Removable singularities and singular solutions of semilinear elliptic equations

Kentaro Hirata

Department of Mathematics, Graduate School of Science, Hiroshima University,
Higashi-Hiroshima 739-8526, Japan
e-mail: hiratake@hiroshima-u.ac.jp

Takayori Ono

Fukuyama University, Fukuyama 729-0292, Japan
e-mail: ono@fuhc.fukuyama-u.ac.jp

Abstract

This note proves that a closed set with appropriate properties is removable for solutions of semilinear elliptic equations satisfying a certain growth condition near that set. Also, we give the existence theorem of positive solutions with singularities on a prescribed compact set, which shows that a growth condition in our removability theorem is optimal.

Keywords: removable singularity, singular solution, semilinear elliptic equation, potential estimate, iteration method, Kato class, fixed point argument.

Mathematics Subject Classification (2010): Primary 35J61; Secondary 31B05, 31B10, 35A01, 35B60.

1 Introduction

During the last three decades, the removability of an isolated singularity and the local behavior near a nonremovable isolated point have been investigated in detail for positive solutions of elliptic equations with a source term

$$-\Delta u = u^p \tag{1.1}$$

and with an absorption term

$$\Delta u = u^p, \tag{1.2}$$

where Δ is the Laplacian on $\mathbb{R}^n$. The investigations have been done for each equation separately, because the case $p \geq n/(n-2)$ has different conclusions between (1.1) and (1.2). That is, any isolated point is always removable for every positive solution of (1.2), whereas there exists a positive solution of (1.1) with an isolated singularity at the origin which behaves like $\|\cdot\|^{-2/(p-1)}$ if $p > n/(n-2)$; $\|\cdot\|^{2-n}(-\log\|\cdot\|)^{-(n-2)/2}$ if $p = n/(n-2)$. See Aviles [4], Brézis and Véron [6], Gidas and Spruck [12]. When $1 < p < n/(n-2)$, both of the above equations have positive solutions with an isolated singularity at the origin which blow up with the same speed as the fundamental solution $\|\cdot\|^{2-n}$ of the Laplace equation. As the removable singularity theorem, it is known that if a positive solution u of (1.1) or (1.2) in the unit ball punctured at the origin 0 satisfies the growth condition

$$u(x) = o(\|x\|^{2-n}) \quad \text{as } x \to 0, \tag{1.3}$$

then 0 is a removable singularity of u. See Lions [19], Vázquez and Véron [27], Véron [29]. Recently, the above results were extended by Cîrstea and Du [7] and Taliaferro [26] to equations or inequalities with more general nonlinearity. See also the references therein for other literature on isolated singularities.

We are interested in the removability of higher dimensional singular sets and the existence of positive solutions of (1.1) or (1.2) with a prescribed singular set. These problems were also investigated in many papers. For instance, Véron [28] developed his technique in the study of a removable isolated singularity to obtain the following result: if E is a compact C^∞-manifold in a domain $\Omega \subset \mathbb{R}^n$ of dimension $m < n-2$ and if

$$p \geq \frac{n-m}{n-m-2}, \tag{1.4}$$

then every solution of $\Delta u = |u|^{p-1}u$ in $\Omega \setminus E$ can be extended to the whole of Ω as its solution. See Grillot [13] for the extension to the framework of Riemannian geometry. Adams–Pierre [3] and Baras–Pierre [5] characterized a removable set for equation $\Delta u = |u|^{p-1}u$ with p satisfying (1.4) as a set with zero capacity associated with an appropriate Sobolev space. In contrast, the removability theorem for equation (1.1) with p satisfying (1.4) is known to hold at the distribution level, which means that any solution on $\Omega \setminus E$ satisfies (1.1) in Ω in the sense of distributions but is not necessarily smooth on the whole of Ω (see Dávila–Ponce [8]). We can know this reason from the following fact: when

$$\frac{n-m}{n-m-2} \leq p < \frac{n-m+2}{n-m-2},$$

there exist positive smooth functions u on $\Omega \setminus E$ satisfying (1.1) in Ω in the sense of distributions and

$$\frac{1}{C} d(x,E)^{-2/(p-1)} \leq u(x) \leq C d(x,E)^{-2/(p-1)} \tag{1.5}$$

near E for some constant $C > 1$. See Fakhi [10], Mazzeo and Pacard [21], Rébaï [25]. For the removability for closed sets with dimension $m > n-2$, we refer to [15].

The purpose of this note is to give an optimal growth condition corresponding to (1.3) under which any closed set lying on an m-dimensional set with appropriate properties is removable for solutions of semilinear elliptic equations like (1.1) and (1.2) when the nonlinear exponent p is smaller than $(n-m)/(n-m-2)$. Such removable sets will be defined in terms of quantitative conditions, which are satisfied for m-dimensional compact Lipschitz manifolds, and are called Lipschitz sets of dimension $m \geq 0$ in this note. See Definition 2.1 below. Also, we discuss the best possibility of our growth condition and the existence of positive solutions with singularities on a prescribed compact set of semilinear elliptic equations with general nonlinearities conditioned in terms of a certain Kato class.

Let Ω be a bounded domain in $\mathbb{R}^n$, let E be a compact set with $E \cap \Omega \neq \emptyset$ and let us consider semilinear elliptic equations of the form

$$-\Delta u = F(x, u, \nabla u), \tag{1.6}$$

where Δ is the Laplacian on $\mathbb{R}^n$ and ∇u is the gradient of u. Assume that F is a Borel function on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ satisfying that for all $(x, t, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$,

$$|F(x,t,\xi)| \leq C d(x,E)^{-\alpha}(1 + |t|^p), \tag{1.7}$$

where C, α and p are some constants and $d(x,E)$ denotes the distance from a point x to E. By saying a solution of (1.6) in Ω, we mean a C^1-function on Ω satisfying (1.6) in Ω in the sense of distributions.

Our main result is as follows.

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n$, where $n \geq 3$, and let E be a compact Lipschitz set in $\mathbb{R}^n$ of dimension $m < n-2$ such that $E \cap \Omega \neq \emptyset$. Assume that F satisfies* (1.7) *for some*

$$0 \leq p < \frac{n-m}{n-m-2} \quad \textit{and} \quad \alpha < \min\{2, n-m-p(n-m-2)\}. \tag{1.8}$$

Let u be a solution of (1.6) *in $\Omega \setminus E$. If u satisfies at each $y \in E \cap \Omega$,*

$$u(x) = o(d(x, E)^{2-n+m}) \quad \text{as } x \to y, \tag{1.9}$$

then u can be extended to the whole of Ω as a solution of (1.6) *in Ω.*

Remark 1.2. In Theorem 1.1, E need not be contained in Ω although we assume in Theorem 1.3 below that $E \subset \Omega$. Moreover, no assumptions on the signs of solutions and their Laplacian are imposed. Our result is applicable to several equations like $\Delta u = V_1 u + V_2 |u|^{p-1} u + V_3/(1 + |\nabla u|) + V_4$, where V_i $(i = 1, 2, 3, 4)$ are Borel functions on Ω satisfying $|V_i(x)| \le C d(x, E)^{-\alpha}$ for all $x \in \Omega$ and some positive constant C. Thus the special case $m = 0$ gives a generalization of the removable isolated singularity theorem for (1.1) and (1.2). Also, it is trivial that any subset of E is removable for solutions satisfying (1.9).

The following theorem shows that condition (1.9) is optimal to obtain Theorem 1.1.

Theorem 1.3. *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, where $n \ge 3$, and let E be a compact Lipschitz set in Ω of dimension $m < n - 2$. Suppose that*

$$p > 0, \quad \alpha < n - m - p(n - m - 2) \quad \text{and} \quad \beta < p + 1.$$

Let $c > 0$ (assumed to be small enough when $p = 1$ only). If a is a locally Hölder continuous function on Ω such that for all $x \in \Omega$,

$$|a(x)| \le c d(x, E)^{-\alpha} d(x, \partial\Omega)^{-\beta},$$

then there exist positive solutions $u \in C^2(\Omega \setminus E)$ of

$$\begin{cases} -\Delta u = a u^p & \text{in } \Omega \setminus E, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.10}$$

satisfying

$$\frac{1}{C} d(x, \partial\Omega) d(x, E)^{2-n+m} \le u(x) \le C d(x, \partial\Omega) d(x, E)^{2-n+m} \tag{1.11}$$

for some constant $C > 1$ and all $x \in \Omega$.

