# EXTENSION OF THE DRASIN–SHEA–JORDAN THEOREM

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Abstract. Passing from regular variation of a function $f$ to regular variation of its integral transform $k * f$ of Mellin-convolution form with kernel $k$ is an Abelian problem; its converse, under suitable Tauberian conditions, is a Tauberian one. In either case, one has a comparison statement that the ratio of $f$ and $k * f$ tends to a constant at infinity. Passing from a comparison statement to a regular-variation statement is a Mercerian problem. The prototype results here are the Drasin-Shea theorem (for non-netative $k$) and Jordan's theorem (for $k$ which may change sign). We free Jordan's theorem from its non-essential technical conditions which reduce its applicability. Our proof is simpler than the counter-parts of the previous results and does not even use the Pólya Peak Theorem which has been so essential before. The usefulness of the extension is highlighted by an application to Hankel transforms.

## 1. Introduction

In [BI1, BI2], we proved Mercerian results for Hankel transforms. In particular, in [BI2] we introduced a method, a type of localization, which seems useful for other problems, too. Here we apply the method to general integral transforms, and thereby extend the theorems of Drasin-Shea [DS] and Jordan [J]. As an application, we give a Mercerian result for Hankel transforms of non-monotone functions.

We recall the setting. Given a measurable kernel $k : (0, \infty) \to \mathbf{R}$, let

$$\check{k}(z) := \int_0^\infty t^{-z} k(t) \frac{dt}{t}$$

be its Mellin transform for $z \in \mathbf{C}$ such that the integral converges absolutely. For suitable functions $f, g : (0, \infty) \to \mathbf{R}$, the Mellin convolution is the function $f * g$ given by

$$(f * g)(x) := \int_0^\infty f(x/t) g(t) \frac{dt}{t}$$

for those $x > 0$ for which the integral converges absolutely. For $\rho \in \mathbf{R}$, we write $R_\rho$ for the class of functions $f$ *regularly varying* (at infinity) with *index* $\rho$: $f$ is measurable, positive for large enough $x$, and

$$f(\lambda x)/f(x) \to \lambda^\rho \qquad (x \to \infty) \qquad \forall \lambda > 0;$$

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see [BGT] for background. We are concerned here with comparisons between the asymptotic behaviour of the function $f$ and that of its transform $k * f$. The simplest such results of this type are *Abelian*, and state that under suitable conditions

$$f(x) \sim x^{\rho}\ell(x) \qquad (x \to \infty) \tag{1.1}$$

with $l \in R_0$ implies

$$(k * f)(x) \sim cx^{\rho}\ell(x) \qquad (x \to \infty), \tag{1.2}$$

where

$$c = \check{k}(\rho). \tag{1.3}$$

*Tauberian* results supply a partial converse under suitable side-conditions (*Tauberian conditions*); see e.g. [BGT, Ch. 4] for a detailed treatment. In these circumstances, one has

$$(k * f)(x)/f(x) \to c \qquad (x \to \infty). \tag{1.4}$$

The question arises of whether one can obtain (1.1) — and so (1.2) — from (1.4), with $\rho$ as in (1.3). Such results are *Mercerian* in character; for a textbook treatment, see e.g. [BGT, Ch. 5]. The prototypes are due to Drasin and Shea ([DS, Th. 6.2]; [BGT, Th. 5.2.1]), with $k$ non-negative, and Jordan ([J, Th. 1, 1a]; [BGT, Th. 5.3.1]), where $k$ can change sign.

This area of Mercerian theorems of Drasin-Shea-Jordan type has long suffered from several outstanding problems. First, the proofs are long, complicated and highly technical, and one seeks to simplify them as much as possible. Next, until recently no such results were available for the important case where the integrals defining $\check{k}$ and $k * f$ are only conditionally rather than absolutely convergent, as is the case for Fourier and Hankel transforms, for example. In two recent works [BI1, BI2] we succeeded both in simplifying the proofs of Drasin-Shea-Jordan theorems and in extending them to Fourier and Hankel transforms.

Here we focus on a third long-standing problem: the need to rid Jordan's theorems [J] of various technical conditions which complicate their statements and reduce their applicability. Our approach here was suggested by the observation [BI1, §8.1] that in the Hankel case, for $\rho$ small enough to give $k * f$ absolutely convergent, one of Jordan's theorems [J, Th. 1] nevertheless did not apply because its conditions exclude kernels such as the Fourier and Hankel ones, which oscillate infinitely often in sign. (However, Jordan's second theorem [J, Th. 1a] applies to some but not all Hankel cases; see §5 for details.) Our results succeed in solving the problem raised there. The main specific contributions are:

(i) freeing Jordan's theorem from its non-essential technical conditions;
(ii) eliminating the need to use Pólya peaks (see below for details);

(iii) extending the results of [BI1, BI2] in the Hankel case with absolute convergence (the motivating situation, as mentioned above) to non-monotone functions.

