Two sided global estimates of heat kernels in Lipschitz domains

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Abstract

This note presents sharp upper and lower bound estimates of the heat kernel in a bounded Lipschitz domain. To this end, we introduce an auxiliary set which is different from Bogdan's set used in the study of the Green function for the Laplace operator. Also, we give global estimates of kernel functions with pole at parabolic boundary points.

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1 Introduction

Let (x,t) denote a typical point in $\mathbb{R}^n \times \mathbb{R}$, where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, and let $\gamma(x,t)$ stand for the fundamental solution of the heat equation given by

$$\gamma(x,t) = \begin{cases} \dfrac{1}{(4\pi t)^{n/2}} \exp\left\{-\dfrac{\|x\|^2}{4t}\right\} & \text{if } t > 0, \\ 0 & \text{if } t \le 0. \end{cases} \tag{1.1}$$

Let Ω be a domain in $\mathbb{R}^n$. We denote by Γ the Green function for $\Omega \times \mathbb{R}$ and the heat operator. If $(y,s) \in \Omega \times \mathbb{R}$ is fixed, then it is represented as

$$\Gamma(x,t;y,s) = \gamma(x-y,t-s) - h_{(y,s)}(x,t) \quad \text{for all } (x,t) \in \Omega \times \mathbb{R},$$

where $h_{(y,s)}$ is the greatest thermic minorant of $\gamma(\cdot - y, \cdot - s)$ on $\Omega \times \mathbb{R}$ (see [19]). In the case $s = 0$, the Green function $\Gamma(\cdot,\cdot;y,0)$ is also referred to as the heat kernel for Ω. It is well known that $\Gamma(x,t;y,0) \le \gamma(x-y,t)$ for all $(x,t) \in \Omega \times \mathbb{R}$, and moreover that if x and y are apart from the boundary $\partial\Omega$ and if they are close to each other, then $\Gamma(x,t;y,0) \ge (Ct)^{-n/2}\exp(-C\|x-y\|^2/t)$ for some constant $C > 1$ (see [4, Theorem 8] for instance). But the global behavior, particularly the boundary behavior, is not well known because it is influenced by the shape of a domain. For the last few decades, many researchers have studied two sided global estimates of heat kernels. The large time behavior of the heat kernel on a bounded Lipschitz domain Ω was established by Davies [8, Theorem 4.2.5]: for any $\varepsilon > 0$, there exists $T > 0$ such that for all $x, y \in \Omega$ and $t \ge T$,

$$(1-\varepsilon)\phi(x)\phi(y)e^{-Et} \le \Gamma(x,t;y,0) \le (1+\varepsilon)\phi(x)\phi(y)e^{-Et},$$

where ϕ is the eigenfunction corresponding to the first eigenvalue E of the minus Laplacian $-\Delta$. The small time behavior is more delicate. For simplicity, we use the notations $a \wedge b = \min\{a,b\}$, $a \vee b = \max\{a,b\}$ and

$$\gamma_C(x,t) = \frac{C}{t^{n/2}} \exp\left\{-\frac{\|x\|^2}{Ct}\right\}.$$

The symbol C stands for an absolute positive constant whose value is unimportant and may vary at each occurrence. Writing $C(a,b,\dots)$ means that a constant C depends only on $a,b,\dots$. By $\delta(x)$, we denote the Euclidean distance in $\mathbb{R}^n$ from a point x to the boundary $\partial\Omega$. Davies [7, Theorem 3] proved that if Ω is a bounded Lipschitz domain, then there exists $C=C(n,\Omega,T)>1$ such that for all $x,y\in\Omega$ and $0<t<T$,

$$\Gamma(x,t;y,0)\le \frac{\phi(x)\phi(y)}{t^\alpha}\gamma_C(x-y,t), \tag{1.2}$$

where $\alpha\ge 1$ is a constant satisfying $\phi(x)\ge C\delta(x)^\alpha$ for all $x\in\Omega$ and some $C>0$. If $\partial\Omega$ is smooth, then we can take $\alpha=1$. In this case, the following sharper estimate was obtained by Hui [12, Lemma 1.3] (upper estimate) and Zhang [21, Theorem 1.1] (lower estimate):

$$\Gamma(x,t;y,0)\le \left(\frac{\delta(x)}{\sqrt{t}}\wedge 1\right)\left(\frac{\delta(y)}{\sqrt{t}}\wedge 1\right)\gamma_C(x-y,t), \tag{1.3}$$

$$\Gamma(x,t;y,0)\ge \left(\frac{\delta(x)}{\sqrt{t}}\wedge 1\right)\left(\frac{\delta(y)}{\sqrt{t}}\wedge 1\right)\gamma_{\frac{1}{C}}(x-y,t). \tag{1.4}$$

Also, Cho [6] obtained these estimates in a bounded $C^{1,a}$ domain with $0<a<1$.

The purpose of this note is to establish lower and upper bound estimates sharper than (1.2) when Ω is a bounded Lipschitz domain. To this end, we introduce an auxiliary set. Let $\kappa>1$ and $T>0$. For $x\in\overline{\Omega}$ and $0<t<T$, we define

$$\mathcal{B}_p(x,t)=\left\{b\in\Omega:\frac{1}{\kappa}\|b-x\|\le\sqrt{t}\le\kappa\delta(b)\right\}.$$

Here the subscript "p" means "parabolic" in order to distinguish from the elliptic case. This set is nonempty if $\kappa\ge\kappa(\Omega,T)$ (see Lemma 2.1). We fix some $\kappa=\kappa(\Omega,T)$ in arguments below. Let x_0 be a fixed point in Ω (which is away from $\partial\Omega$) and let $G(x,y)$ denote the Green function for Ω and the Laplace operator. Instead of the eigenfunction ϕ, we use the truncated Green function

$$g(x)=G(x,x_0)\wedge 1.$$

The main result is as follows.

Theorem 1.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ $(n\ge 2)$ and let $T>0$. Then there exists $C=C(n,\Omega,T)>1$ such that for all $x,y\in\Omega$ and $0<t<T$,*

$$\Gamma(x,t;y,0)\le \frac{g(x)g(y)}{g(b_x)g(b_y)}\gamma_C(x-y,t), \tag{1.5}$$

$$\Gamma(x,t;y,0)\ge \frac{g(x)g(y)}{g(b_x)g(b_y)}\gamma_{\frac{1}{C}}(x-y,t), \tag{1.6}$$

where $b_x\in\mathcal{B}_p(x,t)$ and $b_y\in\mathcal{B}_p(y,t)$.

