Boundary behavior of superharmonic functions satisfying nonlinear inequalities in uniform domains *

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Abstract

In a uniform domain Ω, we investigate the boundary behavior of positive superharmonic functions u satisfying the nonlinear inequality

$$-\Delta u(x) \leq c\delta_\Omega(x)^{-\alpha} u(x)^p \quad \text{for a.e. } x \in \Omega$$

with some constants $c > 0$, $\alpha \in \mathbb{R}$ and $p > 0$, where Δ is the Laplacian and $\delta_\Omega(x)$ is the distance from a point x to the boundary of Ω. In particular, we present a Fatou type theorem concerning the existence of nontangential limits and a Littlewood type theorem concerning the nonexistence of tangential limits.

Keywords: nontangential limit, minimal fine limit, superharmonic function, nonlinear elliptic equation, uniform domain

Mathematics Subject Classifications (2000): Primary 31B25; Secondary 31B05, 31A05, 31A20, 31C45, 35J60

1 Introduction

The study of the boundary behavior of harmonic, superharmonic and subharmonic functions and solutions of linear elliptic and parabolic equations has a long history. In 1906, Fatou [20] proved that every positive harmonic function in the unit disk has finite boundary limits almost everywhere along nontangential approach regions. The higher dimensional analogue was established by Bray and Evans [10]. Littlewood [37] showed the best possibility of nontangential approach regions in the following sense: there exists a bounded harmonic function which diverges almost everywhere along tangential curves. This was improved by Aikawa [1, 2] showing "everywhere" divergence.

*This work was partially supported by Grant-in-Aid for Young Scientists (B) (No. 19740062), Japan Society for the Promotion of Science.

See also [17]. Carleson [12] obtained a local version of the Fatou theorem by establishing a boundary estimate for positive harmonic functions vanishing on a portion of the boundary. As an application of the study of maximal functions, Nagel and Stein [47] showed that all positive harmonic functions have finite boundary limits along some tangential sequences. For some generalizations, see [4, 39]. Also, the boundary behavior of harmonic functions with finite weighted Dirichlet integrals was investigated by many authors (see [25, 43, 46] etc.). Furthermore, the Fatou theorem was extended to two directions: one is to nonsmooth domains and trees. See [32, 57] for Lipschitz domains, [33, 51] for nontangentially accessible domains (abbreviated to NTA), [3] for uniform domains and [5, 16] for trees. Another is to solutions of several equations. See [27, 35] and references therein for the Laplace equation associated with the Bergman metric on the unit ball of $\mathbb{C}^n$, [9, 24, 29, 36] for the boundary behavior at infinity of solutions of the Helmholtz equation $\Delta u = \lambda u$ in $\mathbb{R}^n$, [8, 44] for α-harmonic functions and [19, 34] for parabolic equations.

In 1928, Littlewood [38] proved that every positive superharmonic function in the unit disk has finite radial limits almost everywhere. The higher dimensional analogue was due to Privalov [49]. Dahlberg [15] introduced the notion of radial limits to extend Littlewood's result to Lipschitz domains. See also [60] for the extension to NTA domains. However, considering a positive superharmonic function which takes $+\infty$ on a countable dense subset of the unit disk, we see that nontangential limits of superharmonic functions do not necessarily exist. Tolsted [52, 53, 54] gave several sufficient conditions for positive superharmonic functions to have nontangential limits. Arsove and Huber [7] and Wu [58] refined his results and obtained the following result. See also [42] for Riesz potentials.

Theorem A. *Let u be a positive superharmonic function on the unit ball B of $\mathbb{R}^n$, $n \geq 2$, whose Riesz measure μ_u is absolutely continuous with respect to Lebesgue measure, say $d\mu_u(x) = f_u(x)dx$. If there is $q > n/2$ such that*

$$\int_B (1 - |x|)^{2q-1} f_u(x)^q dx < \infty,$$

then u has finite nontangential limits almost everywhere on the boundary. In particular, if there exists a constant A such that

$$f_u(x) \leq \frac{A}{(1 - |x|)^2} \quad \text{for almost every } x \in B,$$

then u has finite nontangential limits almost everywhere on the boundary.

In contrast to the above concrete results, Naïm [45] studied the boundary behavior of Green potentials in general domains by introducing the notion of minimal fine limit. In [18], Doob obtained a result for minimal fine limits of superharmonic functions. These results were further extended by Gowrisankaran [22, 23] to general axiomatic situations. The relation between nontangential limits and minimal fine limits of harmonic functions in the unit ball was investigated in [11].

By the way, we can know from Theorem A the existence of nontangential limits for solutions of the Poisson equation $-\Delta u = f$ with a suitable density f. However, such classical results are not (directly) applicable to solutions of nonlinear elliptic equations. Recently, the existence of positive solutions of nonlinear elliptic equations of the form $-\Delta u = Vu^p$ was studied widely by many mathematicians, using the method of not only partial differential equations but also the probabilistic and classical potential theories. See [13, 28, 50, 55, 59, 61] and references therein. But, because of the difficulty of analysis involving the nonlinearities, it seems that there is no potential theoretic investigation for solutions of such nonlinear equations.

The purpose of this paper is to investigate the boundary behavior of positive solutions of nonlinear elliptic equations and, more generally, positive superharmonic functions satisfying nonlinear inequalities. To state our results, we need to prepare some notations and terminology. Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $\delta_\Omega(x)$ stand for the distance from a point x to the boundary $\partial\Omega$ of Ω. A lower semicontinuous function $u : \Omega \to (-\infty, +\infty]$, where $u \not\equiv +\infty$, is called *superharmonic* on Ω if it satisfies the mean value inequality

$$u(x) \geq \frac{1}{\nu_n r^n} \int_{B(x,r)} u(y)dy \quad \text{for } 0 < r < \delta_\Omega(x),$$

where $B(x,r)$ denotes the open ball of center x and radius r and ν_n is the volume of the unit ball. Let Δ be the Laplace operator on $\mathbb{R}^n$. It is well known that if u is a superharmonic function on Ω, then there exists a unique (Radon) measure μ_u on Ω such that

$$\int_\Omega \phi(x)d\mu_u(x) = -\int_\Omega u(x)\Delta\phi(x)dx \quad \text{for all } \phi \in C_0^\infty(\Omega),$$

where $C_0^\infty(\Omega)$ is the collection of all infinitely differentiable functions vanishing outside a compact set in Ω (see [6, Section 4.3]). The measure μ_u is called the *Riesz measure* associated with u. If μ_u is absolutely continuous with respect to Lebesgue measure and $d\mu_u(x) = f_u(x)dx$ with f_u being a nonnegative locally integrable function on Ω, then we call f_u the *Riesz function* associated with u for convenience. It is clear that $f_u = -\Delta u$ for $u \in C^2(\Omega)$.

In the previous papers [30, 31], the author studied positive superharmonic functions u on Ω having an associated Riesz function f_u and satisfying the nonlinear inequality

$$f_u(x) \leq c\delta_\Omega(x)^{-\alpha}u(x)^p \quad \text{for almost every } x \in \Omega, \tag{1.1}$$

where $c > 0$, $\alpha \geq 0$ and $p > 0$ are constants. The following boundary growth estimate was proved.

