Invariant harmonic functions in the unit ball of $\mathbb{C}^n$ and Martin kernels of general domains in $\mathbb{R}^n$

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Preface

This doctoral dissertation consists of Part I and Part II. In Part I, we treat the Laplace-Beltrami operator on the unit ball of the n-dimensional complex space associated with the Bergman metric; in Part II, we treat the Laplace operator on the n-dimensional real space associated with the Euclidean metric.

Part I includes two chapters, Chapters 1 and 2. In Chapter 1, we discuss the boundary behavior of invariant harmonic functions on the unit ball. In 1969, Korányi introduced an approach region to guarantee the existence of boundary limits of Poisson-Szegö integrals. Our main purpose is to show the best possibility of the Korányi approach region. The result is stronger than the earlier work by Hakim and Sibony. In Chapter 2, we give characterizations of the invariant harmonic α-Bloch space and the invariant harmonic BMO space by using the spherical integral of compositions with Möbius transformations. We also apply these characterizations to show inclusion relationships among the α-Bloch space, the weighted Dirichlet space and the BMO space.

Part II includes three chapters, Chapters 3, 4 and 5. Chapter 3 is a joint work with H. Aikawa and T. Lundh about minimal Martin boundary points of a John domain. We show that the number of minimal Martin boundary points at each Euclidean boundary point is estimated by the John constant. For a class of John domains represented as the union of convex sets, we give a sufficient condition for the Martin compactification to be homeomorphic to the Euclidean closure. In Chapter 4, we study the boundary behavior of the quotient of Martin kernels of given intersecting domains. The main tool is a new characterization of the minimal thinness for a difference of two subdomains. As a consequence, we obtain the boundary growth of the Martin kernel in a Lipschitz domain. In Chapter 5, we give comparison estimates for the Green function and the Martin kernel in a uniform domain. These estimates enable us to show the equivalence of ordinary thinness and minimal thinness of a set contained in a non-tangential cone.

Keywords : invariant harmonic function, boundary behavior, Korányi approach region, α-Bloch space, BMO space, Green function, Martin kernel, minimal Martin boundary point, John domain, minimal thinness, comparison estimate

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This thesis consists of two papers and three manuscripts.

[H1] K. Hirata, *Sharpness of the Korányi approach region*, Proc. Amer. Math. Soc., to appear.

[H2] K. Hirata, *Characterizations of $\mathcal{M}$-harmonic α-Bloch and BMO functions on the unit ball of* $\mathbb{C}^n$, Complex Variables, **49** (2004), no. 7–9, 583–594.

[AHL] H. Aikawa, K. Hirata and T. Lundh, *Martin boundary points of a John domain and unions of convex sets*, preprint.

[H3] K. Hirata, *Boundary behavior of quotients of Martin kernels*, in preparation.

[H4] K. Hirata, *An estimate for the product of the Green function and the Martin kernel*, in preparation.

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Contents

Part I

Invariant harmonic functions in the unit ball of $\mathbb{C}^n$

Introduction

This introduction includes consistent notations and terminologies employed in Part I. Let $\mathbb{C}^n$ be the n-dimensional complex space with inner product $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w}_j$ and norm $|z| = \sqrt{\langle z, z\rangle}$. In Chapters 1 and 2, we will discuss solutions of the Laplace-Beltrami equation in the unit ball B of $\mathbb{C}^n$ associated with the Bergman metric. *The Laplace-Beltrami operator on B associated with the Bergman metric* is defined by

$$\widetilde{\Delta} = \frac{4}{n+1}(1-|z|^2)\sum_{j,k=1}^n (\delta_{j,k} - \overline{z}_j z_k)\frac{\partial^2}{\partial z_k \partial \overline{z}_j},$$

where $\delta_{j,k} = 0$ ($j \neq k$) and $\delta_{j,j} = 1$. The group of holomorphic automorphisms of B, denoted by $\mathrm{Aut}(B)$, plays an important role in the invariant harmonic function theory. Every holomorphic automorphism of B can be represented as the composition of a unitary transformation on $\mathbb{C}^n$ and a Möbius transformation of B. *A Möbius transformation of B* is defined for $a, z \in B$ by

$$\varphi_a(z) = \frac{a - P_a z - \sqrt{1-|a|^2}(z - P_a z)}{1 - \langle z, a\rangle},$$

where $P_a z = \langle z, a\rangle a|a|^{-2}$ ($a \neq 0$) and $P_0 z = 0$. We note that the operator $\widetilde{\Delta}$ is invariant under $\mathrm{Aut}(B)$: that is, $\widetilde{\Delta}(f \circ \psi) = (\widetilde{\Delta} f) \circ \psi$ for each $f \in C^2(B)$ and $\psi \in \mathrm{Aut}(B)$. For this reason, a C^2-solution of the equation

$$\widetilde{\Delta} f = 0 \quad \text{in } B$$

is called *an invariant harmonic function* (or an $\mathcal{M}$-harmonic function) on B. We note in the case $n = 1$ that invariant harmonic functions are just harmonic functions for the (usual) Laplace operator.

The gradient operator on B associated with the Bergman metric is denoted by $\widetilde{\nabla}$. It satisfies that for $f \in C^1(B)$,

$$|\widetilde{\nabla} f|^2 = \frac{2}{n+1}(1-|z|^2)\sum_{j,k=1}^n (\delta_{j,k} - \overline{z}_j z_k)\left(\overline{\frac{\partial f}{\partial z_j}}\frac{\partial f}{\partial z_k} + \frac{\partial f}{\partial \overline{z}_j}\overline{\frac{\partial f}{\partial \overline{z}_k}}\right),$$

and $|\widetilde{\nabla}(f \circ \psi)| = |(\widetilde{\nabla} f) \circ \psi|$ for $\psi \in \mathrm{Aut}(B)$. The gradient $\widetilde{\nabla} f$ is called *the invariant gradient* of f on B.

Let ν denote the Lebesgue measure on $\mathbb{C}^n$ normalized so that $\nu(B) = 1$. The measure λ on B defined by $d\lambda(z) = (1-|z|^2)^{-(n+1)}d\nu(z)$ is useful in our study because it is invariant under $\mathrm{Aut}(B)$: that is, for a measurable subset U of B, an integrable function f on B with respect to λ and $\psi \in \mathrm{Aut}(B)$, we have

$$\int_{\psi(U)} f(z)d\lambda(z) = \int_U f(\psi(z))d\lambda(z).$$

We call λ *the invariant measure* on B.

In the invariant harmonic function theory, the Hardy space is fundamental and important. For $1 \le p < \infty$, *the p-th Hardy space $\mathcal{H}^p$* is defined as the collection of all invariant harmonic functions f on B for which

$$\|f\|_{\mathcal{H}^p} := \sup_{0<r<1} \left(\int_S |f(r\zeta)|^p d\sigma(\zeta) \right)^{1/p} < \infty,$$

where S is the unit sphere and σ is the surface measure on S normalized so that $\sigma(S) = 1$. We also denote by $\mathcal{H}^\infty$ the collection of all bounded invariant harmonic functions on B. It is easy to show, using Hölder's inequality, that $\mathcal{H}^q \subset \mathcal{H}^p$ if $1 \le p < q \le \infty$. We should note that each element in $\mathcal{H}^p$ can be represented as a Poisson-Szegö integral: that is, if $f \in \mathcal{H}^p$, then there exists a complex measure μ on S such that

$$f(z) = \int_S \mathcal{P}(z,\zeta)d\mu(\zeta) \quad \text{for } z \in B,$$

where $\mathcal{P}(z,\zeta)$ is *the Poisson-Szegö kernel of B* defined by

$$\mathcal{P}(z,\zeta) = \frac{(1-|z|^2)^n}{|1-\langle z,\zeta\rangle|^{2n}}.$$

In particular, if $p > 1$, then μ is absolutely continuous with respect to σ and is represented as $d\mu = f^* d\sigma$ for some p-th integrable function f^* on S. Then we write $\mathcal{P}[f^*]$ and call it *the Poisson-Szegö integral of f^**. Conversely, Jensen's inequality shows that the Poisson-Szegö integral of a complex measure (resp. a p-th integrable function) on S belongs to $\mathcal{H}^1$ (resp. $\mathcal{H}^p$).

In Chapter 1, we will discuss the boundary behavior of invariant harmonic functions in the Hardy space. In 1969, Korányi introduced an approach region to guarantee the existence of boundary limits of invariant harmonic functions in the Hardy space. Our main purpose is to show the best possibility of the Korányi approach region. The result is stronger than the earlier work due to Hakim and Sibony (1983).

In Chapter 2, we will discuss characterizations of certain spaces of invariant harmonic functions. A motivation of the characterization is to clarify the inclusion among spaces. For $0 < p < \infty$ and $\alpha \in \mathbb{R}$, *the weighted Dirichlet space $\mathcal{D}^p_\alpha$* is defined as the collection of all invariant harmonic functions f on B for which

$$\|f\|_{\mathcal{D}^p_\alpha} := \left(\int_B |\widetilde{\nabla} f(z)|^p (1-|z|^2)^\alpha d\lambda(z) \right)^{1/p} < \infty.$$

In 1993, Stoll gave a characterization of the Hardy space and clarify the inclusion between the Hardy space and the weighted Dirichlet space:

$$
\begin{aligned}
&\mathcal{D}_n^p \subset \mathcal{H}^p && \text{for } 1 \leq p < 2; \\
&\mathcal{D}_n^p = \mathcal{H}^p && \text{for } p = 2; \\
&\mathcal{H}^p \subset \mathcal{D}_n^p && \text{for } 2 < p < \infty; \\
&\mathcal{D}_\alpha^p \subset \mathcal{H}^p && \text{for } \alpha < n \text{ and } 1 < p < \infty.
\end{aligned}
$$

Our main purpose is to give characterizations of the α-Bloch space and the BMO space (whose definitions will be described later), and to clarify inclusion relationships among the α-Bloch space, the weighted Dirichlet space and the BMO space.

Throughout Part I, we use the symbol A to denote an absolute positive constant whose value is unimportant and may change from line to line. If necessary, we write $A(a, b, \cdots)$ for a constant depending on $a, b, \cdots$.

Chapter 1

Boundary behavior of invariant harmonic functions

This chapter is based on the paper [H1].

1.1 Historical survey

Many investigations of the boundary behavior of (invariant) harmonic functions on the unit disc D of $\mathbb{C}$ would be motivated by the following result due to Schwarz [19] in 1872.

Theorem (Schwarz). *Let f be a continuous function on the unit circle ∂D. Then its Poisson-Szegö integral $\mathcal{P}[f]$ is extended continuously to ∂D and has values f on ∂D.*

This result means that the Dirichlet problem with continuous boundary data f,

$$\begin{cases} \widetilde{\Delta} h = 0 & \text{in } D, \\ \lim\limits_{z \to \xi, z \in D} h(z) = f(\xi) & \text{for all } \xi \in \partial D, \end{cases}$$

has a unique solution. Moreover, the solution is given by the Poisson-Szegö integral of the boundary data. However, if omitting continuity of boundary data, then the above problem does not have a solution in general. So it is important to consider what approach region is admissible for Poisson-Szegö integrals to have boundary limits.

In 1906, Fatou [4] considered a non-tangential approach region to guarantee the existence of boundary limits of bounded harmonic functions on D. For $\xi \in \partial D$ and $\alpha > 1$, *a non-tangential approach region* at ξ is defined by $\{z \in D : |z - \xi| < \alpha(1 - |z|)\}$.

Theorem (Fatou). *If h is a bounded harmonic function on D, then h has a non-tangential limit at almost every point of ∂D.*

The best possibility of the non-tangential approach region was first established in 1927 by Littlewood [13] in the following sense.

Theorem (Littlewood). *Let γ_0 be a tangential curve in D which ends at $z = 1$, and let γ_θ be the curve γ_0 rotated about the origin through an angle θ, so that γ_θ touches ∂D internally at $e^{i\theta}$. Then there exists a bounded harmonic function on D which admits no limits as $z \to e^{i\theta}$ along γ_θ for almost every θ in $[0, 2\pi)$.*

In 1990, Aikawa [1] improved no convergence "almost everywhere" to "everywhere".

Theorem (Aikawa). *Under the same assumption as in Littlewood's theorem, there exists a bounded harmonic function on D which admits no limits as $z \to e^{i\theta}$ along γ_θ for every θ in $[0, 2\pi)$.*

Remark 1.1.1. Fatou's theorem can be extended to the upper half space of the n-dimensional real space. See [20, Chapter VII]. The best possibility of the non-tangential approach region in the upper half space was proved by Aikawa [2]. In 1984, Nagel and Stein [14] obtained the marvelous result that Poisson integrals in the upper half space have boundary limits at almost every point of the boundary within an approach region which is not contained in any non-tangential approach regions.

The extension of Fatou's theorem to the unit ball B of $\mathbb{C}^n$ was achieved by Koyányi [11]. He considered the following approach region. For $\xi \in S$ and $\alpha > 1$, we let

$$\mathcal{A}_\alpha(\xi) = \left\{ z \in B : |1 - \langle z, \xi \rangle| < \frac{\alpha}{2}(1 - |z|^2) \right\}.$$

We note that this approach region is non-tangential in the special real direction and is tangential in the complex tangential directions. In the sequel, we will say $\mathcal{A}_\alpha(\xi)$ *the Korányi approach region* at ξ. Korányi's result is stated as follows.

Theorem (Korányi). *If f is an integrable function on S, then its Poisson-Szegö integral $\mathcal{P}[f]$ has the boundary limit $f(\xi)$ as $z \to \xi$ within $\mathcal{A}_\alpha(\xi)$ at almost every point ξ of S.*

1.2 Sharpness of the Korányi approach region

The best possibility of the Korányi approach region for Poisson-Szegö integrals to have boundary limits was proved in 1983 by Hakim and Sibony [8] in the following sense.

Theorem (Hakim - Sibony). *Suppose that $n \geq 2$. Let $\alpha > 1$ and let $g : (0, 1] \to [\alpha, \infty)$ be a decreasing function such that*

$$\lim_{t \to 0+} g(t) = \infty. \tag{1.1}$$

For $\xi \in S$, we define

$$D_{\alpha,g}(\xi) = \left\{ z \in B : \begin{array}{l} |1 - \langle z, \xi \rangle| \leq \alpha(1 - |\langle z, \xi \rangle|), \\ |1 - \langle z, \xi \rangle| \leq g(|1 - \langle z, \xi \rangle|)(1 - |z|) \end{array} \right\}.$$

Then there exists a bounded holomorphic function on B which admits no limits as $z \to \xi$ within $D_{\alpha,g}(\xi)$ at almost every point ξ of S.

We now compare $D_{\alpha,g}(\xi)$ with $\mathcal{A}_\alpha(\xi)$. To this end, we may assume by unitary invariance that $\xi = e_1 = (1, 0, \cdots, 0)$ for simplicity. In view of (1.1), the set of points satisfying the second inequality defining $D_{\alpha,g}(e_1)$ is quite wider than the Korányi approach region $\mathcal{A}_\alpha(e_1)$ near e_1 in all directions. On the other hand, the first inequality is $|1 - z_1| \le \alpha(1 - |z_1|)$ and provides the restriction in the z_1-plane only. From these, we see that $D_{\alpha,g}(\xi)$ is wider than any Korányi approach regions in the complex tangential directions, and is the same, non-tangential, in the special real direction. See Figure 1.1.

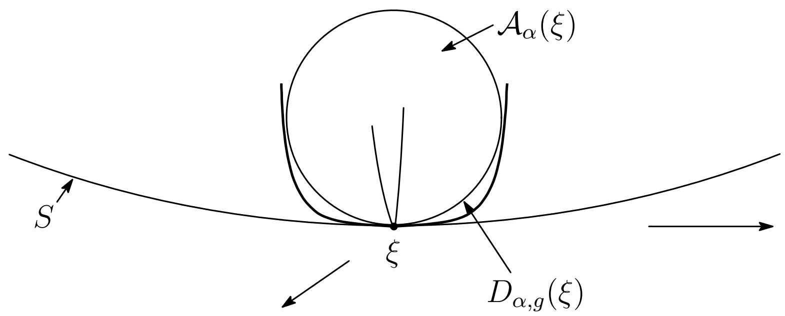

Figure 1.1: Difference between $D_{\alpha,g}(\xi)$ and $\mathcal{A}_\alpha(\xi)$.

In 1986, Sueiro [23] proved a result similar to Nagel-Stein's theorem. He actually studied in more general domain, the so-called space of homogeneous type, than the unit ball.

Theorem (Sueiro). *If f is an integrable function on S, then its Poisson-Szegö integral $\mathcal{P}[f]$ has the boundary limit $f(\xi)$ at almost every point ξ of S within a certain approach region which is not contained in any Korányi approach regions at ξ.*

The purpose of this chapter is to show the best possibility of the Korányi approach region in the Littlewood sense. We consider a curve γ in B which ends at e_1 and satisfies that

(1.2)
$$\lim_{z \to e_1,\, z \in \gamma} \frac{|1 - \langle z, e_1 \rangle|}{1 - |z|^2} = \infty.$$

This means that for each $\alpha > 1$, points of γ near e_1 lie outside the Korányi approach region $\mathcal{A}_\alpha(e_1)$. Let $\mathcal{U}$ denote the group of unitary transformations of $\mathbb{C}^n$ and write $U\gamma$ for the image of γ through $U \in \mathcal{U}$. Since unitary transformations preserve inner products, we see that $U\gamma$ touches S internally at Ue_1 and lies outside the Korányi approach region $\mathcal{A}_\alpha(Ue_1)$ near Ue_1 for each $\alpha > 1$.

Our result is as follows.

Theorem 1.2.1. *Let γ be a curve in B which ends at e_1 and satisfies the property* (1.2). *Then there exists a real valued bounded function f on S of which Poisson-Szegö integral $\mathcal{P}[f]$ admits no limits as $|z| \to 1$ along $U\gamma$ for every $U \in \mathcal{U}$, that is,*

$$\liminf_{|z| \to 1,\, z \in U\gamma} \mathcal{P}[f](z) \neq \limsup_{|z| \to 1,\, z \in U\gamma} \mathcal{P}[f](z) \quad \textit{for every } U \in \mathcal{U}.$$

Remark 1.2.2. In addition, we can make f satisfy

$$\liminf_{|z|\to 1,\, z\in U\gamma} \mathcal{P}[f](z) = \inf_{\zeta\in S} f(\zeta) \quad \text{and} \quad \limsup_{|z|\to 1,\, z\in U\gamma} \mathcal{P}[f](z) = \sup_{\zeta\in S} f(\zeta)$$

for every $U \in \mathcal{U}$.

Remark 1.2.3. In view of Sueiro's theorem, the limit in (1.2) can not be replaced by the upper limit.

Remark 1.2.4. Since $\mathcal{U}$ acts transitively on S, for each $\xi \in S$ there is $U_\xi \in \mathcal{U}$ such that $\xi = U_\xi e_1$. Therefore Theorem 1.2.1 implies that there exists a real valued bounded invariant harmonic function on B which admits no limits as $z \to \xi$ along $U_\xi\gamma$ at *every* point ξ of S. We note that the Poisson-Szegö integral in Theorem 1.2.1 may not be pluriharmonic. However, Theorem 1.2.1 is stronger than Hakim-Sibony's theorem in the following points:

- It improves no convergence "almost everywhere" to "everywhere".
- It establishes that a tangential approach in the special real direction can not be allowed in Korányi's theorem.
- The existence of a bounded invariant harmonic function which fails to have boundary limits is ensured even if we replace $D_{\alpha,g}(e_1)$ by much smaller curve γ satisfying the property (1.2).

Our method is different from Hakim and Sibony's. Their proof is based on a higher dimensional Blaschke product. However, we will prove Theorem 1.2.1 in Section 1.4 by constructing a bounded function on S and using lower and upper estimates of the Poisson-Szegö integral in Section 1.3. In the proofs we adapt ideas from [1, 2]. Whereas the polar (resp. the Euclidean) coordinate was used to construct a bounded function on the unit circle (resp. $\mathbb{R}^n$) in [1, 2], they are not applicable in our case. This is an important difference between [1, 2] and our case.

1.3 Lower and upper estimates for Poisson-Szegö integrals

We begin with introducing a non-isotropic ball in S. We observe that the function $d(z,w) = |1 - \langle z, w\rangle|^{1/2}$ satisfies the triangle inequality on $B \cup S$, and defines a metric on S (cf. [22, Lemma 7.3]). For $\xi \in S$ and $r > 0$, we write $Q(\xi, r) = \{\zeta \in S : d(\zeta,\xi) < r\}$, the non-isotropic ball of center ξ and radius r. Note that, to emphasize the metric d, we use the slightly different definition from [22]. We observe from [22, p. 84] that $\sigma(Q(U\xi, r)) = \sigma(Q(\xi, r))$ for any unitary transformations U and that

$$\lim_{r\to 0} \frac{\sigma(Q(\xi,r))}{r^{2n}} = \frac{2^n}{4\sqrt{\pi}} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2}+1)}. \tag{1.3}$$

Moreover, there is a constant $A_0 > 1$ depending only on the dimension n such that

$$A_0^{-1} r^{2n} \leq \sigma(Q(\xi, r)) \leq A_0 r^{2n} \tag{1.4}$$

for $\xi \in S$ and $0 \leq r \leq \operatorname{diam} S = \sqrt{2}$. Here $\operatorname{diam} E = \sup\{d(\eta, \zeta) : \eta, \zeta \in E\}$ for $E \subset S$.

Let $T > 0$ and $\xi \in S$. For an integrable function g on S, we define the truncated maximal function at ξ by

$$\mathcal{M}_T[g](\xi) = \sup_{r \geq T} r^{-2n} \int_{Q(\xi,r)} |g(\zeta)| d\sigma(\zeta).$$

Lemma 1.3.1. *There exists a positive constant A_1 depending only on the dimension n such that if g is an integrable function on S and $C > 0$, then*

$$|\mathcal{P}[g](t\xi)| \leq A_1 \left((1-t)^{-n} \int_{Q(\xi, C\sqrt{1-t})} |g(\zeta)| d\sigma(\zeta) + C^{-2n} \mathcal{M}_{C\sqrt{1-t}}[g](\xi) \right)$$

for $\xi \in S$ and $0 < t < 1$.

Proof. Let $\xi \in S$ and $0 < t < 1$ be fixed, and put

$$\begin{aligned} V_0 &= Q(\xi, C\sqrt{1-t}), \\ V_j &= Q(\xi, 2^j C\sqrt{1-t}) \setminus Q(\xi, 2^{j-1} C\sqrt{1-t}) \qquad (j = 1, \cdots, N), \end{aligned}$$

where N is the smallest integer such that $2^N C\sqrt{1-t} > \sqrt{2}$. Then

$$|\mathcal{P}[g](t\xi)| \leq \sum_{j=0}^{N} \int_{V_j} \frac{(1-t^2)^n}{|1 - \langle t\xi, \zeta \rangle|^{2n}} |g(\zeta)| d\sigma(\zeta).$$

Since $|1 - \langle t\xi, \zeta \rangle| \geq 1 - t$ for $\zeta \in S$, it follows that

$$\int_{V_0} \frac{(1-t^2)^n}{|1 - \langle t\xi, \zeta \rangle|^{2n}} |g(\zeta)| d\sigma(\zeta) \leq \frac{2^n}{(1-t)^n} \int_{Q(\xi, C\sqrt{1-t})} |g(\zeta)| d\sigma(\zeta).$$

Let $j = 1, \cdots, N$. By the triangle inequality, we have for $\zeta \in V_j$,

$$2^{j-1} C\sqrt{1-t} \leq d(\xi, \zeta) \leq d(\xi, t\xi) + d(t\xi, \zeta) \leq 2d(t\xi, \zeta) = 2|1 - \langle t\xi, \zeta \rangle|^{1/2}.$$

Hence it follows that

$$\begin{aligned} \int_{V_j} \frac{(1-t^2)^n}{|1 - \langle t\xi, \zeta \rangle|^{2n}} |g(\zeta)| d\sigma(\zeta) &\leq \frac{2^{9n}}{2^{4nj} C^{4n} (1-t)^n} \int_{Q(\xi, 2^j C\sqrt{1-t})} |g(\zeta)| d\sigma(\zeta) \\ &\leq \frac{2^{9n}}{2^{2nj} C^{2n}} \mathcal{M}_{C\sqrt{1-t}}[g](\xi). \end{aligned}$$

Since $\sum_{j=1}^{N} 2^{-2nj} < 1$, we obtain the lemma with $A_1 = 2^{9n}$. □

As a consequence of Lemma 1.3.1, we obtain the following lower and upper estimates.

Lemma 1.3.2. *The following statements hold.*

(i) *If g is an integrable function on S, then*

$$|\mathcal{P}[g](t\xi)| \le A_2 \mathcal{M}_{\sqrt{1-t}}[g](\xi) \quad \textit{for } \xi \in S \textit{ and } 0 < t < 1,$$

where A_2 is a positive constant depending only on the dimension n.

(ii) *Let $\xi \in S$, $0 < r < 1$ and $C > 0$. If g is a measurable function on S such that $g = 1$ on $Q(\xi, C\sqrt{1-r})$ and $|g| \le 1$ on S, then*

$$\mathcal{P}[g](t\xi) \ge 1 - \frac{A_3}{C^{2n}} \quad \textit{for } r \le t < 1,$$

where A_3 is a positive constant depending only on the dimension n.

Proof. Putting $C = 1$ in Lemma 1.3.1, we obtain (i) with $A_2 = 2A_1$. Let us show (ii). We put $h = (1-g)/2$. Then $h = 0$ on $Q(\xi, C\sqrt{1-r})$ and $|h| \le 1$ on S. Applying Lemma 1.3.1 to h, we obtain from (1.4) that for $r \le t < 1$,

$$\mathcal{P}[h](t\xi) \le \frac{A_1}{C^{2n}} \mathcal{M}_{C\sqrt{1-t}}[h](\xi) \le \frac{A_1}{C^{2n}} \sup_{\rho \ge C\sqrt{1-t}} \frac{\sigma(Q(\xi,\rho))}{\rho^{2n}} \le \frac{A_0 A_1}{C^{2n}}.$$

Since $\mathcal{P}[g] = 1 - 2\mathcal{P}[h]$, we obtain (ii) with $A_3 = 2A_0A_1$. □

1.4 Proof of Theorem 1.2.1

Let π be the radial projection to S defined by $\pi(z) = z/|z|$ for $z \ne 0$. We note that (1.2) implies

(1.5)
$$\lim_{z \to e_1, z \in \gamma} \frac{d(z, e_1)}{d(z, \pi(z))} = \infty,$$

since $1 - |z|^2 \ge 1 - |z| = d(z, \pi(z))^2$ for $z \in B \setminus \{0\}$.

Lemma 1.4.1. *Let γ be the curve as in Theorem 1.2.1. Then there exist sequences of positive numbers $\{a_j\}_{j=1}^{\infty}$, $\{b_j\}_{j=1}^{\infty}$ and subcurves $\{\gamma_j\}_{j=1}^{\infty}$ of γ with the following properties:*

(i) $0 < a_j < b_j < a_{j+1} < b_{j+1} < 1$ *and* $\lim_{j\to\infty} a_j = 1$*;*

(ii) $a_j \le |z| \le b_j$ *for* $z \in \gamma_j$*;*

(iii) $\operatorname{diam} \pi(\gamma_j) \le \sqrt{1 - b_{j-1}}$ *if* $j \ge 2$*;*

(iv) $\displaystyle \lim_{j\to\infty} \frac{\operatorname{diam} \pi(\gamma_j)}{\sqrt{1-a_j}} = \infty.$

Proof. Let $\alpha_j > 1$ be such that $\alpha_j \to \infty$ as $j \to \infty$. We shall choose $\{a_j\}$, $\{b_j\}$ and $\{\gamma_j\}$, inductively. By (1.5), we find a_1 with $\inf_{z\in\gamma} |z| < a_1 < 1$ and

$$d(z, e_1) \geq \alpha_1 d(z, \pi(z)) \quad \text{for } z \in \gamma \cap \{|z| \geq a_1\}.$$

Let γ' be the connected component of $\gamma \cap \{|z| \geq a_1\}$ which ends at e_1. Since there is $z_0 \in \gamma' \cap \{|z| = a_1\}$, we have from the triangle inequality that

$$\begin{aligned}\operatorname{diam} \pi(\gamma') &\geq d(\pi(z_0), e_1)\\ &\geq d(z_0, e_1) - d(z_0, \pi(z_0))\\ &\geq (\alpha_1 - 1) d(z_0, \pi(z_0))\\ &= (\alpha_1 - 1)\sqrt{1 - a_1}.\end{aligned}$$

Let γ'' be a subcurve of γ' connecting a point in $\{|z| = a_1\}$ and a point near e_1 such that

$$\operatorname{diam} \pi(\gamma'') \geq \frac{1}{2} \operatorname{diam} \pi(\gamma').$$

We take b_1 so that $\sup_{z\in\gamma''} |z| < b_1 < 1$, and let γ_1 be the connected component of $\gamma \cap \{a_1 \leq |z| \leq b_1\}$ containing γ''. Then

$$\operatorname{diam} \pi(\gamma_1) \geq \operatorname{diam} \pi(\gamma'') \geq \frac{\alpha_1 - 1}{2}\sqrt{1 - a_1}.$$

We next choose a_2, b_2 and γ_2 as follows. Let a_2 be such that $b_1 < a_2 < 1$ and

$$\frac{1}{4}\sqrt{1 - b_1} \geq d(z, e_1) \geq \alpha_2 d(z, \pi(z)) \quad \text{for } z \in \gamma \cap \{|z| \geq a_2\}. \tag{1.6}$$

By repeating the above procedure, we can find b_2 and γ_2 with $a_2 < b_2 < 1$ and $a_2 \leq |z| \leq b_2$ for $z \in \gamma_2$, and

$$\operatorname{diam} \pi(\gamma_2) \geq \frac{\alpha_2 - 1}{2}\sqrt{1 - a_2}.$$

It also follows from (1.6) and $\alpha_2 > 1$ that

$$d(\pi(z), e_1) \leq d(z, e_1) + d(z, \pi(z)) \leq \frac{1}{2}\sqrt{1 - b_1} \quad \text{for } z \in \gamma_2,$$

and so $\operatorname{diam} \pi(\gamma_2) \leq \sqrt{1 - b_1}$ by the triangle inequality.

Continuing this procedure, we obtain the required sequences. □

In the rest of this section, we suppose that $\{a_j\}$, $\{b_j\}$ and $\{\gamma_j\}$ are as in Lemma 1.4.1, and put

$$\ell_j = \frac{\operatorname{diam} \pi(\gamma_j)}{4}, \quad c_j = \left(\frac{\operatorname{diam} \pi(\gamma_j)}{\sqrt{1 - a_j}}\right)^{1/2} \quad \text{and} \quad \rho_j = c_j\sqrt{1 - a_j}$$

to simplify the notation. We note from Lemma 1.4.1 that

$$\lim_{j\to\infty} \ell_j = 0, \quad \lim_{j\to\infty} \frac{\rho_j}{\ell_j} = 0 \quad \text{and} \quad \lim_{j\to\infty} c_j = \infty. \tag{1.7}$$

Therefore, taking a subsequence if necessary, we may assume, in the argument below, that $\rho_j < \ell_j$ for every $j \in \mathbb{N}$.