For equation $\Delta u = u^p + Vu$ with V being bounded, it is known that there are positive singular solutions satisfying (1.5) or (1.11) near E when E is a compact smooth manifold of dimension $m < n - 2$ and $1 < p < (n - m)/(n - m - 2)$. See Delanoë [9], Finn–McOwen [11], Grillot [13] and McOwen [22]. Note that we assume no restriction on the sign of Δu in Theorem 1.3. We will obtain the existence theorem of singular solutions for equation $-\Delta u + Vu = f(x, u)$ (see Theorems 4.2 and 4.3 below).

The plan of this note is as follows. In Section 2, we present notation and elementary lemmas which will be used in the proof of the theorems. The proof of Theorem 1.1 is given in Section 3 by modifying a method used in a parabolic problem [16]. In Section 4, we establish the existence theorem of singular solutions of semilinear elliptic equations with general nonlinearities conditioned in terms of a certain Kato class and apply it to prove Theorem 1.3 in Section 5.

2 Preliminaries

A typical point in $\mathbb{R}^n$ is denoted by x and its Euclidean norm by $\|x\|$. We write $d(x, E)$ for the Euclidean distance from a point x to a set E in $\mathbb{R}^n$. Also, for $r > 0$, we write

$$E(r) := \{x \in \mathbb{R}^n : d(x, E) \le r\}.$$

The n-dimensional Lebesgue measure and the m-dimensional Hausdorff measure on $\mathbb{R}^n$ of a Borel set E are denoted by $|E|$ and $\mathcal{H}^m(E)$, respectively. If $m = 0$, then $\mathcal{H}^0$ is interpreted as the counting measure. By $B(x,r)$, we denote the open ball in $\mathbb{R}^n$ of center x and radius $r>0$. The symbol C stands for an absolute positive constant whose value is unimportant and may vary at each occurrence. If necessary, we write $C_1, C_2, \dots$ to specify them.

Definition 2.1. Let E be a set in $\mathbb{R}^n$ and let $0 \le m < n$. We say that E is a *Lipschitz set of dimension* m if E is $\mathcal{H}^m$-measurable and there exist positive constants r_0 and $C > 1$ such that for all $x \in E$, $0 < r < r_0$ and $0 < R < r_0$,

$$\frac{1}{C} r^m \le \mathcal{H}^m(E \cap B(x,r)) \le C r^m \tag{2.1}$$

and

$$|E(r) \cap B(x,R)| \le C r^{n-m} R^m. \tag{2.2}$$

One example is a compact Lipschitz manifold in $\mathbb{R}^n$ of dimension m ($1 \le m \le n-1$) defined as follows. A subset E of $\mathbb{R}^n$ is a *Lipschitz manifold* of dimension m if for every $x \in E$ there exist an open neighborhood U of x in $\mathbb{R}^n$ and a bi-Lipschitz mapping $\phi : U \to \phi(U) \subset \mathbb{R}^n$ such that $\phi(E \cap U) = \mathbb{R}^m_0 \cap \phi(U)$, where $\mathbb{R}^m_0 := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_{m+1} = \dots = x_n = 0\}$. If E is a compact Lipschitz manifold of dimension m, then $\mathcal{H}^m(E \cap B(x,r))$ is comparable to $\mathcal{H}^m(\mathbb{R}^m_0 \cap \phi(B(x,r)))$ and the standard finite covering argument yields (2.2).

In what follows, we suppose that Ω is a bounded domain in $\mathbb{R}^n$, where $n \ge 3$, and that E is a compact Lipschitz set in $\mathbb{R}^n$ of dimension $m < n-2$ such that $E \cap \Omega \ne \emptyset$. Moreover, r_0 is the constant in the definition of a Lipschitz set E. We start with elementary estimates.

Lemma 2.2. *The following statements hold:*

(i) *If $m - n < \lambda < 0$, then there exists a positive constant C such that for all $x \in E$, $0 < r < r_0$ and $0 < R < r_0$,*

$$\int_{E(r) \cap B(x,R)} d(y,E)^\lambda \, dy \le C r^{\lambda+n-m} R^m. \tag{2.3}$$

Moreover, we have

$$\int_\Omega d(y,E)^\lambda \, dy < \infty. \tag{2.4}$$

(ii) *There exists a positive constant C such that for all $x \in E$, $0 < r < r_0$ and $0 < R < r_0$,*

$$\int_{E(r) \cap B(x,R)} \log \frac{1}{d(y,E)} \, dy \le C r^{n-m} (1 - \log r) R^m. \tag{2.5}$$

Proof. Let $x \in E$, $0 < r < r_0$ and $0 < R < r_0$ be fixed. Since the function $(y,t) \mapsto t^{\lambda-1} \chi_{\{(y,t): d(y,E) \le t\}}(y,t)$ is nonnegative and measurable with respect to $(n+1)$-dimensional Lebesgue measure on $(E(r) \cap B(x,R)) \times (0,r)$, we obtain, as a consequence of the Fubini-Tonelli theorem, that if $\lambda < 0$, then

$$\int_{E(r) \cap B(x,R)} d(y,E)^\lambda \, dy = r^\lambda |E(r) \cap B(x,R)| - \lambda \int_0^r t^{\lambda-1} |E(t) \cap B(x,R)| \, dt,$$

and if $\lambda = 0$, then

$$\int_{E(r) \cap B(x,R)} \log \frac{1}{d(y,E)} \, dy = \int_0^r \frac{1}{t} |E(t) \cap B(x,R)| \, dt - |E(r) \cap B(x,R)| \log r.$$

Hence (2.3) and (2.5) follow from (2.2).

Next, we show (2.4). Since E is compact, we observe that $E(r_0/2)$ is covered by finitely many balls of radius r_0 and center lying in E. By (2.3), we have

$$\int_{E(r_0/2)} d(y,E)^\lambda \, dy < \infty.$$

Since Ω is bounded, we also have $\int_{\Omega \setminus E(r_0/2)} d(y,E)^\lambda \, dy < \infty$. Thus (2.4) follows. $\square$

Lemma 2.3. *Let $m - n < \lambda < 0$. Then there exists a positive constant C such that for all $x \in \Omega$,*

$$\int_\Omega \|x-y\|^{2-n} d(y,E)^\lambda \, dy \le \begin{cases} Cd(x,E)^{2+\lambda} & \text{if } \lambda < -2, \\ C\left(1 + \log^+ \dfrac{1}{d(x,E)}\right) & \text{if } \lambda = -2, \\ C & \text{if } \lambda > -2, \end{cases} \tag{2.6}$$

where $\log^+ t = \max\{0, \log t\}$.

Proof. Let $x \in \Omega \setminus E$ and let $R := \operatorname{diam} \Omega$, the diameter of Ω. Then the left hand side of (2.6) is equal to

$$R^{2-n} \int_\Omega d(y,E)^\lambda dy + (n-2) \int_0^R r^{1-n} \int_{\Omega \cap B(x,r)} d(y,E)^\lambda \, dy dr. \tag{2.7}$$

By (2.4), the first integral in (2.7) is finite. If $r \ge d(x,E)/2$, then $B(x,r) \subset B(x^*, 3r)$ for some $x^* \in E$. Letting $r_1 := \min\{r_0/3, d(x,E)/2\}$, we see from Lemma 2.2 (i) that the second integral in (2.7) is estimated by

$$Cd(x,E)^\lambda \int_0^{r_1} r \, dr + C \int_{r_1}^{r_0/3} r^{1+\lambda} \, dr + C.$$

Computing this yields (2.6). □

We define

$$\rho(x) := C_1 \int_E \|x-y\|^{2-n} \, d\mathcal{H}^m(y) \quad \text{for } x \in \mathbb{R}^n, \tag{2.8}$$

where the constant C_1 is chosen so that $-\Delta(C_1 \| \cdot \|^{2-n}) = \delta_0$ (the Dirac measure at the origin) in the sense of distributions. Since E is compact, ρ is superharmonic on $\mathbb{R}^n$ and harmonic on $\mathbb{R}^n \setminus E$.