Estimates of this kind in the elliptic case were given by Aikawa [1, Section 3] and Bogdan [5]. For each pair of points $x,y\in\Omega$, we let

$$\mathcal{B}_e(x,y)=\left\{b\in\Omega:\frac{1}{\kappa}(\|b-x\|\vee\|b-y\|)\le\|x-y\|\le\kappa\delta(b)\right\}. \tag{1.7}$$

Here the subscript "e" means "elliptic". This definition is slightly different from theirs, but is essentially the same (see [11]). Then there exists $C=C(n,\Omega)>1$ such that for each $x,y\in\Omega$ and $b\in\mathcal{B}_e(x,y)$,

$$\frac{1}{C}\mathcal{G}(x,y)\le G(x,y)\le C\mathcal{G}(x,y), \tag{1.8}$$

where

$$\mathcal{G}(x,y)=\begin{cases}\dfrac{g(x)g(y)}{g(b)^2}\left(1+\log^+\dfrac{\delta(x)\wedge\delta(y)}{\|x-y\|}\right) & \text{if } n=2,\\[2ex] \dfrac{g(x)g(y)}{g(b)^2}\|x-y\|^{2-n} & \text{if } n\ge 3.\end{cases}$$

Here $\log^+ f = (\log f) \vee 0$. Note that the auxiliary sets are quite different between the elliptic and parabolic cases, because $\mathcal{B}_e(x,y)$ is determined by two points $x, y \in \Omega$, whereas $\mathcal{B}_p(x,t)$ by only one point $(x,t) \in \Omega \times (0,T)$.

Remark 1.2. Recently, Gyrya and Saloff-Coste [16] obtained two sided estimates of heat kernels in "unbounded" inner uniform domains. They used the quantity

$$\sqrt{\int_{B(x,\sqrt{t})\cap\Omega} h(z)^2\,dz \int_{B(y,\sqrt{t})\cap\Omega} h(z)^2\,dz}$$

with a harmonic profile h instead of our $t^{n/2}g(b_x)g(b_y)$. Also, this quantity is comparable to $t^{n/2}h(b_x)h(b_y)$, where $\mathcal{B}_p(x,t)$ is defined with respect to the internal metric. (see [16, pp. 103–104]). Our proof is based merely on the so-called local comparison principle for temperatures and the boundary Harnack principle for harmonic functions, and is simpler than theirs.

As a consequence of Theorem 1.1, we obtain the following improvement of (1.2).

Corollary 1.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ $(n \geq 2)$ and let $T > 0$. Then there exist $C = C(n,\Omega,T) > 1$, $\alpha = \alpha(n,\Omega) > 0$ and $\beta = \beta(n,\Omega) > 0$ with $\beta \leq 1 \leq \alpha$ such that for all $x, y \in \Omega$ and $0 < t < T$,*

$$\Gamma(x,t;y,0) \leq \left(\frac{\delta(x)}{\sqrt{t}} \wedge 1\right)^{\beta} \left(\frac{\delta(y)}{\sqrt{t}} \wedge 1\right)^{\beta} \gamma_C(x-y,t), \tag{1.9}$$

$$\Gamma(x,t;y,0) \geq \left(\frac{\delta(x)}{\sqrt{t}} \wedge 1\right)^{\alpha} \left(\frac{\delta(y)}{\sqrt{t}} \wedge 1\right)^{\alpha} \gamma_{\frac{1}{C}}(x-y,t). \tag{1.10}$$

Moreover, if Ω is a Liapunov-Dini domain, we can take $\alpha = \beta = 1$.

Remark 1.4. See Widman [20] for the definition of Liapunov-Dini domains. Note that bounded $C^{1,a}$ domains with $0 < a \leq 1$ are Liapunov-Dini domains.

This note is organized as follows. Section 2 collects some elementary lemmas concerning the set $\mathcal{B}_p(x,t)$ and the function g. Proofs of Theorem 1.1 and Corollary 1.3 are given in Sections 3 and 4, respectively. As a consequence of Theorem 1.1, we establish upper and lower bound estimates of kernel functions with pole at parabolic boundary points in Section 5.

2 Preliminaries

A bounded domain Ω in $\mathbb{R}^n$ is called a *Lipschitz domain* with localization radius $r_0 > 0$ and Lipschitz constant $L > 0$ if for each $\xi \in \partial\Omega$ there exist a local Cartesian coordinate system $(x_1,\ldots,x_n) = (x', x_n)$ and a function $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying the Lipschitz condition $|\psi(x') - \psi(y')| \leq L\|x' - y'\|$ such that

$$\Omega \cap B(\xi, r_0) = \{(x', x_n) : x_n > \psi(x')\} \cap B(\xi, r_0).$$

Then we see that for each $\xi \in \partial\Omega$, there is a point $z \in \mathbb{R}^n$ such that the truncated circular cone $\{x : \angle x\xi z < \theta, \|x - \xi\| < r_0\}$ is contained in Ω, where $\theta = \arctan(1/L)$. Therefore, if $0 < r < r_0/2$, then the point, denoted by ξ_r, in the intersection of the axis $\overline{z\xi}$ and $\partial B(\xi, r) \cap \Omega$ satisfies $\delta(\xi_r) \geq r\sin\theta$. Also, the notation $C(\Omega)$ (which has already used in the introduction) means $C(L, r_0, \operatorname{diam}\Omega)$.

In the rest of this note, we suppose that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ $(n \geq 2)$ with localization radius $r_0 > 0$ and Lipschitz constant $L > 0$ and that $\delta(x_0) \geq r_0/2$. Also, $T > 0$ is fixed. We start with some elementary lemmas.