For each $j \in \mathbb{N}$, let us choose finitely many points $\{\eta_j^\nu\}_\nu$ in S such that

(P1) $S = \bigcup_{\nu} Q(\eta_j^\nu, \ell_j)$,

(P2) $\{Q(\eta_j^\nu, \ell_j/2)\}_\nu$ are mutually disjoint.

This is possible. In fact, we first take an arbitrary $\eta_j^1 \in S$, and take $\eta_j^\mu \in S \setminus \bigcup_{\nu=1}^{\mu-1} Q(\eta_j^\nu, \ell_j)$ inductively as long as $S \setminus \bigcup_{\nu=1}^{\mu-1} Q(\eta_j^\nu, \ell_j) \neq \emptyset$. Since S is compact, we can get finitely many points $\{\eta_j^\nu\}_\nu$ satisfying (P1). It also fulfills that $d(\eta_j^\nu, \eta_j^\mu) \geq \ell_j$ if $\nu \neq \mu$. Hence (P2) follows from the triangle inequality.

We put

$$M_j = \bigcup_{\nu} \{\zeta \in S : d(\zeta, \eta_j^\nu) = \ell_j\}.$$

Then $\pi(U\gamma_j) \cap M_j \neq \emptyset$ for any unitary transformations U. Indeed, (P1) shows that $\pi(U\gamma_j) \cap Q(\eta_j^\nu, \ell_j) \neq \emptyset$ for some ν. Since $\operatorname{diam} \pi(U\gamma_j) = \operatorname{diam} \pi(\gamma_j) = 4\ell_j$ and $\operatorname{diam} Q(\eta_j^\nu, \ell_j) \leq 2\ell_j$, we have $\pi(U\gamma_j) \cap \{\zeta \in S : d(\zeta, \eta_j^\nu) = \ell_j\} \neq \emptyset$, and so $\pi(U\gamma_j) \cap M_j \neq \emptyset$.

Let G_j be the subset of B given by

$$G_j = \{z \in B : a_j \leq |z| \leq b_j \text{ and } \pi(z) \in M_j\}.$$

Since $U\gamma_j \subset \{a_j \leq |z| \leq b_j\}$ by Lemma 1.4.1 (ii), it follows that $U\gamma_j \cap G_j \neq \emptyset$. We also put

$$E_j = \bigcup_{\nu} R_j^\nu,$$

where $R_j^\nu = \{\zeta \in S : \ell_j - \rho_j < d(\zeta, \eta_j^\nu) < \ell_j + \rho_j\}$, the non-isotropic ring. See Figure 1.2. Since the value $\sigma(R_j^\nu)$ is independent of η_j^ν by unitary invariance, we write κ_j for this value.

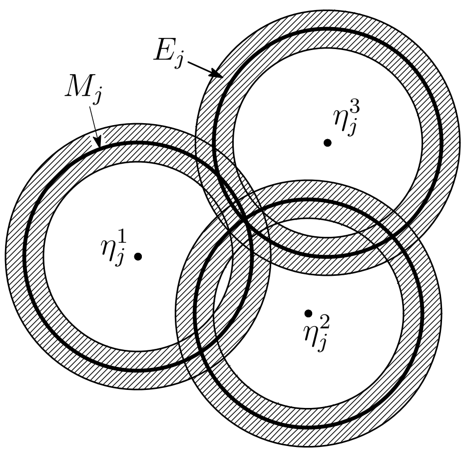

Figure 1.2: M_j and E_j.

We note that

(1.8) $$\lim_{j\to\infty} \frac{\kappa_j}{\ell_j^{2n}} = 0.$$

In fact, we obtain from (1.3) and (1.7) that for $\eta \in S$,

$$
\begin{aligned}
\frac{\kappa_j}{\ell_j^{2n}} &= \frac{\sigma(Q(\eta, \ell_j + \rho_j)) - \sigma(Q(\eta, \ell_j - \rho_j))}{\ell_j^{2n}} \\
&= \left(\frac{\ell_j + \rho_j}{\ell_j}\right)^{2n} \frac{\sigma(Q(\eta, \ell_j + \rho_j))}{(\ell_j + \rho_j)^{2n}} - \left(\frac{\ell_j - \rho_j}{\ell_j}\right)^{2n} \frac{\sigma(Q(\eta, \ell_j - \rho_j))}{(\ell_j - \rho_j)^{2n}} \\
&\to 0 \quad \text{as } j \to \infty.
\end{aligned}
$$

Lemma 1.4.2. *Let $\{E_j\}$ be as above, and let χ_{E_j} denote the characteristic function of E_j. The following properties hold.*

(i) $\displaystyle\lim_{j\to\infty} \left(\sup_{|z| \le b_{j-1}} \mathcal{P}[\chi_{E_j}](z) \right) = 0.$

(ii) $\displaystyle\lim_{j\to\infty} \sigma(E_j) = 0.$

Proof. Let $z \in B$ be such that $|z| \le b_{j-1}$. By Lemma 1.3.2 (i), we have

$$
\begin{aligned}
\mathcal{P}[\chi_{E_j}](z) &\le A_2 \mathcal{M}_{\sqrt{1-|z|}}[\chi_{E_j}](\pi(z)) \\
&\le A_2 \sup_{r \ge \sqrt{1-|z|}} r^{-2n} \sum_{\nu} \sigma(R_j^{\nu} \cap Q(\pi(z), r)) \\
&\le A_2 \sup_{r \ge \sqrt{1-|z|}} r^{-2n} N_j(z, r) \kappa_j,
\end{aligned}
$$

where $N_j(z, r)$ is the number of ν such that $R_j^{\nu} \cap Q(\pi(z), r) \neq \emptyset$. Since $\sqrt{1-|z|} \ge \operatorname{diam} \pi(\gamma_j)$ by Lemma 1.4.1 (iii), we observe from $\rho_j < \ell_j \le r/4$ that if $R_j^{\nu} \cap Q(\pi(z), r) \neq \emptyset$, then $Q(\eta_j^{\nu}, \ell_j/2) \subset Q(\pi(z), 2r)$. Therefore it follows from (1.4) and (P2) that $N_j(z, r) \le A_4 (r/\ell_j)^{2n}$ with a positive constant A_4 depending only on the dimension n. Hence we obtain

$$
\mathcal{P}[\chi_{E_j}](z) \le A_2 A_4 \frac{\kappa_j}{\ell_j^{2n}},
$$

so that (i) follows from (1.8).

Taking $z = 0$ in (i), we obtain

$$
\sigma(E_j) = \mathcal{P}[\chi_{E_j}](0) \to 0 \quad \text{as } j \to \infty,
$$

and thus (ii) follows. □

We now construct a bounded function f on S satisfying the property in Theorem 1.2.1.

Proof of Theorem 1.2.1. In view of Lemma 1.4.2, we may assume, taking a subsequence of j if necessary, that

(1.9) $$\mathcal{P}[\chi_{E_j}](z) \le 2^{-j} \quad \text{for } |z| \le b_{j-1},$$

and $\sigma(E_j) \le 2^{-j}$. Then $\sigma(\bigcap_k \bigcup_{j=k}^\infty E_j) = 0$. Let

$$f_j(\zeta) = \begin{cases} (-1)^{I_j(\zeta)} & \text{if } \zeta \in \bigcup_{i=1}^j E_i, \\ 0 & \text{if } \zeta \notin \bigcup_{i=1}^j E_i, \end{cases}$$

where $I_j(\zeta)$ is the maximum integer i such that $\zeta \in E_i$ for $\zeta \in \bigcup_{i=1}^j E_i$. Then we observe that f_j converges almost everywhere on S to

$$f(\zeta) = \begin{cases} (-1)^{I(\zeta)} & \text{if } \zeta \in \bigcup_{j=1}^\infty E_j \setminus \bigcap_k \bigcup_{j=k}^\infty E_j, \\ 0 & \text{if } \zeta \notin \bigcup_{j=1}^\infty E_j \text{ or } \zeta \in \bigcap_k \bigcup_{j=k}^\infty E_j, \end{cases}$$

where $I(\zeta)$ is the maximum integer i such that $\zeta \in E_i$ for $\zeta \in \bigcup_{j=1}^\infty E_j \setminus \bigcap_k \bigcup_{j=k}^\infty E_j$. We also see that

(a) $f_j = (-1)^j$ on E_j, and $|f_j| \le 1$ on S;

(b) $|f_{j+1} - f_j| \le 2\chi_{E_{j+1}}$;

(c) $\mathcal{P}[f_j]$ converges to $\mathcal{P}[f]$ on B.

Let U be a unitary transformation. Since $U\gamma$ intersects G_j for all j as stated in the paragraph defining G_j, we take $z_j \in U\gamma \cap G_j$. Note that $a_j \le |z_j| \le b_j$ and $Q(\pi(z_j), c_j\sqrt{1-a_j}) \subset E_j$. If j is even, then it follows from Lemma 1.3.2 (ii), Lemma 1.4.1 (i) and (1.9) that

$$\begin{aligned} \mathcal{P}[f](z_j) &= \mathcal{P}[f_j](z_j) + \sum_{k=j}^\infty \mathcal{P}[f_{k+1} - f_k](z_j) \\ &\ge \mathcal{P}[f_j](z_j) - 2\sum_{k=j}^\infty \mathcal{P}[\chi_{E_{k+1}}](z_j) \\ &\ge 1 - \frac{A_3}{c_j^{2n}} - 2^{1-j}. \end{aligned}$$

Similarly, if j is odd, then

$$\mathcal{P}[f](z_j) \le -1 + \frac{A_3}{c_j^{2n}} + 2^{1-j}.$$

Hence we conclude from (1.7) that

$$\liminf_{|z|\to 1,\, z\in U\gamma} \mathcal{P}[f](z) = -1 < 1 = \limsup_{|z|\to 1,\, z\in U\gamma} \mathcal{P}[f](z).$$

Thus the proof of Theorem 1.2.1 is complete. $\square$

Chapter 2

Characterizations of function spaces

This chapter is based on the paper [H2].

2.1 Background and Motivation

There are several characterizations of spaces of holomorphic functions on the unit ball B of $\mathbb{C}^n$. Choa and Choe [3] and Jevtić [9, 10] gave characterizations of the BMOA in terms of Carleson measures. In [21], Stoll characterized the p-th Hardy space by

$$\int_B G(z,0)|f(z)|^{p-2}|\widetilde{\nabla} f(z)|^2 d\lambda(z) < \infty, \tag{2.1}$$

where G is the Green function for $\widetilde{\Delta}$ (whose definition will be described in Section 2.2). Ouyang, Yang and Zhao [16] and Nowak [15] also characterized the weighted Bergman space and the Bloch space in terms of several finite integrals similar to (2.1) involving Möbius transformations. The hyperbolic Hardy space was characterized by Kwon [12].

The purpose of this chapter is to give characterizations of the invariant harmonic α-Bloch space and the invariant harmonic BMO space. Our characterization of the α-Bloch space is motivated by the classical Hardy-Littlewood theorem in one dimension and its extension to higher dimensions due to Pavlović [17]: Let $0 < p < \infty$ and $\alpha > 0$. An invariant harmonic function f on B satisfies the property

$$\left(\int_S |\widetilde{\nabla} f(r\zeta)|^p d\sigma(\zeta)\right)^{1/p} = O((1-r^2)^{-\alpha}) \quad \text{as } r \to 1, \tag{2.2}$$

if and only if f satisfies the property

$$\left(\int_S |f(r\zeta)|^p d\sigma(\zeta)\right)^{1/p} = O((1-r^2)^{-\alpha}) \quad \text{as } r \to 1. \tag{2.3}$$

Since the α-Bloch space consists of functions with a property stronger than (2.2), it may be interesting to characterize the space using the spherical integral like to (2.3). To this end, we shall consider compositions with Möbius transformations. We shall also characterize the invariant harmonic BMO space in terms of boundedness of the p-th spherical integral of compositions with Möbius transformations and in terms of the BMO property with respect to "the invariant measure" on B. As corollaries, we shall obtain characterizations similar to (2.1) for the little α-Bloch space and the BMO space.

Throughout this chapter, we consider real valued invariant harmonic functions on B.

2.2 Characterizations of the α-Bloch space

Let $\alpha \in \mathbb{R}$. *The invariant harmonic α-Bloch space,* written $\mathcal{B}_\alpha$, is defined as the collection of all (real valued) invariant harmonic functions f on B for which

$$\|f\|_{\mathcal{B}_\alpha} := \sup_{z \in B} (1-|z|^2)^\alpha |\widetilde{\nabla} f(z)| < \infty.$$

We recall that $\varphi_a(z)$ is the Möbius transformation of B. Let $E(a,r) = \{z \in B : |\varphi_a(z)| < r\}$.

Our characterizations for the invariant harmonic α-Bloch space are as follows.

Theorem 2.2.1. *The following statements hold.*

(i) *If $\alpha < -1$, then $\mathcal{B}_\alpha$ consists only of constant functions.*

(ii) *Let $1 \le p < \infty$ and set*

$$\rho_{\alpha,p}(a,r) = \begin{cases} (1-|a|^2)^\alpha & \text{if } -n < \alpha p < 0, \\ (1-|a|^2)^\alpha (1-r)^{-\alpha - n/p} & \text{if } \alpha p < -n, \\ (1-|a|^2)^\alpha \left(\log \dfrac{1}{1-r}\right)^{-1} & \text{if } \alpha p = -n \text{ or } \alpha = 0, \\ (1-|a|^2)^\alpha (1-r)^\alpha & \text{if } \alpha > 0. \end{cases}$$

The following properties for an invariant harmonic function f on B are equivalent:

(a) $f \in \mathcal{B}_\alpha$;

(b) $H_{\alpha,p}(f) := \sup\limits_{\substack{0<r<1 \\ a \in B}} \rho_{\alpha,p}(a,r) \left(\int_S |f \circ \varphi_a(r\zeta) - f(a)|^p d\sigma(\zeta) \right)^{1/p} < \infty$;

(c) $I_{\alpha,p}(f) := \sup\limits_{\substack{0<r<1 \\ a \in B}} \rho_{\alpha,p}(a,r) \left(\dfrac{1}{\lambda(E(a,r))} \int_{E(a,r)} |f(z) - f(a)|^p d\lambda(z) \right)^{1/p} < \infty$;

(d) there exists $0 < r_0 < 1$ such that

$$J_{\alpha,p}(f) := \sup_{a \in B} (1-|a|^2)^\alpha \left(\int_{E(a,r_0)} |f(z) - f(a)|^p d\lambda(z) \right)^{1/p} < \infty.$$

Moreover, the quantities $\|f\|_{\mathcal{B}_\alpha}$, $H_{\alpha,p}(f)$, $I_{\alpha,p}(f)$ and $J_{\alpha,p}(f)$ are comparable to each other with a constant depending only on p, α, r_0 and the dimension n.

Corollary 2.2.2. *Let $1 \le p < \infty$ and let f be an invariant harmonic function on B. If there exist $0 < r_0 < 1$ and $p < \beta < \infty$ such that*

$$\int_{E(a,r_0)} |f(z) - f(a)|^p d\lambda(z) = O((1-|a|^2)^\beta) \quad \text{as } |a| \to 1,$$

then f is constant.

For $z \in B$ and $0 < r < 1$, let

$$g(r,z) = \frac{n+1}{2n} \int_{|z|}^{r} \frac{(1-t^2)^{n-1}}{t^{2n-1}} dt,$$

and let $g(z) = g(1,z)$ for simplicity. The Green function for $\widetilde{\Delta}$ is defined by

$$G(z,w) = g(\varphi_w(z)) \quad \text{for } z, w \in B.$$

As another consequence of Theorem 2.2.1, we obtain a characterization similar to (2.1) for the little α-Bloch space.

Corollary 2.2.3. *Let $-1 \le \alpha < 0$, $1 < p < -n/\alpha$ and let f be an invariant harmonic function on B. The following properties are equivalent:*

(i) $f \in \mathcal{B}_\alpha$;

(ii) $\sup_{a \in B}(1-|a|^2)^{\alpha p} \int_B G(z,a)|\widetilde{\nabla} f(z)|^2 |f(z) - f(a)|^{p-2} d\lambda(z) < \infty.$

The proofs of Theorem 2.2.1 and Corollary 2.2.2 will be given in Section 2.4. We shall prove Corollary 2.2.3 in Section 2.6.

2.3 Characterizations of the BMO space

For $0 < p < \infty$, *the invariant harmonic BMO space*, written BMOH_p, is defined as the collection of every invariant harmonic function f on B which is represented as the Poisson-Szegö integral of a function of bounded mean oscillation on S. That is, each element f in BMOH_p is of the form

$$f(z) = \int_S \mathcal{P}(z,\zeta) f^*(\zeta) d\sigma(\zeta)$$

with a corresponding function f^* integrable on S for which

$$\|f^*\|_{\mathrm{BMO}_p(\sigma)} := \sup_{\substack{0<r\le\sqrt{2} \\ \xi \in S}} \left(\frac{1}{\sigma(Q(\xi,r))} \int_{Q(\xi,r)} |f^*(\zeta) - f^*_{\xi,r}|^p d\sigma(\zeta) \right)^{1/p} < \infty,$$

where $f^*_{\xi,r} = \sigma(Q(\xi,r))^{-1} \int_{Q(\xi,r)} f^* d\sigma$, the average of f^* over $Q(\xi,r)$. Here $Q(\xi,r) = \{\zeta \in S : |1 - \langle \zeta, \xi \rangle|^{1/2} < r\}$, the non-isotropic ball of center ξ and radius r.

Our characterizations for the invariant harmonic BMO space are as follows.

Theorem 2.3.1. *Let $1 < p < \infty$ and let f be an invariant harmonic function on B. The following properties are equivalent:*

(i) $f \in \mathrm{BMOH}_p$;

(ii) $\|f\|_{S_p} := \sup\limits_{\substack{0<r<1\\ a\in B}} \left(\int_S |f \circ \varphi_a(r\zeta) - f(a)|^p d\sigma(\zeta) \right)^{1/p} < \infty$;

(iii) $\|f\|_{\mathrm{BMO}_p(\lambda)} := \sup\limits_{\substack{0<r<1\\ a\in B}} \left(\frac{1}{\lambda(E(a,r))} \int_{E(a,r)} |f(z) - f(a)|^p d\lambda(z) \right)^{1/p} < \infty.$

Moreover, the quantities $\|f^*\|_{\mathrm{BMO}_p(\sigma)}$, $\|f\|_{S_p}$ and $\|f\|_{\mathrm{BMO}_p(\lambda)}$ are comparable to each other with a constant depending only on p and the dimension n.

The interesting points of the above characterization of BMOH_p are that a solution of the Dirichlet problem for $\widetilde{\Delta}$ with boundary data of bounded mean oscillation also has bounded mean oscillation with respect to the invariant measure λ on B, and that conversely an invariant harmonic function on B of bounded mean oscillation with respect to λ can be represented as the Poisson-Szegö integral of a function of bounded mean oscillation on S.

Remark 2.3.2. If f is the Poisson-Szegö integral of an integrable function on S, then Theorem 2.3.1 holds for $1 \le p < \infty$. Furthermore, if f is the Poisson-Szegö integral and holomorphic on B, then Theorem 2.3.1 holds for $0 < p < \infty$. We note, in this case, that the equivalence of (i) and (ii) was proved by Ouyang, Yang and Zhao [16].

As a consequence of Theorem 2.3.1, we obtain a characterization similar to (2.1) for the BMO space.

Corollary 2.3.3. *Let $1 < p < \infty$ and let f be an invariant harmonic function on B. The following properties are equivalent:*

(i) $f \in \mathrm{BMOH}_p$;

(ii) $\sup\limits_{a\in B} \int_B G(z,a) |\widetilde{\nabla} f(z)|^2 |f(z) - f(a)|^{p-2} d\lambda(z) < \infty.$

The proofs of Theorem 2.3.1 and Corollary 2.3.3 will be given in Section 2.5 and Section 2.6, respectively.

2.4 Proof of Theorem 2.2.1

For a real valued C^1 function f on B and $j = 1, \cdots, n$, we let

$$X_j f(z) = \frac{\partial f}{\partial z_j}(z) - \overline{z}_j \sum_{k=1}^{n} \overline{z}_k \frac{\partial f}{\partial \overline{z}_k}(z).$$

Then we observe from [22, Lemma 10.5] that for $z \in B$,

$$|\widetilde{\nabla} f(z)|^2 \leq \frac{4}{n+1} \sum_{j=1}^{n} |X_j f(z)|^2 \leq \frac{(1+|z|^2)^2}{(1-|z|^2)^2} |\widetilde{\nabla} f(z)|^2, \tag{2.4}$$

and from [22, Proposition 10.4] that if f is invariant harmonic on B, then $X_j f$ is so.

Proof of Theorem 2.2.1 (i). Let $\alpha < -1$ and $f \in \mathcal{B}_\alpha$. Then, for each $j = 1, \cdots, n$, it follows from (2.4) that

$$|X_j f(z)| \leq A \frac{|\widetilde{\nabla} f(z)|}{1-|z|^2} \leq A \|f\|_{\mathcal{B}_\alpha} (1-|z|^2)^{-\alpha-1}.$$

Since the right hand side tends to zero as $|z| \to 1-$, the maximum principle yields that $X_j f \equiv 0$ for every $j = 1, \cdots, n$, and so $|\widetilde{\nabla} f| \equiv 0$ by (2.4). Hence f is constant. $\square$

In the proof of Theorem 2.2.1 (ii), we use the following known lemmas.

Lemma 2.4.1 ([22, Lemma 10.8**]).** *Let f be a real valued C^1 function on B and $a \in B$. Then for each $\zeta \in S$ and $0 < r < 1$, we have*

$$|f \circ \varphi_a(r\zeta) - f(a)| \leq \sqrt{n+1} \int_0^r \frac{|\widetilde{\nabla} f(\varphi_a(t\zeta))|}{1-t^2}\, dt.$$

Lemma 2.4.2 ([22, Proposition 8.18**]).** *Let $\beta \in \mathbb{R}$. Then there exists a positive constant A depending only on the dimension n such that for $z \in B$,*

$$\int_S \frac{1}{|1-\langle z, \zeta \rangle|^{n+\beta}} d\sigma(\zeta) \leq \begin{cases} A(1-|z|^2)^{-\beta} & \text{if } \beta > 0, \\ A \log \dfrac{1}{1-|z|^2} & \text{if } \beta = 0, \\ A & \text{if } \beta < 0. \end{cases}$$

Lemma 2.4.3 ([22, Proposition 10.1 and 10.2**]).** *Let $0 < p < \infty$ and let f be an invariant harmonic function on B. Then for $a \in B$ and $0 < r < 1$, we have*

$$|f(a)|^p \leq A(n,p,r) \int_{E(a,r)} |f(z)|^p d\lambda(z),$$

and

$$|\widetilde{\nabla} f(a)|^p \leq A(n,p,r) \int_{E(a,r)} |f(z)|^p d\lambda(z).$$

Let us prove Theorem 2.2.1 (ii).

Proof of Theorem 2.2.1 (ii). We first show that (a) implies (b). Suppose $f \in \mathcal{B}_\alpha$, and let $a \in B$, $\zeta \in S$ and $0 < r < 1$. Since

$$1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z, a \rangle|^2},$$

we have by Lemma 2.4.1

$$|f\circ\varphi_a(r\zeta)-f(a)|\le A\int_0^r\frac{|\widetilde{\nabla}f(\varphi_a(t\zeta))|}{1-t^2}dt\le A\|f\|_{\mathcal{B}_\alpha}\int_0^r\frac{(1-|\varphi_a(t\zeta)|^2)^{-\alpha}}{1-t^2}dt$$
$$=A\|f\|_{\mathcal{B}_\alpha}(1-|a|^2)^{-\alpha}\int_0^r\frac{|1-\langle ta,\zeta\rangle|^{2\alpha}}{(1-t^2)^{\alpha+1}}dt.$$

Hence it follows from Minkowski's integral inequality that

$$\begin{aligned}&\left(\int_S|f\circ\varphi_a(r\zeta)-f(a)|^pd\sigma(\zeta)\right)^{1/p}\\ &\le A\|f\|_{\mathcal{B}_\alpha}(1-|a|^2)^{-\alpha}\left(\int_S\left(\int_0^r\frac{|1-\langle ta,\zeta\rangle|^{2\alpha}}{(1-t^2)^{\alpha+1}}dt\right)^pd\sigma(\zeta)\right)^{1/p}\\ &\le A\|f\|_{\mathcal{B}_\alpha}(1-|a|^2)^{-\alpha}\int_0^r(1-t^2)^{-\alpha-1}\left(\int_S|1-\langle ta,\zeta\rangle|^{2\alpha p}d\sigma(\zeta)\right)^{1/p}dt.\end{aligned}\tag{2.5}$$

Using Lemma 2.4.2, we now calculate the integral

$$F(a,r):=\int_0^r(1-t^2)^{-\alpha-1}\left(\int_S|1-\langle ta,\zeta\rangle|^{2\alpha p}d\sigma(\zeta)\right)^{1/p}dt.$$

If $-n<\alpha p<-n/2$, then

$$F(a,r)\le A\int_0^1(1-t^2)^{-\alpha-1}(1-t^2)^{(n+2\alpha p)/p}dt=A\int_0^1(1-t^2)^{\alpha-1+n/p}dt<\infty.$$

If $\alpha p=-n/2$, then

$$F(a,r)\le A\int_0^1(1-t^2)^{-\alpha-1}\left(\log\frac{1}{1-t^2}\right)^{1/p}dt<\infty.$$

If $-n/2<\alpha p$, then

$$F(a,r)\le A\int_0^r(1-t^2)^{-\alpha-1}dt\le\begin{cases}A & \text{if } -n/2<\alpha p<0,\\ A\log\dfrac{1}{1-r} & \text{if } \alpha=0,\\ A(1-r)^{-\alpha} & \text{if } \alpha>0.\end{cases}$$

Hence it follows from (2.5) that for $a\in B$ and $0<r<1$,

$$\left(\int_S|f\circ\varphi_a(r\zeta)-f(a)|^pd\sigma(\zeta)\right)^{1/p}\le\begin{cases}A\|f\|_{\mathcal{B}_\alpha}(1-|a|^2)^{-\alpha} & \text{if } -n<\alpha p<0,\\ A\|f\|_{\mathcal{B}_\alpha}(1-|a|^2)^{-\alpha}\log\dfrac{1}{1-r} & \text{if } \alpha=0,\\ A\|f\|_{\mathcal{B}_\alpha}(1-|a|^2)^{-\alpha}(1-r)^{-\alpha} & \text{if } \alpha>0.\end{cases}$$

If $\alpha p=-n$, then

$$F(a,r)\le A\int_0^r(1-t^2)^{-1}dt\le A\log\frac{1}{1-r},$$

so that for $a \in B$ and $0 < r < 1$,

$$\left(\int_S |f \circ \varphi_a(r\zeta) - f(a)|^p d\sigma(\zeta)\right)^{1/p} \leq A\|f\|_{\mathcal{B}_\alpha}(1-|a|^2)^{-\alpha} \log \frac{1}{1-r}.$$

If $\alpha p < -n$, then

$$F(a,r) \leq A \int_0^r (1-t^2)^{\alpha-1+n/p} dt \leq A(1-r)^{\alpha+n/p},$$

so that for $a \in B$ and $0 < r < 1$,

$$\left(\int_S |f \circ \varphi_a(r\zeta) - f(a)|^p d\sigma(\zeta)\right)^{1/p} \leq A\|f\|_{\mathcal{B}_\alpha}(1-|a|^2)^{-\alpha}(1-r)^{\alpha+n/p}.$$

Hence, taking the supremum over $0 < r < 1$ and $a \in B$, we obtain (b).

We next show that (b) implies (c). Let $a \in B$ and $0 < r < 1$. Since $\rho_{\alpha,p}(a,r)$ is positive and non-increasing function of r, we have by integration in polar coordinates

$$\begin{aligned}\int_{E(a,r)} |f(z) - f(a)|^p d\lambda(z) &= 2n \int_0^r \frac{t^{2n-1}}{(1-t^2)^{n+1}} \int_S |f \circ \varphi_a(t\zeta) - f(a)|^p d\sigma(\zeta) dt \\ &\leq H_{\alpha,p}(f)^p \rho_{\alpha,p}(a,r)^{-p} \lambda(E(a,r)),\end{aligned}$$

and (c) follows.

We easily show that (c) implies (d). In fact, for any $a \in B$ and $0 < r_0 < 1$, we have

$$(1-|a|^2)^\alpha \left(\int_{E(a,r_0)} |f(z) - f(a)|^p d\lambda(z)\right)^{1/p} \leq A(n,p,\alpha,r_0) I_{\alpha,p}(f).$$

We finally show that (d) implies (a). Let $a \in B$. Then it follows from Lemma 2.4.3 with $r := r_0$ and $f := f - f(a)$ that

$$|\widetilde{\nabla} f(a)|^p \leq A(n,p,r_0) \int_{E(a,r_0)} |f(z) - f(a)|^p d\lambda(z) \leq A J_{\alpha,p}(f)^p (1-|a|^2)^{-\alpha p},$$

and so $f \in \mathcal{B}_\alpha$. Thus the proof of Theorem 2.2.1 is complete. $\square$

2.5 Proof of Theorem 2.3.1

We recall the Poisson-Szegö kernel of B:

$$\mathcal{P}(z,\zeta) = \frac{(1-|z|^2)^n}{|1-\langle z,\zeta\rangle|^{2n}}.$$

The following change of variables formula is found in [18, Remark in page 44]:

$$\int_S \mathcal{P}(z,\zeta) f(\zeta) d\sigma(\zeta) = \int_S f(\varphi_z(\zeta)) d\sigma(\zeta). \tag{2.6}$$

To prove Theorem 2.3.1, we need the following characterization in terms of the Garsia norm.

Lemma 2.5.1. *Let $1 \leq p < \infty$ and let f be the Poisson-Szegö integral of an integrable function f^* on S. The following properties are equivalent:*

(i) $f \in \mathrm{BMOH}_p$*;*

(ii) $\|f\|_{G_p} := \sup_{a\in B}\left(\int_S \mathcal{P}(a,\zeta)|f^*(\zeta)-f(a)|^p d\sigma(\zeta)\right)^{1/p} < \infty.$

Moreover, the quantities $\|f^*\|_{\mathrm{BMO}_p(\sigma)}$ and $\|f\|_{G_p}$ are comparable with a constant depending only on p and the dimension n.