This last is to be expected: monotonicity was used to ensure convergence in the conditionally convergent case.

We recall ([BGT, §2.1.2]) the Matuszewska indices of a positive function $f$. The *upper Matuszewska index* $\alpha(f)$ is the infimum of those $\alpha$ for which there exists a constant $C = C(\alpha)$ such that for each $\Lambda > 1$,

$$f(\lambda x)/f(x) \leq C\{1 + o(1)\}\lambda^{\alpha} \quad (x \to \infty) \quad \text{uniformly in } \lambda \in [1, \Lambda],$$

the *lower Matuszewska index* $\beta(f)$ is the supremum of those $\beta$ for which, for some constant $D = D(\beta) > 0$ and all $\Lambda > 1$,

$$f(\lambda x)/f(x) \geq D\{1 + o(1)\}\lambda^{\beta} \quad (x \to \infty) \quad \text{uniformly in } \lambda \in [1, \Lambda].$$

One says $f$ has *bounded increase*, $f \in BI$, if $\alpha(f) < \infty$, *bounded decrease*, $f \in BD$, if $\beta(f) > -\infty$.

The *upper order* $\rho(f)$ of a positive function $f$ is defined by

$$\rho(f) := \limsup_{x\to\infty} \frac{\log f(x)}{\log x}.$$

## 2. Results

**Theorem 1.** *Let $k$ be a real kernel such that $\check{k}(z)$ converges absolutely for $a < \Re z < b$. Assume also*

$$\check{k}(z) \neq \check{k}(\rho) \quad \textit{for} \quad \Re z = \rho \textit{ and } z \neq \rho, \tag{2.1}$$

$$|\check{k}'(\rho)| + |\check{k}''(\rho)| > 0. \tag{2.2}$$

*Let $f$ be non-negative, measurable, and locally bounded on $[0, \infty)$, vanish in a neighbourhood of zero, have finite upper order $\rho \in (a, b)$, and $f \in BD \cup BI$. If*

$$(k * f)(x)/f(x) \to c \neq 0 \quad (x \to \infty), \tag{2.3}$$

*then $c = \check{k}(\rho)$ and $f \in R_{\rho}$.*

For non-negative $k$, the theorem above coincides with that of Drasin and Shea [DS] (see also [BGT, Th. 5.2.1]). The point of the theorem is that it dispenses with some extra assumptions of Jordan [J] (see also [BGT, Th. 5.3.1]). These extra conditions are of two kinds. The first one is that in an appropriate subinterval of $(a, b)$ the Mellin transform $\check{k}$ is monotone. The second one is the restriction of the behaviour of $k(t)$ for either large or small $t$. As Jordan wrote himself (in [J, p. 180]), the second one does not seem to be so restrictive since few kernels of normal interest fail to satisfy it. On the other hand, the first one can be quite restrictive for some kernels (see §5).

The proof of Theorem 1 will be given in §3. As we indicated above, the key to the proof is the localization technique introduced in [BI2]. The idea is simple. There are two kinds of similar integral transforms $f \mapsto E_1 * f$ and $f \mapsto E_2 * f$ (see §3) which have been already used in, e.g., [DS], [J] and [BI1]. Instead of such separate use, we apply both transforms at the same time: $f \mapsto E_1 * E_2 * f$. Then this has the effect of localizing the problem completely, that is, it is enough to restrict to the narrow strip $\rho - \epsilon < \Re z < \rho + \epsilon$ for $\epsilon > 0$ small enough instead of the intermediate strips $\rho - \epsilon < \Re z < b$ or $a < \Re z < \rho + \epsilon$ used in the previous works. The usefulness of the idea is shown, for example, by the fact that, by virtue of it, we can avoid the use of the Pólya Peak Theorem of Drasin and Shea, which has been so essential before but does not work well enough in our problem (see [DS], [BGT, §§2.5, 5.2.3]).

We apply Theorem 1 to Hankel transforms. There the kernel $k$ to be considered is

$$k_\nu(x) := x^{-3/2} J_\nu(1/x), \tag{2.4}$$

where $\nu > -\frac{1}{2}$ and $J_\nu$ is the Bessel function. Here we treat only absolutely convergent integrals. The Mellin transform $\check{k}_\nu(z)$ of $k_\nu$ converges absolutely for $-\nu - \frac{3}{2} < \Re z < -1$ (so $a = -\nu - \frac{3}{2}$, $b = -1$), and is given by Weber's integral:

$$\check{k}_\nu(z) = 2^{z+\frac{1}{2}} \frac{\Gamma(\frac{3}{4} + \frac{1}{2}\nu + \frac{1}{2}z)}{\Gamma(\frac{1}{4} + \frac{1}{2}\nu - \frac{1}{2}z)}. \tag{W}$$