Theorem B. *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 2$. Suppose that*

$$0 < p \leq \frac{n+1}{n-1} \quad \textit{and} \quad \alpha \leq n + 1 - p(n-1).$$

If u is a positive superharmonic function on Ω having an associated Riesz function f_u which satisfies (1.1)*, then there exists a constant A depending only on u, c, α, p and Ω such that*

$$u(x) \leq \frac{A}{\delta_\Omega(x)^{n-1}} \quad \textit{for } x \in \Omega.$$

Now, let $0 < p \leq 2/(n-1)$ and $\alpha \leq 2 - p(n-1)$. If u is a positive superharmonic function on the unit ball B having an associated Riesz function f_u which satisfies (1.1), then Theorem B implies that

$$\begin{aligned} f_u(x) &\leq c(1-|x|)^{-\alpha} u(x)^p \\ &\leq \frac{A}{(1-|x|)^{p(n-1)+\alpha}} \leq \frac{A}{(1-|x|)^2} \quad \text{for almost every } x \in B. \end{aligned}$$

Hence, this together with Theorem A yields the following.

Theorem C. *Let u be a positive superharmonic function on B having an associated Riesz function f_u which satisfies* (1.1) *with $\Omega = B$ and some constants*

$$0 < p \leq \frac{2}{n-1} \quad \textit{and} \quad \alpha \leq 2 - p(n-1).$$

Then u has finite nontangential limits almost everywhere on ∂B.

The main result (Theorem 1.1) in this paper asserts that the conclusion of Theorem C is valid for the almost sharp range $0 < p < n/(n-2)$ and $\alpha \leq 2$. We will show this in uniform domains, introduced by Martio and Sarvas [41] in their study of approximation and injectivity properties of mappings. See also [21, 40, 56]. A proper subdomain Ω of $\mathbb{R}^n$ is called a *uniform domain* if there exists a constant A such that each pair of points $x, y \in \Omega$ can be connected by a rectifiable curve γ in Ω with the following properties:

$$\ell(\gamma) \leq A|x-y|, \tag{1.2}$$

$$\min\{\ell(\gamma(x,z)), \ell(\gamma(z,y))\} \leq A\delta_\Omega(z) \quad \text{for all } z \in \gamma, \tag{1.3}$$

where $\ell(\gamma(x,z))$ denotes the length of the subarc $\gamma(x,z)$ of γ from x to z. A uniform domain is a domain satisfying only the interior conditions for an NTA domain (see [26, 33]). Note that the conditions (1.2) and (1.3) can be extended to $x, y \in \overline{\Omega}$, and therefore the nontangential set with aperture θ and vertex at $\xi \in \partial\Omega$ defined by

$$\Gamma_\theta(\xi) = \{x \in \Omega : |x - \xi| < \theta\delta_\Omega(x)\}$$

is nonempty and ξ is accessible from $\Gamma_\theta(\xi)$ whenever θ is sufficiently large, say $\theta \geq \theta_\Omega$. A function u on Ω is said to have *nontangential limit* ℓ at $\xi \in \partial\Omega$ if $u(x)$ tends to ℓ as x approaches to ξ within $\Gamma_\theta(\xi)$ for any $\theta \geq \theta_\Omega$. We define $\mathcal{NS}_{p,\alpha}(\Omega)$ as the collection of every positive superharmonic function u on Ω having an associated Riesz function f_u and satisfying

$$\lim_{r \to 0} \left(\operatorname*{ess\,sup}_{x \in \Gamma_\theta(\xi) \cap B(\xi, r)} \frac{f_u(x)}{\delta_\Omega(x)^{-\alpha} u(x)^p} \right) < \infty \tag{1.4}$$

for each $\xi \in \partial\Omega$ and $\theta \geq \theta_\Omega$. The Fatou type theorem in our context is stated as follows.

Theorem 1.1. *Let Ω be a uniform domain in $\mathbb{R}^n$, $n \geq 3$. Suppose that*

$$0 < p < \frac{n}{n-2} \quad \text{and} \quad \alpha \leq 2.$$

If $u \in \mathcal{NS}_{p,\alpha}(\Omega)$, then u has finite nontangential limits on $\partial\Omega$ except for a set of harmonic measure zero.

For two dimensional result, we require the following condition on Ω: At $\xi \in \partial\Omega$, the Green function $G_\Omega(x, y)$ for Ω satisfies

$$\limsup_{\Gamma_\theta(\xi) \ni x \to \xi} \left(\sup_{y \in B(x, \delta_\Omega(x)/2)} \frac{G_\Omega(x, y)}{\log(\delta_\Omega(x)/|x - y|)} \right) < \infty \tag{1.5}$$

for each $\theta \geq \theta_\Omega$. The elementary geometrical sufficient condition for this is the exterior corkscrew condition: there exist constants $r_\xi > 0$ and $A_\xi > 1$ such that for each $0 < r < r_\xi$, there is a ball of radius r/A_ξ contained in $B(\xi, r) \setminus \Omega$. Therefore, every boundary point of an NTA domain in $\mathbb{R}^2$ satisfies (1.5).

Theorem 1.2. *Let Ω be a uniform domain in $\mathbb{R}^2$ such that $\mathbb{R}^2 \setminus \Omega$ is non-polar. Suppose that*

$$0 < p < \infty \quad \text{and} \quad \alpha < 2.$$

If $u \in \mathcal{NS}_{p,\alpha}(\Omega)$, then u has finite nontangential limits on $\partial\Omega$ except for a set of harmonic measure zero. Furthermore, if we assume that every point of $\partial\Omega$ satisfies (1.5)*, then the conclusion is valid for $\alpha = 2$ as well.*

If Ω is a Lipschitz domain, then its harmonic measure is absolutely continuous with respect to the surface area measure on $\partial\Omega$ (see [14]). Thus the following corollary holds.

Corollary 1.3. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, and let $0 < p < n/(n-2)$ and $\alpha \leq 2$. If $u \in \mathcal{NS}_{p,\alpha}(\Omega)$, then u has finite nontangential limits almost everywhere on $\partial\Omega$.*

Note that the bounds $p < n/(n-2)$ and $\alpha \leq 2$ are almost sharp in Corollary 1.3.

Theorem 1.4. *Let B be the unit ball of $\mathbb{R}^n$, $n \geq 2$, and let $c > 0$ be a constant. Assume either*

$$p > \frac{n}{n-2} \quad \text{or} \quad \alpha > 2.$$

Then there exists a positive function $u \in C^2(B)$ satisfying

$$0 \leq -\Delta u(x) \leq c(1 - |x|)^{-\alpha} u(x)^p \quad \text{in } B$$

such that u fails to have nontangential limits everywhere on ∂B. In particular,

$$\limsup_{\Gamma_\theta(\xi) \ni x \to \xi} u(x) = \infty$$

for every $\xi \in \partial B$ and $\theta > 1$.

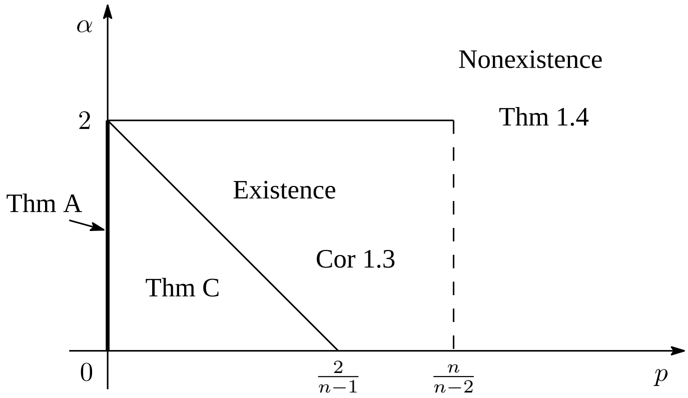

Figure 1: The ranges of $p \geq 0$ and $\alpha \geq 0$.