Lemma 2.4. *There exists a constant $C > 1$ such that for all $x \in \Omega$,*

$$\frac{1}{C} d(x,E)^{2-n+m} \le \rho(x) \le Cd(x,E)^{2-n+m}. \tag{2.9}$$

Proof. Since Ω is bounded, it suffices to show (2.9) for $x \in E(r_0/4) \setminus E$. Take $x^* \in E$ with $\|x^* - x\| = d(x,E)$. Then $B(x^*, r/2) \subset B(x,r)$ for all $r \ge 2d(x,E)$. Letting $R := r_0 + \operatorname{diam} E$, we have by (2.1)

$$\begin{aligned} \frac{\rho(x)}{C_1} &= R^{2-n} \mathcal{H}^m(E) + (n-2) \int_{d(x,E)}^R r^{1-n} \mathcal{H}^m(E \cap B(x,r)) \, dr \\ &\ge \frac{1}{C} \int_{2d(x,E)}^{r_0} r^{1-n} \mathcal{H}^m(E \cap B(x^*, r/2)) \, dr \\ &\ge \frac{1}{C} \int_{2d(x,E)}^{r_0} r^{1-n+m} \, dr \ge \frac{1}{C} d(x,E)^{2-n+m}. \end{aligned}$$

On the other hand, we have $B(x,r) \cap E \subset B(x^*, 2r) \cap E$ for all $r > 0$, and so

$$\frac{\rho(x)}{C_1} \le C \int_{d(x,E)}^{r_0/2} r^{1-n} \mathcal{H}^m(E \cap B(x^*, 2r)) \, dr + C \le Cd(x,E)^{2-n+m}.$$

Thus the lemma is proved. □

3 Proof of Theorem 1.1

This section presents the proof of Theorem 1.1. By G_D we denote the Green function for an open set D and the Laplace operator.

Lemma 3.1. *Assumptions are the same as Theorem 1.1. Let ρ be as in* (2.8)*. Then there exists a positive constant C such that for all $x \in \Omega$,*

$$\int_\Omega G_\Omega(x,y)d(y,E)^{-\alpha}\rho(y)^p\,dy \le \begin{cases} Cd(x,E)^{2-\alpha+p(2-n+m)} & \textit{if } p(2-n+m)-\alpha < -2, \\ C\left(1+\log^+ \dfrac{1}{d(x,E)}\right) & \textit{if } p(2-n+m)-\alpha = -2, \\ C & \textit{if } p(2-n+m)-\alpha > -2. \end{cases} \tag{3.1}$$

Moreover, $\int_\Omega G_\Omega(x,y)d(y,E)^{-\alpha}\rho(y)^p\,dy$ is superharmonic on Ω and is a C^1-function on $\Omega \setminus E$.

Proof. Since $-\alpha + p(2-n+m) > m-n$ and $G_\Omega(x,y) \le C_1\|x-y\|^{2-n}$, we obtain (3.1) from Lemmas 2.3 and 2.4. Also, the local boundedness of the density function $d(y,E)^{-\alpha}\rho(y)^p$ on $\Omega \setminus E$ implies the C^1-regularity of the Green potential. See [23, Theorem 6.6 in p. 119]. □

Lemma 3.2. *Assumptions are the same as Theorem 1.1. Let D be a bounded open set such that $\overline{D} \subset \Omega$. Then there exists a harmonic function h on D such that for all $x \in D \setminus E$,*

$$u(x) = h(x) + \int_{D\setminus E} G_D(x,y)F(y,u(y),\nabla u(y))\,dy. \tag{3.2}$$

Proof. Let $\varepsilon > 0$ be small. Taking a bounded open set ω such that $\overline{D} \subset \omega$ and $\overline{\omega} \subset \Omega$ if necessary, we may assume that u is continuous on $\overline{\Omega} \setminus E$ and, by (1.9) and Lemma 2.4, there is a positive constant $r_\varepsilon < 1/2$ such that

$$|u(x)| \le \varepsilon\rho(x) \quad \text{for all } x \in (E(r_\varepsilon) \cap \Omega) \setminus E, \tag{3.3}$$

where ρ was defined in (2.8). This is possible because the finite covering argument guarantees the local uniform convergence of (1.9). In particular, the case $\varepsilon = 1$ implies that

$$|u(x)| \le C\rho(x) \quad \text{for all } x \in \Omega \setminus E. \tag{3.4}$$

Therefore, by (1.7), we find a positive constant C_2 independent of ε such that

$$|F(x,u(x),\nabla u(x))| \le C_2 d(x,E)^{-\alpha}\rho(x)^p \quad \text{for all } x \in \Omega \setminus E. \tag{3.5}$$

Let

$$v(x) := C_2\int_\Omega G_\Omega(x,y)d(y,E)^{-\alpha}\rho(y)^p\,dy \quad \text{for } x \in \Omega$$

and consider the function defined by

$$u_\varepsilon(x) := u(x) + v(x) + \varepsilon\rho(x) \quad \text{for } x \in \Omega \setminus E.$$

Then u_ε is continuous on $\Omega \setminus E$ and, by (3.5), we have

$$-\Delta u_\varepsilon = F(\cdot,u,\nabla u) + C_2 d(\cdot,E)^{-\alpha}\rho^p \ge 0 \quad \text{in } \Omega \setminus E$$

in the sense of distributions. Therefore u_ε is superharmonic on $\Omega \setminus E$. Furthermore, (3.3) and the continuity of u imply that u_ε is bounded below on $\Omega \setminus E$. Since E is a polar set by Lemma 2.4, the classical removability theorem for superharmonic functions implies that u_ε has a superharmonic extension $\overline{u}_\varepsilon$ to Ω. Then there exists a unique Radon measure μ_ε on Ω such that $-\Delta\overline{u}_\varepsilon = \mu_\varepsilon$ in Ω in the sense of distributions. Also, by the Riesz decomposition theorem, we have for all $x \in D$,

$$\overline{u}_\varepsilon(x) = h_\varepsilon(x) + \int_D G_D(x,y)\,d\mu_\varepsilon(y), \tag{3.6}$$

where h_ε is the greatest harmonic minorant of $\overline{u}_\varepsilon$ on D.

We look for a limit function of each term in (3.6) as $\varepsilon \to 0$. For all $x \in (E(r_\varepsilon) \cap \Omega) \setminus E$, we have by (3.3) and Lemma 2.4

$$|u_\varepsilon(x)| \le C\varepsilon d(x,E)^{2-n+m} + v(x)$$

and by Lemma 3.1

$$v(x) \le \begin{cases} Cd(x,E)^{2-\alpha+p(2-n+m)} & \text{if } p(2-n+m)-\alpha < -2, \\ C\left(1+\log^+ \dfrac{1}{d(x,E)}\right) & \text{if } p(2-n+m)-\alpha = -2, \\ C & \text{if } p(2-n+m)-\alpha > -2. \end{cases}$$

Let $0 < r < \min\{r_\varepsilon, d(\overline{D}, \partial\Omega)/2\}$ and $z \in E \cap \overline{D}$. Then $B(z,2r) \subset \Omega$. Take a nonnegative function $\phi \in C_0^\infty(B(z,2r))$ satisfying $\phi = 1$ on $B(z,r)$ and $|\Delta\phi| \le C/r^2$ on $B(z,2r)$. Then, by Lemma 2.2,

$$\begin{aligned} \mu_\varepsilon(B(z,r)) &\le \int_{B(z,2r)} \phi\, d\mu_\varepsilon = \int_{B(z,2r)} (-\Delta\phi)\overline{u}_\varepsilon\, dx \le \frac{C}{r^2}\int_{B(z,2r)\setminus E} |u_\varepsilon|\, dx \\ &\le \begin{cases} C(\varepsilon r^m + r^{n-\alpha+p(2-n+m)}) & \text{if } p(2-n+m)-\alpha < -2, \\ C(\varepsilon r^m - r^{n-2}\log r) & \text{if } p(2-n+m)-\alpha = -2, \\ C(\varepsilon r^m + r^{n-2}) & \text{if } p(2-n+m)-\alpha > -2. \end{cases} \end{aligned}$$

By the covering lemma, we find N-points z_j in $E \cap \overline{D}$ such that $\{B(z_j, r/5)\}_{j=1}^N$ are mutually disjoint and $E \cap \overline{D} \subset \bigcup_{j=1}^N B(z_j, r)$. Noting $N \le Cr^{-m}$ by (2.2), we obtain

$$\begin{aligned} \mu_\varepsilon(E \cap D) &\le \sum_{j=1}^N \mu_\varepsilon(B(z_j, r)) \\ &\le \begin{cases} C(\varepsilon + r^{n-m-\alpha+p(2-n+m)}) & \text{if } p(2-n+m)-\alpha < -2, \\ C(\varepsilon - r^{n-m-2}\log r) & \text{if } p(2-n+m)-\alpha = -2, \\ C(\varepsilon + r^{n-m-2}) & \text{if } p(2-n+m)-\alpha > -2. \end{cases} \end{aligned}$$

