Removable sets for continuous solutions of semilinear elliptic equations *

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Abstract

We discuss the possible removability of sets for continuous solutions of semilinear elliptic equations of the form $-\Delta u = F(x,u)$. In particular, we show that a set E in $\mathbb{R}^n$ is removable for α-Hölder continuous solutions of such equations if and only if $n-2+\alpha$-dimensional Hausdorff measure of E is zero.

Keywords: removable set, Zygmund class, Hölder continuous, semilinear elliptic equation
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1 Introduction

Throughout this paper, let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ and let E be a compact subset of Ω. By $\mathcal{H}_\beta(E)$ we denote the β-dimensional Hausdorff measure of E. It is well known that if the capacity of E is zero, then every bounded harmonic function on $\Omega \setminus E$ can be extended to Ω as a harmonic function. Then E is said to be *removable* for bounded harmonic functions. In 1963, Carleson [6] have investigated removable sets for Hölder continuous harmonic functions. Namely, he proved that if $\mathcal{H}_{n-2+\alpha}(E) = 0$ with $0 < \alpha \leq 1$, then E is removable for α-Hölder continuous harmonic functions. Moreover, if $\mathcal{H}_{n-2+\alpha}(E) > 0$ with $0 < \alpha < 1$, then there exists an α-Hölder continuous function on Ω which is harmonic on $\Omega \setminus E$, but does not have a harmonic extension to Ω. Note that the last statement for the case $\alpha = 1$ fails to hold in general. Indeed, Uy [18] constructed a compact set E with $\mathcal{H}_{n-1}(E) > 0$ such that E is removable for Lipschitz continuous harmonic functions. After that, Ullrich [17] considered the Zygmund class instead of the Lipschitz class to obtain a necessary and sufficient result in the case $\alpha = 1$: E is removable for harmonic functions in the Zygmund class if and only if $\mathcal{H}_{n-1}(E) = 0$. Abidi [1] obtained a similar result for the Zygmund class of order α with $0 < \alpha < 2$. Also, removability theorems for

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subharmonic functions conditioned by the growth of mean oscillation were given by Shapiro [16] and Kaufman and Wu [8].

Some of the above results were extended to p-harmonic functions (i.e., continuous solutions of the p-Laplace equation). In this case, the size of removable sets depends on p as well. The result that compact sets with p-capacity zero are removable for bounded p-harmonic functions was due to Serrin [14, 15]. Kilpeläinen and Zhong [9] established the removability theorem corresponding to Carleson's: E is removable for α-Hölder continuous p-harmonic functions if and only if $\mathcal{H}_{n-p+\alpha(p-1)}(E) = 0$. See also [5, 12] for extensions to metric spaces. Recently, Ono [13] obtained a similar result for Hölder continuous solutions of quasilinear elliptic equations with lower order terms. The model equation is $\Delta_p u = V|u|^{p-2}u$, where Δ_p is the p-Laplacian and V is nonnegative and bounded.

Also, there are investigations concerning a removable isolated singularity for semilinear elliptic equations with nonlinear terms. Brezis and Veron [4] proved that if $p \geq n/(n-2)$, then any isolated point is removable for every solution of $\Delta u = |u|^{p-1}u$, where Δ is the Laplacian on $\mathbb{R}^n$. Lions [11] studied positive solutions of $-\Delta u = u^p$ and showed that the equation can be extended up to an isolated point when $p \geq n/(n-2)$. For the case $1 < p < n/(n-2)$, it was also proved that any isolated point is removable for bounded positive solutions of $-\Delta u = u^p$. Baras and Pierre [3] characterized removable sets for such equations in terms of the Sobolev $W^{2,p'}$-capacity, where $p' = p/(p-1)$. See also [10, 19]. However, the Carleson type removability theorem for Hölder continuous solutions of semilinear elliptic equations is not known. We will prove, for instance, the following theorem.

Theorem 1.1. *Let $p > 1$ and $0 < \alpha < 1$. Then E is removable for α-Hölder continuous solutions of $-\Delta u = |u|^{p-1}u$ if and only if $\mathcal{H}_{n-2+\alpha}(E) = 0$.*

The size of removable sets in the above theorem is independent of nonlinear exponent p. This means that results can be obtained for more general nonlinearity. Also, it might be interesting to investigate the relation between a general modulus of continuity and Hausdorff measure with respect to a general function. We will state general results in the next section.

2 Notation and results

To state generalizations of Theorem 1.1, we prepare some notation. The symbol C stands for an absolute positive constant whose value is unimportant and may change from line to line. If necessary, we use $C_1, C_2, \dots,$ to specify them. Let Ψ_H be the family of positive increasing functions ψ on $(0, \infty)$ such that

(A1) $\psi(t) \to 0$ as $t \to +0$,

and that there exists a constant C_1 with the following properties:

(A2) $\psi(2t) \leq C_1\psi(t)$ for all $t > 0$, (A3) for all $0 < r < 5\operatorname{diam}\Omega$,

$$\int_0^r \frac{\psi(t)}{t}\,dt \le C_1\psi(r),$$

(A4) for all $r > 0$,

$$\int_r^\infty \frac{\psi(t)}{t^2}\,dt \le C_1\frac{\psi(r)}{r}.$$

Also, Ψ_Z denotes the family of positive increasing functions ψ on $(0,\infty)$ with (A1)–(A3) and

(A5) for all $r > 0$,

$$\int_r^\infty \frac{\psi(t)}{t^3}\,dt \le C_1\frac{\psi(r)}{r^2}.$$

For $\psi \in \Psi_H$, we denote by $\mathscr{C}^{0,\psi}(\Omega)$ the class of all continuous functions u on Ω such that for all $x, y \in \Omega$,

$$|u(x) - u(y)| \le C\psi(\|x - y\|).$$

For $\psi \in \Psi_Z$, the *ψ-Zygmund class* $\mathscr{Z}^\psi(\Omega)$ consists of all continuous functions u on Ω satisfying

$$|u(x - y) - 2u(x) + u(x + y)| \le C\psi(\|y\|),$$

whenever $x, x \pm y \in \Omega$. Observe that if $\psi \in \Psi_H$, then (A4) implies (A5), and so $\Psi_H \subset \Psi_Z$ and $\mathscr{C}^{0,\psi}(\Omega) \subset \mathscr{Z}^\psi(\Omega)$.