We write $F_\nu$, or simply $F$, for the Hankel transform $k_\nu * f$ of $f$:

$$F_\nu(x) := \int_0^\infty k_\nu(x/t) f(t) \frac{dt}{t} \qquad (0 < x < \infty). \tag{2.5}$$

We note that, in the usual terminology, it is not $F_\nu$ but $x^{-1}F_\nu(1/x)$ which is called the Hankel transform of $f$:

$$x^{-1}F_\nu(1/x) = \int_0^\infty (xt)^{1/2} J_\nu(xt) f(t) dt \qquad (0 < x < \infty).$$

**Theorem 2.** *Let $-1/2 < \nu < \infty$. Let $t^{\nu+\frac{1}{2}} f(t) \in L^1_{\mathrm{loc}}[0, \infty)$, $f$ be eventually positive, have finite upper order $\rho := \rho(f) \in (-\nu - \frac{3}{2}, -1)$, $f \in BD \cup BI$, and let $f$ have Hankel transform $F_\nu$. If*

$$F_\nu(x)/f(x) \to c \neq 0 \qquad (x \to \infty), \tag{2.6}$$

*then $c = \check{k}_\nu(\rho)$ and $f \in R_\rho$.*

The theorem above is an analogue of the results of [BI1, BI2]. The difference is that in Theorem 2 the monotonicity assumption of [BI1, BI2] is weakened at the cost of absolute convergence of integrals. For example, if we consider Fourier transforms of radial functions on $\mathbf{R}^n$, then we naturally meet Hankel transforms of order $\nu = \frac{1}{2}(n-2)$, by Bochner's theorem (see e.g. [BC, II.7, Th. 40]). In such a situation, the monotonicity assumption of [BI1, BI2] will sometimes be too strong.

Since we consider only absolutely convergent integrals here, we cannot include the case $\nu = -\frac{1}{2}$, that is, the Fourier cosine transform. For, the Mellin transform of the cosine kernel $k_{-1/2}(x) = (2/\pi)^{1/2} x^{-1} \cos(1/x)$ has no strip of absolute convergence. To consider the cosine transform, it is indispensable to treat conditionally convergent integrals as in [BI1, BI2]. On the other hand, the Fourier sine transform has strips of both absolute and conditional convergence; our results here apply only to the first, those of [BI1, BI2] to both.

As we have shown in [BI1, BI2], for all $\nu > -\frac{1}{2}$, $\check{k}_\nu$ satisfies both the conditions (2.1) and (2.2). So for $f$ as in Theorem 1, Theorem 2 is an immediate consequence of Theorem 1. On the other hand, the Drasin-Shea theorem (the case $k(\cdot) \geq 0$) cannot be applied to the kernel $k_\nu$ as it changes sign. Further, if $\nu$ is large enough, then neither Theorem 1 nor Theorem 1a of Jordan [J] can be applied to $k_\nu$ (see §5).

The vanishing of $f$ near zero as in Theorem 1 will be too restrictive, and actually is not assumed in Theorem 2. Since this is the point of the proof of Theorem 2 in §4, we discuss it briefly. Clearly, we could drop the vanishing of $f$ near zero if we were able to show

$$\left| \int_0^1 f(u) k_\nu(x/u) \frac{du}{u} \right| = o(f(x)) \qquad (x \to \infty). \tag{2.7}$$

However, it seems difficult to prove this directly. In fact, we can show

$$\left| \int_0^1 f(u) k_\nu(x/u) \frac{du}{u} \right| \leq c_1 x^{-(\nu+\frac{3}{2})}$$

for some $c_1 > 0$, and the consequence $f \in R_\rho$ of Theorem 2 certainly implies $x^{-(\nu+\frac{3}{2})} = o(f(x))$ as $x \to \infty$. Unfortunately, we do not know how to prove the last assertion directly, that is, without Theorem 2. For this reason, in the proof of Theorem 2, we follow the line of [BI1] and use Theorem 1 to reduce its complicated arguments rather than use the theorem directly.

## 3. Proof of Theorem 1

For brevity, we will as far as possible keep, step by step, to the proof of [BGT, Theorem 5.2.1] (the Drasin-Shea theorem). By (2.3), $f$ is eventually positive. We note a crude but useful bound. For $\gamma \in \mathbf{R}$, $G : (0,\infty) \to \mathbf{R}$ measurable such that $\check{G}(\gamma)$ converges absolutely, and $f$ non-negative and measurable with bound

$f(x) \le d(\gamma)x^\gamma$ $(x > 0)$, the Mellin convolution $G * f$ exists and satisfies the bound

$$|G * f(x)| \le d(\gamma)x^\gamma \int_0^\infty t^{-\gamma}|G(t)|dt/t = d(\gamma)|G|\check{}(\gamma)x^\gamma \qquad (0 < x < \infty).$$

*Steps 1 and 2* are exactly as those of the proof of [BGT, Theorem 5.3.1] (Jordan's theorem), so $\check{k}(\rho) = c$. Here we can assume $f \in BI$ instead of $f \in BD$; see [BGT, §5.2.4].