The above results are figured as follows:

Also, we shall obtain the following Littlewood type theorem concerning the best possibility of nontangential approaches. By $\mathcal{O}$, we denote the group of all orthogonal transformations on $\mathbb{R}^n$.

Theorem 1.5. *Let B be the unit ball of $\mathbb{R}^n$, $n \geq 2$, and let $p > 0$, $\alpha < 2$ and $c > 0$ be constants. Assume that $c > 0$ is sufficiently small only when $p = 1$, and that V is a measurable function on B such that*

$$|V(x)| \leq c(1 - |x|)^{-\alpha} \quad \textit{for almost every } x \in B.$$

Suppose that γ is a curve in B ending at $e = (1, 0, \cdots, 0)$ such that

$$\lim_{\gamma \ni x \to e} \frac{|x - e|}{1 - |x|} = \infty. \tag{1.6}$$

Then there exists a positive bounded distributional solution $u \in C(B)$ of

$$-\Delta u = V u^p \quad \textit{in } B$$

such that

$$\liminf_{|x| \to 1, x \in O\gamma} u(x) \neq \limsup_{|x| \to 1, x \in O\gamma} u(x) \quad \textit{for all } O \in \mathcal{O}.$$

The rest of the paper is organized as follows. In Section 2, we recall the known results concerning minimal fine topology. The proofs of Theorems 1.1 and 1.2 are presented in Section 3. Theorem 1.4 is proved in Section 4. After showing the existence of positive solutions of $-\Delta u = V u^p$ which are comparable to a given positive harmonic function, we prove Theorem 1.5 in Section 5.

2 Preliminary

Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$. When $n = 2$, we require that $\mathbb{R}^2 \setminus \Omega$ is non-polar. Thus the Green function $G_\Omega(x, y)$ for Ω always exists. Let $x_0 \in \Omega$ be fixed. If $\{y_j\}$ is a sequence in Ω with no limit point in Ω, then we observe from the Harnack principle that $\{G_\Omega(\cdot, y_j)/G_\Omega(x_0, y_j)\}$ converges locally uniformly to a positive harmonic function on Ω. By $\Delta(\Omega)$, we denote the collection of all positive harmonic functions on Ω obtained in such a way. Inducing a suitable metric, we see that $\Omega \cup \Delta(\Omega)$ becomes a metric compactification of Ω. Then $\Omega \cup \Delta(\Omega)$ and $\Delta(\Omega)$ are called the *Martin compactification* and the *Martin boundary* of Ω, respectively. See [6, Chapter 8] for the details. Let $K_\Omega(\cdot, \xi)$ be the *Martin kernel* of Ω associated with $\xi \in \Delta(\Omega)$ and let E be a subset of Ω. By $\widehat{R}^E_{K_\Omega(\cdot,\xi)}$, we denote the lower semicontinuous regularization of the reduced function of $K_\Omega(\cdot, \xi)$ relative to E in Ω defined by

$$R^E_{K_\Omega(\cdot,\xi)}(x) = \inf_u u(x),$$

where the infimum is taken over all nonnegative superharmonic functions u on Ω satisfying $K_\Omega(\cdot, \xi) \leq u$ on E. In general, $\widehat{R}^E_{K_\Omega(\cdot,\xi)} \leq K_\Omega(\cdot, \xi)$. A set E is called *minimally thin* at ξ (with respect to Ω) if

$$\widehat{R}^E_{K_\Omega(\cdot,\xi)}(x) < K_\Omega(x, \xi) \quad \text{for some } x \in \Omega.$$

We say that a function f on Ω has *minimal fine limit* ℓ at ξ if there exists a subset E of Ω, which is minimally thin at ξ, such that

$$\lim_{\Omega\setminus E \ni x \to \xi} f(x) = \ell.$$

For a measure μ on Ω, a function $x \mapsto \int_\Omega G_\Omega(x, y)d\mu(y)$ is called a *Green potential* of μ if $\int_\Omega G_\Omega(x_0, y)d\mu(y) < \infty$ for some $x_0 \in \Omega$. Let ν_1 be the measure on $\Delta(\Omega)$ corresponding to the constant function 1 in the Martin representation

$$1 = \int_{\Delta(\Omega)} K(x, \xi)d\nu_1(\xi).$$

The following general result was proved by Doob [18] and Naïm [45].

Lemma 2.1. *Let u be a nonnegative superharmonic function on Ω. Then u has finite minimal fine limits ν_1-almost everywhere on $\Delta(\Omega)$. Moreover, if u is a Green potential on Ω, then u has minimal fine limit 0 ν_1-almost everywhere on $\Delta(\Omega)$.*

Next, let us recall Aikawa's results [3]. If Ω is a bounded uniform domain, then the boundary Harnack principle guarantees that the Martin compactification $\Omega \cup \Delta(\Omega)$ is homeomorphic to the Euclidean closure $\overline{\Omega}$, so that the Martin boundary $\Delta(\Omega)$ can be identified with the Euclidean boundary $\partial\Omega$. Moreover, the following result holds (see [3, Lemma 5]).

Lemma 2.2. *Let Ω be a uniform domain and let $\xi \in \partial\Omega$. Suppose that $0 < \beta < 1$ and that $\{x_j\}$ is a sequence converging to ξ within $\Gamma_\theta(\xi)$ for some $\theta \geq \theta_\Omega$. Then the bubble set $\bigcup_j B(x_j, \beta\delta_\Omega(x_j))$ is not minimally thin at ξ.*

Observe that ν_1 and the harmonic measure of Ω are mutually absolutely continuous. The following Fatou theorem for harmonic functions was obtained from Lemmas 2.1 and 2.2 (see [3, Theorem 4] and its proof).

Lemma 2.3. *Let Ω be a uniform domain and let h be a positive harmonic function on Ω. If h has minimal fine limit ℓ at $\xi \in \partial\Omega$, then h has nontangential limit ℓ at ξ. Furthermore, h has finite nontangential limits on $\partial\Omega$ except for a set of harmonic measure zero.*

3 Proofs of Theorems 1.1 and 1.2

Throughout this section, we suppose that Ω is a uniform domain in $\mathbb{R}^n$, $n \geq 2$. The symbol A stands for an absolute positive constant whose value is unimportant and may change from line to line. We start with the following extension of the last assertion of Theorem A.

Proposition 3.1. *Let $\xi \in \partial\Omega$, $\theta \geq \theta_\Omega$ and $\alpha \leq 2$. If $n = 2$ and* (1.5) *does not hold at ξ, we assume $\alpha < 2$. Suppose that f is a nonnegative measurable function on Ω such that*

$$f(x) \leq \frac{A}{\delta_\Omega(x)^\alpha} \quad \text{for almost every } x \in \Gamma_{2\theta}(\xi) \cap B(\xi, \rho) \tag{3.1}$$

with some constants $A > 0$ and $\rho > 0$. If the Green potential $\int_\Omega G_\Omega(\cdot, y)f(y)dy$ has minimal fine limit 0 at ξ, then it has limit 0 at ξ along $\Gamma_\theta(\xi)$.