Recall ϕ-Hausdorff measure. Let $B(x,r)$ denote the open ball of center x and radius r. For a positive increasing function ϕ on $(0,\infty)$ such that $\phi(t) \to 0$ ($t \to +0$) and $0 < \rho \le \infty$, we let

$$\mathcal{H}_\phi^{(\rho)}(E) = \inf \sum_j \phi(r_j),$$

where the infimum is taken over all possible coverings of E by a countable collection of balls $B(x_j, r_j)$ such that $r_j < \rho$. Since $\mathcal{H}_\phi^{(\rho)}(E)$ is decreasing as a function of ρ, we define

$$\mathcal{H}_\phi(E) = \lim_{\rho \to +0} \mathcal{H}_\phi^{(\rho)}(E).$$

This is called the *ϕ-Hausdorff measure* of E. When $\phi(t) = t^\beta$, we write $\mathcal{H}_\beta(E)$ for $\mathcal{H}_\phi(E)$ as above. Also, $\mathscr{P}^\phi(\Omega)$ stands for the class of all measurable functions V on Ω with $\|V\|_{\mathscr{P}^\phi(\Omega)} < \infty$, where

$$\|V\|_{\mathscr{P}^\phi(\Omega)} = \sup_{\substack{x \in \mathbb{R}^n \\ 0 < r < 2\operatorname{diam}\Omega}} \frac{1}{\phi(r)} \int_{\Omega \cap B(x,r)} |V(y)|\,dy.$$

As nonlinearity, we consider a measurable function F on $\Omega \times \mathbb{R}$ for which there are nonnegative functions $V \in \mathscr{P}^\phi(\Omega)$ and $f \in \mathscr{C}(\mathbb{R})$ such that

$$|F(x,t)| \le V(x)f(t) \quad \text{for all } x \in \Omega \text{ and } t \in \mathbb{R}, \tag{2.1}$$

and discuss continuous solutions of semilinear elliptic equations of the form

$$-\Delta u = F(x, u), \tag{2.2}$$

where Δ is the Laplacian and the equation is understood in the sense of distributions.

Our results are stated as follows.

Theorem 2.1. *Let $\phi(t) = t^{n-2}\psi(t)$, where $\psi \in \Psi_Z$, and let F be a measurable function on $\Omega \times \mathbb{R}$ satisfying* (2.1) *for some nonnegative functions $V \in \mathscr{P}^{\phi}(\Omega)$ and $f \in \mathscr{C}(\mathbb{R})$. Suppose that $u \in \mathscr{X}^{\psi}(\Omega)$ is a solution of* (2.2) *in $\Omega \setminus E$. If $\mathcal{H}_\phi(E) = 0$, then u satisfies* (2.2) *in the whole of Ω.*

Remark 2.2. Observe that ϕ is a positive increasing function satisfying (A1) and (A2) with $\psi = \phi$. Also, it follows from (A5) that for all $0 < r < 1$,

$$\int_1^\infty \frac{\psi(t)}{t^3}\,dt \le \int_r^\infty \frac{\psi(t)}{t^3}\,dt \le C_1\frac{\psi(r)}{r^2}.$$

Since the left hand side is positive, we have $r^n \le C\phi(r)$, and so $\mathcal{H}_n(E) \le C\mathcal{H}_\phi(E)$.

A sharpness of $\mathcal{H}_\phi(E) = 0$ is shown under additional weak conditions on F. Note that $\mathscr{P}^{\phi}(\Omega) \subset \mathscr{L}^1(\Omega)$. Denote by ν_n the volume of the unit ball of $\mathbb{R}^n$.

Theorem 2.3. *Let $\phi(t) = t^{n-2}\psi(t)$, where $\psi \in \Psi_Z$, and let F be a measurable function on $\Omega \times \mathbb{R}$ satisfying* (2.1) *for some nonnegative functions $V \in \mathscr{P}^{\phi}(\Omega)$ and $f \in \mathscr{C}(\mathbb{R})$. In addition, we assume that*

(i) *for each $x \in \Omega$, $F(x, \cdot)$ is nonnegative on $(0, \infty)$ and $F(x, \cdot) \in \mathscr{C}(0, \infty)$,*

(ii) *there are numbers $m_0 > 0$ and $\varepsilon > 0$ such that for each $0 < m \le m_0$, we find $M > m$ with*

$$\|f\|_{\mathscr{L}^\infty[m,M]} Q_V \le M - \varepsilon, \tag{2.3}$$

where

$$Q_V = \begin{cases} \dfrac{C_1\psi(1)\|V\|_{\mathscr{P}^{\phi}(\Omega)}}{n\nu_n} + \dfrac{\|V\|_{\mathscr{L}^1(\Omega)}}{n(n-2)\nu_n} & (n \ge 3), \\ \dfrac{C_1\psi(5\operatorname{diam}\Omega)}{2\pi}\|V\|_{\mathscr{P}^{\phi}(\Omega)} & (n = 2). \end{cases}$$

If $\mathcal{H}_\phi(E) > 0$, then there exists $u \in \mathscr{X}^{\psi}(\Omega)$ which satisfies (2.2) *in $\Omega \setminus E$, but not in the whole of Ω. Moreover, if $\psi \in \Psi_H$ and $\mathcal{H}_\phi(E) > 0$, then there exists $u \in \mathscr{C}^{0,\psi}(\Omega)$ which satisfies* (2.2) *in $\Omega \setminus E$, but not in the whole of Ω.*

Note that condition (2.3) is satisfied for many semilinear equations. If f is increasing, then $\|f\|_{\mathscr{L}^\infty[m,M]} = f(M)$ for any $m \le M$. Thus the following hold:

- *The case* $f(t)/t \to 0$ $(t \to +0)$: We find $m_0 > 0$ such that for $0 < t \le 2m_0$,

$$\frac{f(t)}{t} Q_V \le \frac{1}{2}. \tag{2.4}$$

  Let $\varepsilon = m_0$ and $M = 2m_0$. Then (2.3) is satisfied for every $V \in \mathscr{P}^{\phi}(\Omega)$.

- *Other case*: Take $m_0 = 1$, $\varepsilon = 1$ and $M = 2$ for instance. If $Q_V \le 1/f(2)$, then (2.3) is satisfied.

If f is any function such that $f(t)/t \to 0$ $(t \to \infty)$, then we find $m_0 > 0$ such that (2.4) holds for all $t \ge m_0$. Let $0 < m \le m_0$. Take $M > \max\{2, m_0\}$ with $\|f\|_{\mathscr{L}^\infty[m,m_0]} Q_V \le M - 1$. Then $\|f\|_{\mathscr{L}^\infty[m,M]} Q_V \le \max\{M-1, M/2\} = M - 1$. Hence (2.3) holds for any $V \in \mathscr{P}^\phi(\Omega)$ if we take $\varepsilon = 1$.

Thus Theorem 2.3 is applicable to semilinear equations $-\Delta u = V|u|^{p-1}u$ $(0 < p \ne 1$, V: any), $-\Delta u = V_1 u + V_2 |u|^{p-1}u$ $(p > 0$, V_1, V_2: small), $-\Delta u = Ve^u$ (V: small), etc. In particular, Theorem 1.1 follows from Theorems 2.1 and 2.3 because $V \equiv 1 \in \mathscr{P}^\phi(\Omega)$.

The plan of this paper is as follows. In Section 3, we prove Theorem 2.1 after discussing removable sets for superharmonic functions in the ψ-Zygmund class. In Sections 4 and 5, the proof of Theorem 2.3 for $\psi \in \Psi_Z$ will be given separately in the cases $n \ge 3$ and $n = 2$. Section 6 provides the proof of Theorem 2.3 for $\psi \in \Psi_H$.

