) \quad \text{and} \quad \lambda = 2\beta p - 2.$$

Then $\gamma < 1$ and

$$\lambda - n + 2 + n\gamma = \beta\left((2-n)p + n\right) < 0. \tag{3.1}$$

For $j \in \mathbb{N}$, let $x_j = (2^j, 0, \ldots, 0)$ and $r_j = 2^{\gamma j - 3}$. Observe that $\{B(x_j, 2r_j)\}_j$ is mutually disjoint. Let $A_1 > 0$ be a constant such that

$$\frac{c\nu_n^p}{2^{(n+4)p+4}} A_1^p \ge A_1, \tag{3.2}$$

where ν_n is the volume of the unit ball in $\mathbb{R}^n$. Let f_j be a nonnegative smooth function on $\mathbb{R}^n$ such that

$$f_j \le A_1 2^{\lambda j} \quad \text{on } \mathbb{R}^n,$$

and

$$f_j = \begin{cases} A_1 2^{\lambda j} & \text{on } B(x_j, r_j), \\ 0 & \text{on } \mathbb{R}^n \setminus B(x_j, 2r_j). \end{cases}$$

Define $f = \sum_{j=1}^{\infty} f_j$. Since

$$1 + |x| \ge 1 + |x_j| - |x - x_j| \ge 2^{j-1} \quad \text{for } x \in B(x_j, 2r_j),$$

it follows from (3.1) that

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{f(x)}{(1+|x|)^{n-2}} dx &= \sum_{j=1}^{\infty} \int_{B(x_j, 2r_j)} \frac{f_j(x)}{(1+|x|)^{n-2}} dx \\ &\le \sum_{j=1}^{\infty} \frac{A_1 2^{\lambda j}}{2^{(j-1)(n-2)}} \nu_n (2r_j)^n \\ &\le \frac{A_1 \nu_n}{2^{n+2}} \sum_{j=1}^{\infty} 2^{j(\lambda - n + 2 + n\gamma)} < +\infty. \end{aligned}$$

Thus the function

$$u(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}} dy$$

is positive and superharmonic on $\mathbb{R}^n$. Since f is bounded and Lipschitz continuous on each compact subset of $\mathbb{R}^n$, it follows from [5, Lemma 4.2] that $u \in C^2(\mathbb{R}^n)$ and $-\Delta u = f$ in $\mathbb{R}^n$. Also, we observe from the mean value property that for $x \in \partial B(x_j, 2r_j)$,

$$u(x) \ge \int_{B(x_j, r_j)} \frac{f_j(y)}{|x-y|^{n-2}} dy = A_1 2^{\lambda j} \frac{\nu_n r_j^n}{|x - x_j|^{n-2}} \ge \frac{A_1 \nu_n}{2^{n+4}} 2^{j(\lambda + 2\gamma)}.$$

By the minimum principle,

$$u(x) \ge \frac{A_1 \nu_n}{2^{n+4}} 2^{j(\lambda + 2\gamma)} \quad \text{for } x \in B(x_j, 2r_j). \tag{3.3}$$

Since $\lambda + 2\gamma = 2\beta$, it follows that

$$\lim_{j \to +\infty} \frac{u(x_j)}{|x_j|^{\beta}} = +\infty.$$

Let us show that

$$f(x) \le \frac{c}{1 + |x|^2} u(x)^p \quad \text{for } x \in \mathbb{R}^n.$$

If $x \notin \bigcup_j B(x_j, 2r_j)$, then

$$\frac{c}{1+|x|^2} u(x)^p \ge 0 = f(x).$$

Let $x \in B(x_j, 2r_j)$. Then

$$1 + |x|^2 \le (1 + |x|)^2 \le (1 + |x_j| + |x - x_j|)^2 \le 2^{2j+4}.$$

Since $p(\lambda + 2\gamma) - 2 = \lambda$, we have by (3.3) and (3.2)

$$\frac{c}{1 + |x|^2} u(x)^p \ge \frac{c}{2^{2j+4}} \left(\frac{A_1 \nu_n}{2^{n+4}} 2^{j(\lambda+2\gamma)} \right)^p \ge A_1 2^{\lambda j} \ge f_j(x) = f(x).$$

Thus Theorem 1.3 is proved.

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