Since $n-m-\alpha+p(2-n+m) > 0$, $n-m-2 > 0$ and $r > 0$ is arbitrary, we have

$$\mu_\varepsilon(E \cap D) \le C\varepsilon.$$

Let $x \in D \setminus E$. Then $G_D(x,\cdot)$ is bounded on $E \cap D$, and so

$$\lim_{\varepsilon \to 0+} \int_{E\cap D} G_D(x,y)\, d\mu_\varepsilon(y) = 0.$$

Also, since $F(\cdot, u, \nabla u) + C_2 d(\cdot, E)^{-\alpha}\rho^p$ is locally bounded on $D \setminus E$ and $\mu_\varepsilon = -\Delta u_\varepsilon = F(\cdot, u, \nabla u) + C_2 d(\cdot, E)^{-\alpha}\rho^p$ in $D \setminus E$ in the sense of distributions, the uniqueness of such a Radon measure implies

$$d\mu_\varepsilon(y) = \{F(y, u(y), \nabla u(y)) + C_2 d(y,E)^{-\alpha}\rho(y)^p\}\, dy \quad \text{on } D \setminus E.$$

Therefore, for all $x \in D \setminus E$,

$$\lim_{\varepsilon\to 0+} \int_D G_D(x,y)\, d\mu_\varepsilon(y) = \int_{D\setminus E} G_D(x,y) F(y, u(y), \nabla u(y))\, dy + v(x) - h_1(x),$$

where h_1 is a harmonic function on D appearing in the Riesz decomposition of v on D.

By the way, we see from (3.4) that $\overline{u}_\varepsilon(x) \ge u(x) \ge -C\rho(x)$ for all $x \in D \setminus E$ and $-C\rho$ is subharmonic on D. Since h_ε is the greatest harmonic minorant of $\overline{u}_\varepsilon$ on D, it follows that $h_\varepsilon \ge -C\rho$ on D, and so h_ε converges decreasingly to a harmonic function h_0 on D as ε decreases to 0. After $\varepsilon \to 0$ in (3.6) and cancelling v from the both sides, we obtain (3.2) with $h = h_0 - h_1$ for all $x \in D \setminus E$. $\square$

Lemma 3.3. *Assumptions are the same as Theorem 1.1. Let D be a bounded open set such that $\overline{D} \subset \Omega$. Then u is bounded on $D \setminus E$.*

Proof. We give the proof only when $1 < p < (n-m)/(n-m-2)$ because the case $0 \le p \le 1$ follows by the similar way (see Remark 3.4 below). Then $0 < n-m-\alpha-p(n-m-2) < 2-\alpha$. Let N be the smallest natural number satisfying

$$N \ge \frac{\log \dfrac{2-\alpha}{n-m-\alpha-p(n-m-2)}}{\log p},$$

which is equivalent to

$$p_N := p^N(2-n+m) + (2-\alpha)p^{N-1} + \cdots + (2-\alpha)p + 2 - \alpha \ge 0.$$

We first consider the case $N \ge 2$. Observe that if $N = 2$, then $m-n < p(2-n+m)-\alpha < -2$, and that if $N \ge 3$, then for $j = 2, \ldots, N-1$,

$$\begin{aligned} m-n &< p(2-n+m)-\alpha \\ &< p^j(2-n+m) + (2-\alpha)p^{j-1} + \cdots + (2-\alpha)p - \alpha < -2. \end{aligned}$$

In the arguments below, we apply Lemma 2.3 repeatedly to get better estimates for u. Take bounded open sets D_j $(j = 1, \ldots, N+2)$ so that

$$\overline{D}_1 \subset \Omega, \quad \overline{D}_j \subset D_{j-1}\, (j = 2, \ldots, N+2) \quad \text{and} \quad D_{N+2} = D.$$

By Lemma 3.2, for each j, there is a harmonic function h_j on D_j such that for all $x \in D_j \setminus E$,

$$u(x) = h_j(x) + \int_{D_j \setminus E} G_{D_j}(x,y) F(y, u(y), \nabla u(y))\, dy.$$

Also, as in (3.5), we have for all $y \in D_1 \setminus E$,

$$|F(y, u(y), \nabla u(y))| \le C d(y,E)^{-\alpha + p(2-n+m)}. \tag{3.7}$$

Since h_1 is bounded on D_2, it follows from Lemma 2.3 that for all $x \in D_2 \setminus E$,

$$|u(x)| \le C + C \int_{D_1} \|x-y\|^{2-n} d(y,E)^{-\alpha+p(2-n+m)}\, dy \le C d(x,E)^{2-\alpha+p(2-n+m)},$$

and so by (1.7),

$$|F(x, u(x), \nabla u(x))| \le C d(x,E)^{-\alpha+(2-\alpha)p+p^2(2-n+m)}. \tag{3.8}$$

Since h_2 is bounded on D_3, it follows from (3.8) and Lemma 2.3 that for all $x \in D_3 \setminus E$,

$$|u(x)| \le C d(x,E)^{2-\alpha+(2-\alpha)p+p^2(2-n+m)},$$

and so by (1.7),

$$|F(x, u(x), \nabla u(x))| \le C d(x,E)^{-\alpha+(2-\alpha)p+(2-\alpha)p^2+p^3(2-n+m)}.$$

Repeating this process $N-1$ times, we obtain for all $x \in D_N \setminus E$,

$$|F(x, u(x), \nabla u(x))| \le C d(x,E)^{p_N - 2}.$$

Again, by the boundedness of h_N on D_{N+1} and Lemma 2.3, we have for all $x \in D_{N+1} \setminus E$,

$$|u(x)| \le \begin{cases} C\left(1 + \log^+ \dfrac{1}{d(x,E)}\right) & \text{if } p_N = 0, \\ C & \text{if } p_N > 0. \end{cases}$$

If $p_N > 0$, then the lemma follows since $D \subset D_{N+1}$. If $p_N = 0$, then we take $\varepsilon > 0$ with $\alpha + \varepsilon < 2$. The above inequality implies that $|u(x)| \le C d(x,E)^{-\varepsilon/p}$, and so $|F(x, u(x), \nabla u(x))| \le C d(x,E)^{-\alpha-\varepsilon}$ for all $x \in D_{N+1} \setminus E$. Lemma 2.3 concludes that u is bounded on $D_{N+2} = D$.

When $N = 1$, at most twice application of Lemma 2.3 yields the boundedness of u, because $p(2-n+m) - \alpha \ge -2$. $\square$

Remark 3.4. When $0 \le p \le 1$, we can take $1 < q < (n-m-\alpha)/(n-m-2)$ because of $\alpha < 2$. Since Ω is bounded, we have $d(x,E)^{-\gamma} \le Cd(x,E)^{-\delta}$ for all $x \in \Omega$ if $0 \le \gamma \le \delta$, and so (3.7) and each estimate for $|u|$ and $|F|$ are valid for q in place of p. Hence Lemma 3.3 is true for $0 \le p \le 1$ as well.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let D be a bounded open set such that $\overline{D} \subset \Omega$. Then (1.7) and Lemma 3.3 imply that $F(\cdot, u, \nabla u)$ is bounded on $D \setminus E$. By the regularity theorem of the Green potential, the integral in (3.2) is a C^1-function on D. Define

$$\overline{u}(x) := \begin{cases} u(x) & \text{for } x \in D \setminus E, \\ h(x) + \displaystyle\int_{D\setminus E} G_D(x,y) F(y, u(y), \nabla u(y))\, dy & \text{for } x \in E, \end{cases}$$

where h is a harmonic function on D given in Lemma 3.2. Then, for all $x \in D$,

$$\overline{u}(x) = h(x) + \int_D G_D(x,y) F(y, \overline{u}(y), \nabla \overline{u}(y))\, dy,$$

and so $\overline{u}$ is a solution of (1.6) in D. Since D is arbitrary, this completes the proof. $\square$

4 Positive solutions with a prescribed singular set

We discuss the existence of positive solutions of semilinear elliptic equations with singularities on a prescribed compact set. Throughout this section, we suppose that Ω is a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, where $n \ge 3$, and that E is a compact Lipschitz set in Ω of dimension $m < n-2$. Let us consider semilinear elliptic equations of the form

$$\begin{cases} -\Delta u + Vu = f(x,u) & \text{in } \Omega \setminus E, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where the equation is understood in the sense of distributions and V and f are Borel functions conditioned in terms of the extended Kato class $\mathcal{K}(\Omega)$. A Borel function ϕ on Ω is said to belong to $\mathcal{K}(\Omega)$ if

$$\lim_{r\to 0+} \left(\sup_{x\in\Omega} \int_{B(x,r)\cap\Omega} \frac{d(y,\partial\Omega)}{d(x,\partial\Omega)} G_\Omega(x,y) |\phi(y)|\, dy \right) = 0.$$