3 Proof of Theorem 2.1

In this section, we let $\psi \in \Psi_Z$ and $\phi(t) = t^{n-2}\psi(t)$. For the proof of Theorem 2.1, we first discuss removable sets for superharmonic functions in $\mathscr{L}^\psi(\Omega)$. The word "measure" means "nonnegative Radon measure". Let G_Ω be the *Green function* for Ω. For a measure μ on Ω, we let

$$G_\Omega \mu(x) = \int_\Omega G_\Omega(x,y)\, d\mu(y).$$

When $d\mu(y) = f(y)dy$, we write $G_\Omega[f]$ for $G_\Omega\mu$. We say that $G_\Omega\mu$ is a *Green potential* of μ on Ω if it is finite at some point in Ω. Then $G_\Omega\mu$ is superharmonic on Ω and harmonic outside the support of μ. Moreover, if Ω is regular for the Dirichlet problem and the support of μ is compact in Ω, then $G_\Omega\mu$ vanishes continuously on $\partial\Omega$. For $u \in \mathscr{L}^1_{loc}(\Omega)$, we write

$$\mathcal{A}(u; x, r) = \frac{1}{\nu_n r^n} \int_{B(x,r)} u(y)\, dy,$$

where ν_n is the volume of the unit ball of $\mathbb{R}^n$. The following lemma is elementary.

Lemma 3.1. *Let $r > 0$ and $x \in \mathbb{R}^n$. If g is a decreasing function on $(0, \infty)$, then*

$$\int_{B(x,r)} g(\|y\|)\, dy \le \int_{B(0,r)} g(\|y\|)\, dy.$$

Proof. Let $Q_1 = B(x,r) \setminus B(0,r)$ and $Q_2 = B(0,r) \setminus B(x,r)$. Consider the mapping $z = x - y$, which maps $y \in Q_1$ onto $z \in Q_2$. Since $\|y\| \ge r > \|z\|$, we have $g(\|y\|) \le g(\|z\|)$. Therefore

$$\int_{Q_1} g(\|y\|)\, dy \le \int_{Q_2} g(\|z\|)\, dz.$$

Thus the lemma follows. □

Lemma 3.2. *If $\mathcal{H}_\phi(E) = 0$, then there exists a Green potential v on Ω, which is harmonic on $\Omega \setminus E$, such that for each $x \in E$,*

$$\limsup_{r\to+0} \frac{v(x) - \mathcal{A}(v; x, r)}{\psi(r)} = \infty. \tag{3.1}$$

Proof. We provide a proof for $n \geq 3$. For the case $n = 2$, we need to change only the fundamental solution of the Laplace equation from $\|\cdot\|^{2-n}$ to $-\log\|\cdot\|$. Let $j \in \mathbb{N}$. By $\mathcal{H}_\phi(E) = 0$ and (A2), we find finitely many points y_{jk} in E and positive numbers r_{jk}, where $k = 1, \cdots, N_j$ say, such that $E \subset \bigcup_k B(y_{jk}, r_{jk})$ and $\sum_k \phi(r_{jk}) \leq 4^{-j}$. Define

$$u(x) = \sum_{j=1}^{\infty} 2^j \sum_{k=1}^{N_j} \phi(r_{jk})\|x - y_{jk}\|^{2-n}.$$

Observe that u is superharmonic on $\mathbb{R}^n$ and harmonic outside E. Let $x \in E$ and $j \in \mathbb{N}$ be fixed. Then $\|x - y_{jk}\| < r_{jk}$ for some $k = k(j, x)$. Take $c > 0$ with $n < 2c^{n-2}$ (when $n = 2$, this is replaced by $\log c > 1/2$). The mean value inequality for superharmonic functions implies that

$$u(x) - \mathcal{A}(u; x, cr_{jk}) \geq 2^j\phi(r_{jk})\big\{\|x - y_{jk}\|^{2-n} - \mathcal{A}(\|\cdot - y_{jk}\|^{2-n}; x, cr_{jk})\big\}.$$

By Lemma 3.1,

$$\mathcal{A}(\|\cdot - y_{jk}\|^{2-n}; x, cr_{jk}) \leq \mathcal{A}(\|\cdot - y_{jk}\|^{2-n}; y_{jk}, cr_{jk}) = \frac{n}{2}(cr_{jk})^{2-n}.$$

Therefore, by (A2),

$$u(x) - \mathcal{A}(u; x, cr_{jk}) \geq 2^j\phi(r_{jk})\Big\{r_{jk}^{2-n} - \frac{n}{2}(cr_{jk})^{2-n}\Big\} \geq \frac{2^j\psi(cr_{jk})}{C}.$$

This shows that (3.1) holds for $v = u$. Observe that the Green potential v appearing in the Riesz decomposition of u on Ω satisfies (3.1) and is harmonic on $\Omega \setminus E$. This completes the proof. $\square$

A function η on $\mathbb{R}^n$ is said to be *symmetric* with respect to $x_0 \in \mathbb{R}^n$ if $\eta(x_0 - y) = \eta(x_0 + y)$ for every $y \in \mathbb{R}^n$.

Lemma 3.3. *Let η be bounded and symmetric with respect to $x_0 \in \Omega$ and let $B(x_0, r) \subset \Omega$. If $u \in \mathscr{Z}^\psi(\Omega)$, then*

$$\left|\int_{B(x_0,r)} \eta(y)\{u(y) - u(x_0)\}\,dy\right| \leq Cr^n\psi(r)\|\eta\|_{\mathscr{L}^\infty(B(x_0,r))}. \tag{3.2}$$

Proof. Making a change of variables and splitting $B(0, r)$ into the upper half and the lower half, we have

$$\begin{aligned}&\int_{B(x_0,r)} \eta(y)\{u(y) - u(x_0)\}\,dy \\ &\qquad = \frac{1}{2}\int_{B(0,r)} \eta(x_0 + y)\{u(x_0 - y) - 2u(x_0) + u(x_0 + y)\}\,dy.\end{aligned}$$

Since $u \in \mathscr{Z}^\psi(\Omega)$ and ψ is increasing, this yields (3.2). $\square$

Let $u : \Omega \to (-\infty, \infty]$ be a function which is locally bounded below. Then the *réduite* of u on Ω is defined by

$$R^u(x) = \inf v(x),$$

where the infimum is taken over all superharmonic functions v on Ω satisfying $v \geq u$ on Ω. Let $\widehat{R}^u$ stand for the lower semicontinuous regularization of R^u, which is called the *balayage* of u on Ω. Then $\widehat{R}^u$ is superharmonic on Ω (see [7, Theorem 8.1]). Also, if $u \in \mathscr{C}(\Omega)$, then $u \leq \widehat{R}^u$ on Ω and $\widehat{R}^u$ is continuous on Ω and harmonic on $\{x \in \Omega : \widehat{R}^u(x) > u(x)\}$. See [7, Theorem 8.14].