Proof. Let $x_0 \in \Omega$ be fixed and let $D = \Omega \setminus \overline{B(x_0, \delta_\Omega(x_0)/2)}$. Since $\int_\Omega G_\Omega(\cdot, y)f(y)dy$ is nonnegative superharmonic on D, it follows from the Riesz decomposition theorem that

$$\int_\Omega G_\Omega(x,y)f(y)dy = h_D(x) + \int_D G_D(x,y)f(y)dy,^\dagger$$

where h_D is a nonnegative harmonic function on D. Let us first show that $\int_D G_D(\cdot, y)f(y)dy$ has limit 0 at ξ along $\Gamma_\theta(\xi)$. For $0 < \varepsilon < \theta/(2\theta+1)$ and $x \in D$, let

$$U_1(x) = \int_{D \setminus B(x, \varepsilon\delta_\Omega(x))} G_D(x,y)f(y)dy,$$
$$U_2(x) = \int_{B(x, \varepsilon\delta_\Omega(x))} G_D(x,y)f(y)dy.$$

Let $\{z_i\}$ be arbitrary sequence in $\Gamma_\theta(\xi) \cap B(\xi, \delta_\Omega(x_0)/4)$ converging to ξ. By Lemma 2.2, we find points $x_i \in B(z_i, \varepsilon\delta_\Omega(z_i)/2)$ such that

$$\lim_{i\to\infty} \int_\Omega G_\Omega(x_i, y)f(y)dy = 0.$$

†This is to guarantee $G_D(x,y) \leq A\log(1/|x-y|)$ in the general case of $n = 2$. If $\mathbb{R}^2 \setminus \overline{\Omega}$ contains a disk, then we need not consider this decomposition.

It follows from the Harnack inequality that

$$U_1(z_i) \le A \int_{D\setminus B(z_i,\varepsilon\delta_\Omega(z_i))} G_D(x_i,y)f(y)dy \le A\int_\Omega G_\Omega(x_i,y)f(y)dy,$$

where A depends only on n. Hence $U_1(z_i) \to 0$ as $i \to \infty$, and so U_1 has limit 0 at ξ along $\Gamma_\theta(\xi)$.

We next consider U_2. Let $x \in \Gamma_\theta(\xi) \cap B(\xi, \delta_\Omega(x_0)/4)$ be sufficiently close to ξ and let $y \in B(x, \varepsilon\delta_\Omega(x))$. Then

$$|y-\xi| \le |y-x| + |x-\xi| < (\varepsilon+\theta)\delta_\Omega(x) \le (\varepsilon+\theta)\frac{\delta_\Omega(y)}{1-\varepsilon} < 2\theta\delta_\Omega(y),$$

and so $y \in \Gamma_{2\theta}(\xi) \cap B(\xi,\rho)$. Observe that

$$G_D(x,y) \le \begin{cases} G_{\mathbb{R}^2\setminus\overline{B(x_0,\delta_\Omega(x_0)/2)}}(x,y) \le A\log\dfrac{1}{|x-y|} & \text{if } n=2 \text{ (general case)},\\ G_\Omega(x,y) \le A\log\dfrac{\delta_\Omega(x)}{|x-y|} & \text{if } n=2 \text{ and (1.5) holds},\\ A|x-y|^{2-n} & \text{if } n\ge 3. \end{cases}$$

Therefore

$$\int_{B(x,\varepsilon\delta_\Omega(x))} G_D(x,y)dy \le \begin{cases} A\delta_\Omega(x)^2\varepsilon^2\left(1+\log\dfrac{1}{\varepsilon\delta_\Omega(x)}\right) & \text{if } n=2 \text{ (general case)},\\ A\delta_\Omega(x)^2\varepsilon^2\left(1+\log\dfrac{1}{\varepsilon}\right) & \text{if } n=2 \text{ and (1.5) holds},\\ A\delta_\Omega(x)^2\varepsilon^2 & \text{if } n\ge 3. \end{cases}$$

Since $f(y) \le A\delta_\Omega(y)^{-\alpha} \le A\delta_\Omega(x)^{-\alpha}$ for a.e. $y \in B(x,\varepsilon\delta_\Omega(x))$ by (3.1), we obtain

$$U_2(x) \le \begin{cases} A\delta_\Omega(x)^{2-\alpha}\varepsilon^2\left(1+\log\dfrac{1}{\varepsilon\delta_\Omega(x)}\right) & \text{if } n=2 \text{ (general case)},\\ A\varepsilon^2\left(1+\log\dfrac{1}{\varepsilon}\right) & \text{if } n=2 \text{ and (1.5) holds},\\ A\varepsilon^2 & \text{if } n\ge 3. \end{cases}$$

Here we used $\alpha \le 2$. Letting $x \to \xi$ within $\Gamma_\theta(\xi)$ and $\varepsilon \to 0$, we see that $\int_D G_D(\cdot,y)f(y)dy$ has limit 0 at ξ along $\Gamma_\theta(\xi)$.

By the same way to U_1, we observe that h_D has limit 0 at ξ along $\Gamma_\theta(\xi)$ as well. This completes the proof. □

In the rest of this section, we suppose that $u \in \mathcal{NS}_{p,\alpha}(\Omega)$ with $p > 0$ and $\alpha \in \mathbb{R}$. By the Riesz decomposition theorem, we have

$$u(x) = h(x) + \int_\Omega G_\Omega(x,y)f_u(y)dy \quad \text{for } x \in \Omega,$$

where h is the greatest harmonic minorant of u on Ω and f_u is the Riesz function associated with u. Then h is nonnegative. By Lemmas 2.1 and 2.3, we observe that there exists a set $E \subset \partial\Omega$ of harmonic measure zero such that for each $\xi \in \partial\Omega \setminus E$,

- h has nontangential limit ℓ at ξ,
- $\int_\Omega G_\Omega(\cdot, y) f_u(y)dy$ has minimal fine limit 0 at ξ.

Here, we state an immediate consequence of Proposition 3.1.

Corollary 3.2. *Let $\xi \in \partial\Omega$, $\theta \geq \theta_\Omega$, $p > 0$ and $\alpha \leq 2$. If $n = 2$ and* (1.5) *does not hold at ξ, we assume $\alpha < 2$. Suppose that $u \in \mathcal{NS}_{p,\alpha}(\Omega)$ is bounded on $\Gamma_{2\theta}(\xi) \cap B(\xi, \rho)$ with some $\rho > 0$ and has minimal fine limit ℓ at ξ. Then u has limit ℓ at ξ along $\Gamma_\theta(\xi)$.*

Proof. By assumption, $u \leq A$ on $\Gamma_{2\theta}(\xi) \cap B(\xi, \rho)$. Therefore, by (1.4),

$$f_u(x) \leq A\delta_\Omega(x)^{-\alpha} u(x)^p \leq A\delta_\Omega(x)^{-\alpha} \quad \text{for almost every } x \in \Gamma_{2\theta}(\xi) \cap B(\xi, \rho).$$

Proposition 3.1 implies that $\int_\Omega G_\Omega(\cdot, y) f_u(y)dy$ has limit 0 at ξ along $\Gamma_\theta(\xi)$. Thus the conclusion follows from this and Lemma 2.3. □

In particular, the following holds.

Corollary 3.3. *Let $p > 0$ and $\alpha \leq 2$. If $n = 2$ and* (1.5) *does not hold at ξ, we assume $\alpha < 2$. Suppose that $u \in \mathcal{NS}_{p,\alpha}(\Omega)$ is bounded on Ω. Then u has finite nontangential limits on $\partial\Omega$ except for a set of harmonic measure zero.*

Remark 3.4. In the above corollaries, we need not impose the upper bound for p although the boundedness of u is required for $n \geq 3$ and $p > n/(n-2)$. See Section 4. As seen in the argument below, we can remove the boundedness of u in the case $p < n/(n-2)$.