*Steps 3–5.* By the Riemann-Lebesgue Lemma and Vitali's theorem in complex analysis [T1, §5.2], we may take $\epsilon > 0$ so small that $[\rho - 2\epsilon, \rho + 2\epsilon] \subset (a, b)$ and that $\check{k}(z)$ takes the value $\check{k}(\rho)$ only at $z = \rho$ in the strip $\rho - 2\epsilon \le \Re z \le \rho + 2\epsilon$ (see [BGT, §5.1.3], Jordan [J, p. 191]). Write $p_1$, $p_2$ for $\rho - \epsilon$, $\rho + \epsilon$, and consider

$$E_1(x) := I_{[1,\infty)}(x)x^{p_1}, \qquad E_2(x) := I_{(0,1]}(x)x^{p_2} \qquad (0 < x < \infty).$$

The convergence strips of $\check{E}_1$, $\check{E}_2$ are $\{\Re z > p_1\}$, $\{\Re z < p_2\}$. Take $\gamma \in (\rho, p_2)$. Then, by definition of $\rho$,

$$f(x) \le d(\gamma)x^\gamma \qquad (0 < x < \infty)$$

for some $d(\gamma) > 0$. Write

$$F(x) := (k * f)(x) \qquad (0 < x < \infty).$$

It is important to consider, instead of $f$ and $F$ themselves, the following regularized versions:

$$h(x) := (E_2 * E_1 * f)(x), \quad H(x) := (E_2 * E_1 * F)(x) \qquad (0 < x < \infty)$$

(since $\gamma \in (p_1, \infty) \cap (-\infty, p_2) \cap (a, b)$, the crude bound above shows successively that the integrals converge absolutely, whence Fubini's theorem gives associativity of the convolutions). Again by Fubini's theorem,

$$H(x) = (E_2 * E_1 * (k * f))(x) = (k * h)(x) \qquad (0 < x < \infty). \tag{3.1}$$

By the following integral representations

$$h(x) = x^{p_2} \int_x^\infty (E_1 * f)(t)dt/t^{1+p_2} = x^{p_1} \int_0^x (E_2 * f)(t)dt/t^{1+p_1},$$

$x^{-p_1}h(x)$ is increasing and $x^{-p_2}h(x)$ is decreasing. So

$$\frac{(xu)^{-p_1}h(ux)}{x^{-p_1}h(x)} \le 1 \qquad (0 < u \le 1,\ x > 0),$$

$$\frac{(xu)^{-p_2}h(ux)}{x^{-p_2}h(x)} \le 1 \qquad (1 \le u < \infty,\ x > 0).$$

Combining, we obtain the key estimate

$$\frac{h(ux)}{h(x)} \le \max(u^{p_1}, u^{p_2}) \quad (0 < u < \infty,\ 0 < x < \infty). \tag{3.2}$$

As in (5.2.2') of [BGT], we have

$$\frac{(E_1 * F)(x)}{(E_1 * f)(x)} = \frac{\int_0^x F(t)dt/t^{1+p_1}}{\int_0^x f(t)dt/t^{1+p_1}} \to \check{k}(\rho) \qquad (x \to \infty)$$

(here we may use $f \in BI$ instead of $f \in BD$ to assure $\int_0^\infty f(t)dt/t^{1+p_1} = \infty$). So

$$\frac{H(x)}{h(x)} = \frac{\int_x^\infty (E_1 * F)(t)dt/t^{1+p_2}}{\int_x^\infty (E_1 * f)(t)dt/t^{1+p_2}} \to \check{k}(\rho) \qquad (x \to \infty). \tag{3.3}$$

We comment briefly on the need to introduce $h$. We will show by standard Tauberian arguments (in Steps 6–10 below) that $f \in R_\rho$ follows from $h \in R_\rho$. Therefore (3.1) and (3.3) imply that the problem has been reduced to that of $h$ from that of $f$. The advantage here is the useful bound (3.2) on $h$ which we cannot expect the original function $f$ to satisfy.