Lemma 2.1. *Let $\theta = \arctan(1/L)$. If $\kappa \geq (r_0/\sqrt{T}) \vee (2\sqrt{T}/r_0 \sin\theta)$, then the set $\mathcal{B}_p(x,t)$ is nonempty for any pair $x \in \overline{\Omega}$ and $0 < t < T$.*

Proof. Let $x \in \overline{\Omega}$ and $0 < t < T$. Put $r = (r_0/2)\sqrt{t/T}$. If $\delta(x) \geq r$, then $x \in \mathcal{B}_p(x,t)$ whenever $\kappa \geq 2\sqrt{T}/r_0$. Consider the case $\delta(x) < r < r_0/2$. Let $\xi \in \partial\Omega$ be a point such that $\|\xi - x\| = \delta(x)$. As mentioned above, we find $\xi_r \in \partial B(\xi, r) \cap \Omega$ such that $\delta(\xi_r) \geq r \sin\theta$. Then

$$\|\xi_r - x\| \leq \|\xi_r - \xi\| + \|\xi - x\| \leq 2r.$$

Therefore, if $\kappa \geq (r_0/\sqrt{T}) \vee (2\sqrt{T}/r_0 \sin\theta)$, then $\xi_r \in \mathcal{B}_p(x,t)$. □

For two positive functions f_1 and f_2, we write $f_1 \approx f_2$ if there is a constant $C \geq 1$ such that $f_1/C \leq f_2 \leq Cf_1$. Then the constant C is called the constant of comparison. The next lemma follows from the Harnack inequality for the Green function G (see [11, Lemma 3.3]).

Lemma 2.2. *Let $\lambda > 0$. If $x, y \in \Omega$ satisfy $\|x - y\| \leq \lambda(\delta(x) \wedge \delta(y))$, then*

$$g(x) \approx g(y),$$

where the constant of comparison depends only on λ, n and Ω.

Lemma 2.3. *Let $\lambda > 0$. If $x \in \Omega$ and $0 < t < T$ satisfy $\delta(x) \geq \lambda\sqrt{t}$, then*

$$g(b) \approx g(x) \quad \text{for all } b \in \mathcal{B}_p(x,t),$$

where the constant of comparison depends only on λ, n, Ω and T.

Proof. Let $b \in \mathcal{B}_p(x,t)$. The assumption and the definition of $\mathcal{B}_p(x,t)$ imply that

$$\|b - x\| \leq \kappa\sqrt{t} \leq \kappa\left(\kappa \vee \frac{1}{\lambda}\right)(\delta(b) \wedge \delta(x)).$$

Hence the conclusion follows from Lemma 2.2. □

The following three lemmas will be used in Section 5.

Lemma 2.4. *There exists $C = C(n, \Omega, T) > 0$ such that if $0 < t < T$, then*

$$g(b) \geq C \quad \text{for all } b \in \mathcal{B}_p(x_0,t).$$

Proof. Let $b \in \mathcal{B}_p(x_0,t)$. Then

$$\|b - x_0\| \leq \left(\kappa^2 \vee \frac{2\operatorname{diam}\Omega}{r_0}\right)(\delta(b) \wedge \delta(x_0)),$$

and so $g(b) \approx g(x_0) = 1$ by Lemma 2.2. □

Lemma 2.5. *There exists $C = C(n, \Omega, T) > 0$ such that if $x \in \Omega$ and $0 < t < T$, then*

$$g(b) \geq C \quad \text{for all } b \in \mathcal{B}_p(x, T + 1 - t).$$

Proof. This follows from $\delta(b) \geq \sqrt{T + 1 - t}/\kappa \geq 1/\kappa$ and the Harnack inequality. □

Lemma 2.6. *Let $x \in \Omega$ and $0 < t < T$. Then*

$$g(b_1) \approx g(b_2) \quad \text{for all } b_1, b_2 \in \mathcal{B}_p(x,t),$$

where the constant of comparison depends only on n, Ω and T.

Proof. Since $\|b_1 - b_2\| \leq \|b_1 - x\| + \|x - b_2\| \leq 2\kappa\sqrt{t} \leq 2\kappa^2(\delta(b_1) \wedge \delta(b_2))$, the conclusion follows from Lemma 2.2. □

3 Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1. A solution of the heat equation on a domain $D \subset \mathbb{R}^{n+1}$ is called a *temperature* on D. The following lemma is a consequence of the parabolic Harnack inequality established by Moser [15].

Lemma 3.1. *Let $\lambda > 0$. Then there exists $C = C(\lambda, n, \Omega) > 0$ such that if u is a nonnegative temperature on $\Omega \times (0, \infty)$, then*

$$u(x, t/2) \le Cu(y, t) \exp\left\{ \frac{C\|x - y\|^2}{t} \right\} \tag{3.1}$$

for any $x, y \in \Omega$ and $t > 0$ satisfying $\delta(x) \wedge \delta(y) \ge \lambda\sqrt{t}$.

Proof. For $t > 0$, we write $r = \lambda\sqrt{t}$. Let $x, y \in \Omega$ satisfy $\delta(x) \wedge \delta(y) \ge r$. If $\|x - y\| \le r/2$, then (3.1) holds by the parabolic Harnack inequality. Consider the case $\|x - y\| > r/2$. Since Ω is Lipschitz, we find a Harnack chain $\{B(z_j, r/C)\}_{j=0}^m$ in Ω such that $z_0 = x$, $z_m = y$ and $z_{j-1} \in B(z_j, r/2C)$ $(j = 1, \cdots, m)$, where $C = C(\Omega) \ge 2$. Moreover, the number m satisfies

$$m \le \frac{C\|x - y\|}{r}$$

for some $C = C(\Omega)$. Let $t_j = (t/2) + (jt/2m)$. Then, by the parabolic Harnack inequality, there is $C = C(\lambda, n, \Omega) > 0$ such that

$$u(z_{j-1}, t_{j-1}) \le C^m u(z_j, t_j) \quad \text{for } j = 1, \cdots, m.$$

Therefore

$$u(x, t/2) \le C^{m^2} u(y, t) \le Cu(y, t) \exp\left\{ \frac{C\|x - y\|^2}{t} \right\}.$$

Thus the lemma is proved. □

The following lemma is elementary and well known.