Lemma 3.4. *Let $u \in \mathscr{L}^\psi(\Omega)$ be superharmonic on $\Omega \setminus E$ and let v be a Green potential on Ω satisfying* (3.1) *for each $x \in E$. Then $u - \widehat{R}^u + v$ is superharmonic on $D = \{x \in \Omega : \widehat{R}^u(x) > u(x)\}$.*

Proof. Let $w = u - \widehat{R}^u + v$. Then w is superharmonic on $D \setminus E$ and lower semicontinuous on D. To show that w is superharmonic on D, it suffices to prove that for each $x \in E \cap D$,

$$\limsup_{r \to +0} \frac{w(x) - \mathcal{A}(w; x, r)}{r^2} \geq 0. \tag{3.3}$$

Let $x \in E \cap D$ and let $r > 0$ be such that $\overline{B(x, r)} \subset D$. Then, by Lemma 3.3 with $\eta \equiv 1$,

$$|u(x) - \mathcal{A}(u; x, r)| \leq C\psi(r).$$

Since $\widehat{R}^u$ is harmonic on D, we have

$$\begin{aligned} \frac{w(x) - \mathcal{A}(w; x, r)}{\psi(r)} &= \frac{u(x) - \mathcal{A}(u; x, r)}{\psi(r)} + \frac{v(x) - \mathcal{A}(v; x, r)}{\psi(r)} \\ &\geq -C + \frac{v(x) - \mathcal{A}(v; x, r)}{\psi(r)}. \end{aligned}$$

Therefore (3.1) implies

$$\limsup_{r \to +0} \frac{w(x) - \mathcal{A}(w; x, r)}{\psi(r)} = \infty,$$

and so (3.3) holds. Hence w is superharmonic on D. $\square$

Lemma 3.5. *Let $u \in \mathscr{L}^\psi(\Omega)$ be superharmonic on $\Omega \setminus E$. If $\mathcal{H}_\phi(E) = 0$, then u is superharmonic on Ω.*

Proof. Let $u \in \mathscr{L}^\psi(\Omega)$ be superharmonic on $\Omega \setminus E$. Without loss of generality, we may assume that Ω is regular for the Dirichlet problem and that $u \in \mathscr{C}(\mathbb{R}^n)$. Then, by [7, Theorem 9.26],

$$\widehat{R}^u = u \quad \text{on } \partial\Omega. \tag{3.4}$$

Let $D = \{x \in \Omega : \widehat{R}^u(x) > u(x)\}$. We claim that $D = \emptyset$. If this is true, then $u = \widehat{R}^u$ on Ω, and so u is superharmonic on Ω. To prove the claim, we suppose to the contrary that $D \neq \emptyset$. Let v be a Green potential on Ω obtained in Lemma 3.2. For $\delta > 0$, we define

$$u_\delta(x) = u(x) - \widehat{R}^u(x) + \delta v(x).$$

Then u_δ is superharmonic on D by Lemma 3.4, and $u_\delta \geq 0$ on ∂D in view of (3.4). The minimum principle shows that $u_\delta \geq 0$ on D. As $\delta \to 0$, we have $\widehat{R}^u \leq u$ on $D \setminus E$ because v is finite there. Hence $u = \widehat{R}^u$ on $D \setminus E$. Since $\mathcal{H}_n(E) = 0$ by Remark 2.2, the continuity implies that $u = \widehat{R}^u$ on D. This is a contradiction. Hence $D = \emptyset$. $\square$

We are now ready to prove Theorem 2.1. For a signed measure ν, we write $|\nu|$ for the total variational measure of ν.

Proof of Theorem 2.1. Let $u \in \mathscr{L}^\psi(\Omega)$ be a solution of (2.2) in $\Omega \setminus E$. Then $f(u) \in \mathscr{C}(\Omega)$. Considering a bounded open set ω with $E \subset \omega$ and $\overline{\omega} \subset \Omega$ instead of Ω, we may assume that $0 \leq f(u) \leq C_2$ on Ω. Then, by (2.1),

$$-C_2 V(x) \leq \inf_{x \in \Omega} F(x, u(x)).$$

We can find a solution $v \in \mathscr{L}^\psi(\Omega)$ of $-\Delta v = C_2 V$ in Ω (distribution)*. Let $w = u + v$. Then $w \in \mathscr{L}^\psi(\Omega)$ and $-\Delta w = F(x, u) + C_2 V$ in $\Omega \setminus E$ (distribution). Thus w is superharmonic on $\Omega \setminus E$. Lemma 3.5 shows that w is superharmonic on Ω, and so there is a unique measure μ on Ω such that $-\Delta w = \mu$ in Ω (distribution). Let

$$d\nu(x) = d\mu(x) - \{F(x, u(x)) + C_2 V(x)\}\, dx.$$

By the uniqueness of μ, we have $|\nu|(\Omega \setminus E) = 0$. We need to show that $|\nu|(E) = 0$. For arbitrary fixed $x_0 \in E$ and $0 < r < \operatorname{dist}(E, \partial\Omega)/2$, we write $B_r = B(x_0, r)$. Let $\eta \in \mathscr{C}_0^\infty(B_{2r})$ be a radial function with respect to x_0 such that $\eta = 1$ on B_r and $0 \leq \eta \leq 1$ and $|\Delta \eta| \leq C/r^2$ on B_{2r}. Note that $\Delta\eta$ is symmetric with respect to x_0. Since $-\Delta(w - w(x_0)) = \mu$ in Ω (distribution), it follows from (2.1), Lemma 3.3, $V \in \mathscr{P}^\phi(\Omega)$ and (A2) that

$$\begin{aligned}
|\nu|(B_r) &\leq \mu(B_r) + \int_{B_r} \{|F(x, u)| + C_2 V\}\, dx \\
&\leq \int_{B_{2r}} \eta\, d\mu + \int_{B_r} \{f(u) + C_2\} V\, dx \\
&\leq \int_{B_{2r}} (-\Delta\eta)(w - w(x_0))\, dx + 2C_2 \int_{B_r} V\, dx \\
&\leq C r^{n-2} \psi(2r) + 2C_2 \|V\|_{\mathscr{P}^\phi(\Omega)} \phi(r) \\
&\leq C\phi(r).
\end{aligned}$$

Let $\varepsilon > 0$. By $\mathcal{H}_\phi(E) = 0$ and (A2), we find sequences of points x_j in E and positive numbers r_j such that $E \subset \bigcup_j B(x_j, r_j)$ and $\sum_j \phi(r_j) < \varepsilon$. Then

$$|\nu|(E) \leq \sum_j |\nu|(B(x_j, r_j)) \leq C \sum_j \phi(r_j) < C\varepsilon.$$

*v is given by the Newtonian (or logarithmic) potential of the density $C_2 V$. Then $v \in \mathscr{L}^\psi(\Omega)$ by Lemma 4.5 (or Lemma 5.2).

As $\varepsilon \to 0$, we have $|\nu|(E) = 0$. Hence $|\nu|(A) = 0$ for any Borel measurable set A in Ω, which concludes that $-\Delta u = F(x,u)$ in Ω (distribution). This completes the proof. $\square$

4 Proof of Theorem 2.3 in the case $\psi \in \Psi_Z$ and $n \geq 3$

Let $\psi \in \Psi_Z$ and $\phi(t) = t^{n-2}\psi(t)$. For a measure μ on Ω, we let

$$\|\mu\|_{\mathscr{P}^\phi(\Omega)} = \sup_{\substack{x\in\mathbb{R}^n \\ 0<r<2\operatorname{diam}\Omega}} \frac{\mu(B(x,r)\cap\Omega)}{\phi(r)}.$$

Then the following lemma holds.