Now we can follow the standard arguments. Choose any sequence $x_n \uparrow \infty$. Consider $j_n(u) := h(x_n u)/h(x_n)$. The functions $u^{-p_1} j_n(u)$ are increasing on $(0, \infty)$ and by (3.2) uniformly bounded on compact $u$-sets in $(0, \infty)$. By Helly selection (of the form as in Widder [Wi, Ch. I, Theorem 16.3] on each $[0, N]$, then diagonalising), we can find a sequence of integers $n' \to \infty$ such that $j_{n'}$ converges pointwise on $(0, \infty)$, to $j$, say. Then $u^{-p_1} j(u)$ is increasing, $j(1) = 1$, and by (3.2)

$$j(u) \leq \max(u^{p_1}, u^{p_2}) \qquad (0 < u < \infty). \tag{3.4}$$

From (3.1), for $r > 0$,

$$\frac{H(rx_{n'})}{h(rx_{n'})} \cdot \frac{h(rx_{n'})}{h(x_{n'})} = \int_0^\infty \frac{h(rx_{n'}/t)}{h(x_{n'})} k(t) \frac{dt}{t}.$$

We now have suitably dominated convergence of the integrand by (3.2) (note that $p_1, p_2 \in (a, b)$), and $H(rx_{n'})/h(rx_{n'}) \to \check{k}(\rho)$ by (3.3). So we obtain the following integral equation for $j$:

$$\check{k}(\rho) j(r) = (k * j)(r) \qquad (0 < r < \infty). \tag{3.5}$$

The equation (3.5) is the same type as those in [DS], [J].

*Step 6.* Write

$$\phi(x) := j(e^x) e^{-\rho x}, \quad K(x) := k(e^x) e^{-\rho x} \qquad (-\infty < x < \infty).$$

Choose $c_1$, $c_2$ such that $0 < \epsilon < c_1 < c_2 < 2\epsilon$. Then by (3.4),

$$\phi(x) \leq \max(e^{-\epsilon x}, e^{\epsilon x}) = e^{\epsilon |x|},$$

whence $e^{-c_1|x|}\phi(x) \in L^2(\mathbf{R})$. On the other hand, $e^{-c_2|x|}K(x) \in L^1(\mathbf{R})$ as

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-c_2|x|}|K(x)|dx &\le \int_{-\infty}^{\infty} e^{-c_2 x}|K(x)|dx \\ &= \int_0^{\infty} t^{-(c_2+\rho)}|k(t)|\frac{dt}{t} = |k|\check{}(c_2+\rho) < \infty\end{aligned}$$

by $c_2 + \rho \in (a, b)$.

We write the Fourier transform as

$$\hat{K}(z) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K(x)e^{-ixz}dx.$$

Then

$$\hat{K}(z) = \frac{1}{\sqrt{2\pi}}\check{k}(\rho + iz) \qquad (|\Im z| \le c_2). \tag{3.6}$$

The equation (3.5) becomes

$$\phi(x) = \frac{1}{\sqrt{2\pi}\hat{K}(0)} \int_{-\infty}^{\infty} K(x-y)\phi(y)dy \qquad (-\infty < x < \infty).$$

Here we note that

$$\sqrt{2\pi}\hat{K}(0) = \check{k}(\rho) = c \ne 0.$$

By (3.6), the transcendental equation $\hat{K}(z) = \hat{K}(0)$ has a unique root $z = 0$ in the strip $|\Im z| \le c_2$, which is at most double by (2.2). By Theorem 146 of Titchmarsh [T2],

$$\phi(x) = a_1 + a_2 x \qquad (x \in \mathbf{R})$$

for some $a_1, a_2 \in \mathbf{C}$. That is,

$$j(x) = x^{\rho}(a_1 + a_2 \log x) \qquad (0 < x < \infty).$$

Since $j$ is real, so are $a_1$, $a_2$. As $j(1) = 1$, we have $a_1 = 1$. Since $j(\cdot) \ge 0$, $a_2 = 0$. Thus $j(x) \equiv x^{\rho}$. Therefore the partial limit $u^{\rho}$ of $h(ux_n)/h(x_n)$ does not depend on the sequence $(x_n)$ chosen. Thus $h(ux)/h(x) \to u^{\rho}$ as $x \to \infty$, so $h \in R^{\rho}$. Since

$$x^{-p_1}(E_1 * f)(x) = \int_0^x f(t)dt/t^{1+p_1},$$

$x^{-p_1}(E_1 * f)(x)$ is increasing. So $\log((E_1 * f)(x)/x^{p_1})$, whence $\log((E_1 * f)(x)/x^{1+p_2})$, is slowly decreasing (see [BGT, §1.7.6]). By the Monotone Density Theorem (see also [BGT, §1.7.6]), we obtain $E_1 * f \in R_{\rho}$ from $h \in R_{\rho}$.

*Steps 7–10* are proved as in the proof of [BGT, Th. 5.2.1]. In them, we deduce a sufficiently strong Tauberian condition on $f$ to pass from $E_1 * f \in R_{\rho}$ to $f \in R_{\rho}$. We note that the Tauberian condition $f \in BI \cup BD$, which is one of the assumptions, does not suffice for this purpose. The main idea here is due to Drasin (see [J, p. 179]). $\square$

## 4. Proof of Theorem 2

In this section, we write, for simplicity, $F$ rather than $F_\nu$. We write $f \asymp g$ if $f = O(g)$ and $g = O(f)$. We first show an analogue of [BGT, Proposition 2.10.3] (an $O$-version of the monotone density theorem) which we will need later.