Lemma 3.2. *Let $\lambda > 0$. Then there exists $C = C(\lambda, n, \Omega) > 1$ such that if $x \in \Omega$ and $t > 0$ satisfy $\delta(x) \ge \lambda\sqrt{t}$, then*

$$\Gamma(x, t; x, 0) \ge \frac{1}{Ct^{n/2}}. \tag{3.2}$$

Proof. For the convenience sake of the reader, we give a proof. Let $x \in \Omega$ and $t > 0$ satisfy $\delta(x) \ge \lambda\sqrt{t}$, and let ϕ be a continuous function on $\mathbb{R}^n$ such that $0 \le \phi \le 1$ and

$$\phi = \begin{cases} 1 & \text{on } B(x, \lambda\sqrt{t}/3), \\ 0 & \text{on } \mathbb{R}^n \setminus B(x, \lambda\sqrt{t}/2). \end{cases}$$

Consider the function u defined on $\Omega \times \mathbb{R}$ by

$$u(z, s) = \begin{cases} \displaystyle\int_\Omega \Gamma(z, s; y, t/2)\phi(y)\, dy & \text{if } s > t/2, \\ 1 & \text{if } s \le t/2. \end{cases}$$

Observe that, on $B(x, \lambda\sqrt{t}/3) \times \mathbb{R}$, it is continuous and satisfies the parabolic mean value equality, and so u is a nonnegative temperature on there (see [18, Theorem 15]). The parabolic Harnack inequality gives

$$1 = u(x, t/2) \le Cu(x, t).$$

Also, the adjoint version of the parabolic Harnack inequality gives

$$\Gamma(x, t; y, t/2) \le C\Gamma(x, t; x, 0) \quad \text{for all } y \in B(x, \lambda\sqrt{t}/2).$$

Hence

$$1 \le Cu(x,t) \le C \int_{B(x,\lambda\sqrt{t}/2)} \Gamma(x,t;y,t/2)\,dy \le Ct^{n/2}\Gamma(x,t;x,0),$$

and so (3.2) follows. □

Lemma 3.3. *Let $\lambda > 0$. Then there exists $C = C(\lambda, n, \Omega) > 1$ such that if $x, y \in \Omega$ and $t > 0$ satisfy $\delta(x) \wedge \delta(y) \ge \lambda\sqrt{t}$, then*

$$\gamma_{\frac{1}{C}}(x-y,t) \le \Gamma(x,t;y,0) \le \gamma_4(x-y,t).$$

Proof. The upper bound estimate always holds. The lower bound estimate follows from Lemmas 3.1 and 3.2:

$$\begin{aligned}\Gamma(x,t;y,0) &\ge \frac{1}{C}\Gamma(y,t/2;y,0)\exp\left\{-\frac{C\|x-y\|^2}{t}\right\}\\ &\ge \frac{1}{Ct^{n/2}}\exp\left\{-\frac{C\|x-y\|^2}{t}\right\}.\end{aligned}$$

□

In what follows, we let

$$\lambda = \frac{1}{50}\left(1 \wedge \frac{r_0}{\sqrt{T}}\right). \tag{3.3}$$

By Lemmas 2.3 and 3.3, we see that (1.5) and (1.6) hold whenever $x, y \in \Omega$ and $0 < t < T$ satisfy $\delta(x) \wedge \delta(y) \ge \lambda\sqrt{t}$. To complete the proof of Theorem 1.1, we consider the case $\delta(x) \wedge \delta(y) < \lambda\sqrt{t}$ in the rest of this section. We use the following local comparison estimate (see Fabes et al. [9, Theorem 1.6]). For $\xi \in \partial\Omega$, $s \in \mathbb{R}$ and $r > 0$, let

$$\begin{aligned}\Psi_r(\xi,s) &= \{(x,t) \in \Omega \times \mathbb{R} : \|x-\xi\| < r, |t-s| < r^2\},\\ \Delta_r(\xi,s) &= \{(x,t) \in \partial\Omega \times \mathbb{R} : \|x-\xi\| < r, |t-s| < r^2\}.\end{aligned}$$

Lemma 3.4 (Local comparison estimate)**.** *Let $\xi \in \partial\Omega$, $s > 0$ and $0 < r < r_0/2$. Suppose that u_1 and u_2 are positive temperatures on $\Psi_{2r}(\xi,s)$ vanishing continuously on $\Delta_{2r}(\xi,s)$. Then there exists $C = C(n,\Omega) \ge 1$ such that*

$$\frac{u_1(x,t)}{u_2(x,t)} \le C\frac{u_1(\xi_r, s+2r^2)}{u_2(\xi_r, s-2r^2)} \quad \text{for all } (x,t) \in \Psi_{r/8}(\xi,s),$$

where ξ_r is the point stated in the first paragraph of Section 2.

Also, we recall the boundary Harnack principle for harmonic functions (see [2]).

Lemma 3.5 (Boundary Harnack principle)**.** *Let $\xi \in \partial\Omega$ and $0 < r < r_0/2$. Suppose that h_1 and h_2 are positive harmonic functions on $\Omega \cap B(\xi,2r)$ vanishing continuously on $\partial\Omega \cap B(\xi,2r)$. Then there exists $C = C(n,\Omega) \ge 1$ such that*

$$\frac{h_1(x)}{h_2(x)} \le C\frac{h_1(y)}{h_2(y)} \quad \text{for all } x, y \in \Omega \cap B(\xi,r).$$

Lemma 3.6. *There exists a constant $C = C(n,\Omega,T) > 1$ such that if $x, y \in \Omega$ and $0 < t < T$ satisfy $\delta(x) \wedge \delta(y) < \lambda\sqrt{t}$, then the upper bound estimate* (1.5) *holds.*

Proof. Since $\Gamma(x,t;y,0) = \Gamma(y,t;x,0)$, we may assume that $\delta(x) \le \delta(y)$. Let $r = 8\lambda\sqrt{t}$. Then $r < r_0/6$ and $t - 4r^2 > 0$. Let $\xi \in \partial\Omega$ be a point such that $\|\xi - x\| = \delta(x) < r/8$. Since the function $v(x,t) = v(x) = G(x,\xi_{3r})$ is a positive temperature on $\Psi_{2r}(\xi,t)$ vanishing continuously on $\Delta_{2r}(\xi,t)$, it follows from Lemmas 3.4 and 3.5 that

$$\frac{\Gamma(x,t;y,0)}{\Gamma(\xi_r,t+2r^2;y,0)} \le C\frac{v(x,t)}{v(\xi_r,t-2r^2)} = C\frac{G(x,\xi_{3r})}{G(\xi_r,\xi_{3r})} \approx \frac{g(x)}{g(\xi_r)}. \tag{3.4}$$