Proof. The lemma will be proved in the same way as in [5, pp. 224–225]. For completeness we give a proof. Let us show first that (i) implies (ii). Let $a \in B$ be fixed, and put $\xi = a/|a|$ and $\rho = (1-|a|)^{1/2}$. As in the proof of Lemma 1.3.1, we split S into

$$V_0 := Q(\xi,\rho) \quad \text{and} \quad V_j := Q(\xi, 2^j\rho) \setminus Q(\xi, 2^{j-1}\rho) \quad (j = 1, \cdots, N),$$

where N is the smallest integer such that $2^N\rho \geq \sqrt{2}$. Then we have

$$\begin{aligned} &|1-\langle a,\zeta\rangle| \geq 1-|a| = \rho^2 \quad \text{for } \zeta \in S, \\ &|1-\langle a,\zeta\rangle|^{1/2} \geq 2^{j-2}\rho \quad \text{for } \zeta \in V_j \text{ with } j \geq 1, \end{aligned}$$

and thus $\mathcal{P}(a,\zeta) \leq A2^{-4nj}\rho^{-2n}$ for $\zeta \in V_j$ with $j \geq 0$. Therefore we have by (1.4) that

$$\begin{aligned} \int_S \mathcal{P}(a,\zeta)|f^*(\zeta)-f^*_{\xi,\rho}|^p d\sigma(\zeta) &= \sum_{j=0}^N \int_{V_j} \mathcal{P}(a,\zeta)|f^*(\zeta)-f^*_{\xi,\rho}|^p d\sigma(\zeta) \\ &\leq \frac{A}{\rho^{2n}}\sum_{j=0}^N 2^{-4nj}\int_{Q(\xi,2^j\rho)} |f^*(\zeta)-f^*_{\xi,\rho}|^p d\sigma(\zeta) \qquad (2.7) \\ &\leq AA_0\|f^*\|^p_{\mathrm{BMO}_p(\sigma)} + 2^pAA_0\sum_{j=1}^N 2^{-2nj}\left(\|f^*\|^p_{\mathrm{BMO}_p(\sigma)} + |f^*_{\xi,2^j\rho}-f^*_{\xi,\rho}|^p\right). \end{aligned}$$

Since

$$\begin{aligned} |f^*_{\xi,2^k\rho}-f^*_{\xi,2^{k-1}\rho}|^p &\leq \frac{1}{\sigma(Q(\xi,2^{k-1}\rho))}\int_{Q(\xi,2^{k-1}\rho)} |f^*(\zeta)-f^*_{\xi,2^k\rho}|^p d\sigma(\zeta) \\ &\leq \frac{2^{2n}A_0^2}{\sigma(Q(\xi,2^k\rho))}\int_{Q(\xi,2^k\rho)} |f^*(\zeta)-f^*_{\xi,2^k\rho}|^p d\sigma(\zeta) \leq 2^{2n}A_0^2\|f^*\|^p_{\mathrm{BMO}_p(\sigma)} \end{aligned}$$

by Jensen's inequality and (1.4), it follows that

$$|f^*_{\xi,2^j\rho}-f^*_{\xi,\rho}|^p \leq j^p\sum_{k=1}^j |f^*_{\xi,2^k\rho}-f^*_{\xi,2^{k-1}\rho}|^p \leq 2^{2n}A_0^2 j^{p+1}\|f^*\|^p_{\mathrm{BMO}_p(\sigma)}.$$

Hence we obtain from (2.7) that

$$\begin{aligned} \int_S \mathcal{P}(a,\zeta)|f^*(\zeta)-f(a)|^p d\sigma(\zeta) &\leq 2^{p+1}\int_S \mathcal{P}(a,\zeta)|f^*(\zeta)-f^*_{\xi,\rho}|^p d\sigma(\zeta) \\ &\leq A(p,n)\|f^*\|^p_{\mathrm{BMO}_p(\sigma)}, \end{aligned}$$

and thus (ii) follows.

We next show that (ii) implies (i). Let $\xi \in S$ and $0 < r \leq \sqrt{2}$, and put $z_{\xi,r} = (1-5^{-1}r^2)\xi$. Since $1 - |z_{\xi,r}|^2 > 5^{-1}r^2$ and

$$|1 - \langle z_{\xi,r}, \zeta\rangle|^{1/2} \leq |1 - \langle z_{\xi,r}, \xi\rangle|^{1/2} + |1 - \langle \xi, \zeta\rangle|^{1/2} < 2r \quad \text{for } \zeta \in Q(\xi, r),$$

we have $\mathcal{P}(z_{\xi,r}, \zeta) \geq 2^{-7n}r^{-2n}$ for $\zeta \in Q(\xi, r)$. Therefore it follows from (1.4) and (2.6) that

$$\begin{aligned}\frac{1}{\sigma(Q(\xi,r))}\int_{Q(\xi,r)} |f^*(\zeta) - f(z_{\xi,r})|^p d\sigma(\zeta) &\leq \frac{A_0}{r^{2n}}\int_{Q(\xi,r)} |f^*(\zeta) - f(z_{\xi,r})|^p d\sigma(\zeta) \\ &\leq 2^{7n}A_0 \int_S \mathcal{P}(z_{\xi,r}, \zeta)|f^*(\zeta) - f(z_{\xi,r})|^p d\sigma(\zeta) \\ &\leq 2^{7n}A_0\|f\|_{G_p}^p.\end{aligned}$$

Hence we obtain $\|f^*\|_{\mathrm{BMO}_p(\sigma)} \leq A(n,p)\|f\|_{G_p}$, and so (i) follows. $\square$

Let us prove Theorem 2.3.1.

Proof of Theorem 2.3.1. We first show that (ii) implies (i). We observe, taking $a = 0$, that $f \in \mathcal{H}^p$, the p-th Hardy space, so that f can be represented as the Poisson-Szegö integral of a p-th integrable function f^* on S. Thus it suffices to show that $\|f^*\|_{\mathrm{BMO}_p(\sigma)} < \infty$. Let $a \in B$ be fixed. By (2.6) we have

$$\int_S \mathcal{P}(a, \zeta)|f^*(\zeta) - f(a)|^p d\sigma(\zeta) = \int_S |f^* \circ \varphi_a(\zeta) - f(a)|^p d\sigma(\zeta).$$

We observe that for almost every point ζ of S,

$$\lim_{r \to 1-} f \circ \varphi_a(r\zeta) - f(a) = f^* \circ \varphi_a(\zeta) - f(a).$$

Indeed, this follows from Korányi's theorem in Section 1.1, since the inequality

$$|1 - \langle \varphi_a(r\zeta), \varphi_a(\zeta)\rangle| = \frac{(1-|a|^2)(1-r)}{|1 - \langle r\zeta, a\rangle||1 - \langle a, \zeta\rangle|} \leq \frac{1}{1-|a|}(1 - |\varphi_a(r\zeta)|^2)$$

implies that $\{\varphi_a(r\zeta) : 0 < r < 1\}$ is contained in the Korányi approach region at $\varphi_a(\zeta)$. Since the function $\int_S |f \circ \varphi_a(r\zeta) - f(a)|^p d\sigma(\zeta)$ is non-decreasing for $0 < r < 1$, we obtain $\|f\|_{G_p} \leq \|f\|_{S_p}$. Hence $f \in \mathrm{BMOH}_p$ by Lemma 2.5.1.

We next show that (iii) implies (ii). Let $a \in B$ be fixed. By the monotonicity of the spherical integral, it is enough to show that

$$\int_S |f \circ \varphi_a(r\zeta) - f(a)|^p d\sigma(\zeta) \leq A\|f\|_{\mathrm{BMO}_p(\lambda)}^p \quad \text{for } \tfrac{1}{2} < r < 1,$$

where A is a constant independent of a and r. Since

$$\lambda(E(a, \tfrac{1+r}{2})) = \frac{(1+r)^{2n}}{(3+r)^n(1-r)^n},$$

it follows from integration in polar coordinates that for $1/2 < r < 1$,

$$\begin{aligned}
\int_S |f\circ\varphi_a(r\zeta) - f(a)|^p d\sigma(\zeta) &\le \frac{2}{1-r}\int_r^{\frac{1+r}{2}}\int_S |f\circ\varphi_a(t\zeta) - f(a)|^p d\sigma(\zeta)dt \\
&\le \frac{2}{1-r}\frac{(1-r^2)^{n+1}}{r^{2n-1}}\int_r^{\frac{1+r}{2}}\frac{t^{2n-1}}{(1-t^2)^{n+1}}\int_S |f\circ\varphi_a(t\zeta) - f(a)|^p d\sigma(\zeta)dt \\
&\le \frac{1}{n}\frac{1}{1-r}\frac{(1-r^2)^{n+1}}{r^{2n-1}}\int_{B(0,\frac{1+r}{2})} |f\circ\varphi_a(z) - f(a)|^p d\lambda(z) \\
&= \frac{1}{n}\frac{(1+r)^{3n+1}}{r^{2n-1}(3+r)^n}\frac{1}{\lambda(E(a,\frac{1+r}{2}))}\int_{E(a,\frac{1+r}{2})} |f(z) - f(a)|^p d\lambda(z) \\
&\le \frac{2^{4n}}{n}\|f\|^p_{\mathrm{BMO}_p(\lambda)}.
\end{aligned}$$

Hence we obtain $\|f\|_{S_p} \le 2^{4n}n^{-1}\|f\|_{\mathrm{BMO}_p(\lambda)}$, and thus (ii) follows.

We finally show that (i) implies (iii). We assume that

$$f(z) = \int_S \mathcal{P}(z,\zeta)f^*(\zeta)d\sigma(\zeta)$$

with $\|f^*\|_{\mathrm{BMO}_p(\sigma)} < \infty$. Let $a \in B$ and $0 < r < 1$ be fixed. We put $\xi = a/|a|$ and $\rho = 1 - |a|$. Since $\mathcal{P}(z,\cdot)d\sigma$ is a probability measure on S, we have by Jensen's inequality, Fubini's theorem and the mean value property

$$\begin{aligned}
\int_{E(a,r)} |f(z) - f(a)|^p d\lambda(z) &= \int_{E(a,r)}\left|\int_S \mathcal{P}(z,\zeta)[f^*(\zeta) - f(a)]d\sigma(\zeta)\right|^p d\lambda(z) \\
&\le \int_{E(a,r)}\int_S \mathcal{P}(z,\zeta)|f^*(\zeta) - f(a)|^p d\sigma(\zeta)d\lambda(z) \\
&= \int_S\left(\int_{E(a,r)}\mathcal{P}(z,\zeta)d\lambda(z)\right)|f^*(\zeta) - f(a)|^p d\sigma(\zeta) \\
&= \lambda(E(a,r))\int_S \mathcal{P}(a,\zeta)|f^*(\zeta) - f(a)|^p d\sigma(\zeta).
\end{aligned}$$

Hence it follows from Lemma 2.5.1 that $\|f\|_{\mathrm{BMO}_p(\lambda)} \le A(n,p)\|f^*\|_{\mathrm{BMO}_p(\sigma)}$, and so (i) follows. Thus Theorem 2.3.1 is proved. □

2.6 Proofs of Corollaries 2.2.2, 2.2.3 and 2.3.3

Proof of Corollary 2.2.2. It follows form Lemma 2.4.3 that

$$|\widetilde{\nabla} f(a)|^p \le A\int_{E(a,r_0)} |f(z) - f(a)|^p d\lambda(z) \le A(1-|a|^2)^\beta,$$

where A is a constant depending only on p, r_0 and the dimension n. Hence we have $f \in \mathcal{B}_{-\beta/p}$, and so f is constant by Theorem 2.2.1 (i). □

Corollaries 2.2.3 and 2.3.3 follow from the following lemma by Kwon [12, Lemma 3.5].

Lemma 2.6.1. *If $1 < p < \infty$ and f is an invariant harmonic function on B, then we have for $0 < r < 1$,*

$$\int_S |f(r\zeta)|^p d\sigma(\zeta) - |f(0)|^p = p(p-1)\int_{rB} g(r,z)|\widetilde{\nabla} f(z)|^2 |f(z)|^{p-2} d\lambda(z). \tag{2.8}$$

Letting $r \to 1-$ in (2.8), it follows from the monotone convergence that

$$\lim_{r\to 1-}\int_S |f(r\zeta)|^p d\sigma(\zeta) - |f(0)|^p = p(p-1)\int_{B} g(z)|\widetilde{\nabla} f(z)|^2 |f(z)|^{p-2} d\lambda(z). \tag{2.9}$$

Proof of Corollary 2.2.3. Multiplying the both sides of (2.9) with $f := f\circ\varphi_a - f(a)$ by $(1-|a|^2)^{\alpha p}$ and taking the supremum over $a \in B$, we obtain Corollary 2.2.3 from Theorem 2.2.1 and the invariance of λ under $\mathrm{Aut}(B)$. $\square$

Proof of Corollary 2.3.3. Let $a \in B$ and apply (2.9) to $f := f\circ\varphi_a - f(a)$. Then, by the change of variable, we have

$$\begin{aligned}\sup_{0<r<1}&\int_S |f\circ\varphi_a(r\zeta) - f(a)|^p d\sigma(\zeta)\\ &= p(p-1)\int_B G(z,a)|\widetilde{\nabla} f(z)|^2 |f(z)-f(a)|^{p-2} d\lambda(z).\end{aligned}$$

Hence, taking the supremum over $a \in B$, we obtain Corollary 2.3.3. $\square$

2.7 Inclusion relationships

Theorem 2.7.1. *Let $2n < p < \infty$. The following statements hold.*

(i) *If $\alpha > 0$, then $\mathcal{D}^p_\alpha \subset \mathcal{B}_{\alpha/p}$.*

(ii) *If $\alpha = 0$, then $\mathcal{D}^p_0 \subset \mathrm{BMOH}_p$. Moreover, this inclusion is strict.*

Proof. Let $f \in \mathcal{D}^p_\alpha$, where $\alpha \geq 0$. Fixing $a \in B$ and $0 < r < 1$, we have by Lemma 2.4.1 and Hölder's inequality

$$\begin{aligned}|f\circ\varphi_a(r\zeta) - f(a)| &\leq A\int_0^r \frac{|\widetilde{\nabla} f(\varphi_a(t\zeta))|}{1-t^2}dt\\ &\leq AA_5\left(\int_0^r t^{2n-1}(1-t^2)^{\alpha-n-1}|\widetilde{\nabla} f(\varphi_a(t\zeta))|^p dt\right)^{1/p},\end{aligned}$$

where

$$A_5 = \left(\int_0^r t^{-\frac{2n-1}{p-1}}(1-t^2)^{-1+\frac{n-\alpha}{p-1}}dt\right)^{(p-1)/p} < \infty.$$

Since

$$1 - |\varphi_a(t\zeta)|^2 = \frac{(1-|a|^2)(1-t^2)}{|1-\langle t\zeta, a\rangle|^2} \geq \frac{1-|a|}{2}(1-t^2),$$

it follows from the invariance of $\widetilde{\nabla}$ and λ under $\mathrm{Aut}(B)$ that

$$\begin{aligned}
&\int_S |f\circ\varphi_a(r\zeta)-f(a)|^p d\sigma(\zeta)\\
&\le A(1-|a|)^{-\alpha}\int_S\int_0^r t^{2n-1}(1-t^2)^{-n-1}(1-|\varphi_a(t\zeta)|^2)^\alpha|\widetilde{\nabla}f(\varphi_a(t\zeta))|^p dtd\sigma(\zeta)\\
&\le A(1-|a|)^{-\alpha}\int_B (1-|z|^2)^\alpha|\widetilde{\nabla}f(z)|^p d\lambda(z).
\end{aligned}$$

Therefore we obtain

$$\sup_{\substack{0<r<1\\ a\in B}}(1-|a|)^{\alpha/p}\left(\int_S |f\circ\varphi_a(r\zeta)-f(a)|^p d\sigma(\zeta)\right)^{1/p}\le A\|f\|_{\mathcal{D}^p_\alpha}.$$

Hence we conclude from Theorem 2.2.1 that $\mathcal{D}^p_\alpha\subset\mathcal{B}_{\alpha/p}$ if $\alpha>0$, and from Theorem 2.3.1 that $\mathcal{D}^p_0\subset\mathrm{BMOH}_p$.

The strictness of the inclusion between $\mathcal{D}^p_0$ and BMOH_p follows from results for the boundary behavior. Indeed, we see from [7] that invariant harmonic functions in $\mathcal{D}^p_0$ have tangential limits at almost every point of S. However, we know from Theorem 1.2.1 that there exists a bounded invariant harmonic function on B which fails to have tangential limits at every point of S. Thus the inclusion is strict. $\square$

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Part II

Martin kernels of general domains in $\mathbb{R}^n$

Introduction

This introduction includes consistent notations and terminologies employed in Part II. In Chapters 3, 4 and 5, we will discuss potential theory on domains in $\mathbb{R}^n$ with $n \geq 2$. More precisely, we will study minimal Martin boundary points of a John domain, the boundary behavior of quotients of Martin kernels, comparison estimates for the Green function and the Martin kernel.

We consider the (usual) *Laplace operator* on $\mathbb{R}^n$:

$$\Delta := \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$

Let Ω be a domain in $\mathbb{R}^n$. A real valued function h on Ω is said to be *harmonic* if $h \in C^2(\Omega)$ and

$$\Delta h = 0 \quad \text{in } \Omega.$$

We say that a function $f : \Omega \to [-\infty, +\infty)$ is *subharmonic* if f is upper semicontinuous on Ω, not identically $-\infty$, and satisfies that for every open ball $B(x,r)$ contained in Ω,

$$f(x) \leq \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)dy,$$

where $|E|$ denotes the Lebesgue measure of a measurable set E. A function $f : \Omega \to (-\infty, +\infty]$ is called *superharmonic* if $-f$ is subharmonic on Ω. It is known that if a superharmonic function f on Ω has a subharmonic minorant on Ω, then the greatest subharmonic minorant of f on Ω exists and is harmonic.

To define the Green function, we recall the fundamental function: for $x, y \in \mathbb{R}^n$,

$$U_y(x) = \begin{cases} -\log|x-y| & (x \neq y,\ n = 2) \\ |x-y|^{2-n} & (x \neq y,\ n \geq 3) \\ +\infty & (x = y). \end{cases}$$

It is easy to check that if y is fixed, then U_y is superharmonic on $\mathbb{R}^n$ and harmonic on $\mathbb{R}^n \setminus \{y\}$. A domain Ω in $\mathbb{R}^n$ is said to be Greenian if, for each $y \in \Omega$, the function U_y has a subharmonic minorant on Ω. We note that if Ω is a Greenian domain, then for each $y \in \Omega$, the function U_y has the greatest harmonic minorant h_y on Ω. The function $G_\Omega : \Omega \times \Omega \to [0, +\infty]$, defined by

$$G_\Omega(x,y) = U_y(x) - h_y(x),$$

is called *the Green function* of Ω (for the Laplace operator). For example, the Green function of $B(z,r)$ is explicitly given as follows: if $n=2$, then

$$G_{B(z,r)}(x,y)=\begin{cases}\log\left(\dfrac{|y-z|}{r}\dfrac{|x-y^*|}{|x-y|}\right) & (x,y\in B(z,r),\ y\neq\{x,z\})\\ \log\left(\dfrac{r}{|x-y|}\right) & (x\in B(z,r)\setminus\{z\},\ y=z)\\ +\infty & (x=y);\end{cases}$$

if $n\geq 3$, then

$$G_{B(z,r)}(x,y)=\begin{cases}|x-y|^{2-n}-\left(\dfrac{|y-z||x-y^*|}{r}\right)^{2-n} & (x,y\in B(z,r),\ y\neq\{x,z\})\\ |x-y|^{2-n}-r^{2-n} & (x\in B(z,r)\setminus\{z\},\ y=z)\\ +\infty & (x=y),\end{cases}$$

where y^* denotes the inverse of a point $y\neq z$ with respect to the sphere $S(z,r)$: that is,

$$y^*=\left(\frac{r}{|y-z|}\right)^2(y-z)+z.$$

We now define the Martin kernel of a Greenian domain Ω, and then define the Martin boundary of Ω. Let $x_0\in\Omega$ be fixed. The function K_Ω, defined on $(\Omega\times\Omega)\setminus\{(x_0,x_0)\}$ by

$$K_\Omega(x,y)=\frac{G_\Omega(x,y)}{G_\Omega(x_0,y)},$$

is called *the Martin kernel* of Ω (relative to x_0). If $y=x_0$, then the above quotient is interpreted as 0. We define *the Martin metric* on $\Omega\times\Omega$ by

$$d(y,z)=\int_\Omega\min\{1,|K_\Omega(x,y)-K_\Omega(x,z)|\}g(x)dx,$$

where $g:\Omega\to(0,1]$ is an integrable function. We can regard Ω as $\{K_\Omega(\cdot,y):y\in\Omega\}$ since the mapping $y\mapsto K_\Omega(\cdot,y)$ is a homeomorphism between them. We then note that the Martin topology on Ω deduced from the metric d coincides with the Euclidean topology on Ω. Let $\{y_j\}$ be a sequence in Ω with no limit point in Ω. Then $\{K_\Omega(\cdot,y_j)\}_{j\geq j_0}$, with j_0 being sufficiently large, is a uniformly bounded sequence of positive harmonic functions on a relatively compact open subset of Ω. Hence the Harnack principle shows that there exists a subsequence $\{K_\Omega(\cdot,y_{j_k})\}$ converging to a positive harmonic function h on Ω, which implies that $\{K_\Omega(\cdot,y_{j_k})\}$ converges to h with respect to the Martin topology. We define $\Delta(\Omega)$ to be the collection of all harmonic functions on Ω that can be obtained as the limit of $\{K_\Omega(\cdot,y_j)\}$ for some sequence $\{y_j\}$ in Ω with no limit point in Ω. We can now extend the Martin metric d to $\Omega\cup\Delta(\Omega)$, and then see that $\Omega\cup\Delta(\Omega)$ is compact with respect to this metric, and that Ω is open and dense in $\Omega\cup\Delta(\Omega)$. Therefore the set $\Omega\cup\Delta(\Omega)$ is a metric compactification of Ω, and $\Delta(\Omega)$ is the boundary of Ω in this compactification. We call $\Omega\cup\Delta(\Omega)$ *the Martin* *compactification* of Ω, and $\Delta(\Omega)$ *the Martin boundary* of Ω. See [8, Chapter 8] for details. To unify the notation, we write $K_\Omega(\cdot,\xi)$ for $\xi \in \Delta(\Omega)$ when we regard ξ as a function.

We are interested in minimal Martin boundary points. We say that $\xi \in \Delta(\Omega)$ is *minimal* if every positive harmonic function on Ω less than or equal to the corresponding Martin kernel $K_\Omega(\cdot,\xi)$ coincides with a constant multiple of $K_\Omega(\cdot,\xi)$. We denote by $\Delta_1(\Omega)$ the collection of all minimal Martin boundary points in $\Delta(\Omega)$. Let $\xi \in \Delta_1(\Omega)$. If $K_\Omega(\cdot,\xi)$ is given as the limit of $\{K_\Omega(\cdot,y_j)\}$ for some sequence $\{y_j\}$ in Ω "converging to $y \in \partial\Omega$", then $K_\Omega(\cdot,\xi)$ and ξ are called *the minimal Martin kernel at* y and *the minimal Martin boundary point at* y, respectively.

The notion of minimal thinness was introduced by Naïm [31], using a regularized reduced function. Let u be a positive superharmonic function on Ω and let E be a subset of Ω. A reduced function of u relative to E on Ω is defined by

$$^{\Omega}R_u^E(x) = \inf\{v(x)\},$$

where the infimum is taken over all positive superharmonic functions v on Ω such that $v \geq u$ on E. By $^{\Omega}\widehat{R}_u^E$, we denote the lower semicontinuous regularization of $^{\Omega}R_u^E$. Observe that $^{\Omega}\widehat{R}_u^E \leq u$ in general. Let $\xi \in \Delta_1(\Delta)$. A set E is said to be *minimally thin* at ξ with respect to Ω if

$$^{\Omega}\widehat{R}^E_{K_\Omega(\cdot,\xi)}(z) < K_\Omega(z,\xi) \quad \text{for some } z \in \Omega.$$

In Chapter 3, we will discuss minimal Martin boundary points of a John domain. In particular, we will show that the number of minimal Martin boundary points at each Euclidean boundary point is estimated in terms of the John constant. For a class of John domains represented as the union of open convex sets, we will give a sufficient condition for the Martin boundary to be homeomorphic to the Euclidean boundary.

In Chapter 4, we investigate the boundary behavior of Martin kernels. Given two intersecting domains, we show the boundary behavior of the quotient of Martin kernels of each domain. To this end, we characterize the minimal thinness for a difference of two subdomains in terms of Martin kernels of each domain. As a consequence, we obtain the boundary growth of the Martin kernel on a Lipschitz domain, which corresponds to earlier results for the boundary decay of the Green function on a Lipschitz domain investigated by Burdzy, Carroll and Gardiner.

In Chapter 5, we will give comparison estimates for the product of the Green function and the Martin kernel in a uniform domain. These comparison estimates will be applied to show the equivalence of ordinary thinness and minimal thinness of a set contained in a non-tangential cone.

Throughout Part II, we use the symbol A to denote an absolute positive constant whose value is unimportant and may change from line to line. If necessary, we write $A(a,b,\cdots)$ for a constant depending on $a,b,\cdots$. For positive functions f_1 and f_2, we write $f_1 \approx f_2$ if there is a constant $A > 1$ such that $A^{-1}f_1 \leq f_2 \leq Af_1$.

Chapter 3

Minimal Martin boundary points of a John domain

This chapter consists of results obtained in a joint work with Hiroaki Aikawa and Torbjörn Lundh [AHL].

3.1 Historical survey and statements of results

In 1941, R. S. Martin [30] introduced an ideal boundary $\Delta(\Omega)$ of a Greenian domain Ω in $\mathbb{R}^n$ to guarantee the integral representation of positive harmonic functions h on Ω:

$$h(x) = \int_{\Delta(\Omega)} K_\Omega(x,\zeta)d\mu(\zeta) \quad \text{for } x \in \Omega,$$

where μ is a measure on $\Delta(\Omega)$ such that $\mu(\Delta(\Omega) \setminus \Delta_1(\Omega)) = 0$. Moreover, the measure μ is uniquely determined by h. From this viewpoint, it is important to show that the Martin boundary is homeomorphic to the Euclidean boundary and all Martin boundary points are minimal. For several domains, this has been proved by many authors: Hunt and Wheeden [25] for Lipschitz domains, Jerison and Kenig [26] for NTA domains (these domains have an exterior condition, which admits the doubling property for harmonic measures), Aikawa [3] for uniform domains (this domain does not have an exterior condition). Ancona [5] studied a bounded domain represented as the union of a family of open balls with the same radius. He gave a sufficient (interior) condition for each Euclidean boundary point to have a unique Martin boundary point.

It is also interesting to estimate the number of minimal Martin boundary points at a Euclidean boundary point where the boundary disperses. Benedicks [10] investigated the number of minimal Martin boundary points at a Euclidean boundary point of a Denjoy domain, a domain whose boundary lies in the hyperplane. See Figure 3.1. He gave an integral criterion involving harmonic measures for a Euclidean boundary point to have one or two minimal Martin boundary points. Ancona [6, 7] and Chevallier [16] also studied a Lipschitz Denjoy

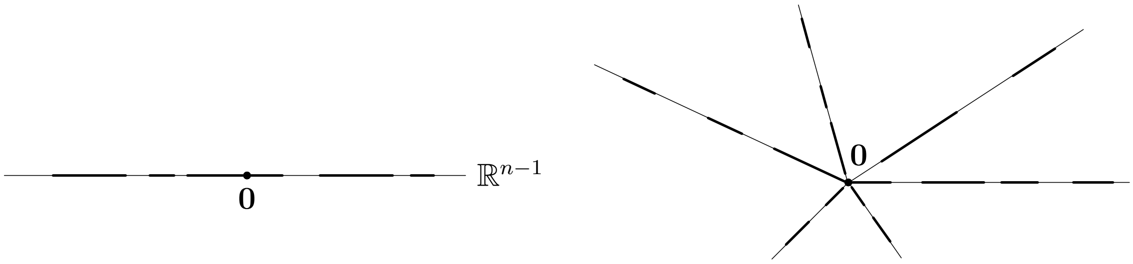

Figure 3.1: Denjoy domain and sectorial domain.

domain, a domain whose boundary lies in a Lipschitz surface. In two dimensions, Cranston and Salisbury [18] considered a sectorial domain, a domain whose boundary lies in the union of m rays emanating from the origin (see Figure 3.1), and gave an integral criterion for determining whether or not there are m minimal Martin boundary points at the origin. A higher dimensional sectorial domain is called a quasi-sectorial domain, for which Lömker [29] gave an integral criterion for the origin to have m minimal Martin boundary points.

As described in [30], there is a domain such that a Euclidean boundary point has infinitely many minimal Martin boundary points. One of the aim in this chapter is to show that every John domain has finitely many minimal Martin boundary points at each Euclidean boundary point. Let Ω be a proper subdomain of $\mathbb{R}^n$, and write $\delta_\Omega(x)$ for the distance from x to the (Euclidean) boundary $\partial\Omega$ of Ω. Suppose that K_0 is a compact subset of Ω. We say that Ω is *a general John domain with John constant c_J, $0 < c_J \le 1$, and John center K_0* if each $x \in \Omega$ can be connected to some point in K_0 by a rectifiable curve γ in Ω such that

$$\delta_\Omega(z) \ge c_J \ell(\gamma(x,z)) \quad \text{for all } z \in \gamma, \tag{3.1}$$

where $\gamma(x,z)$ is the subarc of γ from x to z and $\ell(\gamma(x,z))$ is the length of $\gamma(x,z)$. We note that K_0 is taken as one point $\{x_0\}$ in the usual definition of a John domain. We can easily show that a general John domain is a usual John domain, but the John constants between them may differ. See the proof of Proposition 3.10.7. We also see that every John domain is bounded since $|x - x_0| \le c_J^{-1}\delta_\Omega(x_0)$ for all $x \in \Omega$. We can also check that all domains (Denjoy domain, Lipschitz-Denjoy domain, sectorial domain, quasi-sectorial domain) stated above are John domains if we restrict them to $B(0,1)$.

Our main result is as follows.

Theorem 3.1.1. *Let Ω be a general John domain in $\mathbb{R}^n$ with John constant c_J. The following statements hold.*

(i) *The number of minimal Martin boundary points at every Euclidean boundary point is bounded by a constant depending only on c_J and the dimension n.*

(ii) *If $c_J > \sqrt{3}/2$, then there are at most two minimal Martin boundary points at every Euclidean boundary point.*

Remark 3.1.2. In two dimensions, we consider $\Omega = B(0,1) \setminus E$, where E is the closed set of three equally distributed rays, with length $1/2$, leaving from the origin. See Figure 3.2. Then Ω is a general John domain with John constant $\sin(\pi/3) = \sqrt{3}/2$ and John center $K_0 = S(0,2/3)$. There are three different minimal Martin boundary points at the origin. This simple example shows that the bound $c_J > \sqrt{3}/2$ in Theorem 3.1.1 (ii) is sharp. An example in higher dimensions may be obtained by the similar matter.

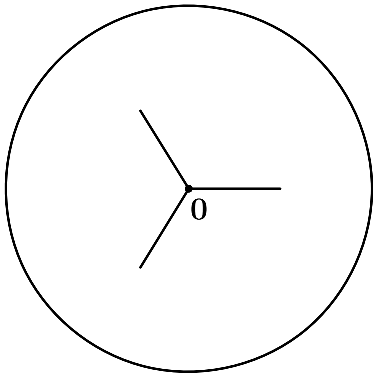

Figure 3.2: Sharpness of the bound $c_J > \sqrt{3}/2$.

Remark 3.1.3. Theorem 3.1.1 generalizes some parts of [6, 7], [10], [16], [18] and [29]. One of the main interests of these papers was to give a criterion for the number of minimal Martin boundary points at a fixed Euclidean boundary point (via Kelvin transform for [10]). Such a criterion seems to be very difficult for a general John domain, since the boundary may disperse at every point.

As a generalization of Ancona's result for the Martin boundary of the union of open balls with the same size, we will consider in Section 3.10 a bounded domain represented as the union of a family of open convex sets, and give a sufficient condition for a Euclidean boundary point to have exactly one Martin boundary point.