This was introduced by Mâagli and Zribi [20]. Note that this condition is weaker near $\partial\Omega$ than the original Kato class condition [2] and that $\mathcal{K}(\Omega)$ is strictly bigger than $L^q(\Omega)$ for $q > n/2$. See their paper for more concrete examples. It is known that if $\phi \in \mathcal{K}(\Omega)$, then the quantity

$$\|\phi\|_{\mathcal{K}(\Omega)} := \sup_{x\in\Omega} \int_\Omega \frac{d(y,\partial\Omega)}{d(x,\partial\Omega)} G_\Omega(x,y) |\phi(y)|\, dy$$

is finite (see [20, Proposition 2]). We impose the following conditions on V and f:

(A1) $V \in \mathcal{K}(\Omega)$ and $\|V\|_{\mathcal{K}(\Omega)} < 1/(4C_3)$ for some sufficiently large constant $C_3 > 0$,

(A2) f is a Borel function on $\Omega \times (0,\infty)$ such that $f(x,\cdot)$ is continuous on $(0,\infty)$ for each $x \in \Omega$,

(A3) there exists a nonnegative Borel function ψ on $\Omega \times (0,\infty)$ such that for each $x \in \Omega$, $\psi(x,\cdot)$ is nondecreasing and $\psi(x,t) \to 0$ as $t \to 0+$ and that

$$|f(x,t)| \le t\psi(x,t) \quad \text{for a.e. } (x,t) \in \Omega \times (0,\infty),$$

(A4) $\psi(\cdot, d(\cdot,\partial\Omega) d(\cdot,E)^{2-n+m}) \in \mathcal{K}(\Omega)$.

Remark 4.1. A constant C_3 in (A1) is the constant appearing in the 3G inequality and depending only on the dimension n and the characters of Ω: for all $x, y, z \in \Omega$,

$$\frac{G_\Omega(x,y)G_\Omega(y,z)}{G_\Omega(x,z)} \le C_3\bigg(\frac{d(y,\partial\Omega)}{d(x,\partial\Omega)}G_\Omega(x,y) + \frac{d(y,\partial\Omega)}{d(z,\partial\Omega)}G_\Omega(y,z)\bigg). \tag{4.2}$$

See Kalton and Verbitsky [17, Lemma 7.1].

We prove the following theorem.

Theorem 4.2. *Suppose that V and f are Borel functions on Ω and on $\Omega \times (0,\infty)$ respectively satisfying* (A1) – (A4)*. Then* (4.1) *has positive solutions $u \in C(\Omega \setminus E)$ satisfying* (1.11) *for some constant $C > 1$ and all $x \in \Omega$ and satisfying the integral equation*

$$u(x) = \lambda \int_E G_\Omega(x,y)\, d\mathcal{H}^m(y) - \int_\Omega G_\Omega(x,y)\{V(y)u(y) - f(y,u(y))\}\, dy \tag{4.3}$$

for some positive constant λ and all $x \in \Omega$.

The special case $m = 0$ generalizes earlier results [18, 20, 24, 31] about the existence of positive solutions blowing up at one point with the same speed as the fundamental solution of the Laplacian. We will see in Section 5 that Theorem 1.3 with $p \ge 1$ is a special case of Theorem 4.2. For the case $0 < p < 1$, we need to replace (A3) by

(A3') there exists a nonnegative Borel function ψ on $\Omega \times (0,\infty)$ such that for each $x \in \Omega$, $\psi(x,\cdot)$ is nonincreasing and $\psi(x,t) \to 0$ as $t \to +\infty$ and that

$$|f(x,t)| \le t\psi(x,t) \quad \text{for a.e. } (x,t) \in \Omega \times (0,\infty).$$

Theorem 4.3. *Suppose that V and f are Borel functions on Ω and on $\Omega \times (0,\infty)$ respectively satisfying* (A1), (A2), (A3') *and* (A4)*. Then* (4.1) *has positive solutions $u \in C(\Omega \setminus E)$ satisfying* (1.11) *and* (4.3) *for some positive constants C, λ and all $x \in \Omega$.*

Theorems 4.2 and 4.3 will be proved by the similar way to [14, 20] using the Schauder fixed point theorem. But our interest is the existence of solutions with singularities on E, not an isolated singularity, and thus an integral operator we consider is different from [14, 20] and further discussions are needed. For completeness, we give a proof after preparing several elementary lemmas.

We recall the following lower and upper estimates of the Green function (see [30, 32]): for all $x, y \in \Omega$,

$$G_\Omega(x,y) \ge \frac{1}{C}\min\bigg\{1, \frac{d(x,\partial\Omega)d(y,\partial\Omega)}{\|x-y\|^2}\bigg\}\|x-y\|^{2-n} \tag{4.4}$$

and

$$G_\Omega(x,y) \le C\min\bigg\{1, \frac{d(x,\partial\Omega)}{\|x-y\|}, \frac{d(x,\partial\Omega)d(y,\partial\Omega)}{\|x-y\|^2}\bigg\}\|x-y\|^{2-n}, \tag{4.5}$$

where the constant $C > 1$ depends only on the characters of Ω and the dimension n. Let

$$\rho_\Omega(x) := \int_E G_\Omega(x,y)\, d\mathcal{H}^m(y) \quad \text{for } x \in \Omega.$$

Note that ρ_Ω is positive and harmonic on $\Omega \setminus E$ and vanishes continuously on $\partial\Omega$ since E is compact in Ω.

Lemma 4.4. *Let $r > 0$ be small. Then there exists a positive constant C depending on r and $\mathcal{H}^m(E)$ such that for all $x \in \Omega \setminus E(r)$,*

$$\rho_\Omega(x) \le C d(x,\partial\Omega).$$

Proof. Let $x \in \Omega \setminus E(r)$. By (4.5), we have

$$\rho_\Omega(x) \le C\int_E \frac{d(x,\partial\Omega)}{\|x-y\|^{n-1}}\, d\mathcal{H}^m(y) \le \frac{C}{r^{n-1}} d(x,\partial\Omega)\mathcal{H}^m(E),$$

as required. $\square$

Lemma 4.5. *There exists a constant $C > 1$ depending on $d(E, \partial\Omega)$ and $\mathcal{H}^m(E)$ such that for all $x \in \Omega$,*

$$\rho_\Omega(x) \geq \frac{1}{C} d(x, \partial\Omega).$$

Proof. Since Ω is bounded, we obtain from (4.4) that for all $x, y \in \Omega$,

$$G_\Omega(x, y) \geq \frac{1}{C} d(x, \partial\Omega) d(y, \partial\Omega), \tag{4.6}$$

and so

$$\rho_\Omega(x) \geq \frac{1}{C} d(x, \partial\Omega) d(E, \partial\Omega) \mathcal{H}^m(E),$$

as required. □

Lemma 4.6. *Let $r > 0$ be small. Then there exists a positive constant C depending on r and $\mathcal{H}^m(E)$ such that for all $x, y \in \Omega \setminus E(r)$,*

$$\rho_\Omega(x)\rho_\Omega(y) \leq C G_\Omega(x, y).$$

In particular,

$$\rho_\Omega(y)^2 \leq C \frac{\rho_\Omega(y)}{\rho_\Omega(x)} G_\Omega(x, y).$$

Proof. This follows from Lemma 4.4 and (4.6). □

Lemma 4.7. *Let $r > 0$ be small. Then there exists a positive constant C depending on r, $d(E, \partial\Omega)$ and $\mathcal{H}^m(E)$ such that for all $x, y \in \Omega$ with $\|x - y\| \geq r$,*

$$G_\Omega(x, y) \leq C \rho_\Omega(x)\rho_\Omega(y).$$

In particular,

$$\frac{\rho_\Omega(y)}{\rho_\Omega(x)} G_\Omega(x, y) \leq C \rho_\Omega(y)^2.$$

Proof. For $x, y \in \Omega$ with $\|x - y\| \geq r$, we have $G_\Omega(x, y) \leq C d(x, \partial\Omega) d(y, \partial\Omega)/r^n$ by (4.5). This lemma follows from Lemma 4.5. □

Lemma 4.8. *There exists a constant $C_4 > 1$ such that for all $x \in \Omega$,*

$$\frac{1}{C_4} d(x, \partial\Omega) d(x, E)^{2-n+m} \leq \rho_\Omega(x) \leq C_4 d(x, \partial\Omega) d(x, E)^{2-n+m}.$$