Lemma 4.1. *Let μ be a measure on Ω with $\|\mu\|_{\mathscr{P}^\phi(\Omega)} < \infty$. Then*

$$\mu(\Omega) \leq \phi(\operatorname{diam}\Omega)\|\mu\|_{\mathscr{P}^\phi(\Omega)} < \infty.$$

Lemma 4.2. *Let μ be a measure on Ω with $\|\mu\|_{\mathscr{P}^\phi(\Omega)} < \infty$, and let μ^* be a measure on $\mathbb{R}^n$ defined by $\mu^*(A) = \mu(A\cap\Omega)$ for Borel measurable sets A in $\mathbb{R}^n$. Then $\|\mu^*\|_{\mathscr{P}^\phi(\mathbb{R}^n)} = \|\mu\|_{\mathscr{P}^\phi(\Omega)}$.*

Proof. By definition, $\mu(B(x,r)\cap\Omega) = \mu^*(B(x,r))$ for $x \in \mathbb{R}^n$ and $r > 0$. Dividing the both sides by $\phi(r)$ and taking the supremum, we have $\|\mu\|_{\mathscr{P}^\phi(\Omega)} \leq \|\mu^*\|_{\mathscr{P}^\phi(\mathbb{R}^n)}$. We need to check the converse inequality. Let $x \in \mathbb{R}^n$ and $r > 0$. If $r < 2\operatorname{diam}\Omega$, then

$$\frac{\mu^*(B(x,r))}{\phi(r)} \leq \|\mu\|_{\mathscr{P}^\phi(\Omega)}.$$

If $r \geq 2\operatorname{diam}\Omega$, then $\phi(\operatorname{diam}\Omega) \leq \phi(r)$, and so

$$\frac{\mu^*(B(x,r))}{\phi(r)} \leq \frac{\mu(\Omega\cap B(y,\operatorname{diam}\Omega))}{\phi(\operatorname{diam}\Omega)} \leq \|\mu\|_{\mathscr{P}^\phi(\Omega)},$$

where y is a point in Ω. These implies that $\|\mu^*\|_{\mathscr{P}^\phi(\mathbb{R}^n)} \leq \|\mu\|_{\mathscr{P}^\phi(\Omega)}$. $\square$

In the rest of this section, we suppose $n \geq 3$. For simplicity, we write

$$G\mu(x) = G_{\mathbb{R}^n}\mu(x) = \frac{1}{a_n}\int_{\mathbb{R}^n} \|x-y\|^{2-n}\,d\mu(y),$$

where $a_n = \nu_n n(n-2)$. Also, let

$$C_3 = \frac{C_1\psi(1)}{n\nu_n},$$

where C_1 is the constant in (A3).

Lemma 4.3. *If μ is a finite measure on $\mathbb{R}^n$ with $\|\mu\|_{\mathscr{P}^\phi(\mathbb{R}^n)} < \infty$, then*

$$\|G\mu\|_{\mathscr{L}^\infty(\mathbb{R}^n)} \leq C_3\|\mu\|_{\mathscr{P}^\phi(\mathbb{R}^n)} + \frac{\mu(\mathbb{R}^n)}{a_n}. \tag{4.1}$$

Proof. Let $x \in \mathbb{R}^n$. Since $\mu(B(x,r)) \le \|\mu\|_{\mathscr{P}^\phi(\mathbb{R}^n)}\phi(r)$ for $r>0$, we have

$$\begin{aligned} G\mu(x) &= \frac{1}{a_n}\int_{\mathbb{R}^n} \|x-y\|^{2-n} d\mu(y) = \frac{1}{\nu_n n}\int_0^\infty r^{1-n}\mu(B(x,r))\,dr \\ &= \frac{1}{\nu_n n}\left\{\int_0^1 r^{1-n}\mu(B(x,r))\,dr + \int_1^\infty r^{1-n}\mu(B(x,r))\,dr\right\} \\ &\le \frac{1}{\nu_n n}\left\{\|\mu\|_{\mathscr{P}^\phi(\mathbb{R}^n)}\int_0^1 \frac{\phi(r)}{r^{n-1}}\,dr + \mu(\mathbb{R}^n)\int_1^\infty r^{1-n}\,dr\right\}. \end{aligned}$$

Since $\phi(r) = r^{n-2}\psi(r)$, we see from (A3) that the brackets in the last is estimated by

$$C_1\|\mu\|_{\mathscr{P}^\phi(\mathbb{R}^n)}\psi(1) + \frac{\mu(\mathbb{R}^n)}{n-2}.$$

Hence (4.1) follows. □

Lemma 4.4. *If μ is a finite measure on $\mathbb{R}^n$ with $\|\mu\|_{\mathscr{P}^\phi(\mathbb{R}^n)} < \infty$, then $G\mu \in \mathscr{C}(\mathbb{R}^n)$.*

Proof. Let $x_0 \in \mathbb{R}^n$ and $\rho > 0$. Write $B_\rho = B(x_0,\rho)$. Observe from (A3) that

$$\begin{aligned} \int_{B_\rho} \|x_0-y\|^{2-n}\,d\mu(y) &= (n-2)\left\{\int_0^\rho r^{1-n}\mu(B_r)\,dr + \mu(B_\rho)\int_\rho^\infty r^{1-n}\,dr\right\} \\ &\le C\|\mu\|_{\mathscr{P}^\phi(\mathbb{R}^n)}\left\{\int_0^\rho \frac{\psi(r)}{r}\,dr + \psi(\rho)\right\} \\ &\le C\|\mu\|_{\mathscr{P}^\phi(\mathbb{R}^n)}\psi(\rho). \end{aligned}$$

Let $x \in B_{\rho/2}$. Since $B_\rho \subset B(x,2\rho)$, we have by (A2)

$$\begin{aligned} &|G\mu(x) - G\mu(x_0)| \\ &\quad\le C\|\mu\|_{\mathscr{P}^\phi(\mathbb{R}^n)}\psi(\rho) + \frac{1}{a_n}\int_{\mathbb{R}^n\setminus B_\rho} \big|\|x-y\|^{2-n} - \|x_0-y\|^{2-n}\big|\,d\mu(y). \end{aligned}$$

By the Lebesgue convergence theorem, the last integral tends to 0 as $x \to x_0$. Therefore (A1) concludes that $G\mu$ is continuous at x_0. □

Lemma 4.5. *If μ is a finite measure on $\mathbb{R}^n$ with $\|\mu\|_{\mathscr{P}^\phi(\mathbb{R}^n)} < \infty$, then $G\mu \in \mathscr{Z}^\psi(\overline{\Omega})$. Moreover, there exists a constant $C>0$ depending only on C_1 and n such that for all $x, y \in \overline{\Omega}$,*

$$|G\mu(x-y) - 2G\mu(x) + G\mu(x+y)| \le C\|\mu\|_{\mathscr{P}^\phi(\mathbb{R}^n)}\psi(\|y\|). \tag{4.2}$$

Proof. For arbitrary fixed $x \in \overline{\Omega}$, let $\mu_x(A) = \mu(\{x-z : z \in A\})$ for Borel measurable sets A in $\mathbb{R}^n$. Then $\|\mu_x\|_{\mathscr{P}^\phi(\mathbb{R}^n)} = \|\mu\|_{\mathscr{P}^\phi(\mathbb{R}^n)}$. Therefore we may prove (4.2) by assuming $x = 0 \in \overline{\Omega}$. For simplicity, we write $B(r) = B(0,r)$. Let $y \in \overline{\Omega}$. By [1, Lemme 1], we have for $z \in \mathbb{R}^n \setminus B(4\|y\|)$,

$$\left|\frac{1}{\|y-z\|^{n-2}} - \frac{2}{\|z\|^{n-2}} + \frac{1}{\|y+z\|^{n-2}}\right| \le C\frac{\|y\|^2}{\|z\|^n}.$$