**Proposition 4.1.** *Let $U(x) = \int_x^\infty u(t)dt$ $(x > 0)$, where $u$ is measurable, eventually positive, and satisfies the weak Tauberian condition $u \in BD \cup BI$. If $\beta(U) > -\infty$ and $\alpha(U) < 0$, then*

$$u(x) \asymp U(x)/x \qquad (x \to \infty).$$

The proof of Proposition 4.1 is quite similar to that of [BGT, Proposition 2.10.3], whence we omit the details.

Before going into details, we explain the proof of Theorem 2 in rough outline. Recall that $-\frac{1}{2} < \nu < \infty$. We write, for $X$ large enough and $0 < x < \infty$,

$$\tilde{f}(x) := I_{[X,\infty)}(x)f(x), \tag{4.1}$$

$$\tilde{g}(x) := (C * \tilde{f})(x), \tag{4.2}$$

$$\tilde{G}(x) := (D * \tilde{f})(x), \tag{4.3}$$

where

$$C(x) := x^{\nu+\frac{1}{2}}e^{-x}, \tag{4.4}$$

$$D(x) := d_\nu \frac{x^{\nu+\frac{3}{2}}}{(1+x^2)^{\nu+\frac{3}{2}}}, \quad d_\nu := \frac{2^{\nu+1}\Gamma(\nu+\frac{3}{2})}{\pi^{1/2}}. \tag{4.5}$$

We will obtain as in [BI1]

$$\tilde{G}(x) = (k_\nu * \tilde{g})(x), \tag{4.6}$$

$$\tilde{G}(x)/\tilde{g}(x) \to c \quad (x \to \infty), \tag{4.7}$$

$$\int_X^\infty f(t)t^{\nu+\frac{1}{2}}dt = \infty. \tag{4.8}$$

We will also obtain $\tilde{g} \in BI$ and $\rho(\tilde{g}) = \rho$. By (4.8) and the monotone convergence theorem,

$$x^{\nu+\frac{3}{2}}\tilde{G}(x) = d_\nu \int_X^\infty \frac{t^{\nu+\frac{1}{2}}f(t)}{\{1+(t/x)^2\}^{\nu+\frac{3}{2}}}dt \to \infty \qquad (x \to \infty),$$

whence by (4.7) we have $x^{-(\nu+\frac{3}{2})} = o(\tilde{g}(x))$ as $x \to \infty$. So (2.7) holds if we replace $f$ by $\tilde{g}$. So this time, by considering $I_{[1,\infty)}\tilde{g}$, we can apply Theorem 1 to obtain $\tilde{g} \in R_\rho$. To deduce $f \in R_\rho$ from this, we will use Karamata's Tauberian theorem. For that, we will need some Tauberian condition on $f$. Here an extra complication, absent in [BI1, BI2], arises. For there, $f$ was assumed to satisfy the strong Tauberian condition of monotonicity, which is not available now. Fortunately, the desired Tauberian condition can be shown by the same arguments as in Steps 7–10 of the proof of [BGT, Theorem 5.2.1], whence the proof will be complete.

Now we are ready to prove Theorem 2.

*Step 1.* Since $f \in BI \cup BD$, by [BGT, Proposition 2.2.1] there exists $X > 0$ such that $f$ and $1/f$ are both positive and locally bounded on $[X, \infty)$. Define $\tilde{f}$ by (4.1) with this $X$. Write, for $x \in (0, \infty)$,

$$\begin{aligned}\tilde{F}(x) &:= (k_\nu * \tilde{f})(x),\\ \bar{f}(x) &:= I_{(0,X)}(x) f(x),\\ \bar{F}(x) &:= (k_\nu * \bar{f})(x).\end{aligned}$$

Pick $\gamma \in (\rho, -1)$. Then there exists $c_1 \in (0, \infty)$ such that

$$0 \le \tilde{f}(x) \le c_1 x^\gamma \qquad (0 < x < \infty), \tag{4.9}$$

whence

$$|\tilde{F}(x)| \le c_1 |k_\nu|\check{}(\gamma) x^\gamma \qquad (0 < x < \infty).$$

On the other hand, since

$$\begin{aligned}J_\nu(x) &\sim (x/2)^\nu / \Gamma(\nu + 1) \qquad (x \to 0+),\\ J_\nu(x) &\sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{(2\nu+1)\pi}{4}\right) \qquad (x \to \infty)\end{aligned}$$

(see Watson [Wa, 7.21]), there exists $c_2$ such that

$$|k_\nu(x)| \le c_2 x^{-(\nu + \frac{3}{2})} \qquad (0 < x < \infty),$$

whence

$$|\bar{F}(x)| \le c_2 x^{-(\nu+\frac{3}{2})} \int_0^X |f(t)| t^{\nu + \frac{1}{2}} dt.$$

As a consequence, for some $c_3 \in (0, \infty)$,

$$|F(x)| \le c_3 \{x^{-(\nu+\frac{3}{2})} + x^\gamma\} \qquad (0 < x < \infty)$$

(recall that $F$ or $F_\nu$ is defined by (2.5)).