In order to prove Theorem 3.1.1, we will introduce a new notion, a system of local reference points of order N. This notion enables us to obtain a Carleson type estimate in a John domain (Section 3.5) using observations in Sections 3.3 and 3.4. The proof of Theorem 3.1.1 (i) will be given in Section 3.6. The proof of Theorem 3.1.1 (ii) will be given in Section 3.8 using a weak boundary Harnack principle in a John domain proved in Section 3.7.

3.2 System of local reference points

Let Ω be a proper subdomain of $\mathbb{R}^n$. For a rectifiable curve $\gamma : [0, \ell(\gamma)] \to \Omega$ and a non-negative Borel function f on Ω, the line integral of f over γ is denoted by

$$\int_\gamma f(z)ds(z) = \int_0^{\ell(\gamma)} f(\gamma(t))dt.$$

The quasi-hyperbolic metric on Ω is defined by

$$k_\Omega(x,y) = \inf_\gamma \int_\gamma \frac{ds(z)}{\delta_\Omega(z)},$$

where the infimum is taken over all rectifiable curve γ in Ω connecting x to y. It is easy to show that if Ω is a John domain, then there exists a positive constant A such that

$$k_\Omega(x,x_0) \le A \log \frac{\delta_\Omega(x_0)}{\delta_\Omega(x)} + A \quad \text{for } x \in \Omega, \tag{3.2}$$

which is called a quasi-hyperbolic boundary condition. We need more precise information on the shape of the boundary. We introduce the following notion.

Definition 3.2.1. Let N be a positive integer and $0 < \eta < 1$. We say that $\xi \in \partial\Omega$ *has a system of local reference points of order* N *with factor* η if there exist $r_\xi > 0$ and $A_\xi > 1$ with the following property: for each positive $r < r_\xi$ there are N points $y_1 = y_1(r), \cdots, y_N = y_N(r)$ in $\Omega \cap S(\xi, r)$ such that $\delta_\Omega(y_j) \ge A_\xi^{-1} r$ for $j = 1, \cdots, N$ and

$$\min_{j=1,\cdots,N} \{k_{\Omega_r}(x, y_j)\} \le A_\xi \log \frac{r}{\delta_\Omega(x)} + A_\xi \quad \text{for } x \in \Omega \cap \overline{B(\xi, \eta r)},$$

where $\Omega_r = \Omega \cap B(\xi, \eta^{-3} r)$. If η is not so important, we simply say that $\xi \in \partial\Omega$ has a system of local reference points of order N.

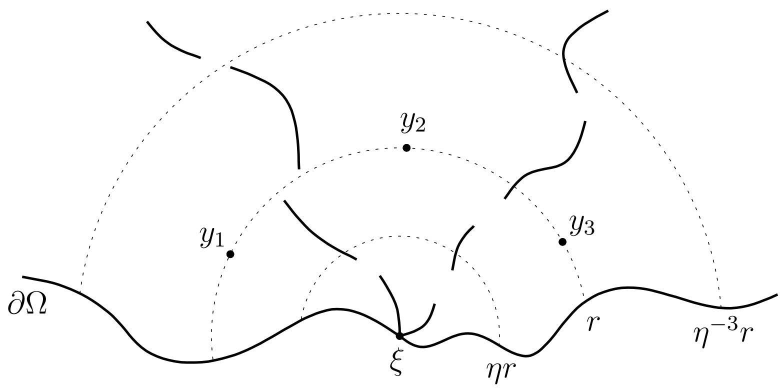

Figure 3.3: System of local reference points of order 3.

Proposition 3.2.2. *Let Ω be a general John domain with John constant c_J and John center K_0. Then every $\xi \in \partial\Omega$ has a system of local reference points of order N with $N \le N(c_J, n) < \infty$. Moreover, if the John constant c_J is greater than $\sqrt{3}/2$, then we can let $N \le 2$ by choosing a suitable factor $0 < \eta < 1$.*

For the proof of the second assertion, we prepare an elementary geometrical observation.

Lemma 3.2.3. *Let w_1, w_2 and w_3 be points on the unit sphere $S(0,1)$. Then*

$$\max \min_{j \neq k} |w_j - w_k| = \sqrt{3},$$

where the maximum is taken over all positions of w_1, w_2 and w_3.

Proof. This is a well-known fact (cf. [21]). For completeness we give a proof. We can easily prove the lemma for $n = 2$. Let $n \geq 3$. We observe from the compactness of $S(0,1)$ that the maximum d is taken by w_1, w_2 and w_3 on $S(0,1)$. There is a unique 2-dimensional plane Π containing w_1, w_2 and w_3, since three distinct points on $S(0,1)$ can not be collinear. Observe that $S(0,1) \cap \Pi$ is circle with radius at most 1. Since w_1, w_2 and w_3 are points on this circle, it follows from the case $n = 2$ that $d \leq \sqrt{3}$. Thus the lemma follows. □

Proof of Proposition 3.2.2. We prove the proposition with $r_\xi = \operatorname{dist}(K_0, \partial\Omega)$. Let $\xi \in \partial\Omega$ and $0 < r < \operatorname{dist}(K_0, \partial\Omega)$. We prove the first assertion with $\eta = 2^{-1}$. Let $x \in \Omega \cap \overline{B(\xi, 2^{-1}r)}$. By definition, there is a rectifiable curve γ in Ω connecting x to some point in K_0 such that (3.1) holds. Then the first hit $y(x)$ of $S(\xi, r)$ along γ satisfies that $2^{-1}c_J r \leq \delta_\Omega(y(x)) \leq r$ and

$$k_{\Omega_r}(x, y(x)) \leq A \log \frac{r}{\delta_\Omega(x)} + A.$$

We associate $y(x)$ with x, although it may not be unique.

Consider, in general, the family of balls $B(y, 4^{-1}c_J r)$ with $y \in S(\xi, r)$. These balls are included in $B(\xi, (4^{-1}c_J + 1)r)$, so that at most $N(c_J, n)$ balls among them can be mutually disjoint. Hence we can find N points $x_1, \cdots, x_N \in \Omega \cap \overline{B(\xi, 2^{-1}r)}$ with $N \leq N(c_J, n)$ such that $\{B(y_1, 4^{-1}c_J r), \cdots, B(y_N, 4^{-1}c_J r)\}$ is maximal, where $y_j = y(x_j) \in \Omega \cap S(\xi, r)$ is the point associated with x_j as above. This means that if $x \in \Omega \cap \overline{B(\xi, 2^{-1}r)}$, then $B(y(x), 4^{-1}c_J r)$ intersects some of $B(y_1, 4^{-1}c_J r), \cdots, B(y_N, 4^{-1}c_J r)$, say $B(y_j, 4^{-1}c_J r)$. Since $B(y(x), 4^{-1}c_J r) \cap B(y_j, 4^{-1}c_J r) \neq \emptyset$ and $B(y(x), 2^{-1}c_J r) \cup B(y_j, 2^{-1}c_J r) \subset \Omega_r$, it follows that $k_{\Omega_r}(y(x), y_j) \leq A$. Hence we have

$$k_{\Omega_r}(x, y_j) \leq k_{\Omega_r}(x, y(x)) + k_{\Omega_r}(y(x), y_j) \leq A \log \frac{r}{\delta_\Omega(x)} + A.$$

Repeating some points, say $y_1 = y(x_1)$, if necessary, we may assume that this property holds with some of N points $y_1, \cdots, y_N$, where N is independent of r and $N \leq N(c_J, n)$. Thus the first assertion follows.

For the proof of the second assertion, let $\sqrt{3}/2 < b' < b < c_J$ and $\eta = 1 - b/c_J > 0$. Let us prove that ξ has a system of local reference points of order at most 2 with factor η. Let $0 < r < \operatorname{dist}(K_0, \partial\Omega)$ and let $x \in \Omega \cap \overline{B(\xi, \eta r)}$. In the same way as in the proof of the first assertion, we can find $y(x) \in \Omega \cap S(\xi, r)$ such that

$$k_{\Omega_r}(x, y(x)) \leq A \log \frac{r}{\delta_\Omega(x)} + A,$$

and

$$\delta_\Omega(y(x)) \geq c_J(1 - \eta)r = br > b'r > \frac{\sqrt{3}}{2}r.$$

Lemma 3.2.3 says that at most two disjoint balls of radius $b'r$ can be placed so that their centers lie on the sphere $S(\xi, r)$. Hence we can choose $x_1, x_2 \in \Omega \cap \overline{B(\xi, \eta r)}$ such that $B(y(x), b'r)$ intersects $B(y_j, b'r)$ for some $j = 1, 2$, where $y_j = y(x_j)$. Since $B(y(x), b'r) \cap B(y_j, b'r) \neq \emptyset$ and $B(y(x), br) \cup B(y_j, br) \subset \Omega_r$, we have $k_{\Omega_r}(y(x), y_j) \leq A$. Hence the second assertion follows. Thus Proposition 3.2.2 is proved. □

Remark 3.2.4. In case $c_J \leq \sqrt{3}/2$, we may have an estimate of N better than the above proof, by considering a lemma similar to Lemma 3.2.3.

3.3 Growth estimate for subharmonic functions

In this section, we refine Domar's theorem [19, Theorem 2] for the boundedness of a subharmonic function majorized by a positive function. Actually, we give a growth estimate for subharmonic functions satisfying a Nevanlinna type integral condition.

Theorem 3.3.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ and let f be a non-negative subharmonic function on Ω. If there is a positive constant ε such that*

$$I := \int_\Omega (\log^+ f(x))^{n-1+\varepsilon} dx < \infty,$$

then there exists a positive constant A depending only on ε and the dimension n such that

$$f(x) \leq \exp\left(2 + A\left(\frac{I}{\delta_\Omega(x)^n}\right)^{1/\varepsilon}\right) \quad \textit{for } x \in \Omega. \tag{3.3}$$

For the proof, we prepare the following lemma.

Lemma 3.3.2. *Let f be a non-negative subharmonic function on $B(x, r)$ and denote $L_n = (e^2/|B(0, 1)|)^{1/n}$. If $f(x) \geq t > 0$ and*

$$r \geq L_n |\{y \in B(x, r) : e^{-1}t < f(y) \leq et\}|^{1/n}, \tag{3.4}$$

then there exists a point $y \in B(x, r)$ such that $f(y) > et$.

Proof. Suppose to the contrary that $f \leq et$ on $B(x, r)$. Since (3.4) is equivalent to

$$\frac{|\{y \in B(x, r) : e^{-1}t < f(y) \leq et\}|}{|B(x, r)|} \leq \frac{1}{e^2},$$

we have by the mean value property of a subharmonic function that

$$\begin{aligned}
t \leq u(x) &\leq \frac{1}{|B(x, r)|} \int_{B(x,r)} f(y) dy \\
&= \frac{1}{|B(x, r)|} \left(\int_{B(x,r) \cap \{f \leq e^{-1}t\}} f(y) dy + \int_{B(x,r) \cap \{e^{-1}t < f(y) \leq et\}} f(y) dy \right) \\
&\leq \frac{t}{e} + \frac{et}{e^2} < t.
\end{aligned}$$

This is a contradiction, and hence the lemma follows. □

Let us prove Theorem 3.3.1.

Proof of Theorem 3.3.1. Since the right hand side of (3.3) is not less than e^2, it suffices to show that

$$\delta_\Omega(x) \le AI^{1/n}(\log f(x))^{-\varepsilon/n}, \quad \text{whenever } f(x) > e^2. \tag{3.5}$$

We fix $x_1 \in \Omega$ with $f(x_1) > e^2$, and let

$$r_j = L_n|\{y \in \Omega : e^{j-2}f(x_1) < f(y) \le e^j f(x_1)\}|^{1/n}.$$

Let us show (3.5) for $x = x_1$. We choose a finite or infinite sequence $\{x_j\}$ in Ω as follows: If $\delta_\Omega(x_1) < r_1$, then we stop. If $\delta_\Omega(x_1) \ge r_1$, then $B(x_1, r_1) \subset \Omega$, so that there exists $x_2 \in B(x_1, r_1)$ such that $f(x_2) > ef(x_1)$ by Lemma 3.3.2 with $t = f(x_1)$. Next we consider $\delta_\Omega(x_2)$. If $\delta_\Omega(x_2) < r_2$, then we stop. If $\delta_\Omega(x_2) \ge r_2$, then $B(x_2, r_2) \subset \Omega$, so that there exists $x_3 \in B(x_2, r_2)$ such that $f(x_3) > e^2 f(x_1)$ by Lemma 3.3.2 with $t = ef(x_1)$. Repeating this procedure, we obtain a finite or infinite sequence $\{x_j\}$. We claim that

$$\delta_\Omega(x_1) \le 2\sum_{j=1}^{\infty} r_j. \tag{3.6}$$

Suppose first that $\{x_j\}_{j=1}^J$ is finite. If $J = 1$, then $\delta_\Omega(x_1) < r_1$, so that (3.6) holds trivially. If $J \ge 2$, then $x_{j+1} \in B(x_j, r_j)$ for $j = 1, \cdots, J-1$ and $\delta_\Omega(x_J) < r_J$ by our choice, and hence

$$\delta_\Omega(x_1) \le |x_1 - x_2| + \cdots + |x_{J-1} - x_J| + \delta_\Omega(x_J) < \sum_{j=1}^{J} r_j.$$

Suppose next that $\{x_j\}$ is infinite. Since $f(x_j) > e^{j-1}f(x_1) \to \infty$, it follows from the local boundedness of a subharmonic function that x_j goes to the boundary. Hence there is an integer $J \ge 2$ such that $\delta_\Omega(x_J) \le \delta_\Omega(x_1)/2$, and then we have

$$\delta_\Omega(x_1) \le |x_1 - x_2| + \cdots + |x_{J-1} - x_J| + \delta_\Omega(x_J) \le \sum_{j=1}^{J-1} r_j + \frac{1}{2}\delta_\Omega(x_1).$$

Thus (3.6) holds.

To obtain (3.5) with $x = x_1$, it is enough to show that

$$\sum_{j=1}^{\infty} r_j \le AI^{1/n}(\log f(x_1))^{-\varepsilon/n}. \tag{3.7}$$

Let j_1 be the integer such that $e^{j_1} < f(x_1) \le e^{j_1+1}$. Then $j_1 \ge 2$ and

$$r_j \le L_n|\{y \in \Omega : e^{j_1+j-2} < f(y) \le e^{j_1+j+1}\}|^{1/n}.$$

Since the family of intervals $\{(e^{j_1+j-2}, e^{j_1+j+1})\}_j$ overlaps at most 3 times, it follows from Hölder's inequality that

$$\begin{aligned}
\sum_{j=1}^{\infty} r_j &\le 3L_n \sum_{j=j_1}^{\infty} |\{y \in \Omega : e^{j-1} < f(y) \le e^j\}|^{1/n} \\
&\le 3L_n \left(\sum_{j=j_1}^{\infty} \frac{1}{j^{(n-1+\varepsilon)/(n-1)}} \right)^{(n-1)/n} \left(\sum_{j=j_1}^{\infty} j^{n-1+\varepsilon} |\{y \in \Omega : e^{j-1} < f(y) \le e^j\}| \right)^{1/n} \\
&\le A j_1^{-\varepsilon/n} \left(\int_{\Omega} (\log^+ f(y))^{n-1+\varepsilon} dy \right)^{1/n} \\
&\le A(\log f(x_1))^{-\varepsilon/n} I^{1/n},
\end{aligned}$$

where A is a constant depending only on ε and n. Thus (3.7) follows, and the proof is complete. $\square$

3.4 Integrability of negative power of the distance function

In [2], the global integrability of negative power of the distance function have been proved in a John domain Ω: there is a positive constant τ such that

$$\int_{\Omega} \delta_{\Omega}(x)^{-\tau} dx < \infty.$$

The purpose of this section is to show the local version.

Theorem 3.4.1. *Let Ω be a general John domain in $\mathbb{R}^n$ with John constant c_J and John center K_0. Then there exist positive constants τ and A depending only on c_J and the dimension n such that*

$$\int_{\Omega \cap B(\xi, r)} \left(\frac{r}{\delta_{\Omega}(x)} \right)^{\tau} dx \le Ar$$

for each $\xi \in \partial\Omega$ and $0 < r < \operatorname{dist}(K_0, \partial\Omega)/2$.

Proof. For each $j \in \mathbb{N} \cup \{0\}$, we let

$$V_j = \left\{ x \in \Omega \cap B(\xi, r + (1 + c_J^{-1}) 2^{1-j} r) : 2^{-j-1} r \le \delta_{\Omega}(x) < 2^{-j} r \right\}.$$

For a moment, we fix $x \in \bigcup_{i=j+1}^{\infty} V_i$. Then there is a rectifiable curve γ in Ω connecting x to some point in K_0 with the property (3.1). Let $y \in \gamma$ be such that $\delta_{\Omega}(y) = 2^{-j} r$. Then $|x - y| \le c_J^{-1} 2^{-j} r$; in other words, $x \in \overline{B(y, c_J^{-1} 2^{-j} r)}$. We observe that

$$|B(y, 5c_J^{-1} 2^{-j} r)| \le A_1 |V_j \cap B(y, c_J^{-1} 2^{-j} r)|, \tag{3.8}$$

where A_1 is a constant depending only on c_J and n. In fact, taking $y' \in B(y, 2^{-j-1} r)$ so that $\delta_{\Omega}(y') = 2^{-1}(2^{-j} + 2^{-j-1}) r$, we have $B(y', 2^{-j-2} r) \subset V_j \cap B(y, 2^{-j} r)$. Thus (3.8) holds.

By the covering lemma, we can find a sequence $\{y_k\}$ such that

$$\bigcup_{i=j+1}^{\infty} V_i \subset \bigcup_k \overline{B(y_k, 5c_J^{-1}2^{-j}r)}$$

and $\{\overline{B(y_k, c_J^{-1}2^{-j}r)}\}_k$ are mutually disjoint. Then we have from (3.8) that

$$\sum_{i=j+1}^{\infty} |V_i| = \left|\bigcup_{i=j+1}^{\infty} V_i\right| \le \sum_k |B(y_k, 5c_J^{-1}2^{-j}r)| \le A_1 \sum_k |V_j \cap B(y_k, c_J^{-1}2^{-j}r)| \le A_1|V_j|.$$

Hence, writing $t = 1 + 2^{-1}A_1^{-1}$, we have

$$\begin{aligned} A_1 \sum_{j=0}^{N} t^{j+1}|V_j| &\ge \sum_{j=0}^{N} \sum_{i=j+1}^{N+1} t^{j+1}|V_i| = \sum_{i=1}^{N+1} \sum_{j=0}^{i-1} t^{j+1}|V_i| \ge \sum_{i=1}^{N} \sum_{j=0}^{i-1} t^{j+1}|V_i| \\ &= \sum_{i=1}^{N} \frac{t^{i+1} - t}{t-1}|V_i| = \frac{1}{t-1} \sum_{i=0}^{N} t^{i+1}|V_i| - \frac{t}{t-1} \sum_{i=0}^{N} |V_i|, \end{aligned}$$

and so

$$\sum_{j=0}^{N} t^{j+1}|V_j| \le \frac{t}{1-(t-1)A_1} \sum_{j=0}^{N} |V_j| = A \sum_{j=0}^{N} |V_j|.$$

Letting $N \to \infty$, we have

$$\sum_{j=0}^{\infty} t^{j+1}|V_j| \le A \sum_{j=0}^{\infty} |V_j| \le A|B(\xi, r + (1 + c_J^{-1})2r)| \le Ar^n.$$

Since $t^j < (r/\delta_\Omega(x))^\tau \le t^{j+1}$ for $x \in V_j$ with $\tau = \log t/\log 2 > 0$, it follows that

$$\int_{\Omega \cap B(\xi,r)} \left(\frac{r}{\delta_\Omega(x)}\right)^\tau dx \le \sum_{j=0}^{\infty} t^{j+1}|V_j| \le Ar^n.$$

Thus the theorem is established. $\square$

3.5 Weak Carleson estimate in a John domain

The purpose of this section is to show a Carleson type estimate in a John domain. The original form obtained by Carleson [13] was stated as follows: Given $\xi \in \partial\Omega$, there exists $y_r \in \Omega \cap S(\xi, r)$, apart from the boundary enough, such that if h is a positive harmonic function on $\Omega \cap B(\xi, 2r)$ vanishing (continuously) on $\partial\Omega \setminus B(\xi, 2^{-1}r)$, then

$$h(x) \le Ah(y_r) \quad \text{for } x \in \Omega \cap S(\xi, r),$$

where A is a constant independent of x, r and h. This estimate may be obtained in a uniform domain. However, it is impossible in a John domain. So we refine this estimate suitably using local reference points introduced in Section 3.2.

Throughout this section, we suppose that Ω is a general John domain. We note from Proposition 3.2.2 that each boundary point has a system of local reference points of order N.

Theorem 3.5.1. *Let $\xi \in \partial\Omega$ have a system of local reference points $y_1, \cdots, y_N \in \Omega \cap S(\xi, r)$ of order N with factor η for $0 < r < r_\xi$. If h is a positive harmonic function on $\Omega \cap B(\xi, \eta^{-3}r)$ which vanishes quasi-everywhere on $\partial\Omega \cap B(\xi, \eta^{-3}r)$ and is bounded on $\Omega \cap B(\xi, \eta r) \setminus \overline{B(\xi, \eta^3 r)}$, then*

$$h(x) \leq A \sum_{j=1}^{N} h(y_j) \quad \text{for } x \in \Omega \cap S(\xi, \eta^2 r), \tag{3.9}$$

where A is a constant independent of x, r and h.

In the proof we use the following material: Let Ω be a domain in $\mathbb{R}^n$ and let D be an open subset of Ω. If h is a positive harmonic function on D which vanishes quasi-everywhere on $\partial D \cap \Omega$ and is bounded near points of $\partial D \cap \Omega$, then we see from [8, Theorem 5.2.1] that h has a subharmonic extension h^* to Ω which is valued 0 quasi-everywhere on $\partial D \cap \Omega$ and everywhere on $\Omega \setminus \overline{D}$.

Proof of Theorem 3.5.1. By Definition 3.2.1 and Corollary 6.1.2, there are positive constants A_2 and λ such that

$$h(x) \leq A_2 \left(\frac{r}{\delta_\Omega(x)} \right)^\lambda \sum_{j=1}^{N} h(y_j) \quad \text{for } x \in \Omega \cap \overline{B(\xi, \eta r)}. \tag{3.10}$$

Let h^* be a subharmonic extension of h to $B(\xi, \eta r) \setminus \overline{B(\xi, \eta^3 r)}$ as above, and apply Theorem 3.3.1 to $\varepsilon = 1$ and $f = h^*/(A_2 \sum_{j=1}^N h(y_j))$ on $B(\xi, \eta r) \setminus \overline{B(\xi, \eta^3 r)}$. Let $\tau > 0$ be as in Theorem 3.4.1. Applying the elementary inequality,

$$(\log t)^n \leq \left(\frac{n}{\tau} \right)^n t^\tau \quad \text{for } t \geq 1,$$

to $t = r/\delta_\Omega(x) \geq 1$ for $x \in \Omega$, we have

$$\left[\log \left(\frac{r}{\delta_\Omega(x)} \right) \right]^n \leq A \left(\frac{r}{\delta_\Omega(x)} \right)^\tau.$$

This, together with (3.10) and Theorem 3.4.1, yields that

$$I = \int_\Omega (\log^+ u)^n dx \leq A \int_{\Omega \cap B(\xi, r)} \left(\frac{r}{\delta_\Omega(x)} \right)^\tau dx \leq A r^n.$$

Hence we conclude from Theorem 3.3.1 that $u \leq \exp(2 + AIr^{-n}) \leq A$ on $S(\xi, \eta^2 r)$, which shows (3.9). Thus the proof is complete. $\square$

Corollary 3.5.2. *Let $\xi \in \partial\Omega$ have a system of local reference points $y_1, \cdots, y_N \in \Omega \cap S(\xi, r)$ of order N with factor η for $0 < r < r_\xi$. If h is a bounded positive harmonic function on $\Omega \cap B(\xi, \eta^{-3}r)$ vanishing quasi-everywhere on $\partial\Omega \cap B(\xi, \eta^{-3}r)$, then*

$$h(x) \leq A \sum_{j=1}^{N} h(y_j) \quad \text{for } x \in \Omega \cap \overline{B(\xi, \eta^2 r)},$$

where A is a constant independent of x, r and h.

Proof. Since h satisfies the assumption in Theorem 3.5.1, we have (3.9). The conclusion follows from the maximum principle. $\square$

Theorem 3.5.1 also yields the growth estimate for kernel functions. For $\xi \in \partial\Omega$, we denote by $\mathcal{H}_\xi$ the collection of all kernel functions at ξ normalized at x_0 (John center in a usual sense), that is, the set of all positive harmonic functions h on Ω such that $h(x_0) = 1$, h vanishes quasi-everywhere on $\partial\Omega$ and is bounded on $\Omega \setminus B(\xi, r)$ for each $r > 0$. We note from (3.2) and Corollary 6.1.2 that there exist positive constants A and λ depending only on c_J and the dimension n such that

$$\frac{h(x)}{h(x_0)} \le A\left(\frac{\delta_\Omega(x_0)}{\delta_\Omega(x)}\right)^\lambda \quad \text{for } x \in \Omega. \tag{3.11}$$

Corollary 3.5.3. *Let $\xi \in \partial\Omega$ have a system of local reference points $y_1, \cdots, y_N \in \Omega \cap S(\xi, r)$ of order N with factor η for $0 < r < r_\xi$. If $h \in \mathcal{H}_\xi$, then*

$$h(x) \le A|x - \xi|^{-\lambda} \quad \textit{for } x \in \Omega,$$

where $\lambda > 0$ is as in (3.11) *and A is a constant independent of x, r and h.*

Proof. Since $h \in \mathcal{H}_\xi$ satisfies the assumption in Theorem 3.5.1, we have (3.9). We also have $h(y_j) \le Ar^{-\lambda}$ by applying (3.11) to each y_j. Since h vanishes quasi-everywhere on $\partial\Omega$ and is bounded near the boundary apart from a neighborhood of ξ, it follows from the maximum principle that

$$h(x) \le A\sum_{j=1}^{N} h(y_j) \le Ar^{-\lambda} \quad \text{for } x \in \Omega \setminus B(\xi, \eta^2 r).$$

Since Ω is bounded, we obtain $h(x) \le A|x - \xi|^{-\lambda}$ for $x \in \Omega$. $\square$

3.6 Proof of Theorem 3.1.1 (i)

Before giving the proof of Theorem 3.1.1 (i), we show that all Martin kernels at ξ belong to $\mathcal{H}_\xi$. The following theorem can be found in [9].

Theorem (Bass and Burdzy). *Let Ω be a John domain in $\mathbb{R}^n$. Suppose that V is an open set and that K is a compact subset of V. Then there exists a constant $A \ge 1$ depending on V, K and Ω such that if u and v are positive harmonic functions on Ω that vanish quasi-everywhere on $\partial\Omega \cap V$ and are bounded near $\partial\Omega \cap V$, then*

$$\frac{u(x)}{v(x)} \le A\frac{u(y)}{v(y)} \quad \textit{for } x, y \in K \cap \Omega.$$

Let $\xi \in \partial\Omega$ and let $\{y_j\}$ be a sequence in Ω converging to ξ. Applying Bass and Burdzy's result, we have that for $x \in \Omega \setminus B(\xi, r)$ and j sufficiently large,

$$K_\Omega(x, y_j) = \frac{G_\Omega(x, y_j)}{G_\Omega(x_0, y_j)} \le A\frac{G_\Omega(x, y_0)}{G_\Omega(x_0, y_0)}$$

where y_0 is a fixed point in $\Omega \cap B(\xi, 2^{-1}r)$. Letting $j \to \infty$, we see that all Martin kernels at ξ are bounded on $\Omega \setminus B(\xi, r)$ for each $r > 0$ and vanish quasi-everywhere on $\partial\Omega$. Thus they belong to $\mathcal{H}_\xi$. In particular, $\mathcal{H}_\xi$ is non-empty.

From this observation, it is enough to show the following proposition in order to prove Theorem 3.1.1 (i).

Proposition 3.6.1. *Let Ω be a general John domain in $\mathbb{R}^n$ with John constant c_J, and let $\xi \in \partial\Omega$. Then the number of minimal functions in $\mathcal{H}_\xi$ is bounded by a constant depending only on c_J and the dimension n.*

For the proof, we prepare the following lemma.

Lemma 3.6.2. *Let Ω be a Greenian domain in $\mathbb{R}^n$ and $\xi \in \partial\Omega$. Suppose that there exist a positive integer M and a positive constant A with the following property: If $h_0, \cdots, h_M \in \mathcal{H}_\xi$, then there is $k \in \{0, \cdots, M\}$ such that*

$$h_k \le A \sum_{j \ne k} h_j \quad \textit{on } D.$$

Then $\mathcal{H}_\xi$ has at most M minimal harmonic functions.

Proof. Suppose that there are $M+1$ different minimal harmonic functions $h_0, \cdots h_M \in \mathcal{H}_\xi$. If necessary relabeling, we may assume by assumption that

$$h_0 \le A \sum_{j=1}^{M} h_j \quad \text{on } D.$$

We may also assume that $A \ge 1$. Then $(A \sum_{j=1}^{M} h_j - h_0)/(AM - 1) \in \mathcal{H}_\xi$. Writing h for this function, we have

$$\frac{1}{AM} h_0 + (1 - \frac{1}{AM}) h = \frac{1}{M} \sum_{j=1}^{M} h_j.$$

Compare the Martin representation measures for the both sides. The measure for the left hand side has at least $1/AM$ mass at h_0, whereas the measure for the right hand side has 0 mass at h_0. This contradicts the uniqueness of the Martin representation. Hence the lemma follows. □

Proof of Proposition 3.6.1. Let $h_1, \cdots, h_M \in \mathcal{H}_\xi$, and let h_j^* be a subharmonic extension of h_j to $\mathbb{R}^n \setminus \{\xi\}$ as mentioned in the previous section. Let H_j be the Kelvin transformation of h_j^* with respect to $S(\xi, 1)$: that is,

$$H_j(x) = |x - \xi|^{2-n} h_j^*(\xi + |x - \xi|^{-2}(x - \xi)).$$

We then observe that H_j is a non-negative subharmonic function on $\mathbb{R}^n$ which is positive and harmonic on the Kelvin image Ω^* of Ω and is equal to 0 quasi-everywhere outside Ω^*. Moreover, Corollary 3.5.3 shows

$$H_j(x) \le A|x - \xi|^{2-n+\lambda}.$$

Thus H_j is of order at most $2-n+\lambda$. We let

$$w = \max_{j=0,\cdots,M}\{H_j - \sum_{k\neq j} H_k\},$$

and let w^+ be the upper semicontinuous regularization of $\max\{w,0\}$. Then w^+ is a non-negative subharmonic function on $\mathbb{R}^n$ of order $2-n+\lambda$. If none of $\{x : H_j(x) > \sum_{k\neq j} H_k(x)\}$ is empty, then w^+ has $M+1$ tracts. Hence [22, Theorem 3] yields that

$$2-n+\lambda \geq \frac{1}{2}\log\left(\frac{M+1}{4}\right) + \frac{3}{2} \quad \text{if } M \geq 3.$$

Hence, if $M > 4\exp(1-2n+2\lambda)-1$, then $\{x : H_j(x) > \sum_{k\neq j} H_k(x)\} = \emptyset$ for some $j = 0,\cdots,M$. This means that $H_j \leq \sum_{k\neq j} H_k$ on Ω^*, so that

$$h_j \leq \sum_{k\neq j} h_k \quad \text{on } \Omega.$$

Hence Lemma 3.6.2 implies that $\mathcal{H}_\xi$ has at most M minimal harmonic functions, or equivalently there are M minimal Martin boundary points at ξ. Thus the number of minimal Martin boundary points at ξ is bounded by $4\exp(1-2n+2\lambda)$. $\square$

3.7 Weak boundary Harnack principle in a John domain

In order to prove Theorem 3.1.1 (ii), we need more concrete discussion. First of all, we note that there is a difference of the behavior of the Green function between $n=2$ and $n\geq 3$: that is, if $n \geq 3$, then

$$G_\Omega(x,y) \approx r^{2-n} \quad \text{for } x \in S(y, 2^{-1}\delta_\Omega(y)) \text{ with } \delta_\Omega(y) \approx r;$$

if $n=2$, then this estimate does not necessarily hold. To avoid this difficulty, we consider the Green function G_r of the intersection $\widetilde{\Omega}_r := \Omega \cap B(\xi, A_3 r)$ with sufficiently large $A_3 > \eta^{-3}$. Then we have for any $n \geq 2$,

$$G_r(x,y) \approx r^{2-n} \quad \text{for } x \in S(y, 2^{-1}\delta_\Omega(y)) \text{ with } \delta_\Omega(y) \approx r, \tag{3.12}$$

where the constant of comparison depends only on Ω and A_3.