Proof. We may assume without loss of generality that $r_0 < d(E, \partial\Omega)/2$. Then $E(r_0)$ is compact in Ω. By the Riesz decomposition, we have for all $x \in E(r_0)$,

$$\rho_\Omega(x) = h(x) + \rho(x),$$

where h is a negative and bounded harmonic function on $E(r_0)$ and ρ is given by (2.8). It follows from Lemma 2.4 that for all $x \in E(r_0/4)$,

$$\frac{1}{C} d(x, E)^{2-n+m} \leq \rho_\Omega(x) \leq C d(x, E)^{2-n+m}.$$

Also, by Lemmas 4.4 and 4.5, we have for all $x \in \Omega \setminus E(r_0/4)$,

$$\frac{1}{C} d(x, \partial\Omega) \leq \rho_\Omega(x) \leq C d(x, \partial\Omega).$$

Since Ω is bounded, these deduce the required estimate. □

Lemma 4.9. *Let $y \in \Omega$. Then $G_\Omega(\cdot, y)/\rho_\Omega \in C(\overline{\Omega} \setminus \{y\})$, where the value of $G_\Omega(\cdot, y)/\rho_\Omega$ on $E \setminus \{y\}$ is interpreted as 0.*

Proof. Since ρ_Ω is positive and harmonic on $\Omega \setminus E$ and vanishes continuously on $\partial\Omega$, it follows, as an application of the boundary Harnack principle, that $G_\Omega(\cdot, y)/\rho_\Omega$ is continuous up to $\partial\Omega$ (see [1]). Therefore $G_\Omega(\cdot, y)/\rho_\Omega \in C(\overline{\Omega} \setminus (E \cup \{y\}))$. Also, the continuity on $E \setminus \{y\}$ follows from Lemma 4.8. $\square$

Lemma 4.10. *If $\phi \in \mathcal{K}(\Omega)$, then we have for each $z \in \overline{\Omega}$,*

$$\lim_{r \to 0+} \left(\sup_{x \in \Omega} \int_{\Omega \cap B(z,r)} \frac{\rho_\Omega(y)}{\rho_\Omega(x)} G_\Omega(x, y) |\phi(y)| \, dy \right) = 0.$$

Moreover, the quantity

$$\|\phi\|_{\rho_\Omega} := \sup_{x \in \Omega} \int_\Omega \frac{\rho_\Omega(y)}{\rho_\Omega(x)} G_\Omega(x, y) |\phi(y)| \, dy$$

is finite and satisfies $\|\phi\|_{\rho_\Omega} \le 2C_3 \|\phi\|_{\mathcal{K}(\Omega)}$ with the constant C_3 in (4.2).

Proof. Since ρ_Ω is a positive superharmonic function on Ω, the lemma follows from [20, Proposition 3]. $\square$

Lemma 4.11. *If $\phi \in \mathcal{K}(\Omega)$, then we have for small $r > 0$,*

$$\int_{\Omega \setminus E(r)} \rho_\Omega(y)^2 |\phi(y)| \, dy < \infty.$$

Proof. Let $x \in \Omega \setminus E(r)$. By Lemma 4.6, we have

$$\int_{\Omega \setminus E(r)} \rho_\Omega(y)^2 |\phi(y)| \, dy \le C \int_{\Omega \setminus E(r)} \frac{\rho_\Omega(y)}{\rho_\Omega(x)} G_\Omega(x, y) |\phi(y)| \, dy \le C \|\phi\|_{\rho_\Omega},$$

and thus the lemma follows from Lemma 4.10. $\square$

Lemma 4.12. *If $\phi \in \mathcal{K}(\Omega)$, then we have for any compact subset K of Ω,*

$$\int_K \rho_\Omega(y) |\phi(y)| \, dy < \infty.$$

Proof. Take $x_1 \in \Omega \setminus E$ so that $G_\Omega(x_1, y)/\rho_\Omega(x_1) \ge C > 0$ for all $y \in K$. Then

$$\int_K \rho_\Omega(y) |\phi(y)| \, dy \le C \int_K \frac{\rho_\Omega(y)}{\rho_\Omega(x_1)} G_\Omega(x_1, y) |\phi(y)| \, dy \le C \|\phi\|_{\rho_\Omega}.$$

Thus the lemma follows from Lemma 4.10. $\square$

The proofs of Theorems 4.2 and 4.3 are similar to each other. We give the proof only for Theorem 4.2. The different parts will be mentioned in Remarks.

Noting from (A1) and Lemma 4.10 that $\|V\|_{\rho_\Omega} < 1/2$, we consider the following function space and integral operator. For $\lambda > 0$, we let

$$W_\lambda := \left\{ w \in C(\overline{\Omega}) : \frac{2(1 - 2\|V\|_{\rho_\Omega})}{3 - 2\|V\|_{\rho_\Omega}} \lambda \le w \le \frac{4}{3 - 2\|V\|_{\rho_\Omega}} \lambda \right\}$$

and define the operator T_λ on W_λ by

$$T_\lambda w(x) := \lambda - \int_\Omega \frac{G_\Omega(x, y)}{\rho_\Omega(x)} \{V(y) w(y) \rho_\Omega(y) - f(y, w(y) \rho_\Omega(y))\} \, dy \quad \text{for } x \in \Omega. \tag{4.7}$$

For simplicity, we write

$$\phi(y) := |V(y)| + \psi(y, d(y, \partial\Omega) d(y, E)^{2-n+m}).$$

Then $\phi \in \mathcal{K}(\Omega)$ by (A1) and (A4). Moreover, if $0 < \lambda \le (3 - 2\|V\|_{\rho\Omega})/(4C_4)$ with C_4 being the constant in Lemma 4.8, then we see from (A3) and Lemma 4.8 that for all $w \in W_\lambda$, the integrand in (4.7) is bounded by

$$\begin{aligned}&\frac{4\lambda}{3 - 2\|V\|_{\rho\Omega}} \cdot \frac{\rho_\Omega(y)}{\rho_\Omega(x)} G_\Omega(x,y) \left\{ |V(y)| + \psi(y, \frac{4\lambda}{3 - 2\|V\|_{\rho\Omega}} \rho_\Omega(y)) \right\} \\ &\le \frac{\rho_\Omega(y)}{\rho_\Omega(x)} G_\Omega(x,y)\phi(y).\end{aligned} \tag{4.8}$$

By Lemma 4.10, T_λ is well-defined for such λ at least.

Remark 4.13. If f satisfies (A3') instead of (A3), then the integrand in (4.7) is bounded by

$$\begin{aligned}&\frac{4\lambda}{3 - 2\|V\|_{\rho\Omega}} \cdot \frac{\rho_\Omega(y)}{\rho_\Omega(x)} G_\Omega(x,y) \left\{ |V(y)| + \psi(y, \frac{(2 - 4\|V\|_{\rho\Omega})\lambda}{3 - 2\|V\|_{\rho\Omega}} \rho_\Omega(y)) \right\} \\ &\le \frac{4\lambda}{3 - 2\|V\|_{\rho\Omega}} \cdot \frac{\rho_\Omega(y)}{\rho_\Omega(x)} G_\Omega(x,y)\phi(y),\end{aligned}$$

whenever $\lambda \ge C_4(3 - 2\|V\|_{\rho\Omega})/(2 - 4\|V\|_{\rho\Omega})$.

In the arguments below, we suppose that $0 < \lambda \le (3 - 2\|V\|_{\rho\Omega})/(4C_4)$. We denote $T_\lambda(W_\lambda) := \{T_\lambda w : w \in W_\lambda\}$.