Observe from (A2) and (A5) that

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus B(4\|y\|)}\left|\frac{1}{\|y-z\|^{n-2}}-\frac{2}{\|z\|^{n-2}}+\frac{1}{\|y+z\|^{n-2}}\right|d\mu(z)\\
&\qquad\le C\int_{\mathbb{R}^n\setminus B(4\|y\|)}\frac{\|y\|^2}{\|z\|^n}\,d\mu(z)\le C\|y\|^2\int_{4\|y\|}^{\infty}\frac{\mu(B(r))}{r^{n+1}}\,dr\\
&\qquad\le C\|\mu\|_{\mathscr{P}^\phi(\mathbb{R}^n)}\|y\|^2\int_{4\|y\|}^{\infty}\frac{\psi(r)}{r^3}\,dr\le C\|\mu\|_{\mathscr{P}^\phi(\mathbb{R}^n)}\psi(\|y\|).
\end{aligned}$$

Also, since

$$\begin{aligned}
&\int_{B(5\|y\|)}\frac{1}{\|z\|^{n-2}}\,d\mu(z)\\
&\quad=(n-2)\left\{\int_0^{5\|y\|}r^{1-n}\mu(B(r))\,dr+\mu(B(5\|y\|))\int_{5\|y\|}^{\infty}r^{1-n}\,dr\right\}\\
&\quad\le C\|\mu\|_{\mathscr{P}^\phi(\mathbb{R}^n)}\left\{\int_0^{5\|y\|}\frac{\psi(r)}{r}\,dr+\psi(5\|y\|)\right\}\\
&\quad\le C\|\mu\|_{\mathscr{P}^\phi(\mathbb{R}^n)}\psi(\|y\|),
\end{aligned}$$

we observe that

$$\int_{B(4\|y\|)}\left|\frac{1}{\|y-z\|^{n-2}}-\frac{2}{\|z\|^{n-2}}+\frac{1}{\|y+z\|^{n-2}}\right|d\mu(z)\le C\|\mu\|_{\mathscr{P}^\phi(\mathbb{R}^n)}\psi(\|y\|).$$

Hence (4.2) follows. □

Lemma 4.6. *Let F be a measurable function on $\Omega\times\mathbb{R}$, satisfying* (2.1) *for some nonnegative functions $V\in\mathscr{P}^\phi(\Omega)$ and $f\in\mathscr{C}(\mathbb{R})$, such that for each $x\in\Omega$, $F(x,\cdot)$ is nonnegative on $(0,\infty)$ and $F(x,\cdot)\in\mathscr{C}(0,\infty)$. Let μ be a measure on Ω with $0<\|\mu\|_{\mathscr{P}^\phi(\Omega)}<\infty$. Put $m=\min_{\partial\Omega}G\mu$. Assume that there is a constant $M>m$ such that*

$$\|f\|_{\mathscr{L}^\infty[m,M]}\left(C_3\|V\|_{\mathscr{P}^\phi(\Omega)}+\frac{\|V\|_{\mathscr{L}^1(\Omega)}}{a_n}\right)+\left(C_3\|\mu\|_{\mathscr{P}^\phi(\Omega)}+\frac{\mu(\Omega)}{a_n}\right)\le M. \tag{4.3}$$

Then there exists a positive solution $u\in\mathscr{Z}^\psi(\overline{\Omega})$ of

$$-\Delta u=F(x,u)+\mu\quad\textit{in }\Omega\quad\textit{(distribution)}. \tag{4.4}$$

Proof. The proof is based on the Schauder fixed point theorem. Instead of F, V and μ, we consider F^* defined by $F^*=F$ on $\Omega\times\mathbb{R}$, $F^*=0$ on $(\mathbb{R}^n\setminus\Omega)\times\mathbb{R}$ and V^* defined by $V^*=V$ on Ω, $V^*=0$ on $\mathbb{R}^n\setminus\Omega$ and μ^* defined by $\mu^*(A)=\mu(A\cap\Omega)$ for Borel measurable sets A in $\mathbb{R}^n$. Note from Lemma 4.2 that $\|V^*\|_{\mathscr{P}^\phi(\mathbb{R}^n)}=\|V\|_{\mathscr{P}^\phi(\Omega)}$ and $\|\mu^*\|_{\mathscr{P}^\phi(\mathbb{R}^n)}=\|\mu\|_{\mathscr{P}^\phi(\Omega)}$. In arguments below, we write F, V and μ for F^*, V^* and μ^*, respectively. Let $m=\min_{\partial\Omega}G\mu$ and let M be a constant satisfying (4.3). Then $m>0$ because $\|\mu\|_{\mathscr{P}^\phi(\Omega)}>0$. Let

$$\mathscr{W}=\{w\in\mathscr{C}(\overline{\Omega}):m\le w\le M\}.$$

This is a nonempty bounded closed convex subset of the Banach space $\mathscr{C}(\overline{\Omega})$. Consider the operator $\mathcal{T}$ on $\mathscr{W}$: for $x \in \mathbb{R}^n$,

$$\mathcal{T}w(x) = G[F(\cdot, w)](x) + G\mu(x).$$

Let $\mathcal{T}(\mathscr{W}) = \{\mathcal{T}w : w \in \mathscr{W}\}$. Note that if $w \in \mathscr{W}$, then

$$\|F(\cdot, w)\|_{\mathscr{P}^\phi(\mathbb{R}^n)} \le \|Vf(w)\|_{\mathscr{P}^\phi(\mathbb{R}^n)} \le \|f\|_{\mathscr{L}^\infty[m,M]}\|V\|_{\mathscr{P}^\phi(\Omega)} < \infty,$$
$$\|F(\cdot, w)\|_{\mathscr{L}^1(\mathbb{R}^n)} \le \|Vf(w)\|_{\mathscr{L}^1(\mathbb{R}^n)} \le \|f\|_{\mathscr{L}^\infty[m,M]}\|V\|_{\mathscr{L}^1(\Omega)} < \infty.$$

From the proof of Lemma 4.4, we observe that $\mathcal{T}(\mathscr{W})$ is equicontinuous on $\overline{\Omega}$. Also, Lemma 4.3 implies

$$\begin{aligned}\mathcal{T}w(x) &\le \|f\|_{\mathscr{L}^\infty[m,M]}\left(C_3\|V\|_{\mathscr{P}^\phi(\Omega)} + \frac{\|V\|_{\mathscr{L}^1(\Omega)}}{a_n}\right) + \left(C_3\|\mu\|_{\mathscr{P}^\phi(\Omega)} + \frac{\mu(\Omega)}{a_n}\right)\\ &\le M.\end{aligned}$$

By $F(\cdot, w) \ge 0$ and the minimum principle, $\mathcal{T}w \ge G\mu \ge m$ on $\overline{\Omega}$. Hence $\mathcal{T}(\mathscr{W}) \subset \mathscr{W}$. The Arzelá-Ascoli theorem implies that $\mathcal{T}(\mathscr{W})$ is relatively compact in $\mathscr{C}(\overline{\Omega})$.