Let U be an open set and let E be a Borel subset of ∂U. By $\omega(x,E,U)$ we denote the harmonic measure of E for U evaluated at x. That is , the Perron-Wiener-Brelot solution of the Dirichlet problem in U with boundary data 1 on E and 0 on $\partial U \setminus E$. We let $U(r) = \{x \in \Omega : \delta_\Omega(x) < r\}$.

Lemma 3.7.1. *Let Ω be a general John domain in $\mathbb{R}^n$ with John center K_0. Then there exists constants $0 < \varepsilon_0 < 1$ and $A_7 \geq 1$ such that if $0 < r < 2^{-1}\operatorname{dist}(K_0, \partial\Omega)$, then*

$$\omega(x, U(r)\cap S(x, A_7 r), U(r)\cap B(x, A_7 r)) \leq \varepsilon_0 \quad \textit{for } x \in U(r).$$

Proof. Let $x \in U(r)$. By the definition, there is a rectifiable curve γ in Ω connecting x to some point in K_0 with the property (3.1). Then we can find a point $z \in \gamma$ such that $\delta_\Omega(z) = 2r$. Since $|x - z| \le \ell(\gamma(x, z)) \le 2c_J^{-1}r$, we have $B(z, r) \subset B(x, 3c_J^{-1}r) \setminus U(r)$. Therefore there is a constant $0 < \varepsilon_0 < 1$ depending only on c_J and the dimension n such that

$$\frac{|U(r) \cap B(x, 3c_J^{-1}r)|}{|B(x, 3c_J^{-1}r)|} \le \varepsilon_0.$$

Let $A_7 = 3c_J^{-1}$. We note that $\omega(\cdot, U(r) \cap S(x, A_7r), U(r) \cap B(x, A_7r))$ has a subharmonic extension ω to $B(x, A_7r)$ with zero values quasi-everywhere on $\partial U(r) \cap B(x, A_7r)$ and everywhere on $B(x, A_7r) \setminus \overline{U(r)}$. Then the mean value property of a subharmonic function yields that

$$\omega(x) \le \frac{1}{|B(x, A_7r)|} \int_{B(x,A_7r)} \omega(y)dy \le \varepsilon_0.$$

Thus the lemma follows. □

Lemma 3.7.2. *Let Ω be a general John domain in $\mathbb{R}^n$ and let A_7 be a constant as in Lemma 3.7.1. Then there exists a positive constant $A_8 \le 1$ such that if $r > 0$ and $\rho > 0$, then*

$$\omega(x, U(\rho) \cap S(x, r), U(\rho) \cap B(x, r)) \le \exp\left(A_7 - A_8\frac{r}{\rho}\right) \quad \text{for } x \in U(\rho). \tag{3.13}$$

Proof. We note that if $r \le A_7\rho$, then (3.13) holds clearly since the right hand side of (3.13) is not less than 1. So we assume that $r > A_7\rho$. Let $k \in \mathbb{N}$ be such that $kA_7\rho < r \le (k+1)A_7\rho$. We claim that for $j = 0, \cdots, k$,

$$\sup_{U(\rho)\cap B(x,r-jA_7\rho)} \omega(\cdot, U(\rho) \cap S(x, r), U(\rho) \cap B(x, r)) \le \varepsilon_0^j, \tag{3.14}$$

where ε_0 is a constant in Lemma 3.7.1. We show this by induction. If $j = 0$, then (3.14) holds clearly. Assuming that (3.14) holds for $j - 1$, we show (3.14) for j. Let $y \in U(\rho) \cap S(x, r - jA_7\rho)$. Since $S(y, A_7\rho) \subset \overline{B(x, r - (j - 1)A_7\rho)}$, it follows from the assumption, the maximum principle and Lemma 3.7.1 that

$$\begin{aligned}\omega(y, U(\rho) \cap S(x, r), U(\rho) \cap B(x, r)) &\le \varepsilon_0^{j-1}\omega(y, U(\rho) \cap S(y, A_7\rho), U(\rho) \cap B(y, A_7\rho)) \\ &\le \varepsilon_0^j.\end{aligned}$$

Since y is an arbitrary point in $U(\rho) \cap S(x, r - jA_7\rho)$, the maximum principle yields (3.14) for j. Finally, noting that $(A_7\rho)^{-1}r \le 2k$, we obtain from (3.14) with $j = k$ that

$$\omega(x, U(\rho) \cap S(x, r), U(\rho) \cap B(x, r)) \le \exp((\varepsilon_0 - 1)k) \le \exp\left(\frac{\varepsilon_0 - 1}{2A_7}\frac{r}{\rho}\right).$$

Thus the lemma is proved. □

In the rest of this section, we suppose that Ω is a general John domain in $\mathbb{R}^n$.

Lemma 3.7.3. *Let $\xi \in \partial\Omega$ have a system of local reference points $y_1, \cdots, y_N \in \Omega \cap S(\xi, r)$ of order N with factor η for $0 < r < r_\xi$. If $x \in \Omega \cap \overline{B(\xi, \eta^3 r)}$, then*

$$\omega(x, \Omega \cap S(\xi, \eta^2 r), \Omega \cap B(\xi, \eta^2 r)) \leq A r^{n-2} \sum_{j=1}^{N} G_r(x, y_j), \tag{3.15}$$

where A is a constant depends only on c_J, A_ξ and the dimension n.

Proof. Let $0 < r < r_\xi$. For each $x \in \Omega \cap \overline{B(\xi, \eta r)}$, there is a local reference point $y(x) \in \{y_1, \cdots, y_N\}$ such that

$$k_{\Omega_r}(x, y(x)) \leq A_\xi \log \frac{r}{\delta_\Omega(x)} + A_\xi.$$

Let $y'(x) \in S(y(x), 2^{-1}\delta_\Omega(y(x)))$. Then we have by Lemma 6.1.3

$$k_{\Omega_r \setminus \{y(x)\}}(x, y'(x)) \leq A_\xi \log \frac{r}{\delta_\Omega(x)} + A_\xi.$$

Letting $f(x) = r^{n-2} \sum_{j=1}^{N} G_r(x, y_j)$, we obtain from (3.12) and Corollary 6.1.2 that

$$f(x) \geq A \left(\frac{\delta_\Omega(x)}{r} \right)^\lambda \quad \text{for } x \in \Omega \cap \overline{B(\xi, \eta r)},$$

where A and λ are positive constants depending only on c_J, A_ξ and the dimension n. Let

$$\Omega_j = \{x \in \widetilde{\Omega}_r : \exp(-2^{j+1}) \leq f(x) < \exp(-2^j)\},$$
$$U_j = \{x \in \widetilde{\Omega}_r : f(x) < \exp(-2^j)\}.$$

Then we see that

$$U_j \cap B(\xi, \eta r) \subset V_j := \left\{ x \in \Omega : \delta_\Omega(x) < Ar \exp\left(-\frac{2^j}{\lambda} \right) \right\}.$$

We now define a decreasing sequence $\{r_j\}$ by $r_0 = \eta^2 r$ and

$$r_j = \left(\eta^2 - \frac{6(\eta^2 - \eta^3)}{\pi^2} \sum_{k=1}^{j} \frac{1}{k^2} \right) r \quad \text{for } j \geq 1.$$

We note that $r_j \to \eta^3 r$. Let $\omega_0 = \omega(\cdot, \Omega \cap S(\xi, \eta^2 r), \Omega \cap B(\xi, \eta^2 r))$ and put

$$d_j = \begin{cases} \sup\limits_{x \in \Omega_j \cap B(\xi, r_j)} \dfrac{\omega_0(x)}{f(x)} & \text{if } \Omega_j \cap B(\xi, r_j) \neq \emptyset, \\ 0 & \text{if } \Omega_j \cap B(\xi, r_j) = \emptyset. \end{cases}$$

It is sufficient to show that d_j is bounded by a constant independent of r and j, since $r_j > \eta^3 r$ for all $j \geq 0$. Let $j > 0$ and let $x \in \Omega_j \cap B(\xi, r_j)$. Then the maximum principle yields that

$$\omega_0(x) \leq \omega(x, U_j \cap S(\xi, r_{j-1}), U_j \cap B(\xi, r_{j-1})) + d_{j-1} f(x). \tag{3.16}$$

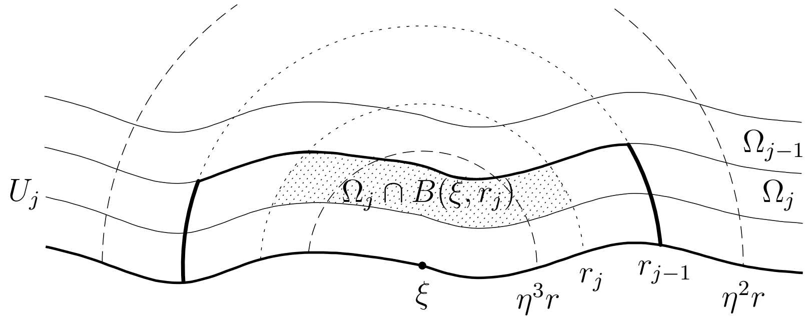

Figure 3.4: Maximum principle on $\Omega_j \cap B(\xi, r_{j-1})$.

Since $B(x, r_{j-1} - r_j) \subset B(\xi, r_{j-1})$, the first term of the right hand side of (3.16) is not greater than

$$\omega(x, V_j \cap S(x, r_{j-1} - r_j), V_j \cap B(x, r_{j-1} - r_j)) \le \exp\left(A_7 - A_8 \frac{r_{j-1} - r_j}{Ar\exp(-2^j\lambda^{-1})}\right)$$

by Lemma 3.7.2. Let us divide the both sides of (3.16) by $u(x)$ and take the supremum over $\Omega_j \cap B(\xi, r_j)$. Then we have

$$d_j \le \exp\left(2^{j+1} + A_7 - A_8 \frac{6(\eta^2 - \eta^3)}{\pi^2} \frac{\exp(2^j\lambda^{-1})}{Aj^2}\right) + d_{j-1}.$$

Since $d_0 \le e^2$, we obtain

$$d_j \le \sum_{j=1}^{\infty} \exp\left(2^{j+1} + A_7 - A_8 \frac{6(\eta^2 - \eta^3)}{\pi^2} \frac{\exp(2^j\lambda^{-1})}{Aj^2}\right) + d_0 < \infty.$$

Thus the lemma is established. □

Lemma 3.7.4. *Let $\xi \in \partial\Omega$ have a system of local reference points $y_1, \cdots, y_N \in \Omega \cap S(\xi, r)$ of order N with factor η for $0 < r < r_\xi$. If $x \in \Omega \cap \overline{B(\xi, \eta^3 r)}$ and $y \in \Omega \cap S(\xi, \eta^{-3} r)$, then*

$$G_r(x, y) \le Ar^{n-2} \sum_{j=1}^{N} G_r(x, y_j) \sum_{k=1}^{N} G_r(y_k, y), \tag{3.17}$$

where A is a constant depends only on c_J, A_ξ and the dimension n.

Proof. Let us apply Corollary 3.5.2 to $h(x) = G_r(x, y)$ with $y \in \Omega \cap S(\xi, \eta^{-3} r)$. Then

$$G_r(x, y) \le A \sum_{j=1}^{N} G_r(y_j, y) \quad \text{for } x \in \Omega \cap S(\xi, \eta^2 r).$$

Hence Lemma 3.7.3 and the maximum principle yields that

$$G_r(x, y) \le Ar^{n-2} \sum_{j=1}^{N} G_r(x, y_j) \sum_{k=1}^{N} G_r(y_k, y) \quad \text{for } x \in \Omega \cap \overline{B(\xi, \eta^3 r)},$$

and thus the lemma follows. □

For further arguments, we need the following improvement of (3.15): If $x \in \Omega \cap S(\xi, \eta^9 r)$ and $y \in \Omega \cap S(\xi, \eta^{-3} r)$, then

$$G_r(x,y) \le A r^{n-2} \sum_{j=1}^{N} G_r(x, y_j) G_r(y_j, y), \tag{3.18}$$

where A is a constant depending only on c_J, A_ξ and the dimension n. We should note that the cross terms $G_r(x, y_j) G_r(y_k, y)$ $(j \ne k)$ disappear from the right hand side of (3.15).

If $N = 1$, then (3.18) is nothing but (3.15). If $N \le 2$, then Ancona's ingenious trick [6, Théorème 7.3] gives (3.18) from (3.15). However, the proof is rather complicated and we postpone the proof to Section 3.9. The remaining arguments are rather easy and hold for arbitrary $N \ge 1$, provided that (3.18) holds. Let us show the weak boundary Harnack principle defined by Ancona [6, Définition 2.3].

Lemma 3.7.5 (Weak boundary Harnack principle). *Let $\xi \in \partial\Omega$ have a system of local reference points $y_1, \cdots, y_N \in \Omega \cap S(\xi, r)$ of order N with factor η for $0 < r < r_\xi$. Moreover, suppose that* (3.18) *holds. Let $h_0, \cdots, h_N \in \mathcal{H}_\xi$. Then*

$$h_0(x) \le A \sum_{j=1}^{N} \frac{h_0(y_j)}{h_j(y_j)} h_j(x) \quad \text{for } x \in \Omega \setminus B(\xi, \eta^9 r), \tag{3.19}$$

where A is a constant depending only on c_J, η, A_ξ and the dimension n.

Proof. In (3.18), we replace the roles of x and y and write z for y. By dilation and changing A_3, we obtain from the symmetry of the Green function that if $x \in \Omega \cap S(\xi, \eta^9 r)$ and $z \in \Omega \cap S(\xi, \eta^{21} r)$, then

$$G_r(x,z) \le A r^{n-2} \sum_{j=1}^{N} G_r(x, z_j) G_r(z_j, z),$$

where $z_1, \cdots, z_N \in \Omega \cap S(\xi, \eta^{12} r)$ are local reference points. Moreover, for each z_j, we can find a local reference point $y_{k(j)} \in \Omega \cap S(\xi, r)$ such that

$$k_{\widetilde{\Omega}_r \setminus \{x,z\}}(z_j, y_{k(j)}) \le A k_{\widetilde{\Omega}_r}(z_j, y_{k(j)}) + A \le A.$$

By Corollary 6.1.2, we have $G_r(x, z_j) \approx G_r(x, y_{k(j)})$ and $G_r(z_j, z) \approx G_r(y_{k(j)}, z)$, whenever $x \in \Omega \cap S(\xi, \eta^9 r)$ and $z \in \Omega \cap S(\xi, \eta^{21} r)$, where the constants of comparison are depend only on η and n. Hence we obtain that if $x \in \Omega \cap S(\xi, \eta^9 r)$ and $z \in \Omega \cap S(\xi, \eta^{21} r)$, then

$$G_r(x,z) \le A r^{n-2} \sum_{j=1}^{N} G_r(x, y_j) G_r(y_j, z). \tag{3.20}$$

Let $\delta = \eta^{-3} r$ and $\rho = \eta^{21} r$. We see that the regularized reduced function ${}^{\widetilde{\Omega}_r}\widehat{R}_{h_0}^{\Omega \cap (S(\xi,\delta) \cup S(\xi,\rho))}$ in $\widetilde{\Omega}_r$ is a Green potential of measures μ concentrated on $\Omega \cap S(\xi, \delta)$ and ν on $\Omega \cap S(\xi, \rho)$

such that ${}^{\widetilde{\Omega}_r}\widehat{R}_{h_0}^{\Omega\cap(S(\xi,\delta)\cup S(\xi,\rho))} = h_0$ on $\Omega \cap B(\xi,\delta) \setminus \overline{B(\xi,\rho)}$. It follows from (3.18) and (3.20) that for $x \in \Omega \cap S(\xi, \eta^9 r)$,

$$
\begin{aligned}
h_0(x) &= \int_{\Omega\cap S(\xi,\delta)} G_r(x,y)d\mu(y) + \int_{\Omega\cap S(\xi,\rho)} G_r(x,z)d\nu(z) \\
&\le Ar^{n-2}\sum_{j=1}^{N}\left(\int_{\Omega\cap S(\xi,\delta)} G_r(x,y_j)G_r(y_j,y)d\mu(y) + \int_{\Omega\cap S(\xi,\rho)} G_r(x,y_j)G_r(y_j,z)d\nu(z)\right) \\
&= Ar^{n-2}\sum_{j=1}^{N} G_r(x,y_j)h_0(y_j).
\end{aligned}
$$

Let $\varepsilon = 1-\eta^9$. We observe from (3.12) and the Harnack inequality that $h_j(y_j)r^{n-2}G_r(x,y_j) \approx h_j(x)$ for $x \in S(y_j, \varepsilon\delta_\Omega(y_j))$, and so $h_j(y_j)r^{n-2}G_r(x,y_j) \le Ah_j(x)$ for $x \in \Omega \cap S(\xi,\eta^9 r) \subset \widetilde{\Omega}_r \setminus B(y_j, \varepsilon\delta_\Omega(y_j))$ by the maximum principle. Hence (3.19) follows for $x \in \Omega \setminus B(\xi, \eta^9 r)$ by the maximum principle. □

3.8 Proof of Theorem 3.1.1 (ii)

In order to prove Theorem 3.1.1 (ii), it is sufficient to show the following proposition.

Proposition 3.8.1. *Let Ω be a general John domain in $\mathbb{R}^n$ and let $\xi \in \partial\Omega$ have a system of local reference points of order N. Suppose that $N \le 2$. Then the number of minimal functions in $\mathcal{H}_\xi$ is at most 2. Furthermore, if $N = 1$, then $\mathcal{H}_\xi$ consists of exactly one minimal function.*

For the proof, we prepare the following lemma.

Lemma 3.8.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $\xi \in \partial\Omega$. If $h \in \mathcal{H}_\xi$, then the measure associated with h in the Martin representation is concentrated on minimal Martin boundary points at ξ.*

Proof. By the Martin representation, there is a unique measure μ on $\Delta(\Omega)$ such that $\mu(\Delta(\Omega) \setminus \Delta_1(\Omega)) = 0$ and

$$
h(x) = \int_{\Delta(\Omega)} K_\Omega(x,\zeta)\,d\mu(\zeta) \quad \text{for } x \in \Omega.
$$

We now write $\Delta(\xi;\Omega)$ for the set of all Martin boundary points at ξ. Let E be a compact subset of $\Delta(\Omega) \setminus \Delta(\xi;\Omega)$ and let $\{E_j\}$ be a decreasing sequence of compact neighborhoods of E in the Martin topology such that $(E_1 \cap \Omega) \cap B(\xi, r_1) = \emptyset$ for some $r_1 > 0$ and $\bigcap_j E_j = E$. Then we have by [8, Corollary 9.1.4]

$$
{}^{\Omega}\widehat{R}_h^{E_j\cap\Omega}(x) = \int_{\Delta_1(\Omega)} {}^{\Omega}\widehat{R}_{K_\Omega(\cdot,\zeta)}^{E_j\cap\Omega}(x)d\mu(\zeta) \quad \text{for } x \in \Omega.
$$

Noting that $\lim_{j\to\infty} {}^{\Omega}\widehat{R}_h^{E_j\cap\Omega}$ is bounded and harmonic on Ω and vanishes quasi-everywhere on $\partial\Omega$ since h is the kernel function at ξ, we have by the monotone convergence

$$0 = \lim_{j\to\infty} {}^{\Omega}\widehat{R}_h^{E_j\cap\Omega}(x_0) = \int_{\Delta_1(\Omega)} \lim_{j\to\infty} {}^{\Omega}\widehat{R}_{K_\Omega(\cdot,\zeta)}^{E_j\cap\Omega}(x_0)d\mu(\zeta). \tag{3.21}$$

Let $\zeta \in E\cap\Delta_1(\Omega)$. Then $E_j\cap\Omega$ is not minimally thin at ζ with respect to Ω for each j (see [8, Lemma 9.1.5]), and so $\lim_{j\to\infty} {}^{\Omega}\widehat{R}_{K_\Omega(\cdot,\zeta)}^{E_j\cap\Omega}(x_0) = K_\Omega(x_0,\zeta) = 1$. Hence $\mu(E) = 0$ by (3.21). Thus the lemma follows. □

Proof of Proposition 3.8.1 for $N = 1$. As stated in the first paragraph of Section 3.6, $\mathcal{H}_\xi$ is non-empty. Let $h_0, h_1 \in \mathcal{H}_\xi$. Let $\{r_j\}$ be a sequence such that $r_j \to 0$ and take a local reference point $y_1^j \in \Omega\cap S(\xi, r_j)$. Then one of the inequalities $h_0(y_1^j) \le h_1(y_1^j)$ and $h_1(y_1^j) \le h_0(y_1^j)$ holds for infinitely many j. Hence Lemma 3.7.5 with $N = 1$ yields that either $h_0 \le Ah_1$ or $h_1 \le Ah_0$ holds on Ω. Moreover we suppose that h_0 and h_1 are minimal. Then $h_0 \equiv h_1$ in any case. This implies that $\mathcal{H}_\xi$ is singleton. Moreover, the Martin representation theorem and Lemma 3.8.2 show that the element in $\mathcal{H}_\xi$ is minimal. □

Proof of Proposition 3.8.1 for $N = 2$. As we shall show in the next section that (3.18) holds for $N = 2$, and hence Lemma 3.7.5 holds for $N = 2$. We follow the proof of Ancona [6, Théoremè 2.5]. We slightly generalize the proof of Proposition 3.8.1 for $N = 1$. Let $h_0, h_1, h_2 \in \mathcal{H}_\xi$, and take a decreasing sequence $\{r_j\}$ such that $r_j \to 0$. For each r_j sufficiently small, we find reference points $y_i^j \in D\cap S(\xi, r_j)$ with $i = 1, 2$. For a moment, we fix j and consider $\max_{0\le k\le 2} h_k(y_1^j)$. Then we find $k(j)$ such that $h_{k(j)} = \max_{0\le k\le 2} h_k(y_1^j)$. This holds for infinitely many j, so that there is $k_1 \in \{0,1,2\}$ such that

$$h_{k_1}(y_1^j) = \max_{0\le k\le 2} h_k(y_1^j) \tag{3.22}$$

for infinitely many j. We also find $k_2 \in \{0,1,2\}$ such that

$$h_{k_2}(y_2^j) = \max_{0\le k\le 2} h_k(y_2^j)$$

for infinitely many j satisfying (3.22). Thus

$$h_k(y_i^j) \le h_{k_i}(y_i^j) \quad \text{for all } k \in \{0,1,2\} \text{ and } i \in \{1,2\}$$

holds for infinitely many j. If necessary relabeling h_0, h_1, h_2, we may assume that $k_1 \ne 0$ and $k_2 \ne 0$. Then Lemma 3.7.5 yields that

$$h_0(x) \le A\sum_{i=1}^{2} \frac{h_0(y_i^j)}{h_{k_i}(y_i^j)} h_{k_i}(x) \le A\sum_{k=1}^{2} h_k(x) \quad \text{for } x \in \Omega\setminus B(\xi, \eta^{-3}r_j).$$

This holds for infinitely many j. Letting $j\to\infty$, we obtain

$$h_0 \le A\sum_{k=1}^{2} h_k \quad \text{on } \Omega.$$

This, together with Lemma 3.6.2, completes the proof. □

3.9 Proof of (3.18) and open problem

Lemma 3.9.1. *Let $\xi \in \partial\Omega$ have a system of local reference points $y_1, y_2 \in \Omega \cap S(\xi, r)$ of order2 with factor η for $0 < r < r_\xi$. If $x \in \Omega \cap S(\xi, \eta^9 r)$ and $y \in \Omega \cap S(\xi, \eta^{-3} r)$, then*

$$G_r(x,y) \le A r^{n-2} \sum_{j=1}^{2} G_r(x, y_j) G_r(y_j, y),$$

where A is a constant depending only on c_J, η, A_ξ and the dimension n.

Proof. Besides the local reference points $y_1, y_2 \in \Omega \cap S(\xi, r)$, we take local reference points $y_1^*, y_2^* \in \Omega \cap S(\xi, \eta^6 r)$ such that $\delta_\Omega(y_j^*) \ge A_\xi \eta^6 r$ for $j = 1, 2$ and

$$\min_{j=1,2}\{k_{\Omega\cap B(\xi,\eta^3 r)}(x, y_j^*)\} \le A_\xi \log \frac{\eta^6 r}{\delta_\Omega(x)} + A_\xi \quad \text{for } x \in \Omega \cap \overline{B(\xi, \eta^7 r)}.$$

See figure 3.5.

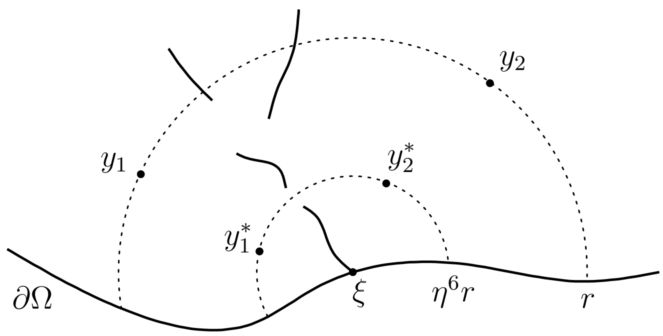

Figure 3.5: Position of y_1^*, y_2^*.

Then, for $j = 1, 2$, we have

$$\min_{k=1,2}\{k_{\Omega_r}(y_j^*, y_k)\} \le A_\xi \log \frac{r}{\delta_\Omega(y_j^*)} + A_\xi \le A.$$

So, we may assume either

(3.23) $$k_{\Omega_r}(y_1^*, y_1) \le A \quad \text{and} \quad k_{\Omega_r}(y_2^*, y_1) \le A,$$

or

(3.24) $$k_{\Omega_r}(y_1^*, y_1) \le A \quad \text{and} \quad k_{\Omega_r}(y_2^*, y_2) \le A,$$

by replacing the roles of y_1 and y_2, if necessary.

We first consider the case when (3.23) holds. Let $x \in \Omega \cap B(\xi, \eta^9 r)$. Then Lemma 3.7.4 for y_1^*, y_2^* and Corollary 6.1.2 (together with Lemma 6.1.3) yield that for $y \in \Omega \cap S(\xi, \eta^3 r)$,

$$G_r(x,y) \le A r^{n-2} \sum_{j,k} G_r(x, y_j^*) G_r(y_k^*, y) \le A r^{n-2} G_r(x, y_1) G_r(y_1, y).$$

By the maximum principle, we have this inequality for $y \in \Omega \cap S(\xi, \eta^{-3}r)$. Hence the lemma follows in this case.

We next consider the case when (3.24) holds. Let $\Phi = \{z \in \widetilde{\Omega}_r : G_r(z, y_1) \geq G_r(z, y_2)\}$. If either $x, y \in \Phi$ or $x, y \in \widetilde{\Omega}_r \setminus \Phi$, then the lemma follows from (3.17). Let us consider the remaining cases. Exchanging the roles of y_1 and y_2 if necessary, we may assume that $x \in \Phi \cap S(\xi, \eta^9 r)$ and $y \in (\widetilde{\Omega}_r \setminus \Phi) \cap S(\xi, \eta^{-3}r)$. Let $E = \Phi \setminus B(\xi, \eta^3 r)$ and consider the regularized reduced function ${}^{\widetilde{\Omega}_r}\widehat{R}^E_{G_r(\cdot,y)}$ in $\widetilde{\Omega}_r$. This function is represented as the Green potential of a measure μ supported in ∂E. For a moment, we let $z \in E$. Then we have from (3.17) for y_1^*, y_2^* and the maximum principle that

$$G_r(x, z) \leq Ar^{n-2} \sum_{j,k} G_r(x, y_j^*) G_r(y_k^*, z). \tag{3.25}$$

By (3.24) and Corollary 6.1.2 (together with Lemma 6.1.3), we have $G_r(x, y_j^*) \leq AG_r(x, y_j)$ for $j = 1, 2$. We also have $G_r(y_k^*, z) \leq AG_r(y_k, z)$ for $k = 1, 2$. In fact, if $z \in B(y_k, 2^{-1}(1 - \eta^6)\delta_\Omega(y_k))$, then

$$G_r(y_k, z) \approx |y_k - z|^{2-n} \geq Ar^{2-n} \geq AG_r(y_k^*, z);$$

if $z \in \widetilde{\Omega}_r \setminus B(y_k, 2^{-1}(1 - \eta^6)\delta_\Omega(y_k))$, then (3.24) and Corollary 6.1.2 (together with Lemma 6.1.3) yield $G_r(y_k^*, z) \approx G_r(y_k, z)$. Hence (3.25) becomes

$$G_r(x, z) \leq Ar^{n-2} \sum_{j,k} G_r(x, y_j) G_r(y_k, z) \leq Ar^{n-2} G_r(x, y_1) G_r(y_1, z)$$

by the definition of Φ. Therefore

$$\begin{aligned} {}^{\widetilde{\Omega}_r}\widehat{R}^E(x) &\leq Ar^{n-2} G_r(x, y_1) \int_E G_r(y_1, z) d\mu(z) \\ &= Ar^{n-2} G_r(x, y_1) {}^{\widetilde{\Omega}_r}\widehat{R}^E_{G_r(\cdot,y)}(y_1) \leq Ar^{n-2} G_r(x, y_1) G_r(y_1, y). \end{aligned} \tag{3.26}$$

Let $v_y = G_r(\cdot, y) - {}^{\widetilde{\Omega}_r}\widehat{R}^E_{G_r(\cdot,y)}$. Then

$$v_y = 0 \quad \text{quasi-everywhere on } E = \Phi \setminus B(\xi, \eta^3 r). \tag{3.27}$$

By (3.17) we have

$$v_y(x) \leq G_r(z, y) \leq Ar^{n-2} G_r(z, y_2) G_r(y_2, y) \quad \text{for } z \in \Omega \cap \partial\Phi \cap \overline{B(\xi, \eta^3 r)}. \tag{3.28}$$

Observe that $\Omega \cap \partial(\Phi \cap B(\xi, \eta^3 r)) \subset (\Phi \setminus B(\xi, \eta^3 r)) \cup (\Omega \cap \partial\Phi \cap \overline{B(\xi, \eta^3 r)})$. Hence (3.27), (3.28) and the maximum principle yield that

$$v_y \leq Ar^{n-2} G_r(\cdot, y_2) G_r(y_2, y) \quad \text{on } \Phi \cap \overline{B(\xi, \eta^3 r)}.$$

This, together with (3.26), implies that

$$G_r(x, y) \leq Ar^{n-2} (G_r(x, y_1) G_r(y_1, y) + G_r(x, y_2) G_r(y_2, y)).$$

Thus the proof of Lemma 3.9.1 is complete. □

When $N \geq 3$, we could not prove Lemma 3.9.1.