Let $b_x \in \mathcal{B}_p(x,t)$. Then

$$\|\xi_r - b_x\| \le \|\xi_r - x\| + \|x - b_x\| \le Cr \le C(\delta(\xi_r) \wedge \delta(b_x)),$$

and so Lemma 2.2 gives

$$g(\xi_r) \approx g(b_x). \tag{3.5}$$

By (3.4) and (3.5), we have

$$\Gamma(x,t;y,0) \le C\frac{g(x)}{g(b_x)}\Gamma(\xi_r, t+2r^2; y, 0). \tag{3.6}$$

We consider two cases: $\delta(y) \ge r/16$ and $\delta(y) < r/16$.

Case 1: $\delta(y) \ge r/16$**.** Let $b_y \in \mathcal{B}_p(y,t)$. Then, by Lemma 2.3,

$$g(b_y) \approx g(y).$$

Since

$$\|\xi_r - y\|^2 \ge \frac{1}{2}\|x-y\|^2 - \|\xi_r - x\|^2 \ge \frac{1}{2}\|x-y\|^2 - Ct,$$

we have

$$\begin{aligned}\Gamma(\xi_r, t+2r^2; y, 0) &\le \frac{1}{\{4\pi(t+2r^2)\}^{n/2}}\exp\left\{-\frac{\|\xi_r - y\|^2}{4(t+2r^2)}\right\} \\ &\le \frac{C}{t^{n/2}}\exp\left\{-\frac{\|x-y\|^2}{Ct}\right\}.\end{aligned}$$

These, together with (3.6), yields (1.5).

Case 2: $\delta(y) < r/16$**.** Let $\eta \in \partial\Omega$ be a point such that $\|\eta - y\| = \delta(y)$ and let $b_y \in \mathcal{B}_p(y,t)$. Applying the adjoint version of the local comparison estimate to $\Gamma(\xi_r, t+2r^2; \cdot, \cdot)$ and $G(\eta_{3r}, \cdot)$, we have by the same reasoning as for (3.6) that

$$\frac{\Gamma(\xi_r, t+2r^2; y, 0)}{\Gamma(\xi_r, t+2r^2; \eta_{r/2}, -r^2/2)} \le C\frac{G(\eta_{3r}, y)}{G(\eta_{3r}, \eta_{r/2})} \approx \frac{g(y)}{g(\eta_{r/2})} \approx \frac{g(y)}{g(b_y)}. \tag{3.7}$$

Since

$$\|\xi_r - \eta_{r/2}\|^2 \ge \frac{1}{2}\|x-y\|^2 - \|\xi_r - x + y - \eta_{r/2}\|^2 \ge \frac{1}{2}\|x-y\|^2 - Ct,$$

we have

$$\Gamma(\xi_r, t+2r^2; \eta_{r/2}, -r^2/2) \le \frac{C}{t^{n/2}}\exp\left\{-\frac{\|x-y\|^2}{Ct}\right\}. \tag{3.8}$$

Hence (1.5) follows from (3.6), (3.7) and (3.8). Thus Lemma 3.6 is proved. □

Lemma 3.7. *There exists $C = C(n,\Omega,T) > 1$ such that if $x, y \in \Omega$ and $0 < t < T$ satisfy $\delta(x) \wedge \delta(y) < \lambda\sqrt{t}$, then the lower bound estimate* (1.6) *holds.*

Proof. The proof is almost the same as that of Lemma 3.6, and we will use the same notations. Replacing the position of $v = G(\cdot, \xi_{3r})$ and $\Gamma(\cdot,\cdot;y,0)$ in (3.4), we have

$$\frac{\Gamma(x,t;y,0)}{\Gamma(\xi_r, t-2r^2; y, 0)} \ge \frac{1}{C}\frac{G(x,\xi_{3r})}{G(\xi_r,\xi_{3r})} \approx \frac{g(x)}{g(b_x)}, \tag{3.9}$$

where $b_x \in \mathcal{B}_p(x,t)$. If $\delta(y) \ge r/16$, then $g(b_y) \approx g(y)$ for $b_y \in \mathcal{B}_p(y,t)$. Since $\|\xi_r - y\|^2 \le 2\|x-y\|^2 + Ct$, we obtain (1.6) from (3.9) and Lemma 3.3.

If $\delta(y) < r/16$, then we can apply the adjoint version of the local comparison estimate to $\Gamma(\xi_r, t-2r^2; \cdot, \cdot)$ and $G(\eta_{3r}, \cdot)$ because $t - 2r^2 > r^2$. Let $b_y \in \mathcal{B}_p(y,t)$. Then

$$\frac{\Gamma(\xi_r, t-2r^2; y, 0)}{\Gamma(\xi_r, t-2r^2; \eta_{r/2}, r^2/2)} \ge \frac{1}{C}\frac{G(\eta_{3r}, y)}{G(\eta_{3r}, \eta_{r/2})} \approx \frac{g(y)}{g(b_y)}. \tag{3.10}$$

Since $\Gamma(\xi_r, t-2r^2; \eta_{r/2}, r^2/2) = \Gamma(\xi_r, t-5r^2/2; \eta_{r/2}, 0)$ and $\|\xi_r - \eta_{r/2}\|^2 \le 2\|x-y\|^2 + Ct$, we obtain (1.6) from (3.9), (3.10) and Lemma 3.3. □

Proof of Theorem 1.1. Let λ be as in (3.3). As mentioned above, Lemmas 2.3 and 3.3 show that (1.5) and (1.6) hold when $\delta(x) \wedge \delta(y) \geq \lambda\sqrt{t}$. Another case was discussed in Lemmas 3.6 and 3.7. Thus the proof is complete. $\square$

Since $\Gamma(x,t;y,s) = \Gamma(x,t-s;y,0)$, we obtain the following corollary.