*Step 2.* Let $C$ and $D$ be as in (4.4) and (4.5). Then as in the proof of [BI1, Lemma 2], we have

$$(C * \bar{F})(x) = (D * \bar{f})(x) \qquad (0 < x < \infty),$$

the integral converging absolutely. By (4.9), $\tilde{f} \in L^1[0, \infty)$, while $C(x\cdot) \in L^1[0, \infty)$ for any $x > 0$. So Parseval's formula (cf. [MO, Theorem III]) yields

$$(C * \tilde{F})(x) = (D * \tilde{f})(x) \qquad (0 < x < \infty).$$

Combining, we obtain

$$(D * f)(x) = (C * F)(x) \qquad (0 < x < \infty), \tag{4.10}$$

the integral converging absolutely.

*Step 3.* Define $\tilde{g}$ and $\tilde{G}$ by (4.2) and (4.3). We write $G(x) := (D * f)(x)$ for $0 < x < \infty$. By (4.10),

$$G(x) = (C * F)(x) \qquad (0 < x < \infty). \tag{4.11}$$

As in [BI1, §3], we have (4.8), whence

$$\tilde{G}(x)/G(x) \to 1 \qquad (x \to \infty).$$

On the other hand, since $1/f$ is locally bounded on $[X, \infty)$, as in [BI1, §3], it follows from (2.7) and (4.11) that

$$G(x)/\tilde{g}(x) \to c \qquad (x \to \infty),$$

whence (4.7) follows.

*Step 4.* Since $|k_\nu|\check{}(\gamma) < \infty$ and $|C|\check{}(\gamma) < \infty$, it follows from (4.9) that

$$|k_\nu| * (|C| * \tilde{f})(x) \le c_1 |k_\nu|\check{}(\gamma)|C|\check{}(\gamma)x^\gamma < \infty.$$

Therefore, by Fubini's theorem,

$$\begin{aligned}\tilde{G}(x) &= (D * \tilde{f})(x) = ((k_\nu * C) * \tilde{f})(x)\\ &= (k_\nu * (C * \tilde{f}))(x) = (k_\nu * \tilde{g})(x),\end{aligned}$$

whence (4.6).

By the definition (4.2), $x^{-(\nu+\frac{1}{2})}\tilde{g}(x)$ is decreasing, whence $\tilde{g} \in BI$. As in [BI1, Lemma 5], $\rho(\tilde{g}) = \rho$.

As we saw above, (2.7) holds if we replace $f$ by $\tilde{g}$:

$$\int_0^1 \tilde{g}(t)k_\nu(x/t)dt/t = o(\tilde{g}(x)) \qquad (x \to \infty).$$

Therefore from (4.7) we have for $\bar{g} := I_{[1,\infty)}\tilde{g}$,

$$(k_\nu * \bar{g})(x)/\bar{g}(x) \to c \qquad (x \to \infty).$$

By [BI1, Proposition 3] and [BI2, §5], the kernel $k_\nu$ satisfies the conditions (2.1) and (2.2). Therefore, by Theorem 1, $c = \check{k}(\rho)$ and $\bar{g} \in R_\rho$, whence $\tilde{g} \in R_\rho$.

*Step 5.* Write

$$U(x) := \int_x^\infty \tilde{f}(t)t^{-(\nu+\frac{3}{2})}dt \qquad (0 < x < \infty).$$

Since

$$\tilde{g}(x) = x^{\nu+\frac{1}{2}} \int_0^\infty e^{-xt} d\{U(1/t)\},$$

by Karamata's Tauberian theorem (cf. [BGT, Theorem 1.7.1']) $\tilde{g} \in R_\rho$ yields $U \in R_{\rho-\nu-\frac{1}{2}}$. In particular, $\beta(U) = \alpha(U) = \rho - \nu - \frac{1}{2} < 0$. So if we write

$$q(x) := x^{\nu+\frac{1}{2}} U(x) \qquad (0 < x < \infty),$$

then Proposition 4.1 yields, for some $M > 1$ and $Y > 0$,

$$M^{-1} \le f(x)/q(x) \le M \qquad (x \ge Y).$$

As a consequence, $f \in OR$ (see [BGT, §2.0.2] for the definition of $OR$).