Open problem. *Let Ω be a John domain in $\mathbb{R}^n$ and let $\xi \in \partial\Omega$ have a system of local reference points of order N with factor $0 < \eta < 1$ for $0 < r < r_\xi$. Does* (3.18) *hold for $N \geq 3$?*

If we can prove this, then the similar argument with the proof of Proposition 3.8.1 for $N = 2$ yields that there are at most N minimal Martin boundary points at ξ.

3.10 Domains represented as union of convex sets

In this section, we consider a class of John domains represented as the union of a family of open convex sets. Especially, we give a sufficient condition for the Martin boundary and the Euclidean boundary to be homeomorphic.

In [5], Ancona considered a bounded domain represented as the union of a family of open balls with the same radius and gave a sufficient condition for the Martin boundary and the Euclidean boundary to be homeomorphic. Let $[x, y]$ stand for the (open) line segment with endpoints x and y. For $0 < \theta < \pi$, we denote by $\Gamma_\theta(x, y)$ the open circular cone $\{z \in \mathbb{R}^n : \angle zxy < \theta\}$ with vertex x, axis $[x, y]$ and aperture θ. Ancona says that a bounded domain Ω in $\mathbb{R}^n$ is *admissible* if

(A1) Ω is the union of a family of open balls with the same radius.

(A2) Let $\xi \in \partial\Omega$. If two balls, say B_1 and B_2, tangent to each other at ξ, then Ω includes a truncated circular cone $\Gamma_\theta(\xi, y) \cap B(\xi, r)$ for some $\theta > 0$, $r > 0$ and y in the hyperplane tangent to B_j at ξ. See Figure 3.6.

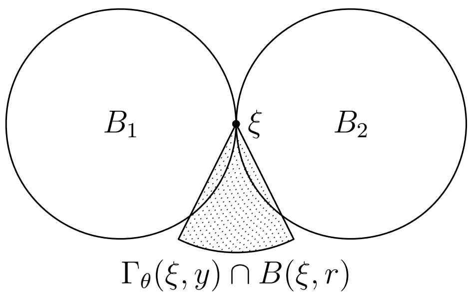

Figure 3.6: Condition (A2).

He proved the following.

Theorem (Ancona). *Let Ω be a bounded admissible domain in $\mathbb{R}^n$. Then every Euclidean boundary point of Ω has exactly one Martin boundary point and it is minimal. Moreover, the Martin boundary of Ω is homeomorphic to the Euclidean boundary.*

We now generalize (A1) and (A2). Let $\kappa \ge 1$ and $\rho_0 > 0$. We consider a bounded domain Ω in $\mathbb{R}^n$ such that

(I) Ω is the union of a family of open convex sets $\{C_\lambda\}_{\lambda\in\Lambda}$ such that

$$B(z_\lambda, \rho_0) \subset C_\lambda \subset B(z_\lambda, \kappa\rho_0) \quad \text{for some } z_\lambda \in C_\lambda. \tag{3.29}$$

Instead of (A2), we consider the following condition at $\xi \in \partial\Omega$.

(II) There exist positive constants $\theta_1 \le \sin^{-1}(1/\kappa)$ and $\rho_1 \le \rho_0 \cos\theta_1$ such that the union $\mathcal{C}(\xi)$ of truncated cones $\Gamma_{\theta_1}(\xi, y) \cap B(\xi, 2\rho_1)$ included in Ω is connected, that is,

$$\mathcal{C}(\xi) = \bigcup_{\substack{y\in\Omega \\ \Gamma_{\theta_1}(\xi,y)\cap B(\xi,2\rho_1)\subset\Omega}} \Gamma_{\theta_1}(\xi, y) \cap B(\xi, 2\rho_1) \quad \text{is connected.}$$

See Figure 3.7.

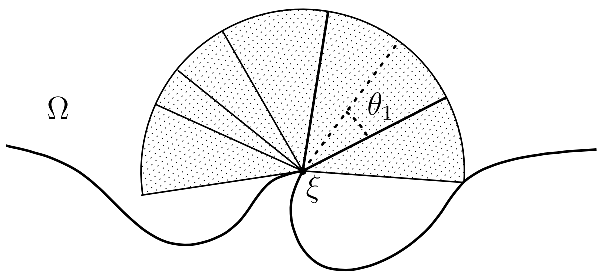

Figure 3.7: Condition (II).

We note that if Ω is a bounded domain represented as the union of a family of open balls with the same radius, then our condition (II) is equivalent to Ancona's condition (A2).

Our result is as follows.

Theorem 3.10.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ satisfying (I). If $\xi \in \partial\Omega$ satisfies the condition (II), then there exists exactly one Martin boundary point at ξ and it is minimal.*

Corollary 3.10.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ satisfying (I). If every $\xi \in \partial\Omega$ satisfies the condition (II), then the Martin boundary of Ω and the minimal Martin boundary of Ω are homeomorphic to the Euclidean boundary.*

Remark 3.10.3. The bounds $\theta_1 \le \sin^{-1}(1/\kappa)$ and $\rho_1 \le \rho_0 \cos\theta_1$ are sharp. See Examples 3.10.4 and 3.10.5 below. Under these assumptions, there exists a truncated circular cone $\Gamma_{\theta_1}(\xi, y) \cap B(\xi, 2\rho_1)$ included in Ω.

We now give examples for sharpness of the bounds $\theta_1 \le \sin^{-1}(1/\kappa)$ and $\rho_1 \le \rho_0 \cos\theta_1$ for when $n = 2$ and $\rho_0 = 1$. Write $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ and $\mathbb{R}^2_- = \{(x, y) \in \mathbb{R}^2 : y < 0\}$, and denote the interior of the convex hull of a set E by $\mathrm{co}(E)$.

Example 3.10.4 (The case $\theta_1 > \sin^{-1}(1/\kappa)$). Let $w_0 = (0, \kappa)$ and consider

$$D = \left(B(0, \kappa + 1) \setminus (\overline{B(0, \kappa - 1) \cap \mathbb{R}^2_+})\right) \cup \mathrm{co}(\{0\} \cup B(w_0, 1)).$$

See Figure 3.8. Then $\mathrm{co}(\{0\} \cup B(w_0, 1))$ does not contain $\Gamma_{\theta_1}(0, y) \cap B(0, 2\rho_1)$ for any $y \in \mathbb{R}^2$ and $\rho_1 > 0$. Therefore $\mathcal{C}(0) = B(0, 2\rho_1) \cap \mathbb{R}^2_-$. However there are two Martin boundary points at 0.

Example 3.10.5 (The case $0 < \theta_1 \le \sin^{-1}(1/\kappa)$ and $\rho_1 > \rho_0 \cos\theta_1$). The example is given in each case of $\kappa \ge 2$ and $1 < \kappa < 2$.

- *The case $\kappa \ge 2$.* Let $w_1 = (0, 1)$, $w_2 = (1, 1)$ and $w_3 = (1, 1 - \kappa)$. Consider

  $$D = \left(B(0, 5) \setminus (\overline{B(0, 3) \cap \mathbb{R}^2_+})\right) \cup B(w_1, 1) \cup \mathrm{co}(\{w_2\} \cup B(w_3, 1)).$$

  See Figure 3.9. Then $\mathrm{co}(\{w_2\} \cup B(w_3, 1)) \cap \mathbb{R}^2_+ \subset B(0, \sqrt{3})$. Since $\rho_1 > \cos\theta_1 \ge \cos(\pi/6) = \sqrt{3}/2$, each $\Gamma_{\theta_1}(0, y) \cap B(0, 2\rho_1)$ intersecting with $\mathrm{co}(\{w_2\} \cup B(w_3, 1)) \cap \mathbb{R}^2_+$ is not contained in D. Clearly, each $\Gamma_{\theta_1}(0, y) \cap B(0, 2\rho_1)$ is not included in $B(w_1, 1)$. Therefore $\mathcal{C}(0) = B(0, 2\rho_1) \cap \mathbb{R}^2_-$. However there are two Martin boundary points at 0.

- *The case $1 < \kappa < 2$.* Put $\varepsilon = 2^{-1}(\kappa + 1)^{-1}\kappa \cos(\sin^{-1}(1/\kappa))$. Let $w_1' = (0, 1)$, $w_2' = (\cos(\sin^{-1}(1/\kappa)) - \varepsilon, 1 - \kappa^{-1})$ and $w_3' = (\cos(\sin^{-1}(1/\kappa)) - \varepsilon, 1 - \kappa^{-1} - \kappa)$. Consider

  $$D = \left(B(0, 5) \setminus (\overline{B(0, 3) \cap \mathbb{R}^2_+})\right) \cup B(w_1', 1) \cup \mathrm{co}(\{w_2'\} \cup B(w_3', 1)).$$

  See Figure 3.10. Then $\mathrm{co}(\{w_2'\} \cup B(w_3', 1)) \cap \mathbb{R}^2_+ \subset B(0, 2\cos(\sin^{-1}(1/\kappa)))$. Since $\rho_1 > \cos\theta_1 \ge \cos(\sin^{-1}(1/\kappa))$, any $\Gamma_{\theta_1}(0, y) \cap B(0, 2\rho_1)$ are not included in $(\mathrm{co}(\{w_2'\} \cup B(w_3', 1)) \cap \mathbb{R}^2_+) \cup B(w_1', 1)$. Therefore $\mathcal{C}(0) = B(0, 2\rho_1) \cap \mathbb{R}^2_-$. However there are two Martin boundary points at 0.

Before proving Theorem 3.10.1, we show that a bounded domain satisfying (I) is a John domain. To this end, we prepare the following elementary lemma.

Lemma 3.10.6. *Let C be an open convex set. Then the following statements hold.*

(i) *δ_C is a concave function on C.*

(ii) *Let $x \in \overline{C}$. If C satisfies that $B(z, \rho_0) \subset C \subset B(z, \kappa\rho_0)$ for some $z \in C$, then*

$$|x - w| \le \kappa\delta_C(w) \quad \textit{for all } w \in [x, z].$$

Proof. We first show (i). Let $x, y \in C$, $w \in B(0, 1)$, and $0 \le t \le 1$. Then the points $x + \delta_C(x)w$ and $y + \delta_C(y)w$ lie in C. Hence the point $(1-t)x + ty + \{(1-t)\delta_C(x) + t\delta_C(y)\}w$ lies in C by convexity. Since w is an arbitrary point in $B(0, 1)$, it follows that

$$\delta_C((1 - t)x + ty) \ge (1 - t)\delta_C(x) + t\delta_C(y).$$

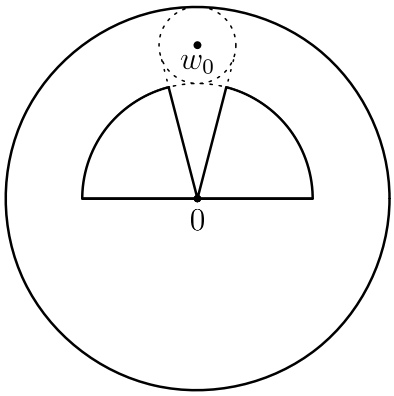

Figure 3.8: Example 3.10.4

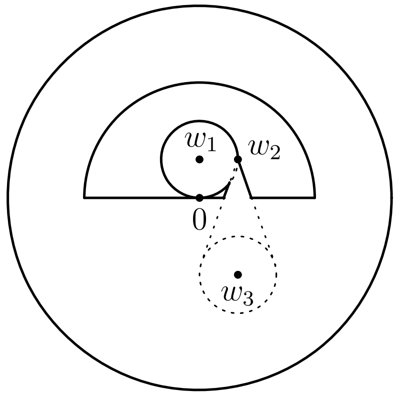

Figure 3.9: Example 3.10.5 for $\kappa \geq 2$

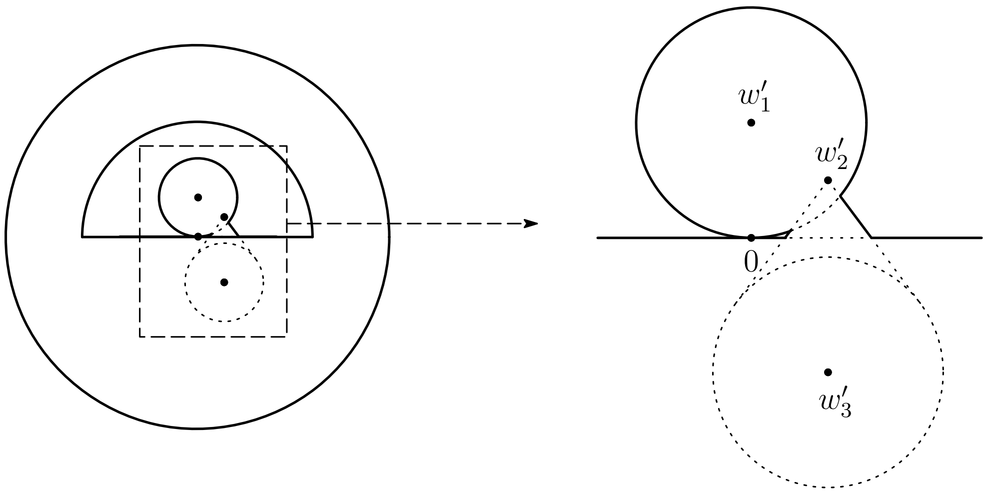

Figure 3.10: Example 3.10.5 for $1 < \kappa < 2$

We next show (ii). Let $x \in \overline{C}$ and $w \in [x, z]$. Writing $w = (1-t)x + tz$ with some $0 \leq t \leq 1$, we have $|x - w| \leq t|x - z| \leq t\kappa\rho_0$. Hence it follows from (i) that

$$\delta_C(w) \geq (1-t)\delta_C(x) + t\delta_C(z) \geq t\delta_C(z) \geq t\rho_0 \geq \kappa^{-1}|x - w|.$$

Thus the lemma is proved. □

Proposition 3.10.7. *Every bounded domain in $\mathbb{R}^n$ satisfying (I) is a John domain.*

Proof. Let Ω be a bounded domain represented as the union of a family $\{C_\lambda\}_{\lambda \in \Lambda}$ of open convex sets satisfying (3.29). We put $K_0 = \overline{\{z_\lambda : \lambda \in \Lambda\}}$, and let $x_0 \in K_0$. We first show that each $z_{\lambda_0} \in K_0$ can be connected to x_0 by a rectifiable curve in Ω satisfying (3.1). Since Ω is connected and δ_Ω is continuous on Ω, there is a positive $r_1 \leq \rho_0$ such that the closed set

$$E := \{x \in \Omega : \delta_\Omega(x) \geq r_1\} \quad \text{is connected.}$$

Then $K_0 \subset E$ by (3.29). By compactness of E, there is a positive integer M_1 such that $E \subset \bigcup_{j=1}^{M_1} B(y_j, 2^{-1}r_1)$, where $y_j \in E$. Let γ_1 be a curve in E connecting z_{λ_0} to x_0. We may assume, by relabeling if necessary, that

$$\begin{aligned}
&\gamma_1 \subset \bigcup_{j=1}^{M_2} B(y_j, 2^{-1}r_1) \quad \text{for some } M_2 \le M_1,\\
&B(y_j, 2^{-1}r_1) \cap B(y_{j+1}, 2^{-1}r_1) \cap E \ne \emptyset \quad \text{for } j = 1, \cdots, M_2 - 1,\\
&z_{\lambda_0} \in B(y_1, 2^{-1}r_1) \cap E \quad \text{and} \quad K_0 \in B(y_{M_2}, 2^{-1}r_1) \cap E.
\end{aligned}$$

Hence we can take a curve γ_2 in $\bigcup_{j=1}^{M_2} B(y_j, 2^{-1}r_1)$ connecting z_{λ_0} to x_0 so that

$$\ell(\gamma_2) \le M_2 r_1 \le M_1 r_1. \tag{3.30}$$

We see that $\bigcup_{j=1}^{M_2} B(y_j, 2^{-1}r_1) \subset \{x \in \Omega : \delta_\Omega(x) \ge 2^{-1}r_1\}$, so that $\delta_\Omega(z) \ge 2^{-1}r_1$ for all $z \in \gamma_2$. Hence we have

$$\ell(\gamma_2(z_{\lambda_0}, z)) \le \ell(\gamma_2) \le 2M_2\delta_\Omega(z) \le 2M_1\delta_\Omega(z) \quad \text{for all } z \in \gamma_2. \tag{3.31}$$

We next show (3.1) for a general point $x \in \Omega$. Let $x \in C_{\lambda_x}$. Applying Lemma 3.10.6 to $C = C_{\lambda_x}$ and $z = z_{\lambda_x}$, we have $|x - w| \le \kappa\delta_\Omega(w)$ for all $w \in [x, z_{\lambda_x}]$. Let γ_3 be a curve in Ω connecting z_{λ_x} to x_0 and satisfying (3.30) and (3.31). We define the curve γ in Ω connecting x to x_0 by

$$\gamma = [x, z_{\lambda_x}] \cup \gamma_3.$$

Then $\ell(\gamma) \le \kappa\rho_0 + M_1r_1$. It suffices to show that γ satisfies (3.1) for $z \in \gamma_3$.
Case 1 : $z \in \gamma_3 \cap B(z_{\lambda_x}, \rho_0/2)$. In this case, we have

$$\delta_\Omega(z) \ge \frac{\rho_0}{2} \ge \frac{\rho_0}{2(\kappa\rho_0 + M_1r_1)}\ell(\gamma(x, z)).$$

Case 2 : $z \in \gamma_3 \setminus B(z_{\lambda_x}, \rho_0/2)$. By the property of γ_3, we have

$$\delta_\Omega(z) \ge \frac{1}{2M_1}\ell(\gamma(z_{\lambda_x}, z) \ge \frac{\rho_0}{4M_1} \ge \frac{\rho_0}{4M_1(\kappa\rho_0 + M_1r_1)}\ell(\gamma(x, z)).$$

Thus the proposition is established. □

In order to prove Theorem 3.10.1, it is enough to show, by Propositions 3.8.1 and 3.10.7, that if $\xi \in \partial\Omega$ satisfies the condition (II), then ξ has a system of local reference points of order 1.

Proposition 3.10.8. *Let Ω be a bounded domain in $\mathbb{R}^n$ satisfying (I). If $\xi \in \partial\Omega$ satisfies the condition (II), then ξ has a system of local reference points of order 1.*

To this end, we prepare some lemmas. We may assume, by translation and dilation, that $\xi = 0$ and $\rho_1 = 1$. The aperture $\theta_1 \le \sin^{-1}(1/\kappa)$ is fixed and we write $\Gamma(x, y)$ for $\Gamma_{\theta_1}(x, y)$ to simplify the notation. Note that $1 = \rho_1 \le \rho_0 \cos\theta_1$, so that $0 < \theta_1 < \pi/2$ and $\rho_0 \ge \sec\theta_1$. Let C_λ be a convex set such that $B(z_\lambda, \rho_0) \subset C_\lambda \subset B(z_\lambda, \kappa\rho_0)$. If $x \in \overline{C}_\lambda \setminus B(z_\lambda, \rho_0)$, then

$$\Gamma(x, z_\lambda) \cap B(x, 2) \subset \mathrm{co}(\{x\} \cup B(z_\lambda, \rho_0)) \subset C_\lambda, \tag{3.32}$$

where $\mathrm{co}(\{x\} \cup B(z_\lambda, \rho_0))$ is the convex hull of $\{x\} \cup B(z_\lambda, \rho_0)$. Let

$$\mathcal{Y} = \{y \in S(0,1) : \Gamma(0,y) \cap B(0,2) \subset \Omega\}.$$

We first show that $\mathcal{Y}$ is non-empty and that the point 0 can be accessible along a ray issuing from the origin toward a point in $\mathcal{Y}$.

Lemma 3.10.9. *There is a positive constant $r_0 < 1$ such that if $C_\lambda \cap B(0, r_0) \neq \emptyset$, then $C_\lambda \cap \mathcal{Y} \neq \emptyset$. In particular, $\mathcal{Y} \neq \emptyset$.*

Proof. Suppose to the contrary that there is a sequence $\{C_{\lambda_j}\}$ with $\mathrm{dist}(0, C_{\lambda_j}) \to 0$ and $C_{\lambda_j} \cap \mathcal{Y} \neq \emptyset$. Let z_{λ_j} be such that $B(z_{\lambda_j}, \rho_0) \subset C_{\lambda_j} \subset B(z_{\lambda_j}, \kappa\rho_0)$. Taking a subsequence if necessary, we may assume that z_j converges, say to z_0. We claim that

$$\Gamma(0, z_0) \cap B(0,2) \subset \bigcup_j C_{\lambda_j}. \tag{3.33}$$

Let $x \in \Gamma(0, z_0) \cap B(0,2)$. Then $\angle x0z_0 < \theta_1$ and $|x| < 2$ by definition. From our assumption, we can find $x_{\lambda_j} \in \partial C_{\lambda_j}$ with $x_{\lambda_j} \to 0$ as $j \to \infty$. Therefore we have by continuity that $\angle x x_{\lambda_j} z_{\lambda_j} < \theta_1$ and $|x - x_{\lambda_j}| < 2$ for j sufficiently large. Then it follows from (3.32) that

$$x \in \Gamma(x_{\lambda_j}, z_{\lambda_j}) \cap B(x_{\lambda_j}, 2) \subset \mathrm{co}(\{x_{\lambda_j}\} \cup B(z_{\lambda_j}, \rho_0)) \subset C_{\lambda_j}.$$

Thus (3.33) follows. Now we let $y_0 = z_0/|z_0|$. Then we have by definition and (3.33) that $y_0 \in \mathcal{Y} \cap \bigcup_j C_{\lambda_j}$. However, this contradicts $C_{\lambda_j} \cap \mathcal{Y} = \emptyset$ for all j. Hence the lemma follows. $\square$

Let C be a convex set. As shown in Lemma 3.10.6, the function δ_C is concave. Therefore we have

$$\delta_C(z) \geq \frac{|z-y|}{|x-y|}\delta_C(x) + \frac{|x-z|}{|x-y|}\delta_C(y) \quad \text{for } z \in [x,y], \tag{3.34}$$

whenever $x \neq y \in \overline{C}$.

Lemma 3.10.10. *Let $0 < r_0 < 1$ be as in Lemma 3.10.9, and let $0 < r < \min\{r_0, 3^{-1}\sin\theta_1\}$. If $C_\lambda \cap B(0,r) \neq \emptyset$ and $y \in C_\lambda \cap \mathcal{Y}$, then there exists a point $w \in C_\lambda \cap \Gamma(0,y) \cap B(0, 3r/\sin\theta_1)$ such that*

$$\delta_{C_\lambda \cap \Gamma(0,y)}(w) \geq \frac{\sin\theta_1}{4} r.$$

Proof. Let $x \in C_\lambda \cap B(0,r)$. Then $[x,y] \subset C_\lambda$. We observe that there is a point $w_1 \in [x,y] \cap \overline{\Gamma(0,y)}$ with $|w_1| \leq r/\sin\theta_1$. In fact, if $x \in \overline{\Gamma(0,y)}$, then we may take $w_1 = x$. Otherwise, letting w_1 be the intersection of $[x,y]$ and $\partial\Gamma(0,y)$, we have

$$r > \mathrm{dist}(x, [0,y]) \geq \mathrm{dist}(w_1, [0,y]) = |w_1| \sin\theta_1,$$

so that $|w_1| \le r/\sin\theta_1$. Since $|w_1 - y| \ge 1 - r/\sin\theta_1$ and $3r/\sin\theta_1 < 1$, we can find a point $w_2 \in [w_1, y] \subset C_\lambda \cap \Gamma(0,y)$ with $|w_1 - w_2| = r/\sin\theta_1$. By (3.34) with $C = \Gamma(0,y)$, we obtain

$$\delta_{\Gamma(0,y)}(w_2) \ge \frac{|w_1 - w_2|}{|w_1 - y|}\delta_{\Gamma(0,y)}(y) \ge \frac{r/\sin\theta_1}{r/\sin\theta_1 + 1}\sin\theta_1 > \frac{r}{2}.$$

Moreover, we have $|w_2| \le 2r/\sin\theta_1$. Since $|w_2 - z_\lambda| \ge \rho_0 - 2r/\sin\theta_1 > r$ by $3r/\sin\theta_1 < 1 \le \rho_0$, we can take a point $w \in [w_2, z_\lambda] \subset C_\lambda$ such that $|w - w_2| = r/4$. Then it follows from (3.34) with $C = C_\lambda$ that

$$\delta_{C_\lambda}(w) \ge \frac{|w - w_2|}{|z_\lambda - w_2|}\delta_{C_\lambda}(z_\lambda) \ge \frac{r/4}{\kappa\rho}\rho_0 \ge \frac{\sin\theta_1}{4}r.$$

Hence we conclude that

$$\delta_{\Gamma(0,y)\cap C_\lambda}(w) \ge \min\left\{\frac{r}{2} - \frac{r}{4}, \frac{\sin\theta_1}{4}r\right\} = \frac{\sin\theta_1}{4}r,$$

and

$$|w| \le |w - w_2| + |w_2 - w_1| + |w_1| \le \frac{r}{4} + \frac{r}{\sin\theta_1} + \frac{r}{\sin\theta_1} < \frac{3r}{\sin\theta_1}.$$

Thus the lemma is proved. □

We fix a point $y_1 \in \mathcal{Y}$, and let $y_r = ry_1$ for $0 < r < 1$. Then $y_r \in \Omega \cap S(0,r)$ and $\delta_\Omega(y_r) \ge r\sin\theta_1$. Let $0 < \eta^3 < 6^{-1}\sin\theta_1$ and write $\Omega_r = \Omega \cap B(0, \eta^{-3}r)$.

Lemma 3.10.11. *Let $0 < r_0 < 1$ be as in Lemma 3.10.9. Then there is a positive constant A such that if $0 < r < r_0$, then*

$$k_{\Omega_r}(ry, y_r) \le A \quad \textit{for } y \in \mathcal{Y}.$$

Proof. Note that $\mathcal{C}(0) \cap S(0,1)$ is connected since the cone $\mathcal{C}(0)$ is connected. We observe that there is a closed connected subset E of $\mathcal{C}(0) \cap S(0,1)$ and $0 < r_1 \le \sin\theta_1$ such that $\mathcal{Y} \subset E$ and $\operatorname{dist}(E, \partial\mathcal{C}(0)) \ge r_1$. Then $y, y_1 \in E$. In view of the compactness of E, we can take a curve γ in $\mathcal{C}(0) \cap S(0,1)$ joining y and y_1 such that $\delta_{\mathcal{C}(0)}(z) \ge 2^{-1}r_1$ for all $z \in \gamma$ and $\ell(\gamma) \le Ar_1$, where A is a constant depending only on a covering constant of E. Let γ_r be the image of γ in $S(0,r)$ under dilation. Then we have

$$k_{\Omega_r}(ry, y_r) \le \int_{\gamma_r} \frac{ds}{\delta_\Omega(z)} \le \frac{Ar_1 r}{2^{-1}r_1 r} = 2A.$$

Thus the lemma follows. □

Let us prove Proposition 3.10.8.

Proof of Proposition 3.10.8. By translation and dilation, we may assume that $\xi = 0$ and $\rho_1 = 1$. Let $0 < r_0 < 1$ be as in Lemma 3.10.9 and suppose that $0 < \eta^3 < 6^{-1}\sin\theta_1$ and $y_r = ry_1$ are as above. Let $0 < r < \min\{r_0, 3^{-1}\sin\theta_1\}$, It is sufficient to show that

$$k_{\Omega_r}(x, y_r) \le A\log\frac{r}{\delta_\Omega(x)} + A \quad \text{for } x \in \Omega \cap \overline{B(0,\eta r)}, \tag{3.35}$$

where A is a constant independent of x and r. Let $x \in \Omega \cap \overline{B(0, \eta r)}$. Then there is a convex set C_λ containing x and there is $y \in C_\lambda \cap \mathcal{Y}$ by Lemma 3.10.9. By Lemma 3.10.10, we find a point $w \in C_\lambda \cap \Gamma(0, y) \cap B(0, 3r/\sin\theta_1)$ such that $\delta_{C_\lambda \cap \Gamma(0,y)}(w) \geq 4^{-1} r \sin\theta_1$. Since

$$\delta_{\Omega_r}(z) \geq \delta_{C_\lambda}(z) \geq \frac{|x-z|}{|x-w|}\delta_{C_\lambda}(w) \geq \frac{\sin^2\theta_1}{16}|x-z| \quad \text{for } z \in [x, w]$$

by $[x, w] \subset B(0, 2^{-1}\eta^{-3}r)$ and (3.34), it follows that

$$k_{\Omega_r}(x, w) \leq \int_{[x,w]} \frac{ds(z)}{\delta_{\Omega_r}(z)} \leq A \log \frac{r}{\delta_\Omega(x)} + A.$$

Since

$$\delta_{\Omega_r}(z) \geq \delta_{\Gamma(0,y)}(z) \geq \frac{|w-z|}{|w-ry|}\delta_{\Gamma(0,y)}(ry) \geq \frac{\sin^2\theta}{4}|x-z| \quad \text{for } z \in [w, ry],$$

it also follows that

$$k_{\Omega_r}(w, ry) \leq \int_{[w,ry]} \frac{ds(z)}{\delta_{\Omega_r}(z)} \leq A \log \frac{r}{\delta_\Omega(x)} + A.$$

Hence we obtain from the triangle inequality and Lemma 3.10.11 that

$$k_{\Omega_r}(x, y_r) \leq k_{\Omega_r}(x, w) + k_{\Omega_r}(w, ry) + k_{\Omega_r}(ry, y_r) \leq A \log \frac{r}{\delta_\Omega(x)} + A,$$

and thus Proposition 3.10.8 is established. $\square$

Chapter 4

Boundary behavior of Martin kernels

This chapter is based on the manuscript [H3].

4.1 Motivation and results

One of the purposes of this chapter is to show the boundary growth of the Martin kernel on a Lipschitz domain. This is motivated by earlier works due to Burdzy [11, 12], Carroll [14, 15] and Gardiner [23]. We write **0** for the origin of $\mathbb{R}^n$ ($n \geq 2$) to distinct from $0 \in \mathbb{R}$, and denote $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and $e = (\mathbf{0}', 1)$. Suppose that $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfies $\phi(\mathbf{0}') = 0$ and the Lipschitz property:

$$|\phi(x') - \phi(y')| \leq L|x' - y'| \quad (x', y' \in \mathbb{R}^{n-1})$$

for some positive constant L. We put $\Omega_\phi = \{(x', x_n) : x_n > \phi(x')\}$ and set

$$I^+ = \int_{\{|x'|<1\}} \frac{\max\{\phi(x'), 0\}}{|x'|^n} dx', \tag{4.1}$$

$$I^- = \int_{\{|x'|<1\}} \frac{\max\{-\phi(x'), 0\}}{|x'|^n} dx'. \tag{4.2}$$

In [11], Burdzy obtained a result on the angular derivative problem of analytic functions in a Lipschitz domain. The following theorem was an important step in his work.