Corollary 3.8. *There exists $C = C(n,\Omega,T) > 1$ such that the following lower and upper bound estimates hold for all $x, y \in \Omega$ and $0 < s < t < T$:*

$$\Gamma(x,t;y,s) \leq \frac{g(x)g(y)}{g(b_x)g(b_y)}\gamma_C(x-y,t-s),$$
$$\Gamma(x,t;y,s) \geq \frac{g(x)g(y)}{g(b_x)g(b_y)}\gamma_{\frac{1}{C}}(x-y,t-s),$$

where $b_x \in \mathcal{B}_p(x,t-s)$ and $b_y \in \mathcal{B}_p(y,t-s)$.

4 Proof of Corollary 1.3

As stated in Section 2, we observe that for each $\xi \in \partial\Omega$ there are circular cones V_1 and V_2 with vertex ξ and aperture θ and $\pi - \theta$, respectively, such that

$$V_1 \cap B(\xi, r_0) \subset \Omega \cap B(\xi, r_0) \subset V_2.$$

The both of V_1 and V_2 have the same axis. It is well known that there exists a unique positive harmonic function h_i on V_i with pole at ∞ which vanishes continuously on ∂V_i and $h_i(\xi_1) = 1$, where ξ_r is the point in the intersection of the axis of V_1 and $\partial B(\xi,r) \cap \Omega$. This function has the form

$$h_i(x) = \|x-\xi\|^{\tau_i} f_i(\xi + \frac{x-\xi}{\|x-\xi\|}) \quad \text{for all } x \in V_i, \tag{4.1}$$

where f_i is a positive function on $\partial B(\xi,1) \cap V_i$ satisfying $f_i(z) \approx \mathrm{dist}(z, \partial V_i)$ and $\tau_i > 0$ is a constant depending only on θ and n. Note that $\tau_2 \leq 1 \leq \tau_1$. It is well known that

$$\frac{1}{C}\delta(x)^{\tau_1} \leq g(x) \leq C\delta(x)^{\tau_2} \quad \text{for all } x \in \Omega,$$

and so

$$\frac{1}{C}\frac{\delta(x)^{\tau_1}}{\delta(y)^{\tau_2}} \leq \frac{g(x)}{g(y)} \leq C\frac{\delta(x)^{\tau_2}}{\delta(y)^{\tau_1}} \quad \text{for all } x, y \in \Omega.$$

Properties of τ_i and the above estimate of g can be found in [14]. We need the following sharper estimate, which is also known as a consequence of the boundary Harnack principle.

Lemma 4.1. *Let $\xi \in \partial\Omega$ and $0 < r < r_0/6$. Then there exist $C = C(n,\Omega) \geq 1$, $\alpha = \alpha(n,\Omega) > 0$ and $\beta = \beta(n,\Omega) > 0$ with $\tau_2 \leq \beta \leq 1 \leq \alpha \leq \tau_1$ such that*

$$\frac{1}{C}\left(\frac{t}{r}\right)^{\alpha} \leq \frac{g(\xi_t)}{g(\xi_r)} \leq C\left(\frac{t}{r}\right)^{\beta} \quad \textit{for all } 0 < t < r.$$

Proof. For the convenience sake of the reader, we give a proof. We use a reduced function of a nonnegative superharmonic function u on D relative to a set $E \subset D$ defined by

$${}^D R_u^E(x) = \inf\{v(x)\},$$

where the infimum is taken over all nonnegative superharmonic functions v on D such that $v \geq u$ on E. Note that ${}^D R_u^E \leq u$ on D. See [3, Section 5.3] for details.

Let $\xi \in \partial\Omega$ and $0 < r < r_0/6$. Now, we adopt $D = V_1 \cap B(\xi, r_0)$, $E = B(\xi_{3r}, r\sin\theta)$ and $u = g/g(\xi_r)$. Then ${}^D R_u^E$ is a positive harmonic function on $D \setminus \overline{E}$ vanishing continuously on ∂D such that ${}^D R_u^E(\xi_r) \approx 1$. The boundary Harnack principle implies that

$$\frac{h_1(\xi_t)}{h_1(\xi_r)} \le C \frac{{}^D R_u^E(\xi_t)}{{}^D R_u^E(\xi_r)} \le C\, {}^D R_u^E(\xi_t) \le C u(\xi_t) \quad \text{for all } 0 < t < r.$$

Using (4.1), we can estimate the left hand side from below by a constant multiple of $(t/r)^{\tau_1}$. Thus the lower bound estimate follows.

To prove the upper bound estimate, we substitute $D = \Omega \cap B(\xi, r_0)$ and $u = h_2/h_2(\xi_r)$ in the above. Then the boundary Harnack principle gives

$$\frac{g(\xi_t)}{g(\xi_r)} \le C \frac{{}^D R_u^E(\xi_t)}{{}^D R_u^E(\xi_r)} \le C\, {}^D R_u^E(\xi_t) \le C u(\xi_t) \le C \left(\frac{t}{r}\right)^{\tau_2}.$$

Thus the lemma is proved. □

Remark 4.2. If Ω is a Liapunov-Dini domain, then we can take $\alpha = \beta = 1$ in Lemma 4.1. Indeed, we know from [17] that the Poisson kernel satisfies

$$P(x, \xi) \approx \frac{\delta(x)}{\|x - \xi\|^n} \quad \text{for all } x \in \Omega \text{ and } \xi \in \partial\Omega.$$

Let $x \in \Omega$ and let $\xi \in \partial\Omega$ be a point such that $\|\xi - x\| = \delta(x)$. Since $g(x)P(x, \xi) \approx \delta(x)^{2-n}$ (see [10, Theorems 1.3 and 1.6]), we have

$$g(x) \approx \delta(x) \quad \text{for all } x \in \Omega.$$

Hence we can take $\alpha = \beta = 1$. Also, when Ω is a bounded Lipschitz domain, there may exist noncircular cones W_1 and W_2 with vertex ξ, whose shapes are independent of ξ, such that

$$V_1 \cap B(\xi, r_0) \subset W_1 \cap B(\xi, r_0) \subset \Omega \cap B(\xi, r_0) \subset W_2 \cap B(\xi, r_0) \subset V_2.$$

Therefore we may take $\alpha \le \tau_1$ and $\beta \ge \tau_2$.