Lemma 4.14. *$T_\lambda(W_\lambda)$ is equicontinuous on $\overline{\Omega}$.*

Proof. Let $\varepsilon > 0$ and $z \in \overline{\Omega}$. By Lemma 4.10, there exists a positive constant r_z such that for all $0 < r < r_z$,

$$\sup_{x \in \Omega} \int_{\Omega \cap B(z,r)} \frac{\rho_\Omega(y)}{\rho_\Omega(x)} G_\Omega(x,y)\phi(y)\,dy \le \varepsilon.$$

Let $0 < r < r_z$ be small enough and let $x_1, x_2 \in B(z, r/2) \cap \Omega$. Then

$$\begin{aligned}|T_\lambda w(x_1) - T_\lambda w(x_2)| \le \varepsilon &+ \int_{E(r)\setminus B(z,r)} \left| \frac{G_\Omega(x_1,y)}{\rho_\Omega(x_1)} - \frac{G_\Omega(x_2,y)}{\rho_\Omega(x_2)} \right| \rho_\Omega(y)\phi(y)\,dy \\ &+ \int_{\Omega\setminus(E(r)\cup B(z,r))} \left| \frac{G_\Omega(x_1,y)}{\rho_\Omega(x_1)} - \frac{G_\Omega(x_2,y)}{\rho_\Omega(x_2)} \right| \rho_\Omega(y)\phi(y)\,dy.\end{aligned}$$

Since $G_\Omega(x,y)/\rho_\Omega(x)$ is bounded for $(x,y) \in (B(z,r/2) \cap \Omega) \times (E(r) \setminus B(z,r))$, it follows from Lemmas 4.9, 4.12 and the Lebesgue convergence theorem that the first integral tends to zero as $\|x_1 - x_2\| \to 0$. Also, by Lemma 4.7, we have for all $y \in \Omega \setminus B(z,r)$,

$$\left| \frac{G_\Omega(x_1,y)}{\rho_\Omega(x_1)} - \frac{G_\Omega(x_2,y)}{\rho_\Omega(x_2)} \right| \le C\rho_\Omega(y),$$

and so Lemmas 4.9, 4.11 and the Lebesgue convergence theorem imply that the second integral tends to zero as $\|x_1 - x_2\| \to 0$. Thus $T_\lambda w$ is continuous at z uniformly for $w \in W_\lambda$. ☐

Lemma 4.15. *There exists a positive constant $\lambda_0 \le (3 - 2\|V\|_{\rho\Omega})/(4C_4)$ such that if $0 < \lambda \le \lambda_0$, then $T_\lambda(W_\lambda) \subset W_\lambda$. Furthermore, $T_\lambda(W_\lambda)$ is relatively compact in $C(\overline{\Omega})$.*

Proof. Let $w \in W_\lambda$. For $0 < \eta < 1$, we define

$$\Psi_\eta(x) := \int_\Omega \frac{\rho_\Omega(y)}{\rho_\Omega(x)} G_\Omega(x,y)\psi(y, \eta\rho_\Omega(y))\,dy \quad \text{for } x \in \Omega.$$

The same arguments as in the proof of Lemma 4.14 shows that $\Psi_\eta \in C(\overline{\Omega})$. Also, (A3) and Lemma 4.10 imply that Ψ_η converges decreasingly to zero function on $\overline{\Omega}$ as η decreases to 0. By the Dini theorem, the convergence is uniform on $\overline{\Omega}$. Therefore there exists a positive constant λ_0 such that for all $0 < \lambda \le \lambda_0$,

$$\sup_{x \in \overline{\Omega}} \Psi_{4\lambda/(3 - 2\|V\|_{\rho\Omega})}(x) \le \frac{1 - 2\|V\|_{\rho\Omega}}{4}. \tag{4.9}$$

By (4.7) and (4.8), we have

$$\begin{aligned}|T_\lambda w(x) - \lambda| &\le \frac{4\lambda}{3 - 2\|V\|_{\rho\Omega}} \left\{ \|V\|_{\rho\Omega} + \Psi_{4\lambda/(3-2\|V\|_{\rho\Omega})}(x) \right\} \\ &\le \frac{1 + 2\|V\|_{\rho\Omega}}{3 - 2\|V\|_{\rho\Omega}} \lambda.\end{aligned}$$

This and $T_\lambda w \in C(\overline{\Omega})$ conclude that $T_\lambda(W_\lambda) \subset W_\lambda$. The relatively compactness follows from Lemma 4.14 and the Ascoli-Arzelá theorem. □

Remark 4.16. If f satisfies (A3') instead of (A3), then (4.9) is replaced by

$$\sup_{x \in \overline{\Omega}} \Psi_{(2-4\|V\|_{\rho\Omega})\lambda/(3-2\|V\|_{\rho\Omega})}(x) \le \frac{1 - 2\|V\|_{\rho\Omega}}{4}$$

for all $\lambda \ge \lambda_0 \ge C_4(3 - 2\|V\|_{\rho\Omega})/(2 - 4\|V\|_{\rho\Omega})$.

Lemma 4.17. *If* $0 < \lambda \le \lambda_0$*, then* T_λ *is continuous on* W_λ*.*

Proof. Let $\{w_j\}$ be a sequence in W_λ converging to $w \in W_\lambda$ with respect to the uniform norm on $C(\overline{\Omega})$. By (A2) and Lemma 4.14, $T_\lambda w_j$ converges pointwisely to $T_\lambda w$ on $\overline{\Omega}$. The relatively compactness of $T_\lambda(W_\lambda)$ implies the uniform convergence. □

Proof of Theorem 4.2. Let $0 < \lambda \le \lambda_0$. Observe from Lemmas 4.15 and 4.17 that W_λ is a nonempty bounded closed convex subset of $C(\overline{\Omega})$ and T_λ is a continuous mapping from W_λ into itself such that $T_\lambda(W_\lambda)$ is relatively compact in $C(\overline{\Omega})$. By the Schauder fixed point theorem, there is $w \in W_\lambda$ such that $T_\lambda w = w$. Let $u := \rho_\Omega w$. Then $u \in C(\Omega \setminus E)$ and u satisfies (1.11) by Lemma 4.8, and so u vanishes continuously on $\partial\Omega$. Also, by the definition of T_λ, we have for all $x \in \Omega$,

$$u(x) = \lambda \rho_\Omega(x) - \int_\Omega G_\Omega(x, y)\{V(y)u(y) - f(y, u(y))\}\, dy.$$

Since ρ_Ω is harmonic on $\Omega \setminus E$, we see that u satisfies $-\Delta u + Vu = f(x, u)$ in $\Omega \setminus E$ in the sense of distributions. This completes the proof. □

5 Proof of Theorem 1.3

We apply Theorems 4.2 and 4.3 to prove Theorem 1.3.

Proof of Theorem 1.3. For simplicity, we write $\gamma := n - m - \alpha + p(2 - n + m)$. Let

$$\phi(x) := d(x, E)^{\gamma-2} d(x, \partial\Omega)^{p-1-\beta} \quad \text{for } x \in \Omega.$$

Taking Theorems 4.2 and 4.3 into account, it suffices to show $\phi \in \mathcal{K}(\Omega)$. Let $0 < r < r_1 := d(E, \partial\Omega)/4$ and let

$$\begin{aligned}\Phi(x, r) &:= \int_{B(x,r)\cap\Omega} \frac{d(y, \partial\Omega)}{d(x, \partial\Omega)} G_\Omega(x, y)\phi(y)\, dy \\ &= \int_{B(x,r)\cap\Omega} \frac{d(y, E)^{\gamma-2} d(y, \partial\Omega)^{p-\beta}}{d(x, \partial\Omega)} G_\Omega(x, y)\, dy.\end{aligned}$$

We consider several cases separately.

Case 1: $2r \le \min\{d(x, E), d(x, \partial\Omega)\}$. We have for all $y \in B(x, r)$,

$$\begin{aligned}\frac{1}{2} d(x, E) &\le d(y, E) \le 2d(x, E), \\ \frac{1}{2} d(x, \partial\Omega) &\le d(y, \partial\Omega) \le 2d(x, \partial\Omega),\end{aligned}$$

and so

$$\Phi(x,r) \le Cd(x,E)^{\gamma-2} d(x,\partial\Omega)^{p-\beta-1} r^2.$$

If $d(x,E) \le r_1$, then $r_1 \le d(x,\partial\Omega) \le \operatorname{diam}\Omega$. Therefore

$$\Phi(x,r) \le Cd(x,E)^{\gamma-2} r^2 \le \begin{cases} Cr^2 & \text{if } \gamma-2 \ge 0, \\ Cr^\gamma & \text{if } \gamma-2 < 0. \end{cases}$$

If $d(x,E) \ge r_1$, then

$$\Phi(x,r) \le Cd(x,\partial\Omega)^{p-\beta-1} r^2 \le \begin{cases} Cr^2 & \text{if } p-\beta-1 \ge 0, \\ Cr^{p-\beta+1} & \text{if } p-\beta-1 < 0. \end{cases}$$

Case 2: $d(x,\partial\Omega) < 2r \le d(x,E)$. We note that for all $y \in B(x,r)$,

$$\begin{aligned} d(y,\partial\Omega) &\le d(x,\partial\Omega) + \|x-y\| \le 3r, \\ d(y,E) &\ge d(E,\partial\Omega) - d(x,\partial\Omega) - \|x-y\| \ge r_1. \end{aligned}$$