Theorem A. *Suppose that I^+ and I^- are as in* (4.1) *and* (4.2). *If $I^+ < \infty$ and $I^- = \infty$, then*

$$\lim_{t \to 0+} \frac{G_{\Omega_\phi}(te, e)}{t} = \infty.$$

Burdzy's approach was based on probabilistic methods and the minimal fine topology. An analytic proof was given by Carroll [14]. Gardiner [23] also gave a simple proof of Theorem A. In [15], Carroll investigated the boundary behavior of $G_{\Omega_\phi}(te, e)/t$ in other cases.

Theorem B. *Suppose that I^+ and I^- are as in* (4.1) *and* (4.2). *The following statements hold.*

(i) *If* $I^+ = \infty$ *and* $I^- < \infty$*, then*

$$\lim_{t \to 0+} \frac{G_{\Omega_\phi}(te, e)}{t} = 0.$$

(ii) *If* $I^+ < \infty$ *and* $I^- < \infty$*, then the limit of* $G_{\Omega_\phi}(te, e)/t$*, as* $t \to 0+$*, exists and*

$$0 < \lim_{t \to 0+} \frac{G_{\Omega_\phi}(te, e)}{t} < \infty.$$

Theorems A and B show the relationship between the convergence of the integrals I^+, I^- and the boundary decay of the Green function of Ω_ϕ. We are now interested in a relationship between the convergence of the integrals I^+, I^- and the boundary growth of the Martin kernel $K_{\Omega_\phi}(\cdot, \mathbf{0})$ of Ω_ϕ with pole at the origin.

Theorem 4.1.1. *Suppose that* I^+ *and* I^- *are as in* (4.1) *and* (4.2)*. The following statements hold.*

(i) *If* $I^+ < \infty$ *and* $I^- = \infty$*, then*

$$\lim_{t \to 0+} t^{n-1} K_{\Omega_\phi}(te, \mathbf{0}) = 0.$$

(ii) *If* $I^+ = \infty$ *and* $I^- < \infty$*, then*

$$\lim_{t \to 0+} t^{n-1} K_{\Omega_\phi}(te, \mathbf{0}) = \infty.$$

(iii) *If* $I^+ < \infty$ *and* $I^- < \infty$*, then the limit of* $t^{n-1} K_{\Omega_\phi}(te, \mathbf{0})$*, as* $t \to 0+$*, exists and*

$$0 < \lim_{t \to 0+} t^{n-1} K_{\Omega_\phi}(te, \mathbf{0}) < \infty.$$

When $I^+ = \infty$ and $I^- = \infty$, the limit of $t^{n-1} K_{\Omega_\phi}(te, \mathbf{0})$ may take any values 0, positive and finite, or ∞, as the following simple example shows.

Example 4.1.2. To simplify the notation, we write $\mathbb{R}^{n-1}_{1+} = \{x' \in \mathbb{R}^{n-1} : x_1 \geq 0\}$ and $\mathbb{R}^{n-1}_{1-} = \{x' \in \mathbb{R}^{n-1} : x_1 \leq 0\}$ in this example.

(i) If $\phi(x')$ is equal to $x_1/2$ on $\mathbb{R}^{n-1}_{1+}$ and x_1 on $\mathbb{R}^{n-1}_{1-}$, then

$$\lim_{t \to 0+} t^{n-1} K_{\Omega_\phi}(te, \mathbf{0}) = 0.$$

(ii) If $\phi(x')$ is equal to x_1 on $\mathbb{R}^{n-1}_{1+}$ and x_1 on $\mathbb{R}^{n-1}_{1-}$, then the limit of $t^{n-1} K_{\Omega_\phi}(te, \mathbf{0})$, as $t \to 0+$, exists and

$$0 < \lim_{t \to 0+} t^{n-1} K_{\Omega_\phi}(te, \mathbf{0}) < \infty.$$

(iii) If $\phi(x')$ is equal to x_1 on $\mathbb{R}^{n-1}_{1+}$ and $x_1/2$ on $\mathbb{R}^{n-1}_{1-}$, then

$$\lim_{t\to 0+} t^{n-1} K_{\Omega_\phi}(te, \mathbf{0}) = \infty.$$

It is easy to check that $I^+ = \infty$ and $I^- = \infty$. The value of the limit in each case follows from [27, Theorems 1 and 2].

Let $\mathbb{R}^n_+ = \{(x', x_n) : x_n > 0\}$. As we will state in Section 4.5, the convergence of the integrals I^+ and I^- is connected with the minimal thinness of the sets $\mathbb{R}^n_+ \setminus \Omega_\phi$ and $\Omega_\phi \setminus \mathbb{R}^n_+$. See Section 4.2 for the definition of minimal thinness. Since $K_{\mathbb{R}^n_+}(te, \mathbf{0}) = t^{1-n}$, Theorem 4.1.1 may be interpreted as the relationship between the minimal thinness of the sets $\mathbb{R}^n_+ \setminus \Omega_\phi$, $\Omega_\phi \setminus \mathbb{R}^n_+$ and the boundary behavior of the quotient of Martin kernels of Ω_ϕ and $\mathbb{R}^n_+$. So, given two intersecting domains Φ and Ψ, it is valuable to investigate a relationship between the minimal thinness of the sets $\Phi \setminus \Psi$, $\Psi \setminus \Phi$ and the boundary behavior of the quotient of Martin kernels of Φ and Ψ.

4.2 Statements for general domains

To state our results for general domains, we need a definition of minimal fine limit. Recall that a subset E on Ω is said to be minimally thin at $\xi \in \Delta_1(\Omega)$ with respect to Ω if

$${}^{\Omega}\widehat{R}^{E}_{K_\Omega(\cdot,\xi)}(z) < K_\Omega(z, \xi) \quad \text{for some } z \in \Omega.$$

Minimal thinness enables us to equip the minimal fine topology in the Martin compactification of Ω. Roughly speaking, the minimal fine topology is the collection of subsets W of the Martin compactification such that $\Omega \setminus W$ is minimally thin at every point of $W \cap \Delta_1(\Omega)$. See [8, Definition 9.2.3] for the precise definition. Let U be a minimal fine neighborhood of $\xi \in \Delta_1(\Omega)$. We say that a function f on U has *minimal fine limit* l at ξ with respect to Ω if there is a subset E on Ω, minimally thin at ξ with respect to Ω, such that $f(x) \to l$ as $x \to \xi$ along $U \setminus E$, and then we write

$$\underset{\Omega}{\mathrm{mf}}\text{-}\lim_{x\to\xi} f(x) = l.$$

We note from the definition that a function is not necessarily defined in whole of a domain when we consider minimal fine limit.

Theorem 4.2.1. *Suppose that Φ and Ψ are Greenian domains in $\mathbb{R}^n$ such that $\Phi \cap \Psi$ is a non-empty domain. Let $\xi \in \Delta_1(\Phi)$, where ξ is in the closure of $\Phi \cap \Psi$ in the Martin compactification of Φ. Let $\zeta \in \Delta_1(\Psi)$, where ζ is in the closure of $\Phi \cap \Psi$ in the Martin compactification of Ψ. If $\Phi \setminus \Psi$ is minimally thin at ξ with respect to Φ, then $K_\Psi(\cdot, \zeta)/K_\Phi(\cdot, \xi)$ has a finite minimal fine limit at ξ with respect to Φ. Furthermore, the following statements hold.*

(i) *If $\Psi \setminus \Phi$ is not minimally thin at ζ with respect to Ψ, then*

$$\underset{\Phi}{\mathrm{mf}}\text{-}\lim_{x\to\xi} \frac{K_\Psi(x,\zeta)}{K_\Phi(x,\xi)} = 0.$$

(ii) *If $\Psi \setminus \Phi$ is minimally thin at ζ with respect to Ψ, where ζ is a point such that*

$$K_\Psi(\cdot,\zeta) - {}^{\Psi}R^{\Psi\setminus\Phi}_{K_\Psi(\cdot,\zeta)} = \alpha\big(K_\Phi(\cdot,\xi) - {}^{\Phi}R^{\Phi\setminus\Psi}_{K_\Phi(\cdot,\xi)}\big) \quad \text{on } \Phi\cap\Psi \tag{4.3}$$

for some positive constant α, then

$$0 < \underset{\Phi}{\mathrm{mf}}\text{-}\lim_{x\to\xi} \frac{K_\Psi(x,\zeta)}{K_\Phi(x,\xi)} < \infty.$$

(iii) *If $\Psi \setminus \Phi$ is minimally thin at ζ with respect to Ψ, where ζ is a point such that* (4.3) *is not satisfied, then*

$$\underset{\Phi}{\mathrm{mf}}\text{-}\lim_{x\to\xi} \frac{K_\Psi(x,\zeta)}{K_\Phi(x,\xi)} = 0.$$

For Lipschitz domains, Theorem 4.2.1 can be restated as follows. We note from [25] that each Euclidean boundary point of a Lipschitz domain has a unique Martin boundary point and it is minimal. So we identify a Martin boundary point with a Euclidean boundary point.

Corollary 4.2.2. *Suppose that Φ and Ψ are Lipschitz domains in $\mathbb{R}^n$ such that $\Phi \cap \Psi$ is also a Lipschitz domain. Let $y \in \partial\Phi \cap \partial\Psi$, and suppose that $\Phi \setminus \Psi$ is minimally thin at y with respect to Φ. The following statements hold.*

(i) *If $\Psi \setminus \Phi$ is not minimally thin at y with respect to Ψ, then*

$$\underset{\Phi}{\mathrm{mf}}\text{-}\lim_{x\to y} \frac{K_\Psi(x,y)}{K_\Phi(x,y)} = 0.$$

(ii) *If $\Psi \setminus \Phi$ is minimally thin at y with respect to Ψ, then*

$$0 < \underset{\Phi}{\mathrm{mf}}\text{-}\lim_{x\to y} \frac{K_\Psi(x,y)}{K_\Phi(x,y)} < \infty.$$

Remark 4.2.3. If $\Phi \setminus \Psi$ is "not" minimally thin at y with respect to Φ and $\Psi \setminus \Phi$ is "not" minimally thin at y with respect to Ψ, then the limit of $K_\Psi(\cdot,y)/K_\Phi(\cdot,y)$ may take any values 0, positive and finite, or ∞. See Example 4.1.2.

4.3 Characterization of minimal thinness for a difference of two subdomains

Naïm [31, Théorème 11] gave a characterization of the minimal thinness for a difference of two subdomains in terms of Green functions of each domain, which played an important role in the proof of Theorems A and B. In order to prove Theorem 4.2.1, we give a new characterization of the minimal thinness for a difference.

Lemma 4.3.1. *Suppose that Ω is a Greenian domain in $\mathbb{R}^n$ and that D is a subdomain of Ω. Let $\xi \in \Delta_1(\Omega)$, where ξ is in the closure of D in the Martin compactification of Ω. The following statements are equivalent.*

(i) *$\Omega \setminus D$ is minimally thin at ξ with respect to Ω;*

(ii) *there exists $\eta \in \Delta_1(D)$ such that*

$$\operatorname*{mf}_{D}\text{-}\lim_{x\to\eta} \frac{K_\Omega(x,\xi)}{K_D(x,\eta)} > 0. \tag{4.4}$$

Furthermore, the point $\eta \in \Delta_1(D)$ in (ii) *is uniquely determined and the corresponding Martin kernel is represented as*

$$K_D(\cdot,\eta) = \alpha\big(K_\Omega(\cdot,\xi) - {}^{\Omega}R^{\Omega\setminus D}_{K_\Omega(\cdot,\xi)}\big) \quad \textit{on } D$$

for some positive constant α.

Remark 4.3.2. We note in Lemma 4.3.1 that the minimal fine limit in (4.4) exists and satisfies that

$$\operatorname*{mf}_{D}\text{-}\lim_{x\to\eta} \frac{K_\Omega(x,\xi)}{K_D(x,\eta)} = \mu^D_{K_\Omega(\cdot,\xi)}(\{\eta\}) = \inf_{x\in D} \frac{K_\Omega(x,\xi)}{K_D(x,\eta)} = \liminf_{x\to\eta} \frac{K_\Omega(x,\xi)}{K_D(x,\eta)} < \infty, \tag{4.5}$$

where $\mu^D_{K_\Omega(\cdot,\xi)}$ is the measure on $\Delta(D)$ associated with $K_\Omega(\cdot,\xi)$ in the Martin representation. See [8, Theorems 9.2.6 and 9.3.3]. Thus the minimal thinness of $\Omega \setminus D$ can be also characterized in terms of any of quantities in (4.5) instead of the minimal fine limit.

For the proof of Lemma 4.3.1, we need the following lemmas. Lemma 4.3.3 can be deduced from [8, Theorems 9.2.6 and 9.3.3]. Lemma 4.3.4 is due to Naïm [31, Théorème 15] (cf. [8, Theorem 9.5.5]).

Lemma 4.3.3. *Let E be a subset of a Greenian domain Ω in $\mathbb{R}^n$ and let $\xi \in \Delta_1(\Omega)$. The following statements are equivalent.*

(i) *E is minimally thin at ξ with respect to Ω;*

(ii) *there exists a positive superharmonic function u on Ω such that*

$$\inf_{x\in\Omega} \frac{u(x)}{K_\Omega(x,\xi)} < \inf_{x\in E} \frac{u(x)}{K_\Omega(x,\xi)}.$$

Lemma 4.3.4. *Suppose that Ω is a Greenian domain in $\mathbb{R}^n$ and that D is a subdomain of Ω. Let $\xi \in \Delta_1(\Omega)$, where ξ is in the closure of D in the Martin compactification of Ω. Assume that $\Omega \setminus D$ is minimally thin at ξ with respect to Ω, and let $\eta \in \Delta_1(D)$ be a point such that*

$$K_D(\cdot,\eta) = \alpha\big(K_\Omega(\cdot,\xi) - {}^{\Omega}R^{\Omega\setminus D}_{K_\Omega(\cdot,\xi)}\big) \quad \textit{on } D$$

for some positive constant α. The following statements for a subset E of D are equivalent.

(i) *E is minimally thin at η with respect to D;*

(ii) *E is minimally thin at ξ with respect to Ω.*

We say that a property holds quasi-everywhere if it holds apart from a polar set. The following lemma is elementary. For the convenience sake of the reader, we give a proof.

Lemma 4.3.5. *Let D be a Greenian domain in $\mathbb{R}^n$ and let $\zeta \in \Delta_1(D)$. Then $K_D(\cdot, \zeta)$ vanishes quasi-everywhere on ∂D.*

Proof. Let V be a Martin topology (closed) neighborhood of ζ with respect to D. Then $V \cap D$ is not minimally thin at ζ with respect to D. Therefore we have from [8, Theorem 6.9.1] that

$$K_D(x, \zeta) = {}^D R^{V\cap D}_{K_D(\cdot,\zeta)}(x) = H^{D\setminus V}_{K_D(\cdot,\zeta)\chi_{\partial V\cap D}}(x) \quad \text{for } x \in D \setminus V,$$

where $H^{D\setminus V}_{K_D(\cdot,\zeta)\chi_{\partial V\cap D}}$ denotes the Perron-Wiener-Brelot solution of the Dirichlet problem in $D \setminus V$ with boundary function $K_D(\cdot, \zeta)$ on $\partial(V \cap D) \cap D$ and 0 on ∂D. Since V is arbitrary, we obtain the lemma. □

Let Ω be a domain in $\mathbb{R}^n$ and let D be a subdomain of Ω. If h is a positive harmonic function on D which vanishes quasi-everywhere on $\partial D \cap \Omega$ and is bounded near each point of $\partial D \cap \Omega$, then we see from [8, Theorem 5.2.1] that h has a subharmonic extension h^* to Ω which is valued 0 quasi-everywhere on $\partial D \cap \Omega$ and everywhere on $\Omega \setminus \overline{D}$. In what follows we use the mark $*$, like as h^*, to denote such a subharmonic extension.

Let us prove Lemma 4.3.1.

Proof of Lemma 4.3.1. By [31, Théorème 12] (cf. [8, Theorem 9.5.5]), we can easily show that (i) implies (ii). In fact, $f := K_\Omega(\cdot, \xi) - {}^\Omega R^{\Omega\setminus D}_{K_\Omega(\cdot,\xi)}$ is a minimal harmonic function on D, and so there exists $\eta \in \Delta_1(D)$ such that $K_D(\cdot, \eta) = f/f(x_0)$ on D. Hence we obtain

$$\inf_{x\in D} \frac{K_\Omega(x, \xi)}{K_D(x, \eta)} \geq f(x_0) > 0,$$

and thus (4.4) follows from (4.5).

We next show that (ii) implies (i). We may assume that $\Omega \setminus D$ is non-polar. Let $\eta \in \Delta_1(D)$ be a point such that

$$\alpha := \underset{D}{\mathrm{mf}}\text{-}\lim_{x\to\eta} \frac{K_\Omega(x, \xi)}{K_D(x, \eta)} > 0.$$

By (4.5), we have $K_D(\cdot, \eta) \leq \alpha^{-1} K_\Omega(\cdot, \xi)$ on D. Also, $K_D(\cdot, \eta)$ vanishes quasi-everywhere on $\partial D \cap \Omega$ by Lemma 4.3.5. Thus $K_D^*(\cdot, \eta)$ is well-defined as a subharmonic function on Ω and is dominated by $\alpha^{-1} K_\Omega(\cdot, \xi)$ on Ω. Let $u = \alpha^{-1} K_\Omega(\cdot, \xi) - K_D^*(\cdot, \eta)$. Then u is superharmonic on Ω. Since $\Omega \setminus D$ is non-polar, there is a point in $\Omega \setminus D$ at which u is positive. Therefore the minimum principle yields that u is positive on Ω. Also, we have that

$$\inf_{x\in\Omega} \frac{u(x)}{K_\Omega(x, \xi)} = \alpha^{-1} - \sup_{x\in D} \frac{K_D(x, \eta)}{K_\Omega(x, \xi)} < \alpha^{-1},$$

$$\inf_{x\in\Omega\setminus(D\cup F)} \frac{u(x)}{K_\Omega(x, \xi)} = \alpha^{-1} - \sup_{x\in\Omega\setminus(D\cup F)} \frac{K_D^*(x, \eta)}{K_\Omega(x, \xi)} = \alpha^{-1},$$

where F is a polar set in $\partial D \cap \Omega$ such that $K_D^*(\cdot, \eta) > 0$ on F. Hence it follows from Lemma 4.3.3 that $\Omega \setminus (D \cup F)$ is minimally thin at ξ with respect to Ω, and so is $\Omega \setminus D$.

We finally show the uniqueness of $\eta \in \Delta_1(D)$. We suppose to the contrary that there exists $\zeta \in \Delta_1(D)$ such that $K_D(\cdot, \zeta) \leq \beta K_\Omega(\cdot, \xi)$ on D and $K_D(\cdot, \zeta)$ is different from $K_D(\cdot, \eta) := \gamma\big(K_\Omega(\cdot, \xi) - {}^\Omega R^{\Omega \setminus D}_{K_\Omega(\cdot, \xi)}\big)$, where β and γ are some positive constants. We may assume that β is the smallest number satisfying $K_D(\cdot, \zeta) \leq \beta K_\Omega(\cdot, \xi)$ on D. Since $\xi \in \Delta_1(\Omega)$, it follows that $\beta K_\Omega(\cdot, \xi)$ is the least harmonic majorant of $K_D^*(\cdot, \zeta)$ on Ω. Let W be a Martin topology neighborhood of ζ with respect to D such that η is apart from W. Then $W \cap D$ is minimally thin at η with respect to D. Thus the minimal thinness of $\Omega \setminus D$ at ξ with respect to Ω, together with Lemma 4.3.4, yields that $W \cap D$ is minimally thin at ξ with respect to Ω.

On the other hand, since $W \cap D$ is not minimally thin at ζ with respect to D, we have

$$K_D(\cdot, \zeta) = {}^D\widehat{R}^{W \cap D}_{K_D(\cdot, \zeta)} \leq \beta \, {}^D\widehat{R}^{W \cap D}_{K_\Omega(\cdot, \xi)} \leq \beta \, {}^\Omega\widehat{R}^{W \cap D}_{K_\Omega(\cdot, \xi)} \quad \text{on } D.$$

Since $\beta K_\Omega(\cdot, \xi)$ is the least one among superharmonic functions u on Ω satisfying $K_D^*(\cdot, \zeta) \leq u$ on Ω, we have ${}^\Omega\widehat{R}^{W \cap D}_{K_\Omega(\cdot, \xi)} = K_\Omega(\cdot, \xi)$ on Ω, so that $W \cap D$ is not minimally thin at ξ with respect to Ω. Thus we obtain a contradiction, and hence the uniqueness of $\eta \in \Delta_1(D)$ is established. The proof of Lemma 4.3.1 is complete. □

4.4 Proof of Theorem 4.2.1

In this section, we give a proof of Theorem 4.2.1.

Proof of Theorem 4.2.1. In order to prove the first assertion, we assume that $\Phi \setminus (\Phi \cap \Psi)$ is minimally thin at ξ with respect to Φ. Let $\eta \in \Delta_1(\Phi \cap \Psi)$ be a point such that $K_{\Phi \cap \Psi}(\cdot, \eta) = \alpha\big(K_\Phi(\cdot, \xi) - {}^\Phi R^{\Phi \setminus \Psi}_{K_\Phi(\cdot, \xi)}\big)$ on $\Phi \cap \Psi$ for some positive constant α. Then we have by Lemma 4.3.1 with $D := \Phi \cap \Psi$ and $\Omega := \Phi$ that

$$0 < \underset{\Phi \cap \Psi}{\mathrm{mf}}\text{-}\lim_{x \to \eta} \frac{K_\Phi(x, \xi)}{K_{\Phi \cap \Psi}(x, \eta)} < \infty. \tag{4.6}$$

It also follows from [8, Theorem 9.3.3] that $K_\Psi(\cdot, \zeta)/K_{\Phi \cap \Psi}(\cdot, \eta)$ has a finite minimal fine limit at η with respect to $\Phi \cap \Psi$. The minimal thinness of $\Phi \setminus (\Phi \cap \Psi)$ at ξ with respect to Φ, together with Lemma 4.3.4 with $D := \Phi \cap \Psi$ and $\Omega := \Phi$, concludes that $K_\Psi(\cdot, \zeta)/K_\Phi(\cdot, \xi)$ has a finite minimal fine limit at ξ with respect to Φ.

To prove (i), we assume in addition that $\Psi \setminus (\Phi \cap \Psi)$ is not minimally thin at ζ with respect to Ψ. Then Lemma 4.3.1 with $D := \Phi \cap \Psi$ and $\Omega := \Psi$ shows that for any $\eta \in \Delta_1(\Phi \cap \Psi)$, the minimal fine limit in (4.4) is zero. Therefore we have

$$\underset{\Phi \cap \Psi}{\mathrm{mf}}\text{-}\lim_{x \to \eta} \frac{K_\Psi(x, \zeta)}{K_{\Phi \cap \Psi}(x, \eta)} = 0.$$

Hence (i) follows from (4.6) and Lemma 4.3.4 with $D := \Phi \cap \Psi$ and $\Omega := \Phi$.

To prove (ii), we assume in addition that $\Psi \setminus (\Phi \cap \Psi)$ is minimally thin at ζ with respect to Ψ, where ζ is a point in $\Delta_1(\Psi)$ such that (4.3) is satisfied. We note from (4.3) that $K_{\Phi\cap\Psi}(\cdot, \eta)$ is also written as $\beta\big(K_\Psi(\cdot, \zeta) - {}^{\Psi}R^{\Psi\setminus\Phi}_{K_\Psi(\cdot,\zeta)}\big)$ on $\Phi \cap \Psi$ for some positive constant β. Then we have by Lemma 4.3.1 with $D := \Phi \cap \Psi$ and $\Omega := \Psi$ that

$$0 < \underset{\Phi\cap\Psi}{\mathrm{mf}}\text{ - }\lim_{x\to\eta} \frac{K_\Psi(x, \zeta)}{K_{\Phi\cap\Psi}(x, \eta)} < \infty.$$

Therefore (ii) follows from (4.6) and Lemma 4.3.4 with $D := \Phi \cap \Psi$ and $\Omega := \Phi$.

To prove (iii), we assume in addition that $\Psi \setminus (\Phi \cap \Psi)$ is minimally thin at ζ with respect to Ψ, where ζ is a point in $\Delta_1(\Psi)$ such that (4.3) is not satisfied. Then the normalization $K_{\Phi\cap\Psi}(\cdot, \omega)$ of $K_\Psi(\cdot, \zeta) - {}^{\Psi}R^{\Psi\setminus\Phi}_{K_\Psi(\cdot,\zeta)}$ at a reference point is a minimal Martin kernel of $\Phi \cap \Psi$, but is different from $K_{\Phi\cap\Psi}(\cdot, \eta)$. We note from the uniqueness in Lemma 4.3.1 that for only $\omega \in \Delta_1(\Phi \cap \Psi)$, $K_\Psi(\cdot, \zeta)/K_{\Phi\cap\Psi}(\cdot, \omega)$ has a positive minimal fine limit at ω with respect to $\Phi \cap \Psi$. Therefore we have

$$\underset{\Phi\cap\Psi}{\mathrm{mf}}\text{ - }\lim_{x\to\eta} \frac{K_\Psi(x, \zeta)}{K_{\Phi\cap\Psi}(x, \eta)} = 0.$$

Hence (iii) follows from (4.6) and Lemma 4.3.4 with $D := \Phi\cap\Psi$ and $\Omega := \Phi$. Thus Theorem 4.2.1 is established. □

4.5 Proof of Theorem 4.1.1

In order to prove Theorem 4.1.1, we collect lemmas on relationships between the convergence of the integrals I^+, I^- in (4.1), (4.2) and the minimal thinness of the differences $\Omega_\phi \setminus \mathbb{R}^n_+, \mathbb{R}^n_+ \setminus \Omega_\phi$. See [23, Lemma 1 and Proof of Theorem 1] for Lemma 4.5.1 and [20, Theorem 4.2] for Lemma 4.5.2.

Lemma 4.5.1. *The following statements hold.*

(i) $I^+ < \infty$ *if and only if* $\mathbb{R}^n_+ \setminus \Omega_\phi$ *is minimally thin at* $\mathbf{0}$ *with respect to* $\mathbb{R}^n_+$*.*

(ii) *If* $I^+ < \infty$ *and* $I^- = \infty$*, then* $\Omega_\phi \setminus \mathbb{R}^n_+$ *is not minimally thin at* $\mathbf{0}$ *with respect to* Ω_ϕ*.*

Lemma 4.5.2. *Let* Ω *be a Greenian domain in* $\mathbb{R}^n$ *containing* $\mathbb{R}^n_+$*. Suppose that* Ω *has a unique Martin boundary point at infinity and it is minimal. If* $\Omega \setminus \overline{\mathbb{R}^n_+}$ *is minimally thin at* ∞ *with respect to* $\mathbb{R}^n_- := \{(x', x_n) : x_n < 0\}$*, then* $\Omega \setminus \mathbb{R}^n_+$ *is minimally thin at* ∞ *with respect to* Ω*.*

Lemma 4.5.3. *If* $I^- < \infty$*, then* $\Omega_\phi \setminus \mathbb{R}^n_+$ *is minimally thin at* $\mathbf{0}$ *with respect to* $\Omega_\phi \cup \mathbb{R}^n_+$*.*

Proof. By Lemma 4.5.1, we see that $\Omega_\phi \setminus \overline{\mathbb{R}^n_+}$ is minimally thin at $\mathbf{0}$ with respect to $\mathbb{R}^n_-$. Since minimal thinness is invariant under the inversion with respect to the unit sphere, it follows from Lemma 4.5.2 that $\Omega_\phi \setminus \mathbb{R}^n_+$ is minimally thin at $\mathbf{0}$ with respect to $\Omega_\phi \cup \mathbb{R}^n_+$. □

Lemma 4.5.4. *If $I^+ < \infty$ and $I^- < \infty$, then $\Omega_\phi \setminus \mathbb{R}^n_+$ is minimally thin at* $\mathbf{0}$ *with respect to* Ω_ϕ.

Proof. We note from Lemma 4.5.3 that $(\Omega_\phi \cup \mathbb{R}^n_+) \setminus \mathbb{R}^n_+$ is minimally thin at $\mathbf{0}$ with respect to $\Omega_\phi \cup \mathbb{R}^n_+$. Since $(\Omega_\phi \cup \mathbb{R}^n_+) \setminus \Omega_\phi$ is minimally thin at $\mathbf{0}$ with respect to $\Omega_\phi \cup \mathbb{R}^n_+$ by Lemmas 4.3.4 and 4.5.1, the lemma follows from Lemma 4.3.4. $\square$

Let us prove Theorem 4.1.1. We note in a Lipschitz domain that the existence of the minimal fine limit of the quotient of positive harmonic functions implies the existence of the non-tangential limit, and the both values coincide, since a non-tangential cone at a boundary point y is not minimally thin at y (cf. [25, Section 5]).

Proof of Theorem 4.1.1. We first show (i). Since $\mathbb{R}^n_+ \setminus \Omega_\phi$ is minimally thin at $\mathbf{0}$ with respect to $\mathbb{R}^n_+$ and $\Omega_\phi \setminus \mathbb{R}^n_+$ is not minimally thin at $\mathbf{0}$ with respect to Ω_ϕ by Lemma 4.5.1, it follows from Corollary 4.2.2 (i) with $\Phi := \mathbb{R}^n_+$ and $\Psi := \Omega_\phi$ that $K_{\Omega_\phi}(\cdot, \mathbf{0})/K_{\mathbb{R}^n_+}(\cdot, \mathbf{0})$ has minimal fine limit 0 at $\mathbf{0}$ with respect to $\mathbb{R}^n_+$, and hence $t^{n-1}K_{\Omega_\phi}(te, \mathbf{0})$ has limit 0 as $t \to 0+$.

We next show (ii). Since $(\Omega_\phi \cup \mathbb{R}^n_+) \setminus \mathbb{R}^n_+$ is minimally thin at $\mathbf{0}$ with respect to $\Omega_\phi \cup \mathbb{R}^n_+$ by Lemma 4.5.3, we have by Lemma 4.3.1 with $D := \mathbb{R}^n_+$ and $\Omega := \Omega_\phi \cup \mathbb{R}^n_+$ that $K_{\Omega_\phi \cup \mathbb{R}^n_+}(\cdot, \mathbf{0})/K_{\mathbb{R}^n_+}(\cdot, \mathbf{0})$ has a positive minimal fine limit at $\mathbf{0}$ with respect to $\mathbb{R}^n_+$, and hence $t^{n-1}K_{\Omega_\phi \cup \mathbb{R}^n_+}(te, \mathbf{0})$ has a positive limit as $t \to 0+$. Also, it follows from Lemmas 4.3.4 and 4.5.1 that $(\Omega_\phi \cup \mathbb{R}^n_+) \setminus \Omega_\phi$ is not minimally thin at $\mathbf{0}$ with respect to $\Omega_\phi \cup \mathbb{R}^n_+$. Therefore we have by Lemma 4.3.1 with $D := \Omega_\phi$ and $\Omega := \Omega_\phi \cup \mathbb{R}^n_+$ that $K_{\Omega_\phi \cup \mathbb{R}^n_+}(\cdot, \mathbf{0})/K_{\Omega_\phi}(\cdot, \mathbf{0})$ has minimal fine limit 0 at $\mathbf{0}$ with respect to Ω_ϕ, and hence $K_{\Omega_\phi}(te, \mathbf{0})/K_{\Omega_\phi \cup \mathbb{R}^n_+}(te, \mathbf{0})$ has limit ∞ as $t \to 0+$. Thus we conclude that $t^{n-1}K_{\Omega_\phi}(te, \mathbf{0})$ has limit ∞ as $t \to 0+$.