Let us continue the proof of Theorems 1.1 and 1.2. In what follows, let $\xi \in \partial\Omega \setminus E$. By Corollary 3.2, it is enough to prove the following proposition.

Proposition 3.5. *Suppose that $p > 0$ and $\alpha \in \mathbb{R}$ are as in Theorem 1.1 or 1.2. Let $\theta \geq \theta_\Omega$. Then there exists $\rho > 0$ such that $u \in \mathcal{NS}_{p,\alpha}(\Omega)$ is bounded on $\Gamma_\theta(\xi) \cap B(\xi, \rho)$.*

Let us prove Proposition 3.5. Let $z \in \Gamma_\theta(\xi) \cap B(\xi, \rho)$ with $\theta \geq \theta_\Omega$ and $0 < \rho < 1$ being sufficiently small. In view of Lemma 2.2, we may assume that

$$^\exists w \in B(z, \delta_\Omega(z)/4) \text{ s.t. } \int_\Omega G_\Omega(w, y) f_u(y)dy \leq 1. \tag{3.2}$$

The following lemma is elementary.

Lemma 3.6. *There exists a constant A depending only on n such that for $z \in \Omega$ and $x, y \in B(z, \delta_\Omega(z)/4)$,*

$$G_\Omega(x, y) \geq \frac{1}{A}|x - y|^{2-n} \geq \frac{1}{A}\delta_\Omega(z)^{2-n}.$$

Proof. Let $x, y \in B(z, \delta_\Omega(z)/4)$ and let $B = B(y, 3\delta_\Omega(z)/4)$. Then $|x - y| < \delta_\Omega(z)/2$ and $B \subset \Omega$. Since $G_\Omega \geq G_B$ on $B \times B$, the required inequality is obtained by using the explicit formula for G_B (see [6, Theorem 4.1.5]). The detail is left to the reader. □

Lemma 3.7. *There exists a constant A depending only on n such that*

$$\delta_\Omega(z)^{2-n}\int_{B(z,\delta_\Omega(z)/4)} f_u(y)dy \le A.$$

Proof. Let $w \in B(z,\delta_\Omega(z)/4)$ be as in (3.2). By Lemma 3.6,

$$1 \ge \int_\Omega G_\Omega(w,y)f_u(y)dy \ge \frac{1}{A\delta_\Omega(z)^{n-2}}\int_{B(z,\delta_\Omega(z)/4)} f_u(y)dy.$$

Thus the lemma is proved. □

For $x, z \in \Omega$ and $j \in \mathbb{N}$, we define

$$H_{z,j}(x) = h(x) + \int_{\Omega\setminus B(z,2^{-j-1}\delta_\Omega(z))} G_\Omega(x,y)f_u(y)dy,$$
$$V_{z,j}(x) = \int_{B(z,2^{-j-1}\delta_\Omega(z))} G_\Omega(x,y)f_u(y)dy.$$

As in the proof of Proposition 3.1, the use of the Harnack inequality leads to the following.

Lemma 3.8. *For each $j \in \mathbb{N}$, the function $z \mapsto H_{z,j}(z)$ has nontangential limit ℓ at ξ. In particular, this function is bounded on $\Gamma_\theta(\xi) \cap B(\xi,\rho)$.*

To obtain Proposition 3.5, it is enough to show the following lemma because of

$$u(z) = H_{z,j}(z) + V_{z,j}(z).$$

Lemma 3.9. *There exist a constant A and $m \in \mathbb{N}$ such that*

$$V_{z,m}(z) \le A \quad \textit{for } z \in \Gamma_\theta(\xi) \cap B(\xi,\rho). \tag{3.3}$$

The proof of Lemma 3.9 is given separately in two dimensions (the case $\alpha < 2$ and $\alpha = 2$) and higher dimensions.

Proof of Lemma 3.9 for $n \ge 3$. Let

$$\max\{1,p\} < q < \frac{n}{n-2} \quad \text{and} \quad N = \left[\frac{\log(q/(q-1))}{\log(q/p)}\right] + 1.$$

Take $z \in \Gamma_\theta(\xi) \cap B(\xi,\rho)$. Since

$$G_\Omega(x,y) \le AG_\Omega(z,y) \quad \text{for } x \in B(z,2^{-j-2}\delta_\Omega(z)) \text{ and } y \in \Omega\setminus B(z,2^{-j-1}\delta_\Omega(z))$$

by the Harnack inequality, it follows from Lemma 3.8 that there exists a constant A (independent of z) such that if $1 \le j \le N$, then

$$H_{z,j}(x) \le AH_{z,j}(z) \le AH_{z,N}(z) \le A \quad \text{for } x \in B(z,2^{-j-2}\delta_\Omega(z)). \tag{3.4}$$

Let $r = \delta_\Omega(z)$ and let $\psi_z(\zeta) = r^2 f_u(z + r\zeta)$. Then $r \le \rho \le 1$. For simplicity, we write $B(r) = B(0, r)$. Noting $G_\Omega(x, y) \le A|x - y|^{2-n}$ and making the change of variables $x = z + r\eta$ and $y = z + r\zeta$, we have

$$V_{z,j}(z + r\eta) \le A \int_{B(2^{-j-1})} \frac{\psi_z(\zeta)}{|\eta - \zeta|^{n-2}} d\zeta. \tag{3.5}$$

Let $1 \le j \le N$ and $\kappa \ge 1$. Applying the Jensen inequality for the probability measure

$$\frac{|\eta - \zeta|^{2-n} d\zeta}{\int_{B(2^{-j-1})} |\eta - \zeta|^{2-n} d\zeta} \quad \text{on } B(2^{-j-1}),$$

we have

$$V_{z,j}(z + r\eta)^\kappa \le A \int_{B(2^{-j-1})} \frac{\psi_z(\zeta)^\kappa}{|\eta - \zeta|^{n-2}} d\zeta.$$

By the Minkowski inequality and $q < n/(n-2)$,

$$\begin{aligned}
&\left(\int_{B(2^{-j-1})} V_{z,j}(z + r\eta)^{\kappa q} d\eta\right)^{1/q} \\
&\quad \le A\left(\int_{B(2^{-j-1})} \left(\int_{B(2^{-j-1})} \frac{\psi_z(\zeta)^\kappa}{|\eta - \zeta|^{n-2}} d\zeta\right)^q d\eta\right)^{1/q} \\
&\quad \le A \int_{B(2^{-j-1})} \left(\int_{B(2^{-j-1})} \frac{d\eta}{|\eta - \zeta|^{q(n-2)}}\right)^{1/q} \psi_z(\zeta)^\kappa d\zeta \\
&\quad \le A \int_{B(2^{-j-1})} \psi_z(\zeta)^\kappa d\zeta.
\end{aligned} \tag{3.6}$$

By (3.4),

$$u(z + r\eta) = H_{z,j}(z + r\eta) + V_{z,j}(z + r\eta) \le A + V_{z,j}(z + r\eta) \quad \text{for } \eta \in B(2^{-j-2}).$$