We show that $\mathcal{T}$ is continuous on $\mathscr{W}$. Take $w_j, w \in \mathscr{W}$ such that $\|w_j - w\|_{\mathscr{L}^\infty(\overline{\Omega})} \to 0$ as $j \to \infty$. Let $x \in \mathbb{R}^n$. Since $F(x, \cdot) \in \mathscr{C}(0, \infty)$, it follows from the Lebesgue convergence theorem that as $j \to \infty$,

$$|\mathcal{T}w_j(x) - \mathcal{T}w(x)| \le G[|F(\cdot, w_j) - F(\cdot, w)|](x) \to 0.$$

The relatively compactness of $\mathcal{T}(\mathscr{W})$ implies that $\|\mathcal{T}w_j - \mathcal{T}w\|_{\mathscr{L}^\infty(\overline{\Omega})} \to 0$ as $j \to \infty$. Hence $\mathcal{T}$ is continuous on $\mathscr{W}$.

Applying the Schauder fixed point theorem, we find $u \in \mathscr{W}$ such that $\mathcal{T}u = u$ on $\overline{\Omega}$. Using the Fubini theorem, we observe that u satisfies (4.4). Also, it follows from Lemma 4.5 that $u \in \mathscr{Z}^\psi(\overline{\Omega})$. This completes the proof. $\square$

Proof of Theorem 2.3. If $\mathcal{H}_\phi(E) > 0$, then there exists a measure μ supported on E such that $\mu(E) > 0$ and $\|\mu\|_{\mathscr{P}^\phi(\Omega)} < \infty$ (see [2, Theorem 5.1.12]). Multiplying μ by a small constant if necessary, we may assume that $m = \min_{\partial\Omega} G\mu \le m_0$ and

$$C_3\|\mu\|_{\mathscr{P}^\phi(\Omega)} + \frac{\mu(\Omega)}{a_n} \le \varepsilon.$$

Let M be a constant satisfying (2.3). Then V, f and μ fulfill (4.3). By Lemma 4.6, there is a positive solution $u \in \mathscr{Z}^\psi(\overline{\Omega})$ of (4.4). Observe that u satisfies $-\Delta u = F(x, u)$ in $\Omega \setminus E$ (distribution), but does not satisfy it in the whole of Ω. $\square$

5 Proof of Theorem 2.3 in the case $\psi \in \Psi_Z$ and $n = 2$

In this section, we suppose that $n = 2$, $R = \operatorname{diam}\Omega$, B_0 is a disk of radius R with $\overline{\Omega} \subset B_0$, and $\phi = \psi \in \Psi_Z$. The proof of Theorem 2.3 for $n = 2$ is similar to that given in the previous section, but we need to consider

$$G\mu(x) = \frac{1}{2\pi}\int_{B_0} \log\frac{5R}{\|x - y\|}\, d\mu(y).$$

If μ is a finite measure on B_0, then $G\mu$ is superharmonic on $\mathbb{R}^2$ and positive on B_0.

Lemma 5.1. *If μ is a measure on B_0 with $\|\mu\|_{\mathscr{P}^\phi(B_0)} < \infty$, then*

$$\|G\mu\|_{\mathscr{L}^\infty(B_0)} \le \frac{C_1\psi(5R)}{2\pi}\|\mu\|_{\mathscr{P}^\phi(B_0)}.$$

Proof. By Lemma 4.2, we may assume that μ is a measure on $\mathbb{R}^2$ supported on B_0 with $\|\mu\|_{\mathscr{P}^\phi(\mathbb{R}^2)} = \|\mu\|_{\mathscr{P}^\phi(B_0)} < \infty$. Let $x \in B_0$. By the change of variable $r = 5Rt$ and (A3), we have

$$\begin{aligned} G\mu(x) &\le \frac{1}{2\pi}\int_0^1 \frac{1}{t}\,\mu(B(x,5Rt))\,dt \\ &\le \frac{\|\mu\|_{\mathscr{P}^\phi(B_0)}}{2\pi}\int_0^{5R}\frac{\psi(r)}{r}\,dr \le \frac{C_1\psi(5R)}{2\pi}\|\mu\|_{\mathscr{P}^\phi(B_0)}. \end{aligned}$$

Thus the lemma follows. □

Lemma 5.2. *If μ is a measure on B_0 with $\|\mu\|_{\mathscr{P}^\phi(B_0)} < \infty$, then $G\mu \in \mathscr{Z}^\psi(B_0)$. Moreover, there exists a constant $C > 0$ depending only on C_1 such that for all $x, y \in B_0$ with $x \pm y \in B_0$,*

$$|G\mu(x-y) - 2G\mu(x) + G\mu(x+y)| \le C\|\mu\|_{\mathscr{P}^\phi(B_0)}\psi(\|y\|). \tag{5.1}$$

Proof. It suffices to prove (5.1) with $x = 0$. We write $B(r)$ for $B(0,r)$ and identify $\mathbb{R}^2$ with $\mathbb{C}$. If $z \in \mathbb{C}\setminus B(2|y|)$, then $|\log|\frac{z^2-y^2}{z^2}|| \le C|\frac{y}{z}|^2$, and so

$$\int_{\mathbb{C}\setminus B(2|y|)}\left|\log\left|\frac{z^2-y^2}{z^2}\right|\right| d\mu(z) \le C|y|^2\int_{2|y|}^\infty \frac{\mu(B(r))}{r^3}\,dr \le C\|\mu\|_{\mathscr{P}^\phi(B_0)}\psi(|y|).$$

If $z \in B(|y|/2)$, then $|\log|\frac{z^2-y^2}{z^2}|| \le C + 2\log|\frac{y}{z}|$ which gives

$$\begin{aligned} \int_{B(|y|/2)}\left|\log\left|\frac{z^2-y^2}{z^2}\right|\right| d\mu(z) &\le C\mu(B(|y|)) + 2\int_0^1 \frac{\mu(B(t|y|))}{t}\,dt \\ &\le C\|\mu\|_{\mathscr{P}^\phi(B_0)}\psi(|y|). \end{aligned}$$

If $z \in B(y, |y|/2)$, then $|y|/2 \le |z| \le 3|y|/2$ and $3|z| \ge |z+y| \ge 2|y| - |z-y| \ge |z|$. Therefore $\frac{1}{2} \le |\frac{y}{z}||\frac{z+y}{z}| \le 6$, and so

$$\left|\log\left|\frac{z^2-y^2}{z^2}\right|\right| \le \left|\log\left|\frac{y}{z}\right|\left|\frac{z+y}{z}\right|\right| + \log\left|\frac{y}{z-y}\right| \le C + \log\left|\frac{y}{z-y}\right|.$$

By the similar way to the above, we obtain

$$\int_{B(y,|y|/2)}\left|\log\left|\frac{z^2-y^2}{z^2}\right|\right| d\mu(z) \le C\|\mu\|_{\mathscr{P}^\phi(B_0)}\psi(|y|).$$