Lemma 4.3. *Let α and β be as in Lemma 4.1. Then there exists $C = C(n, \Omega, T) \ge 1$ such that if $x \in \Omega$ and $0 < t < T$ satisfy $\delta(x) \le \sqrt{t}$, then*

$$\frac{1}{C}\left(\frac{\delta(x)}{\sqrt{t}}\right)^{\alpha} \le \frac{g(x)}{g(b)} \le C\left(\frac{\delta(x)}{\sqrt{t}}\right)^{\beta}, \tag{4.2}$$

where $b \in \mathcal{B}_p(x, t)$. Moreover, there exists $C = C(n, \Omega) \ge 1$ such that

$$\frac{1}{C}\delta(x)^{\alpha} \le g(x) \le C\delta(x)^{\beta} \quad \text{for all } x \in \Omega. \tag{4.3}$$

Proof. Let $r = \sqrt{t}$ and $b \in \mathcal{B}_p(x, t)$. Then $\delta(b) \ge r/\kappa$. Take $\xi \in \partial\Omega$ with $\|\xi - x\| = \delta(x) \le r$.

If $r < r_0/6$, then we have by Lemma 4.1

$$\frac{1}{C}\left(\frac{\delta(x)}{r}\right)^{\alpha} \le \frac{g(x)}{g(\xi_r)} \le C\left(\frac{\delta(x)}{r}\right)^{\beta}.$$

Since

$$\|\xi_r - b\| \le \|\xi_r - \xi\| + \|\xi - x\| + \|x - b\| \le Cr \le C(\delta(\xi_r) \wedge \delta(b)),$$

it follows from Lemma 2.2 that $g(\xi_r) \approx g(b)$. Thus (4.2) holds in this case.

If $\delta(x) < r_0/6 \le r$, then Lemma 4.1 gives

$$\frac{1}{C}\delta(x)^{\alpha} \le \frac{g(x)}{g(\xi_{r_0/6})} \le C\delta(x)^{\beta}.$$

Since $\delta(b) \ge r_0/6\kappa$, we have $g(\xi_{r_0/6}) \approx 1 \approx g(b)$, and so (4.2) follows.

If $\delta(x) \ge r_0/6$, then $g(x) \approx 1 \approx g(b)$. Therefore we can obtain (4.2) easily.

Also, the similar consideration to the last two cases yields (4.3). Thus the lemma is proved. □

Proof of Corollary 1.3. Let $x, y \in \Omega$ and $0 < t < T$. Consider four cases: $\delta(x) \vee \delta(y) \leq \sqrt{t}$; $\delta(x) \leq \sqrt{t} < \delta(y)$; $\delta(y) \leq \sqrt{t} < \delta(x)$; $\delta(x) \wedge \delta(y) > \sqrt{t}$. Then (1.9) and (1.10) follows from Theorem 1.1 and Lemmas 2.3 and 4.3. □

5 Global estimates for kernel functions with pole at boundary points

This section presents global estimates of kernel functions with pole at parabolic boundary points. We write $\Omega_T = \Omega \times (0, T)$ and $\partial_p \Omega_T = (\partial\Omega \times [0, T)) \cup (\Omega \times \{0\})$ the parabolic boundary of Ω_T. Let $(y, s) \in \partial_p \Omega_\infty$. We say that a nonnegative function $K(\cdot, \cdot; y, s)$ on Ω_∞ is a *kernel function* at (y, s) normalized at (x_0, T_0) if the following conditions are fulfilled:

(i) $K(\cdot, \cdot; y, s)$ is temperature on Ω_∞;

(ii) for each $(z, q) \in \partial_p \Omega_\infty \setminus \{(y, s)\}$,

$$\lim_{\Omega_\infty \ni (x,t) \to (z,q)} K(x, t; y, s) = 0;$$

(iii) $K(x_0, T_0; y, s) = 1$.

In arguments below, we let $T_0 = T+1$. As shown in [9, 13], there exists a unique kernel function at each point of $\partial_p \Omega_T$ if Ω is a bounded Lipschitz domain. Also, in these papers, the kernel function was obtained by considering quotients of caloric measures. The following lemma shows that the kernel function can be obtained as a limit function of quotients of the Green functions.

Lemma 5.1. *Let $y \in \partial\Omega$ and $0 \leq s < T$. Then there exists a sequence $\{y_j\}$ in Ω converging to y such that*

$$K(x, t; y, s) = \lim_{j \to \infty} \frac{\Gamma(x, t; y_j, s)}{\Gamma(x_0, T_0; y_j, s)}. \tag{5.1}$$

Proof. Let $y \in \partial\Omega$ and $0 \leq s < T$. In view of [19, Theorem 6], we find a sequence $\{y_j\}$ in Ω converging to y such that the ratio $\Gamma(x, t; y_j, s)/\Gamma(x_0, T_0; y_j, s)$ converges to a nonnegative temperature $h(x, t)$ in Ω_∞ with $h(x_0, T_0) = 1$. If $t \leq s$, then $h(x, t) = 0$ for all $x \in \Omega$. We show that h vanishes continuously at $(z, q) \in \partial_p \Omega_\infty \setminus \{(y, s)\}$, where $q \geq s$. Let $r > 0$ be sufficiently small such that $(z, q) \notin \Psi_{10r}(y, s)$ and let $(x, t) \in \Psi_r(z, q)$. The adjoint version of the local comparison estimate implies that for sufficiently large j,

$$\frac{\Gamma(x, t; y_j, s)}{\Gamma(x_0, T_0; y_j, s)} \leq C \frac{\Gamma(x, t; y_r, s - 2r^2)}{\Gamma(x_0, T_0; y_r, s + 2r^2)}.$$

Letting $j \to \infty$, we have

$$h(x, t) \leq C \frac{\Gamma(x, t; y_r, s - 2r^2)}{\Gamma(x_0, T_0; y_r, s + 2r^2)}.$$

If $(x, t) \to (z, q)$, then $\Gamma(x, t; y_r, s - 2r^2) \to 0$, and so $h(x, t) \to 0$. Therefore h is a kernel function at (y, s) normalized at (x_0, T_0). The uniqueness implies that $h = K(\cdot, \cdot; y, s)$. Thus the lemma is proved. □

As a consequence of Corollary 3.8, we obtain the following estimates.