Since $\delta_\Omega(z + r\eta) \approx \delta_\Omega(z) = r \le 1$ for $\eta \in B(1/2)$, it follows from (1.4) and $\alpha \le 2$ that

$$\begin{aligned}
\psi_z(\eta) = r^2 f_u(z + r\eta) &\le A r^{2-\alpha} u(z + r\eta)^p \\
&\le A + A V_{z,j}(z + r\eta)^p \quad \text{for a.e. } \eta \in B(2^{-j-2}).
\end{aligned}$$

Here we used the inequality $(a + b)^t \le 2^t(a^t + b^t)$ for $a, b, t > 0$. Let $s = q/p > 1$. Then, by (3.6),

$$\begin{aligned}
\int_{B(2^{-j-2})} \psi_z(\eta)^{\kappa s} d\eta &\le A + A \int_{B(2^{-j-1})} V_{z,j}(z + r\eta)^{\kappa q} d\eta \\
&\le A + A\left(\int_{B(2^{-j-1})} \psi_z(\zeta)^\kappa d\zeta\right)^q.
\end{aligned}$$

We use this inequality N times to obtain

$$\begin{aligned}\int_{B(2^{-N-2})} \psi_z(\zeta)^{s^N} d\zeta &\le A + A\biggl(\int_{B(2^{-N-1})} \psi_z(\zeta)^{s^{N-1}} d\zeta\biggr)^q \\ &\le A + A\biggl(\int_{B(2^{-N})} \psi_z(\zeta)^{s^{N-2}} d\zeta\biggr)^{q^2} \\ &\le \cdots \\ &\le A + A\biggl(\int_{B(1/4)} \psi_z(\zeta) d\zeta\biggr)^{q^N}.\end{aligned}$$

Our choice of N implies that $s^N \ge q/(q-1)$, equivalent to $s^N \le (s^N-1)q$. Therefore

$$\frac{s^N}{s^N-1}(n-2) \le q(n-2) < n.$$

By (3.5) and the Hölder inequality,

$$\begin{aligned}V_{z,N+1}(z) &\le A \int_{B(2^{-N-2})} \frac{\psi_z(\zeta)}{|\zeta|^{n-2}} d\zeta \le A\biggl(\int_{B(2^{-N-2})} \psi_z(\zeta)^{s^N} d\zeta\biggr)^{1/s^N} \\ &\le A + A\biggl(\int_{B(1/4)} \psi_z(\zeta) d\zeta\biggr)^{p^N}.\end{aligned}$$

Therefore Lemma 3.7 yields that

$$V_{z,N+1}(z) \le A + A\biggl(\int_{B(z,\delta_\Omega(z)/4)} \delta_\Omega(z)^{2-n} f_u(y) dy\biggr)^{p^N} \le A,$$

where A is independent of z. Hence (3.3) with $m = N+1$ holds in the case $n \ge 3$. □

Proof of Lemma 3.9 for $n = 2$; general case. We show (3.3) with $m = 2$. By considering $\Omega \setminus \overline{B(x_0, \delta_\Omega(x_0)/2)}$ instead of Ω if necessary (see the proof of Proposition 3.1), we may assume that

$$G_\Omega(x, y) \le A \log \frac{1}{|x-y|} \quad \text{for } x, y \in \Omega \cap B(\xi, 1). \tag{3.7}$$

The proof is similar to one of higher dimensions, so the detail of computations below is left to the reader. Let $j = 1, 2$. Let $r = \delta_\Omega(z)$ and $\psi_z(\zeta) = r^2 f_u(z + r\zeta)$. Making the change of variables $x = z + r\eta$ and $y = z + r\zeta$, we have by (3.7)

$$V_{z,j}(z + r\eta) \le A \int_{B(2^{-j-1})} \psi_z(\zeta) \log \frac{1}{r|\eta - \zeta|} d\zeta.$$

Let $q > \max\{1, p\}$ and $s = q/p > 1$. By the Minkowski inequality,

$$\biggl(\int_{B(1/4)} V_{z,1}(z + r\eta)^q d\eta\biggr)^{1/q} \le A\biggl(1 + \log \frac{1}{r}\biggr) \int_{B(1/4)} \psi_z(\zeta) d\zeta.$$

Since

$$\psi_z(\eta) \le Ar^{2-\alpha} + Ar^{2-\alpha} V_{z,1}(z + r\eta)^p \quad \text{for a.e. } \eta \in B(1/8),$$

it follows that

$$\int_{B(1/8)} \psi_z(\eta)^s d\eta \le Ar^{s(2-\alpha)} + Ar^{s(2-\alpha)} \left(1 + \log \frac{1}{r}\right)^q \left(\int_{B(1/4)} \psi_z(\zeta) d\zeta\right)^q.$$

By the Hölder inequality,

$$\begin{aligned} V_{z,2}(z) &\le A \int_{B(1/8)} \psi_z(\zeta) \log \frac{1}{r|\zeta|} d\zeta \\ &\le A\left(1 + \log \frac{1}{r}\right) \left(\int_{B(1/8)} \psi_z(\zeta)^s d\zeta\right)^{1/s} \\ &\le A\left(1 + \log \frac{1}{r}\right) \left(r^{2-\alpha} + r^{2-\alpha} \left(1 + \log \frac{1}{r}\right)^p \left(\int_{B(1/4)} \psi_z(\zeta) d\zeta\right)^p\right). \end{aligned}$$

If $\alpha < 2$, then

$$r^{2-\alpha}\left(1 + \log \frac{1}{r}\right) \le r^{2-\alpha}\left(1 + \log \frac{1}{r}\right)^{p+1} \le A \quad \text{for } 0 < r < 1.$$

This and Lemma 3.7 yield (3.3) with $m = 2$. □

Proof of Lemma 3.9 for $n = 2$; (1.5) holds. Assume that ξ satisfies (1.5). Let $z \in \Omega$ be sufficiently close to ξ and let $x, y \in B(z, \delta_\Omega(z)/4)$. Then (1.5) implies

$$G_\Omega(x, y) \le A \log \frac{\delta_\Omega(z)}{|x - y|}.$$

Letting $r = \delta_\Omega(z)$ and $\psi_z(\zeta) = r^2 f_u(z + r\zeta)$, we have

$$V_{z,j}(z + r\eta) \le A \int_{B(2^{-j-1})} \psi_z(\zeta) \log \frac{1}{|\eta - \zeta|} d\zeta,$$

and so by the similar way to the above general case of $n = 2$,

$$\int_{B(1/8)} \psi_z(\eta)^s d\eta \le A + A\left(\int_{B(1/4)} \psi_z(\zeta) d\zeta\right)^q.$$

Hence the Hölder inequality and Lemma 3.7 give

$$\begin{aligned} V_{z,2}(z) &\le A \int_{B(1/8)} \psi_z(\zeta) \log \frac{1}{|\zeta|} d\zeta \\ &\le A\left(\int_{B(1/8)} \psi_z(\zeta)^s d\zeta\right)^{1/s} \le A. \end{aligned}$$

This completes the proof. □

4 Proof of Theorem 1.4

In this section, we show Theorem 1.4 by constructing an unbounded function. The proof is given separately for $n \geq 3$ and $n = 2$. Denote $B(r) = B(0, r)$.