Also, we observe that

$$\int_{B(-y,|y|/2)} \left|\log\left|\frac{z^2-y^2}{z^2}\right|\right| d\mu(z) \le C\|\mu\|_{\mathscr{P}^\phi(B_0)}\psi(|y|).$$

If $y \in A := B(2|y|)\backslash(B(|y|/2)\cup B(y,|y|/2)\cup B(-y,|y|/2))$, then $|\log|\frac{z^2-y^2}{z^2}|| \le C$, and so

$$\int_A \left|\log\left|\frac{z^2-y^2}{z^2}\right|\right| d\mu(z) \le C\|\mu\|_{\mathscr{P}^\phi(B_0)}\psi(|y|).$$

Combining these yields (5.1). □

Lemma 5.3. *For* $0 < r < 1$, *we have*

$$\psi(r)\log\frac{1}{r} \le 2C_1\psi(\sqrt{r}).$$

Proof. Let $0 < r < 1$. Since $\psi > 0$ is increasing, it follows from (A3) that

$$C_1\psi(\sqrt{r}) \ge \int_r^{\sqrt{r}} \frac{\psi(t)}{t}\,dt \ge \psi(r)\cdot\frac{1}{2}\log\frac{1}{r}.$$

□

Lemma 5.4. *If* μ *is a measure on* B_0 *with* $\|\mu\|_{\mathscr{P}^\phi(B_0)} < \infty$, *then* $G\mu \in \mathscr{C}(B_0)$.

Proof. Let $x_0 \in B_0$ and let $\rho > 0$ be small enough so that $B_\rho := B(x_0,\rho) \subset B_0$. Then

$$\begin{aligned}\int_{B_\rho} \log\frac{5R}{\|x_0-y\|}\,d\mu(y) &= \int_0^1 \frac{1}{t}\,\mu(B_\rho\cap B_{5Rt})\,dt \\ &\le \|\mu\|_{\mathscr{P}^\phi(B_0)}\left\{\int_0^{\rho/5R}\frac{\psi(5Rt)}{t}\,dt + \psi(\rho)\int_{\rho/5R}^1\frac{1}{t}\,dt\right\} \\ &\le \|\mu\|_{\mathscr{P}^\phi(B_0)}\left\{C_1\psi(\rho)+\psi(\rho)\log\frac{5R}{\rho}\right\}.\end{aligned}$$

Let $x \in B_{\rho/2}$. Then $B_\rho \subset B(x,2\rho)$. Therefore

$$\begin{aligned}&|G\mu(x)-G\mu(x_0)| \\ &\quad\le 2\|\mu\|_{\mathscr{P}^\phi(B_0)}\left\{C_1\psi(2\rho)+\psi(2\rho)\log\frac{5R}{2\rho}\right\} + \int_{B_0\backslash B_\rho}\left|\log\frac{\|x_0-y\|}{\|x-y\|}\right| d\mu(y).\end{aligned}$$

Since the last integral tends to 0 as $x \to x_0$, it follows from (A1) and Lemma 5.3 that $G\mu$ is continuous at x_0. □

Repeating arguments similar to Lemma 4.6, we obtain the following lemma.

Lemma 5.5. *Let F be a measurable function on $\Omega \times \mathbb{R}$, satisfying* (2.1) *for some nonnegative functions $V \in \mathscr{P}^{\phi}(\Omega)$ and $f \in \mathscr{C}(\mathbb{R})$, such that for each $x \in \Omega$, $F(x,\cdot)$ is nonnegative on $(0,\infty)$ and $F(x,\cdot) \in \mathscr{C}(0,\infty)$. Let μ be a measure on Ω with $0 < \|\mu\|_{\mathscr{P}^{\phi}(\Omega)} < \infty$. Put $m = \min_{\partial\Omega} G\mu$. Assume that there is a constant $M > m$ such that*

$$\frac{C_1}{2\pi}\psi(5R)\big(\|f\|_{\mathscr{L}^\infty[m,M]}\|V\|_{\mathscr{P}^{\phi}(\Omega)} + \|\mu\|_{\mathscr{P}^{\phi}(\Omega)}\big) \le M.$$

Then there exists a positive solution $u \in \mathscr{L}^{\psi}(\overline{\Omega})$ of

$$-\Delta u = F(x,u) + \mu \quad \text{in } \Omega \quad \text{(distribution)}.$$

The rest of the proof of Theorem 2.3 for $n = 2$ is the same as that for $n \ge 3$.

6 Proof of Theorem 2.3 in the case $\psi \in \Psi_H$

In this section, we let $\psi \in \Psi_H$ and $\phi(t) = t^{n-2}\psi(t)$.

Lemma 6.1. *Let $G\mu$ denote a potential in Section 4 or 5. If μ is a measure on Ω with $\|\mu\|_{\mathscr{P}^{\phi}(\Omega)} < \infty$, then $G\mu \in \mathscr{C}^{0,\psi}(\overline{\Omega})$.*

Proof. Let us prove this lemma for $n \ge 3$. The proof for $n = 2$ is similar. We may assume $\|\mu\|_{\mathscr{P}^{\phi}(\mathbb{R}^n)} < \infty$. Let $x, x_0 \in \overline{\Omega}$ and let $r = \|x - x_0\|$. Then

$$G\mu(x) - G\mu(x_0) = \frac{1}{\nu_n n}(I_1 + I_2),$$

where

$$I_1 = \int_0^{2r} t^{1-n}\mu(B(x,t))\,dt - \int_0^{2r} t^{1-n}\mu(B(x_0,t))\,dt,$$
$$I_2 = \int_{2r}^{\infty} t^{1-n}\mu(B(x,t))\,dt - \int_{2r}^{\infty} t^{1-n}\mu(B(x_0,t))\,dt.$$

Observe from (A2) and (A3) that

$$I_1 \le 2\|\mu\|_{\mathscr{P}^{\phi}(\mathbb{R}^n)} \int_0^{2r} t^{1-n}\phi(t)\,dt \le C\|\mu\|_{\mathscr{P}^{\phi}(\mathbb{R}^n)}\psi(r).$$

As in [6, p.16], we have

$$I_2 \le \int_r^{\infty} \{t^{1-n} - (t+r)^{1-n}\}\mu(B(x,t))\,dt.$$

Since $t^{1-n} - (t+r)^{1-n} \le Cr/t^n$, this and (A4) give

$$I_2 \le C\|\mu\|_{\mathscr{P}^{\phi}(\mathbb{R}^n)} r \int_r^{\infty} \frac{\phi(t)}{t^n}\,dt \le C\|\mu\|_{\mathscr{P}^{\phi}(\mathbb{R}^n)}\psi(r).$$

Combining these yields $G\mu(x) - G\mu(x_0) \le C\|\mu\|_{\mathscr{P}^{\phi}(\mathbb{R}^n)}\psi(r)$. Since x and x_0 can be interchanged, it follows that $G\mu \in \mathscr{C}^{0,\psi}(\overline{\Omega})$. □

Proof of Theorem 2.3 in the case $\psi \in \Psi_H$. Observe from Lemma 6.1 that we can find $u \in \mathscr{C}^{0,\psi}(\overline{\Omega})$ in Lemma 4.6 and 5.5. Repeating the same arguments completes the proof of Theorem 2.3. □

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