Theorem 5.2. *There exists $C = C(n, \Omega, T) > 1$ such that for all $(y, s) \in \partial_p \Omega_T$ and $(x, t) \in \Omega_T$ with $t > s$,*

$$K(x, t; y, s) \leq \frac{g(x)}{g(b_x) g(b_y)} \gamma_C(x - y, t - s), \tag{5.2}$$

$$K(x, t; y, s) \geq \frac{g(x)}{g(b_x) g(b_y)} \gamma_{\frac{1}{C}}(x - y, t - s), \tag{5.3}$$

where $b_x \in \mathcal{B}_p(x, t - s)$ and $b_y \in \mathcal{B}_p(y, t - s)$.

Proof. We show (5.2) only, because the proof of (5.3) is similar. We first consider the case $y \in \partial\Omega$ and $0 \le s < T$. Let $(x,t) \in \Omega_T$ with $t > s$ and let $\{y_j\}$ be a sequence in Ω converging to y such that (5.1) holds. Observe from Corollary 3.8 that the ratio $\Gamma(x,t;y_j,s)/\Gamma(x_0,T_0;y_j,s)$ is bounded above by

$$\begin{aligned} &C\frac{g(x)}{g(b_x)}\frac{g(b_0)}{g(x_0)}\frac{g(b_j^0)}{g(b_j)}\left(\frac{T_0-s}{t-s}\right)^{n/2}\exp\left\{-\frac{\|x-y_j\|^2}{C(t-s)}+\frac{C\|x_0-y_j\|^2}{T_0-s}\right\} \\ &\le C\frac{g(x)}{g(b_x)g(b_j)}\frac{1}{(t-s)^{n/2}}\exp\left\{-\frac{\|x-y_j\|^2}{C(t-s)}\right\}, \end{aligned}$$

where $b_x \in \mathcal{B}_p(x,t-s)$, $b_0 \in \mathcal{B}_p(x_0,T_0-s)$, $b_j^0 \in \mathcal{B}_p(y_j,T_0-s)$ and $b_j \in \mathcal{B}_p(y_j,t-s)$. Here the last inequality follows by Lemmas 2.4, 2.5 and $\|x_0-y_j\| \le \operatorname{diam}\Omega$. Since there is a subsequence of $\{b_j\}$ converging to some $b_y \in \mathcal{B}_p(y,t-s)$, we obtain from (5.1) that

$$K(x,t;y,s) \le C\frac{g(x)}{g(b_x)g(b_y)}\frac{1}{(t-s)^{n/2}}\exp\left\{-\frac{\|x-y\|^2}{C(t-s)}\right\}.$$

Note from Lemma 2.6 that this inequality is valid for any $b_y \in \mathcal{B}_p(y,t-s)$. Hence (5.2) holds when $y \in \partial\Omega$ and $0 \le s < T$. If $y \in \Omega$ and $s = 0$, then

$$K(x,t;y,0) = \frac{\Gamma(x,t;y,0)}{\Gamma(x_0,T_0;y,0)},$$

and so (5.2) follows from Theorem 1.1. □

Corollary 5.3. *There exists $C = C(n,\Omega,T) > 1$ such that for all $(y,s) \in \partial_p\Omega_T$ and $(x,t) \in \Omega_T$ with $t > s$,*

$$\begin{aligned} K(x,t;y,s) &\le \left(\frac{\delta(x)}{\sqrt{t-s}}\wedge 1\right)^{\beta}\frac{1}{(\delta(y)\vee\sqrt{t-s})^{\alpha}}\gamma_C(x-y;t-s), \\ K(x,t;y,s) &\ge \left(\frac{\delta(x)}{\sqrt{t-s}}\wedge 1\right)^{\alpha}\frac{1}{(\delta(y)\vee\sqrt{t-s})^{\beta}}\gamma_{\frac{1}{C}}(x-y;t-s), \end{aligned}$$

where α and β are the constants given in Lemma 4.1. Moreover, if Ω is a Liapunov-Dini domain, we can take $\alpha = \beta = 1$.

Proof. Let $(y,s) \in \partial_p\Omega_T$ and let $(x,t) \in \Omega_T$ with $t > s$. Let $b_x \in \mathcal{B}_p(x,t-s)$. By Lemma 2.3 and (4.2), we have

$$\frac{1}{C}\left(\frac{\delta(x)}{\sqrt{t-s}}\wedge 1\right)^{\alpha} \le \frac{g(x)}{g(b_x)} \le C\left(\frac{\delta(x)}{\sqrt{t-s}}\wedge 1\right)^{\beta}. \tag{5.4}$$

Let $b_y \in \mathcal{B}_p(y,t-s)$. If $y \in \partial\Omega$, then

$$\delta(b_y) \le \|b_y-y\| \le \kappa\sqrt{t-s} \le \kappa^2\delta(b_y),$$

and so (4.3) gives

$$\frac{1}{C}(t-s)^{\alpha/2} \le \frac{1}{C}\delta(b_y)^{\alpha} \le g(b_y) \le C\delta(b_y)^{\beta} \le C(t-s)^{\beta/2}.$$

Consider the case $y \in \Omega$. Then $s = 0$. If $\delta(y) \le \sqrt{t}$, then

$$\frac{1}{\kappa}\sqrt{t} \le \delta(b_y) \le \delta(y) + \|y-b_y\| \le (1+\kappa)\sqrt{t},$$

and so

$$\frac{1}{C}t^{\alpha/2} \le g(b_y) \le Ct^{\beta/2}.$$

If $\delta(y) > \sqrt{t}$, then $g(b_y) \approx g(y)$ by Lemma 2.3, and so

$$\frac{1}{C}\delta(y)^\alpha \le g(b_y) \le C\delta(y)^\beta.$$

Therefore, combining all cases gives

$$\frac{1}{C}(\delta(y) \vee \sqrt{t-s})^\alpha \le g(b_y) \le C(\delta(y) \vee \sqrt{t-s})^\beta. \tag{5.5}$$

Hence the corollary follows from Theorem 5.2, (5.4) and (5.5). □

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