Proof of Theorem 1.4 for $n \geq 3$. Let $\kappa > 1$ be such that

$$\kappa > 1 + \frac{2-\alpha}{p(n-2)-n} \quad \text{if } p > \frac{n}{n-2},$$
$$\kappa < 1 + \frac{\alpha-2}{n-p(n-2)} \quad \text{if } p < \frac{n}{n-2} \text{ and } \alpha > 2.$$

Let $\lambda = n\kappa - n + 2$. Note that for any pair $(p, \alpha) \notin (-\infty, n/(n-2)] \times (-\infty, 2]$,

$$\alpha + p(\lambda - 2\kappa) - \lambda > 0. \tag{4.1}$$

For simplicity, we write $a_i = 2^{i^3}$. Let $1 < \theta < 2$. For each $i \in \mathbb{N}$, we take N_i-points $\{x_{i,j}\}_{j=1}^{N_i}$ in $\partial B(1 - a_i^{-1})$ such that

- $\{B(x_{i,j}, 5^{-1}a_i^{-1}(\theta - 1))\}_{j=1}^{N_i}$ is mutually disjoint,
- $\partial B(1 - a_i^{-1}) \subset \bigcup_{j=1}^{N_i} B(x_{i,j}, a_i^{-1}(\theta - 1))$.

Observe that for each $\xi \in \partial B$, the nontangential set $\Gamma_\theta(\xi)$ contains infinitely many points $x_{i,j}$ and that

$$N_i \leq A a_i^{n-1}. \tag{4.2}$$

Let $r_i = a_i^{-\kappa}(\theta - 1)$. Then $B(x_{i,j}, r_i) \subset B$ and $B(x_{i,j}, r_i/5) \cap B(x_{k,l}, r_k/5) = \emptyset$ if $(i, j) \neq (k, l)$. Let f_i be a nonnegative smooth function on B such that $f_i \leq a_i^\lambda / i^2$ on B and

$$f_i = \begin{cases} \dfrac{a_i^\lambda}{i^2} & \text{on } \bigcup_{j=1}^{N_i} B(x_{i,j}, r_i/10), \\ 0 & \text{on } B \setminus \bigcup_{j=1}^{N_i} B(x_{i,j}, r_i/5). \end{cases}$$

Let A_1 be a constant such that

$$G_B(x, y) \geq \frac{1}{A_1}|x - y|^{2-n} \quad \text{for } y \in B \text{ and } x \in B(y, (1 - |y|)/2). \tag{4.3}$$

By (4.1), we can take $i_0 \in \mathbb{N}$ such that

$$\frac{c}{2^{|\alpha|}} \left(\frac{\nu_n(\theta - 1)^2}{2^n 5^2 A_1} \right)^p \frac{a_i^{\alpha + p(\lambda - 2\kappa) - \lambda}}{i^{2(p-1)}} \geq 1 \quad \text{for } i \geq i_0, \tag{4.4}$$

where ν_n is the volume of B. Define $f = \sum_{i=i_0}^\infty f_i$. Then (4.2) and $n - 2 + \lambda - n\kappa = 0$

imply that

$$
\begin{aligned}
\int_B (1-|y|)f(y)dy &= \sum_{i=i_0}^{\infty}\int_B (1-|y|)f_i(y)dy \\
&\le A\sum_{i=i_0}^{\infty}\sum_{j=1}^{N_i}\int_{B(x_{i,j},r_i/5)}(1-|x_{i,j}|)f_i(y)dy \\
&\le A\sum_{i=i_0}^{\infty}\frac{a_i^{-1+\lambda-n\kappa}}{i^2}N_i \le A\sum_{i=i_0}^{\infty}\frac{a_i^{n-2+\lambda-n\kappa}}{i^2}<\infty.
\end{aligned}
$$

Thus $u := \int_B G_B(\cdot,y)f(y)dy$ is well defined on B. Since f is locally Hölder continuous on B, it follows from [48, Theorem 6.6] that $u \in C^2(B)$ is a positive solution of $-\Delta u = f$ in B. Also, the mean value property and (4.3) imply that for $x \in \partial B(x_{i,j}, r_i/5)$,

$$
\begin{aligned}
u(x) &\ge \int_{B(x_{i,j},r_i/10)} G_B(x,y)f_i(y)dy \\
&= \frac{a_i^{\lambda}}{i^2}\nu_n\frac{r_i^n}{10^n}G_B(x,x_{i,j}) \ge \frac{\nu_n(\theta-1)^2}{2^n5^2A_1}\frac{a_i^{\lambda-2\kappa}}{i^2}.
\end{aligned}
$$

By the minimum principle,

$$
u(x) \ge \frac{\nu_n(\theta-1)^2}{2^n5^2A_1}\frac{a_i^{\lambda-2\kappa}}{i^2} \quad \text{for } x \in B(x_{i,j}, r_i/5). \tag{4.5}
$$

In particular,

$$
u(x_{i,j}) \ge \frac{\nu_n(\theta-1)^2}{2^n5^2A_1}\frac{a_i^{\lambda-2\kappa}}{i^2}.
$$

Since $\lambda - 2\kappa > 0$ and $\Gamma_\theta(\xi)$ contains infinitely many points $x_{i,j}$, it follows that

$$
\limsup_{\Gamma_\theta(\xi)\ni x\to\xi} u(x) = \infty,
$$

and so u fails to have nontangential limits everywhere on ∂B.

We finally show that $-\Delta u \le c(1-|x|)^{-\alpha}u^p$ in B. If $x \notin \bigcup_{i=i_0}^{\infty}\bigcup_{j=1}^{N_i} B(x_{i,j}, r_i/5)$, then

$$
c(1-|x|)^{-\alpha}u(x)^p \ge 0 = f(x) = -\Delta u(x).
$$

Let $x \in B(x_{i,j}, r_i/5)$. Then, by (4.5) and (4.4),

$$
\begin{aligned}
c(1-|x|)^{-\alpha}u(x)^p &\ge \frac{c}{2^{|\alpha|}}\left(\frac{\nu_n(\theta-1)^2}{2^n5^2A_1}\right)^p\frac{a_i^{\alpha+p(\lambda-2\kappa)}}{i^{2p}} \\
&\ge \frac{a_i^{\lambda}}{i^2} \ge f(x) = -\Delta u(x).
\end{aligned}
$$

Hence $-\Delta u(x) \le c(1-|x|)^{-\alpha}u(x)^p$ for $x \in B$. This completes the proof of Theorem 1.4 for $n \ge 3$. $\square$

Proof of Theorem 1.4 for $n = 2$. The proof for $n = 2$ is parallel to that for $n \geq 3$. We need to consider only the case $p \in \mathbb{R}$ and $\alpha > 2$. Let $1 < \kappa < \alpha/2$. Then $\lambda = 2\kappa < \alpha$. Take $\{x_{i,j}\}$ and define f_i and f in the same way. The same reasoning implies that $u = \int_B G_B(\cdot, y) f(y) dy$ is a positive C^2-solution of $-\Delta u = f$ in B. But, instead of (4.3), we use

$$G_B(x,y) \geq \frac{1}{A_1} \log \frac{1-|y|}{|x-y|} \quad \text{for } y \in B \text{ and } x \in B(y, (1-|y|)/2).$$

Then (4.5) is replaced by

$$u(x) \geq \frac{\pi(\theta-1)^2(\kappa-1)\log 2}{100A_1}\, i \quad \text{for } x \in B(x_{i,j}, r_i/5),$$

because $\log a_i^{\kappa-1} = i^3(\kappa-1)\log 2$. Hence u fails to have nontangential limits everywhere on ∂B. Also, this inequality implies that for $x \in B(x_{i,j}, r_i/5)$,

$$\begin{aligned} c(1-|x|)^{-\alpha} u(x)^p &\geq \frac{c}{2^\alpha} \left(\frac{\pi(\theta-1)^2(\kappa-1)\log 2}{100A_1} \right)^p a_i^{\alpha-\lambda} i^{p+2} \cdot \frac{a_i^\lambda}{i^2} \\ &\geq \frac{a_i^\lambda}{i^2} \geq f(x) = -\Delta u(x), \end{aligned}$$

whenever $i \geq i_0$ is sufficiently large. Hence Theorem 1.4 is valid for $n = 2$. □

5 Proof of Theorem 1.5

First, we show the existence of positive solutions of $-\Delta u = V u^p$, which are comparable to a given positive harmonic function.