*Step 6.* For any fixed $B > 1$,

$$\int_0^B f(u) k_\nu(x/u) du/u = o(q(x)) \qquad (x \to \infty).$$

For, as $x \to \infty$,

$$\left| \int_0^B f(u) k_\nu(x/u) du/u \right| \le c_2 x^{-(\nu+\frac{3}{2})} \int_0^B |f(t)| t^{\nu+\frac{1}{2}} dt = o(q(x)).$$

Then, as in Steps 7–9 of the proof of [BGT, Theorem 5.2.1], we obtain

$$\lim_{\lambda \downarrow 1} \liminf_{x \to \infty} \inf_{\sigma \in [1,\lambda)} f(\sigma x)/f(x) = 1,$$

which implies that $\log f$ is slowly decreasing. Thus, by the Monotone Density Theorem, $U \in R_{\rho-\nu-\frac{1}{2}}$ implies $f \in R_\rho$, as desired. This completes the proof. □

## 5. Remarks

In this section, we show that the kernel $k_\nu$ defined by (2.4) fails to satisfy the conditions of Jordan [J, Theorems 1 and 1a] for some $\rho$ if $\nu$ is large enough. Recall the Mellin transform $\check{k}_\nu$ of our kernel $k_\nu$ from (W).

Write

$$\Psi(x) = \Gamma'(x)/\Gamma(x) \qquad (0 < x < \infty)$$

for the logarithmic derivative of the gamma function (digamma function). As is well known,

$$\Psi(x) = -\gamma - \frac{1}{x} + x \sum_{n=1}^{\infty} \frac{1}{n(x+n)} \qquad (0 < x < \infty),$$

where $\gamma$ is Euler's constant. Since $\Psi'(x) = \sum_{n=0}^{\infty} (x+n)^{-2}$, $\Psi$ is increasing on $(0, \infty)$. Now

$$\log 2 + \frac{1}{2}\Psi(\tfrac{1}{2}) + \frac{1}{2}\Psi(1) = -\gamma < 0,$$
$$\log 2 + \frac{1}{2}\Psi(\tfrac{3}{4}) + \frac{1}{2}\Psi(\tfrac{5}{4}) = -\gamma - 2\log 2 + 2 > 0.$$

So we may define $\nu_1 \in (\frac{1}{2}, 1)$ by

$$\log 2 + \frac{1}{2}\Psi(\tfrac{1}{4} + \tfrac{1}{2}\nu_1) + \frac{1}{2}\Psi(\tfrac{3}{4} + \tfrac{1}{2}\nu_1) = 0$$

($\nu_1 = 0.9616\cdots$ by calculation using *Mathematica*).

For $x \in (-\nu - \frac{3}{2}, \nu + 1)$,

$$\log \check{k}_\nu(x) = (x + \tfrac{1}{2})\log 2 + \log\Gamma(\tfrac{3}{4} + \tfrac{1}{2}\nu + \tfrac{1}{2}x) - \log\Gamma(\tfrac{1}{4} + \tfrac{1}{2}\nu - \tfrac{1}{2}x),$$

whence

$$\check{k}_\nu'(x)/\check{k}_\nu(x) = \log 2 + \frac{1}{2}\Psi(\tfrac{3}{4} + \tfrac{1}{2}\nu + \tfrac{1}{2}x) + \frac{1}{2}\Psi(\tfrac{1}{4} + \tfrac{1}{2}\nu - \tfrac{1}{2}x).$$

So

$$\check{k}_\nu'(-1) = \check{k}_\nu(-1)\left\{\log 2 + \frac{1}{2}\Psi(\tfrac{1}{4} + \tfrac{\nu}{2}) + \frac{1}{2}\Psi(\tfrac{3}{4} + \tfrac{1}{2}\nu)\right\}.$$

Since k oscillates infinitely often near zero (and also $\check{k}_\nu(-1)$ is finite), Jordan's Theorem 1 cannot be applied to $k_\nu$ at all. If $\nu \le \nu_1$, then $\check{k}_\nu'(-1) \le 0$, whence by [BI1, §5], $\check{k}_\nu$ is decreasing on $(-\nu - \frac{3}{2}, -1)$. Further, $|\check{k}_\nu((-\nu - \frac{3}{2})+)| = \infty$. So Jordan's Theorem 1a applies to $k_\nu$ for all $\rho \in (-\nu - \frac{3}{2}, -1)$ if $-\frac{1}{2} < \nu \le \nu_1$.

On the other hand, if $\nu > \nu_1$, then $\check{k}_\nu'(-1) < 0$, whence by [BI2, §5] there exists $b_1 \in (-\nu - \frac{3}{2}, -1)$ such that $\check{k}_\nu$ is decreasing on $(-\nu - \frac{3}{2}, b_1]$ and increasing on $[b_1, -1)$. So Jordan's Theorem 1a still applies to $k_\nu$ if $\nu > \nu_1$ and $\rho \in (-\nu - \frac{3}{2}, b_1]$. However, if $\nu > \nu_1$ and $\rho \in (b_1, -1)$, then neither of Jordan's theorems apply to $k_\nu$. Our Theorems 1 and 2 cover this last case.

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