Let $0 < \varepsilon < \min\{1, p-\beta+1\}$. We see from (4.5) that

$$G_\Omega(x,y) \le C \frac{d(x,\partial\Omega) d(y,\partial\Omega)^{1-\varepsilon}}{\|x-y\|^{n-\varepsilon}}.$$

Therefore

$$\Phi(x,r) \le C \int_{B(x,r)} d(y,\partial\Omega)^{p-\beta+1-\varepsilon} \|x-y\|^{\varepsilon-n}\, dy \le Cr^{p-\beta+1}.$$

Case 3: $d(x,E) < 2r \le d(x,\partial\Omega)$. We note that for all $y \in B(x,r)$,

$$d(y,E) \le 3r \quad \text{and} \quad 2d(x,\partial\Omega) \ge d(y,\partial\Omega) \ge r_1.$$

Therefore we have by Lemma 2.2

$$\begin{aligned} \Phi(x,r) &\le C \int_{B(x,r)} d(y,E)^{\gamma-2} \|x-y\|^{2-n}\, dy \\ &\le C\left(r^{2-n} \int_{B(x,r)} d(y,E)^{\gamma-2}\, dy + (n-2) \int_0^r t^{1-n} \int_{B(x,t)} d(y,E)^{\gamma-2}\, dy dt \right) \\ &\le Cr^\gamma. \end{aligned}$$

Note that the case $\max\{d(x,E), d(x,\partial\Omega)\} < 2r$ does not occur by our choice of r_1. Hence we obtain

$$\lim_{r\to 0+} \left(\sup_{x\in\Omega} \Phi(x,r) \right) = 0,$$

and thus $\phi \in \mathcal{K}(\Omega)$.

Finally, we apply Theorems 4.2 or 4.3 to complete the proof.
Case 1: $p \ne 1$. Since $V \equiv 0$ and $f(x,t) = a(x)t^p$ fulfill (A1), (A2), (A4) and either (A3) or (A3'), there are positive solutions $u \in C(\Omega \setminus E)$ of (1.10) satisfying (1.11) and

$$u(x) = \lambda \rho_\Omega(x) + \int_\Omega G_\Omega(x,y) a(y) u(y)^p\, dy \quad \text{for } x \in \Omega.$$

Then it follows from [23, Theorem 6.6 in p. 119] that $u \in C^2(\Omega \setminus E)$ and (1.10) is satisfied in the classical sense.
Case 2: $p = 1$. Since $0 < \|\phi\|_{\mathcal{K}(\Omega)} < \infty$, we have $\|a\|_{\mathcal{K}(\Omega)} < 1/(4C_3)$ whenever $0 < c < 1/(4C_3\|\phi\|_{\mathcal{K}(\Omega)})$. Therefore we can apply Theorem 4.2 with $V = -a$ and $f \equiv 0$ to obtain the result. $\square$

Acknowledgment

The first named author was partially supported by Grant-in-Aid for Young Scientists (B) (No. 22740081), Japan Society for the Promotion of Science.

References

[1] H. Aikawa, *Boundary Harnack principle and Martin boundary for a uniform domain*, J. Math. Soc. Japan **53** (2001), no. 1, 119–145.

[2] M. Aizenman and B. Simon, *Brownian motion and Harnack inequality for Schrödinger operators*, Comm. Pure Appl. Math. **35** (1982), no. 2, 209–273.

[3] D. R. Adams and M. Pierre, *Capacitary strong type estimates in semilinear problems*, Ann. Inst. Fourier (Grenoble) **41** (1991), no. 1, 117–135.

[4] P. Aviles, *On isolated singularities in some nonlinear partial differential equations*, Indiana Univ. Math. J. **32** (1983), no. 5, 773–791.

[5] P. Baras and M. Pierre, *Singularités éliminables pour des équations semi-linéaires*, Ann. Inst. Fourier (Grenoble) **34** (1984), no. 1, 185–206.

[6] H. Brézis and L. Véron, *Removable singularities for some nonlinear elliptic equations*, Arch. Rational Mech. Anal. **75** (1980/81), no. 1, 1–6.

[7] F. C. Cîrstea and Y. Du, *Asymptotic behavior of solutions of semilinear elliptic equations near an isolated singularity*, J. Funct. Anal. **250** (2007), no. 2, 317–346.

[8] J. Dávila and A. C. Ponce, *Variants of Kato's inequality and removable singularities*, J. Anal. Math. **91** (2003), 143–178.

[9] P. Delanoë, *Generalized stereographic projections with prescribed scalar curvature*, Contemp. Math. **127** (1992), 17–25.

[10] S. Fakhi, *Positive solutions of $\Delta u + u^p = 0$ whose singular set is a manifold with boundary*, Calc. Var. Partial Differential Equations **17** (2003), no. 2, 179–197.

[11] D. L. Finn and R. C. McOwen, *Singularities and asymptotics for the equation $\Delta_g u - u^q = Su$*, Indiana Univ. Math. J. **42** (1993), no. 4, 1487–1523.

[12] B. Gidas and J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math. **34** (1981), no. 4, 525–598.

[13] M. Grillot, *Solutions of some nonlinear elliptic problems, which are singular on a submanifold*, J. Math. Pures Appl. (9) **76** (1997), no. 9, 757–776.

[14] K. Hirata, *Sharp estimates for the Green function, 3G inequalities, and nonlinear Schrödinger problems in uniform cones*, J. Anal. Math. **99** (2006), 309–332.

[15] K. Hirata, *Removable sets for continuous solutions of semilinear elliptic equations*, Manuscripta Math. **135** (2011), no. 1-2, 245–262.

[16] K. Hirata, *Removable singularities of semilinear parabolic equations*, Proc. Amer. Math. Soc. **142** (2014), no. 1, 157–171.

[17] N. J. Kalton and I. E. Verbitsky, *Nonlinear equations and weighted norm inequalities*, Trans. Amer. Math. Soc. **351** (1999), no. 9, 3441–3497.

[18] Y. Li and J. Santanilla, *Existence and nonexistence of positive singular solutions for semilinear elliptic problems with applications in astrophysics*, Differential Integral Equations **8** (1995), no. 6, 1369–1383.

[19] P. L. Lions, *Isolated singularities in semilinear problems*, J. Differential Equations **38** (1980), no. 3, 441–450.

[20] H. Mâagli and M. Zribi, *On a new Kato class and singular solutions of a nonlinear elliptic equation in bounded domains of* $\mathbb{R}^n$, Positivity **9** (2005), no. 4, 667–686.

[21] R. Mazzeo and F. Pacard, *A construction of singular solutions for a semilinear elliptic equation using asymptotic analysis*, J. Differential Geom. **44** (1996), no. 2, 331–370.

[22] R. C. McOwen, *Singularities and the conformal scalar curvature equation*, Lecture Notes in Pure and Appl. Math. **144** (1993), 221–233.

[23] S. C. Port and C. J. Stone, *Brownian motion and classical potential theory*, Academic Press, 1978.

[24] L. Riahi, *Singular solutions of a semi-linear elliptic equation on nonsmooth domains*, Int. J. Evol. Equ. **2** (2008), no. 4, 387–396.

[25] Y. Rébaï, *Weak solutions of nonlinear elliptic equations with prescribed singular set*, J. Differential Equations **127** (1996), no. 2, 439–453.

[26] S. D. Taliaferro, *Isolated singularities of nonlinear elliptic inequalities. II. Asymptotic behavior of solutions*, Indiana Univ. Math. J. **55** (2006), no. 6, 1791–1811.

[27] J. L. Vázquez and L. Véron, *Isolated singularities of some semilinear elliptic equations*, J. Differential Equations **60** (1985), no. 3, 301–321.

[28] L. Véron, *Singularités éliminables d'équations elliptiques non linéaires*, J. Differential Equations **41** (1981), no. 1, 87–95.

[29] L. Véron, *Singular solutions of some nonlinear elliptic equations*, Nonlinear Anal. **5** (1981), no. 3, 225–242.

[30] K. O. Widman, *Inequalities for the Green function and boundary continuity of the gradient of solutions of elliptic differential equations*, Math. Scand. **21** (1967), 17–37.

[31] Q. S. Zhang and Z. Zhao, *Singular solutions of semilinear elliptic and parabolic equations*, Math. Ann. **310** (1998), no. 4, 777–794.

[32] Z. Zhao, *Green function for Schrödinger operator and conditioned Feynman-Kac gauge*, J. Math. Anal. Appl. **116** (1986), no. 2, 309–334.