Lemma 5.1. *Let $p > 0$, $\alpha < 2$ and $c > 0$ be constants. Assume that $c > 0$ is sufficiently small only when $p = 1$. Suppose that V is a measurable function on the unit ball B such that*

$$|V(x)| \leq c(1-|x|)^{-\alpha} \quad \textit{for almost every } x \in B.$$

Let h be a harmonic function on B such that

$$0 < \inf_B h \leq \sup_B h < \infty.$$

Then there exist a constant $\lambda > 0$ and a positive distributional solution $u \in C(B)$ of

$$-\Delta u = V u^p \quad \textit{in } B$$

such that

$$\frac{\lambda}{2} h(x) \leq u(x) \leq \frac{3\lambda}{2} h(x) \quad \textit{for } x \in B. \tag{5.1}$$

Proof. For $\lambda > 0$, let

$$W_\lambda = \left\{ w \in C(B) : \frac{\lambda}{2} \leq w(x) \leq \frac{3\lambda}{2} \right\}.$$

Define the operator on W_λ by

$$T_\lambda w(x) = \lambda + \frac{1}{h(x)} \int_B G_B(x,y) V(y) h(y)^p w(y)^p dy \quad \text{for } x \in B.$$

Now, let $g(x) = \int_B G_B(x,y)(1-|y|)^{-\alpha} dy$. Observe that

$$g(z) = g(x) \quad \text{for } x \in B \text{ and } z \in \partial B(|x|).$$

By the mean value inequality and $\alpha < 2$, we have

$$g(x) = \frac{1}{\sigma(\partial B(|x|))} \int_{\partial B(|x|)} g(z) d\sigma(z) \leq g(0) < \infty \quad \text{for } x \in B \setminus \{0\},$$

where σ is the surface area measure on $\partial B(|x|)$. Let $\lambda > 0$ satisfy

$$\max\left\{ p, \frac{3}{2} \right\} \left(\frac{3\lambda}{2} \right)^{p-1} \frac{(\sup_B h)^p}{\inf_B h} c g(0) \leq \frac{1}{2} \quad \text{if } p \neq 1. \tag{5.2}$$

Also, if $p = 1$, then we assume that $c > 0$ is sufficiently small so that (5.2) holds. Let $w \in W_\lambda$. Take $x, z \in B$ and $r > 0$ (small). Then

$$\begin{aligned} &\int_{B(z,r)} \left| \frac{G_B(x,y)}{h(x)} - \frac{G_B(z,y)}{h(z)} \right| |V(y)| |h(y)|^p |w(y)|^p dy \\ &\quad \leq A \int_{B(z,r)} \left\{ G_B(x,y) + G_B(z,y) \right\} (1-|y|)^{-\alpha} dy \to 0 \quad (r \to 0). \end{aligned}$$

Also, since there is a constant A depending on r such that

$$G_B(x,y) \leq A(1-|z|)(1-|y|) \quad \text{for } x \in B(z, r/2) \text{ and } y \notin B(z,r),$$

it follows from the Lebesgue convergence theorem that

$$\int_{B \setminus B(z,r)} \left| \frac{G_B(x,y)}{h(x)} - \frac{G_B(z,y)}{h(z)} \right| |V(y)| |h(y)|^p |w(y)|^p dy \to 0 \quad (x \to z).$$

Therefore

$$|T_\lambda w(x) - T_\lambda w(z)| \to 0 \quad (x \to z),$$

and so $T_\lambda w \in C(B)$. Moreover, by (5.2),

$$|T_\lambda w(x) - \lambda| \leq \left(\frac{3\lambda}{2} \right)^p \frac{(\sup_B h)^p}{\inf_B h} c g(0) \leq \frac{\lambda}{2}.$$

Hence $T_\lambda w \in W_\lambda$. If $w_1, w_2 \in W_\lambda$, then it follows from the mean value theorem and (5.2) that

$$\begin{aligned}
\|T_\lambda w_1 - T_\lambda w_2\|_\infty &\le \frac{(\sup_B h)^p}{\inf_B h} cg(0)\|w_1^p - w_2^p\|_\infty \\
&\le p\left(\frac{3\lambda}{2}\right)^{p-1} \frac{(\sup_B h)^p}{\inf_B h} cg(0)\|w_1 - w_2\|_\infty \\
&\le \frac{1}{2}\|w_1 - w_2\|_\infty.
\end{aligned}$$

Therefore T_λ is a contraction mapping on the Banach space W_λ. By the fixed point theorem, we find $w \in W_\lambda$ such that $T_\lambda w = w$. Let $u(x) = h(x)w(x)$. Then

$$u(x) = h(x)T_\lambda w(x) = \lambda h(x) + \int_B G_B(x,y)V(y)u(y)^p dy.$$

The Fubini theorem implies that

$$-\int_B u(x)\Delta\phi(x)dx = \int_B V(y)u(y)^p\phi(y)dy \quad \text{for } \phi \in C_0^\infty(B),$$

and so u is a distributional solution of $-\Delta u = Vu^p$ in B. Moreover,

$$\frac{\lambda}{2}h(x) \le u(x) \le \frac{3\lambda}{2}h(x).$$

Thus Lemma 5.1 is proved. □

It is well known that bounded harmonic functions do not necessarily have tangential limits. The following result was proved by Aikawa [1, 2]. See also [27].

Lemma 5.2. *Let γ be a curve in B ending at $e = (1, 0, \cdots, 0)$ and satisfying* (1.6). *Let $a, b \in \mathbb{R}$ be such that $a < b$. Then there exists a bounded harmonic function h on B such that $a \le h \le b$ on B and*

$$\liminf_{|x|\to 1, x\in O\gamma} h(x) = a < b = \limsup_{|x|\to 1, x\in O\gamma} h(x) \quad \textit{for all } O \in \mathcal{O},$$

where $\mathcal{O}$ denotes the group of all orthogonal transformations on $\mathbb{R}^n$.

Now, Theorem 1.5 is proved immediately.

Proof of Theorem 1.5. Let a, b be positive numbers such that $3a < b$ and let h be a harmonic function on B obtained in Lemma 5.2. By Lemma 5.1, we find a positive distributional solution u of $-\Delta u = Vu^p$ in B satisfying (5.1). Then

$$\liminf_{|x|\to 1, x\in O\gamma} u(x) \le \frac{3\lambda}{2} \liminf_{|x|\to 1, x\in O\gamma} h(x) = \frac{3\lambda}{2}a < \frac{\lambda}{2}b$$

and

$$\limsup_{|x|\to 1, x\in O\gamma} u(x) \ge \frac{\lambda}{2} \limsup_{|x|\to 1, x\in O\gamma} h(x) = \frac{\lambda}{2}b.$$

Thus Theorem 1.5 is proved. □

Acknowledgment

The author is grateful to the referee for valuable suggestions and comments.

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