We finally show (iii). Since $\mathbb{R}^n \setminus \Omega_\phi$ is minimally thin at $\mathbf{0}$ with respect to $\mathbb{R}^n_+$ by Lemma 4.5.1 and $\Omega_\phi \setminus \mathbb{R}^n_+$ is minimally thin at $\mathbf{0}$ with respect to Ω_ϕ by Lemma 4.5.4, we have by Corollary 4.2.2 (ii) with $\Phi := \mathbb{R}^n_+$ and $\Psi := \Omega_\phi$ that $K_{\Omega_\phi}(\cdot, \mathbf{0})/K_{\mathbb{R}^n_+}(\cdot, \mathbf{0})$ has a positive and finite minimal fine limit at $\mathbf{0}$ with respect to $\mathbb{R}^n_+$, and hence $t^{n-1}K_{\Omega_\phi}(te, \mathbf{0})$ has a positive and finite limit as $t \to 0+$. $\square$

Chapter 5

Comparison estimates for the Green function and the Martin kernel

This chapter is based on the manuscript [H4].

5.1 Statements of results

We give comparison estimates of the Green function and the Martin kernel in a uniform domain. A proper subdomain Ω of $\mathbb{R}^n$ is said to be uniform if there exists a positive constant A such that each pair of points x and y in Ω can be connected by a rectifiable curve γ in Ω for which

$$\begin{aligned}&\ell(\gamma) \le A|x-y|,\\ &\min\{\ell(\gamma(x,z)), \ell(\gamma(z,y))\} \le A\delta_\Omega(z) \quad \text{for all } z \in \gamma.\end{aligned}$$

It was proved by Aikawa [3] that the Martin compactification of a bounded uniform domain is homeomorphic to the Euclidean closure. Moreover, all Martin boundary points are minimal. In the sequel, we identify the Martin compactification with the Euclidean closure. Furthermore, we denote a unique Martin boundary point at $\xi \in \partial\Omega$ by the same symbol ξ. For $\xi \in \partial\Omega$ and $\alpha > 1$, we write

$$\Gamma_\alpha(\xi) = \{x \in \Omega : |x-\xi| < \alpha\delta_\Omega(x)\}$$

for the non-tangential cone at ξ with aperture α.

Our result in higher dimensions is as follows.

Theorem 5.1.1. *Let Ω be a bounded uniform domain in $\mathbb{R}^n$ with $n \ge 3$, and let $\xi \in \partial\Omega$ and $\alpha > 1$. Then we have*

$$G_\Omega(x,x_0)K_\Omega(x,\xi) \approx |x-\xi|^{2-n} \quad \textit{for } x \in \Gamma_\alpha(\xi) \cap B(\xi, 2^{-1}\delta_\Omega(x_0)),$$

where the constant of comparison depends only on α and Ω.

We note that Theorem 5.1.1 does not hold in general when $n = 2$.

Example 5.1.2. Let $n = 2$. We consider $\Omega = B(\mathbf{0}, 1) \setminus \{\mathbf{0}\}$ and $x_0 = (1/2, 0)$. Then $K_\Omega(x, \mathbf{0}) = -(\log 2)^{-1} \log |x|$ and

$$G_\Omega(x, x_0) = G_{B(\mathbf{0},1)}(x, x_0) = \log \left(\frac{1}{2} \frac{|x - 4x_0|}{|x - x_0|} \right).$$

Hence we obtain

$$G_\Omega(x, x_0) K_\Omega(x, \mathbf{0}) \approx \log \frac{1}{|x|} \quad \text{for } x \in B(\mathbf{0}, 1/4) \setminus \{\mathbf{0}\}.$$

Let $\xi \in \partial\Omega$. We say that ξ satisfies the exterior condition if there exists a positive constant A_1 such that for each $r > 0$ sufficiently small, there is a point $z_r \in B(\xi, r) \setminus \overline{\Omega}$ such that $B(z_r, A_1 r) \subset \mathbb{R}^n \setminus \overline{\Omega}$.

Our result in two dimensions is as follows.

Theorem 5.1.3. *Let Ω be a bounded uniform domain in $\mathbb{R}^2$, and let $\alpha > 1$. The following statements hold.*

(i) *If $\xi \in \partial\Omega$ satisfies the exterior condition, then*

$$G_\Omega(x, x_0) K_\Omega(x, \xi) \approx 1 \quad \textit{for } x \in \Gamma_\alpha(\xi) \cap B(\xi, 2^{-1}\delta_\Omega(x_0)),$$

where the constant of comparison depends only on α and Ω.

(ii) *If $\xi \in \partial\Omega$ is an isolated point, then there exists $\delta > 0$ such that*

$$G_\Omega(x, x_0) K_\Omega(x, \xi) \approx \log \frac{1}{|x - \xi|} \quad \textit{for } x \in B(\xi, \delta) \setminus \{\xi\},$$

where the constant of comparison is independent of x.

We may deduce from Theorems 5.1.1 and 5.1.3 the following relationship between the boundary decay of the Green function and the boundary growth of the Martin kernel.

Corollary 5.1.4. *Let Ω be a bounded uniform domain in $\mathbb{R}^n$. Let $\xi \in \partial\Omega$, $\alpha > 1$ and $\beta > 0$. Suppose that $n \geq 3$. The following relationships hold.*

(i) $\displaystyle\lim_{x \to \xi, x \in \Gamma_\alpha(\xi)} \frac{G_\Omega(x, x_0)}{|x - \xi|^\beta} = \infty$ *if and only if* $\displaystyle\lim_{x \to \xi, x \in \Gamma_\alpha(\xi)} |x - \xi|^{n+\beta-2} K_\Omega(x, \xi) = 0.$

(ii) $\displaystyle\lim_{x \to \xi, x \in \Gamma_\alpha(\xi)} \frac{G_\Omega(x, x_0)}{|x - \xi|^\beta} = 0$ *if and only if* $\displaystyle\lim_{x \to \xi, x \in \Gamma_\alpha(\xi)} |x - \xi|^{n+\beta-2} K_\Omega(x, \xi) = \infty.$

Moreover, if we assume the exterior condition at ξ, then these relationships hold for $n \geq 2$.

5.2 Proofs of Theorems 5.1.1 and 5.1.3

Theorems 5.1.1 and 5.1.3 will be established by showing the propositions below.

Proposition 5.2.1. *Let Ω be a bounded uniform domain in $\mathbb{R}^n$ with $n \geq 3$. Then we have for $x \in \Omega \setminus B(x_0, 2^{-1}\delta_\Omega(x_0))$ and $y \in \overline{\Omega}$,*

$$G_\Omega(x, x_0)K_\Omega(x, y) \leq A|x - y|^{2-n}, \tag{5.1}$$

where A is a constant depending only on Ω.

This proposition follows from the 3G inequality. The 3G inequality was firstly proved in a Lipschitz domain by Cranston, Fabes and Zhao [17]. Aikawa and Lundh [4] extended it to a uniformly John domain. A uniformly John domain is more general than a uniform domain. We may state the 3G inequality in a uniform domain as follows.

Lemma (3G inequality). *Let Ω be a bounded uniform domain in $\mathbb{R}^n$ with $n \geq 3$. Then*

$$\frac{G_\Omega(x, y)G_\Omega(x, z)}{G_\Omega(y, z)} \leq A\big(|x - y|^{2-n} + |x - z|^{2-n}\big) \quad \text{for } x, y, z \in \Omega.$$

Now, applying the 3G inequality with $z = x_0$, we have by the continuity of $K_\Omega(x, \cdot)$ on $\overline{\Omega}$ that

$$K_\Omega(x, y)G_\Omega(x, x_0) \leq A\big(|x - y|^{2-n} + |x - x_0|^{2-n}\big) \quad \text{for } x \in \Omega \text{ and } y \in \overline{\Omega}.$$

Since

$$|x - y| \leq (\operatorname{diam}\Omega)\frac{2|x - x_0|}{\delta_\Omega(x_0)} \quad \text{for } x \in \Omega \setminus B(x_0, 2^{-1}\delta_\Omega(x_0)),$$

we obtain Proposition 5.2.1.

Proposition 5.2.2. *Let Ω be a bounded uniform domain in $\mathbb{R}^n$ with $n \geq 2$. Let $\xi \in \partial\Omega$, $\alpha > 1$ and $\kappa \geq 1$. Then we have for $x \in \Gamma_\alpha(\xi) \cap B(\xi, (2\kappa)^{-1}\delta_\Omega(x_0))$ and $y \in \overline{\Omega} \cap B(\xi, \kappa|x - \xi|)$,*

$$G_\Omega(x, x_0)K_\Omega(x, y) \geq A|x - y|^{2-n}, \tag{5.2}$$

where A is a constant depending only on α, κ and Ω.

For the proof of Proposition 5.2.2, we prepare some materials: the boundary Harnack principle proved in [3] and a lower estimate of the Green function.

Lemma 5.2.3. *Let Ω be a bounded uniform domain in $\mathbb{R}^n$ with $n \geq 2$. Then there exist constants $r_0 > 0$ and $A_2 > 1$ depending only on Ω with the following property: Let $\xi \in \partial\Omega$ and $0 < r < r_0$. Suppose that h_1 and h_2 are bounded positive harmonic functions on $\Omega \cap B(\xi, A_2 r)$ vanishing quasi-everywhere on $\partial\Omega \cap B(\xi, A_2 r)$. Then*

$$\frac{h_1(y)}{h_2(y)} \approx \frac{h_1(y')}{h_2(y')} \quad \text{for } y, y' \in \Omega \cap B(\xi, r),$$

where the constant of comparison depends only on Ω.

A uniform domain can be characterized in terms of the quasi-hyperbolic metric. Gehring and Osgood [24] showed that Ω is a uniform domain if and only if

$$k_\Omega(x,y) \le A\log\left[\left(\frac{|x-y|}{\delta_\Omega(x)}+1\right)\left(\frac{|x-y|}{\delta_\Omega(y)}+1\right)\right]+A \quad \text{for } x,y\in\Omega. \tag{5.3}$$

Lemma 5.2.4. *Let Ω be a uniform domain in $\mathbb{R}^n$ with $n\ge 2$ and let $x,y\in\Omega$ satisfy*

$$|x-y|\le A_3\min\{\delta_\Omega(x),\delta_\Omega(y)\}.$$

Then there exists a positive constant A depending only on A_3 and Ω such that

$$G_\Omega(x,y)\ge A|x-y|^{2-n}.$$

Proof. We may assume, without loss of generality, that $\delta_\Omega(x)\le\delta_\Omega(y)$ and $|x-y|\ge 2^{-1}\delta_\Omega(x)$. Take $w\in S(x,2^{-1}\delta_\Omega(x))$. Then $|y-w|\le 2|x-y|$, so that (5.3), Corollary 6.1.2 and Lemma 6.1.3 yield that

$$G_\Omega(x,y)\approx G_\Omega(x,w)\ge G_{B(x,\delta_\Omega(x))}(x,w)\approx\delta_\Omega(x)^{2-n}\ge A|x-y|^{2-n},$$

as required. □

Let us prove Proposition 5.2.2. If necessary, we write $A(a,b,\cdots)$ for a constant depending on $a,b,\cdots$.

Proof of Proposition 5.2.2. Let $x\in\Gamma_\alpha(\xi)\cap B(\xi,(2\kappa)^{-1}\delta_\Omega(x_0))$ and $y\in\Omega\cap B(\xi,\kappa|x-\xi|)$. Then $x,y\notin B(x_0,2^{-1}\delta_\Omega(x_0))$. Let A_4 be a constant sufficiently large so that $A_4\ge 2A_2$ and $A_4^{-1}\delta_\Omega(x_0)<r_0$, where A_2 and r_0 are constants appearing in Lemma 5.2.3. Then $r:=A_4^{-1}\delta_\Omega(x)<r_0$. We consider two cases.
Case 1 : $\delta_\Omega(y)<r$. Let $y^*\in\partial\Omega$ be such that $|y-y^*|=\delta_\Omega(y)$. Then $|x-y^*|\ge\delta_\Omega(x)\ge A_2 r$ and $|x_0-y^*|\ge\delta_\Omega(x_0)\ge\delta_\Omega(x)\ge A_2 r$. Therefore Lemma 5.2.3 yields that

$$K_\Omega(x,y)=\frac{G_\Omega(x,y)}{G_\Omega(x_0,y)}\approx\frac{G_\Omega(x,y_r)}{G_\Omega(x_0,y_r)},$$

where $y_r\in S(y^*,r)\cap\Omega$ is such that $\delta_\Omega(y_r)\approx r$. Since $y_r\notin B(x_0,2^{-1}\delta_\Omega(x_0))$ and $|x-y_r|\le A(\kappa,A_4,\alpha)r$, it follows from (5.3), Corollary 6.1.2, Lemma 6.1.3 and Lemma 5.2.4 that

$$G_\Omega(x,x_0)\approx G_\Omega(y_r,x_0)\quad\text{and}\quad G_\Omega(x,y_r)\ge A|x-y_r|^{2-n}\ge A|x-y|^{2-n},$$

so that

$$G_\Omega(x,x_0)K_\Omega(x,y)\approx G_\Omega(x,y_r)\ge A|x-y|^{2-n}.$$

Case 2 : $\delta_\Omega(y)\ge r$. Since $|x-y|\le A(\kappa,A_4,\alpha)r$, it follows from (5.3), Corollary 6.1.2, Lemma 6.1.3 and Lemma 5.2.4 that

$$G_\Omega(x,x_0)\approx G_\Omega(y,x_0)\quad\text{and}\quad G_\Omega(x,y)\ge A|x-y|^{2-n},$$

and so (5.2) holds in this case.

Finally, letting y to the boundary, we also obtain (5.2) for $y\in\partial\Omega\cap B(\xi,\kappa|x-\xi|)$. Thus Proposition 5.2.2 is proved. □

Proposition 5.2.5. *Let Ω be a bounded uniform domain in $\mathbb{R}^2$. The following statements hold.*

(i) *If $\xi \in \partial\Omega$ satisfies the exterior condition, then*

$$G_\Omega(x, x_0)K_\Omega(x, \xi) \le A \quad \textit{for } x \in \Gamma_\alpha(\xi) \cap B(\xi, 2^{-1}\delta_\Omega(x_0)),$$

where A is a constant depending only on α and Ω.

(ii) *If $\xi \in \partial\Omega$ is an isolated point, then there exists $\delta > 0$ such that*

$$G_\Omega(x, x_0)K_\Omega(x, \xi) \approx \log \frac{1}{|x - \xi|} \quad \textit{for } x \in B(\xi, \delta) \setminus \{\xi\},$$

where the constant of comparison is independent of x.

In the proof of Proposition 5.2.5, we use the following lemma.

Lemma 5.2.6. *Let Ω be a domain in $\mathbb{R}^2$ and let $\alpha > 1$. Suppose that $\xi \in \partial\Omega$ satisfies the exterior condition. Then there exists a positive constant A depending only on α and A_1 such that*

$$G_\Omega(x, y) \le A \quad \textit{for } x \in \Gamma_\alpha(\xi) \textit{ and } y \in \Omega \setminus B(x, 2^{-1}\delta_\Omega(x)).$$

Proof. Let $x \in \Gamma_\alpha(\xi)$ and put $r = |x - \xi|$. By our assumption, there is $z_r \in B(\xi, r) \setminus \overline{\Omega}$ such that $B(z_r, A_1 r) \subset \mathbb{R}^2 \setminus \overline{\Omega}$. We now write y^* for the inverse of y with respect to $S(z_r, A_1 r)$. Then we obtain that for $y \in S(x, 2^{-1}\delta_\Omega(x))$,

$$G_\Omega(x, y) \le G_{\mathbb{R}^2 \setminus \overline{B(z_r, A_1 r)}}(x, y) = \log\left(\frac{|y - z_r|}{A_1 r}\frac{|x - y^*|}{|x - y|}\right) \le A(\alpha, A_1).$$

Hence the maximum principle yields the lemma. $\square$

Proof of Proposition 5.2.5. We first show (i). Let $x \in \Gamma_\alpha(\xi) \cap B(\xi, 2^{-1}\delta_\Omega(x_0))$ and put $r = A_4^{-1}\delta_\Omega(x)$ as in the proof of Proposition 5.2.2. Repeating the argument in Case 1 in the proof of Proposition 5.2.2 and using the same symbol, we have for $y \in \Omega$ sufficiently near ξ,

$$K_\Omega(x, y) \approx \frac{G_\Omega(x, y_r)}{G_\Omega(x_0, y_r)}.$$

Since $G_\Omega(x, x_0) \approx G_\Omega(y_r, x_0)$, Lemma 5.2.6 yields that

$$G_\Omega(x, x_0)K_\Omega(x, y) \le A.$$

Tending y to ξ, we obtain (i).

We next show (ii). Let $\xi \in \partial\Omega$ be an isolated point and let $\delta = 2^{-1}\min\{1, \operatorname{dist}(\xi, \partial\Omega \setminus \{\xi\}), \delta_\Omega(x_0)\}$. We have that for $x \in B(\xi, \delta)$,

$$\begin{aligned} K_\Omega(x, \xi) &= \beta_1 G_{\Omega \cup \{\xi\}}(x, \xi) \ge \beta_1 G_{B(\xi, 2\delta)}(x, \xi) = \beta_1 \log \frac{2\delta}{|x - \xi|} \\ &\ge 2\beta_1 \delta \log \frac{1}{|x - \xi|}, \end{aligned}$$

where β_1 is some positive constant. On the other hand, since $(B(\xi, \operatorname{diam}\Omega) \setminus \{\xi\}) \setminus \Omega$ is minimally thin at ξ with respect to $B(\xi, \operatorname{diam}\Omega)$, it follows from Lemma 4.3.1 and (4.5) that

$$K_\Omega(x,\xi) \le \beta_2 K_{B(\xi,\operatorname{diam}\Omega)\setminus\{\xi\}}(x,\xi) = \beta_2 \log \frac{\operatorname{diam}\Omega}{|x-\xi|} \le A(\delta,\Omega) \log \frac{1}{|x-\xi|}.$$

We also have by the Harnack inequality that for $x \in B(\xi,\delta)$,

$$G_\Omega(x,x_0) = G_{\Omega\cup\{\xi\}}(x,x_0) \approx G_{\Omega\cup\{\xi\}}(\xi,x_0).$$

Hence we obtain (ii). $\square$

5.3 Equivalence between ordinary thinness and minimally thinness

Throughout this section, we suppose that $n \ge 3$. Let E be a subset of $\mathbb{R}^n$ and let $\xi \in \mathbb{R}^n$ be a limit point of E. We write $E_j = \{x \in E : 2^{-j-1} \le |x-\xi| \le 2^{-j}\}$, and denote by $\widehat{R}_1^E$ the regularized reduced function of the constant function 1 relative to E on $\mathbb{R}^n$. By Wiener's criterion, we can define thinness of a set as follows: a set E is thin at ξ if and only if $\sum_{j=1}^\infty \widehat{R}_1^{E_j}(\xi) < +\infty$ (see [8, Theorem 7.7.2]), which is also equivalent to there exists a positive superharmonic function u on $\mathbb{R}^n$ such that $u(\xi) < +\infty$ and $u(x) \to +\infty$ as $x \to \xi$ along E (see [8, Theorem 7.2.3]). By [8, Theorem 9.2.7], the minimal thinness is characterized as follows: let $E \subset \Omega$ and let ξ be a minimal Martin boundary point of Ω, which is a Martin topology limit point of E. Then E is minimally thin at ξ with respect to Ω if and only if there exists a Green potential $G_\Omega\mu$ on Ω such that $\int K_\Omega(x,\xi)d\mu(x) < +\infty$ and

$$\lim_{y\to\xi,\, y\in E} \frac{G_\Omega\mu(y)}{G_\Omega(x_0,y)} = +\infty.$$

Let E be a set contained in a non-tangential cone at a boundary point ξ. In [28], Lelong-Ferrand proved in the half space that E is thin at ξ if and only if E is minimally thin at ξ. Aikawa [1] proved this equivalence in a Lipschitz domain. The purpose of this section is to extend this result to a uniform domain using Propositions 5.2.1 and 5.2.2. We note again that the minimal Martin boundary of a bounded uniform domain coincides with its Euclidean boundary.

Theorem 5.3.1. *Let Ω be a bounded uniform domain in $\mathbb{R}^n$ with $n \ge 3$, and let $\xi \in \partial\Omega$ and $\alpha > 1$. Suppose that $E \subset \Gamma_\alpha(\xi)$. Then E is thin at ξ if and only if E is minimally thin at ξ with respect to Ω.*

Proof. We may assume, without loss of generality, that ξ is a limit point of E and $E \subset B(\xi, 6^{-1}\delta_\Omega(x_0))$. We first show the necessity. Let E_j be a set defined as above. Since E is thin at ξ, there exists a sequence of positive numbers $\{a_j\}$ such that $a_j \to +\infty$ and

$\sum_{j=1}^{\infty} a_j \widehat{R}_1^{E_j}(\xi) < +\infty$. Let μ_j be the Riesz measure associated with $\widehat{R}_1^{E_j}$, and let $d\nu_j(x) = G_\Omega(x, x_0) d\mu_j(x)$. It then follows from Proposition 5.2.2 with $\kappa = 3$ that for $y \in E_j$,

$$\widehat{R}_1^{E_j}(y) = \int |x - y|^{2-n} d\mu_j(x) \leq A \int K_\Omega(x, y) d\nu_j(x),$$

so that

$$\frac{1}{A} \leq \frac{G_\Omega \nu_j(y)}{G_\Omega(x_0, y)} \quad \text{for quasi-every } y \in E_j.$$

Let $u(y) = \sum_{j=1}^{\infty} a_j G_\Omega \nu_j(y)$. Then u is a Green potential on Ω satisfying that

$$\lim_{y \to \xi,\, y \in E \setminus F} \frac{u(y)}{G_\Omega(x_0, y)} = +\infty,$$

where F is a polar set. Also, we have by Proposition 5.2.1

$$\sum_{j=1}^{\infty} a_j \int K_\Omega(x, \xi) d\nu_j(x) \leq A \sum_{j=1}^{\infty} a_j \widehat{R}_1^{E_j}(\xi) < +\infty.$$

Hence $E \setminus F$ is minimally thin at ξ with respect to Ω, and so is E.

We next show the sufficiency. Since E is minimally thin at ξ with respect to Ω, there exists a Green potential $G_\Omega \mu$ with $\operatorname{supp} \mu \subset \overline{\Gamma_\alpha(\xi)}$ such that $\int K_\Omega(x, \xi) d\mu(x) < +\infty$ and

$$\lim_{y \to \xi,\, y \in E} \frac{G_\Omega \mu(y)}{G_\Omega(x_0, y)} = +\infty.$$

Let $d\nu(x) = G_\Omega(x, x_0)^{-1} d\mu(x)$. It then follows from Proposition 5.2.1 that

$$\frac{G_\Omega \mu(y)}{G_\Omega(x_0, y)} = \int K_\Omega(x, y) d\mu(x) \leq A \int |x - y|^{2-n} d\nu(x),$$

so that

$$\lim_{y \to \xi,\, y \in E} \int |x - y|^{2-n} d\nu(x) = +\infty.$$

Also, Proposition 5.2.2 yields that

$$\int |x - \xi|^{2-n} d\nu(x) \leq A \int K_\Omega(x, \xi) d\mu(x) < +\infty.$$

Thus E is thin at ξ. The proof is complete. $\square$

Chapter 6

Appendix

6.1 Quasi-hyperbolic metric and Harnack's inequality

We show a relationship between the quasi-hyperbolic metric and Harnack's inequality for positive harmonic functions. Recall the definition of the quasi-hyperbolic metric on Ω:

$$k_\Omega(x,y) = \inf_\gamma \int_\gamma \frac{ds(z)}{\delta_\Omega(z)},$$

where the infimum is taken over all rectifiable curve γ in Ω connecting x to y.

We say that a finite sequence of balls $\{B(x_j, 2^{-1}\delta_\Omega(x_j))\}_{j=1}^N$ in Ω is a Harnack chain between x and y if $x_1 = x$, $x_N = y$, and $x_{j+1} \in B(x_j, 2^{-1}\delta_\Omega(x_j))$ for $j = 1, \cdots, N-1$. The number N is called the length of the Harnack chain. As shown in the following lemma, the shortest length of Harnack chain is estimated by the quasi-hyperbolic metric. By $[x,y]$ we denote the line segment between x and y.

Lemma 6.1.1. *Let Ω be a proper subdomain of $\mathbb{R}^n$ and $x, y \in \Omega$. Then the shortest length of the Harnack chain between x and y is comparable to $k_\Omega(x,y)+1$.*

Proof. Let $\{B(x_j, 2^{-1}\delta_\Omega(x_j))\}_{j=1}^N$ be a Harnack chain between x and y. Since $x_{j+1} \in B(x_j, 2^{-1}\delta_\Omega(x_j))$, we have $\operatorname{dist}([x_j, x_{j+1}], \partial\Omega) \geq 2^{-1}\delta_\Omega(x_j)$. Let $\gamma = \bigcup_{j=1}^{N-1}[x_j, x_{j+1}]$. Then

$$\int_\gamma \frac{ds(z)}{\delta_\Omega(z)} = \sum_{j=1}^{N-1} \int_{[x_j, x_{j+1}]} \frac{ds(z)}{\delta_\Omega(z)} \leq N.$$

Hence we obtain $k_\Omega(x,y) \leq \min N$.

Conversely, letting $l = k_\Omega(x,y)$, we can find a rectifiable curve γ in Ω such that

$$\int_\gamma \frac{ds(z)}{\delta_\Omega(z)} < 2l.$$

Let M be the smallest integer such that $2l/(\log(3/2)) \leq M$. Then we can take M points $x_1, \cdots, x_M$ in γ so that $x_1 = x$, $x_M = y$ and

$$\int_{\gamma(x_j, x_{j+1})} \frac{ds(z)}{\delta_\Omega(z)} < \log\frac{3}{2} \quad \text{for } j = 1, \cdots, M-1. \tag{6.1}$$

Since $\delta_\Omega(z) \le \delta_\Omega(x_j) + \ell(\gamma(x_j, z))$ for $z \in \gamma(x_j, x_{j+1})$, the left hand side of (6.1) is bounded from below by

$$\int_0^{\ell(\gamma(x_j,x_{j+1}))} \frac{ds}{\delta_\Omega(x_j)+s} = \log\left(1 + \frac{\ell(\gamma(x_j,x_{j+1}))}{\delta_\Omega(x_j)}\right).$$

This shows that $|x_j - x_{j+1}| \le \ell(\gamma(x_j, x_{j+1})) < 2^{-1}\delta_\Omega(x_j)$; that is, $x_{j+1} \in B(x_j, 2^{-1}\delta_\Omega(x_j))$. Hence $\{B(x_j, 2^{-1}\delta_\Omega(x_j))\}_{j=1}^M$ is the Harnack chain between x and y with length M, for which

$$M \le \frac{2l}{\log(3/2)} + 1 \le \frac{2}{\log(3/2)}(k_\Omega(x,y)+1).$$

Thus the lemma is proved. □

Lemma 6.1.1 and Harnack's inequality yield the following corollary.

Corollary 6.1.2. *Let Ω be a proper subdomain of $\mathbb{R}^n$. Then there exists a constant $A > 1$ depending only on the dimension n such that if $x, y \in \Omega$, then*

$$\exp(-A(k_\Omega(x,y)+1)) \le \frac{h(x)}{h(y)} \le \exp(A(k_\Omega(x,y)+1))$$

for every positive harmonic function h on Ω.

In order to apply Corollary 6.1.2 to the Green function, the following lemma is needed.

Lemma 6.1.3. *Let Ω be a proper subdomain of $\mathbb{R}^n$ and $z \in \Omega$. Then*

$$k_{\Omega\setminus\{z\}}(x,y) \le 3k_\Omega(x,y) + 6\pi \quad \textit{for } x, y \in \Omega \setminus B(z, 2^{-1}\delta_\Omega(z)).$$

Proof. We first claim that if $w \in \Omega$ satisfies $3^{-1}\delta_\Omega(w) > \delta_{\Omega\setminus\{z\}}(w)$, then $w \in B(z, 2^{-1}\delta_\Omega(z))$. Indeed,

$$3|z-w| = 3\delta_{\Omega\setminus\{z\}}(w) < \delta_\Omega(w) \le \delta_\Omega(z) + |z-w|,$$

so that $|z - w| < 2^{-1}\delta_\Omega(z)$.

Let γ be a rectifiable curve in Ω connecting x to y. If $\gamma \cap \partial B(z, 2^{-1}\delta_\Omega(z)) = \emptyset$, then the claim shows that $k_{\Omega\setminus\{z\}}(x,y) \le 3k_\Omega(x,y)$. We consider the case when γ intersects with $\partial B(z, 2^{-1}\delta_\Omega(z))$. We write w_1 and w_2 for points of the first hit and the last hit, respectively, i.e. $\gamma(x, w_1) \cap \partial B(z, 2^{-1}\delta_\Omega(z)) = \emptyset$ and $\gamma(w_2, y) \cap \partial B(z, 2^{-1}\delta_\Omega(z)) = \emptyset$. Let γ_1 be a curve in $\partial B(z, 2^{-1}\delta_\Omega(z))$ connecting w_1 to w_2 such that $\ell(\gamma_1) \le \pi\delta_\Omega(z)$, and let $\gamma' = \gamma(x, w_1) \cup \gamma_1 \cup \gamma(w_2, y)$. It follows from the above claim that if $w \in \gamma \setminus \gamma_1$, then $\delta_\Omega(w) \le 3\delta_{\Omega\setminus\{z\}}(w)$, so that

$$\begin{aligned}\int_\gamma \frac{ds(w)}{\delta_\Omega(w)} &\ge \int_{\gamma(x,w_1)\cup\gamma(w_2,y)} \frac{ds(w)}{\delta_\Omega(w)} \\ &\ge \frac{1}{3}\int_{\gamma(x,w_1)\cup\gamma(w_2,y)} \frac{ds(w)}{\delta_{\Omega\setminus\{z\}}(w)} + \int_{\gamma_1} \frac{ds(w)}{\delta_{\Omega\setminus\{z\}}(w)} - \int_{\gamma_1} \frac{ds(w)}{\delta_{\Omega\setminus\{z\}}(w)} \\ &\ge \frac{1}{3}\int_{\gamma'} \frac{ds(w)}{\delta_{\Omega\setminus\{z\}}(w)} - 2\pi.\end{aligned}$$

Thus we have

$$k_{\Omega\setminus\{z\}}(x,y) \leq 3\int_\gamma \frac{ds(w)}{\delta_\Omega(w)} + 6\pi.$$

Since γ is arbitrary curve, we obtain the lemma. $\square$

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