RATIO MERCERIAN AND TAUBERIAN THEOREMS

AKIHIKO INOUE

1. Introduction

This lecture[1] reviews the work on the title, mostly done jointly with Nick Bingham.

We start with a rough description of the developments of the work. In 1995 when I was visiting London, Nick and I succeeded in proving a Mercerian[2] theorem for Fourier cosine and sine transforms ([BI1]). In the course of our efforts to prove it, we found a theorem of B. I. Korenblum [K1, K2] on Beurling algebras very useful. His theorem has played an important role in this and subsequent work of ours.

Fourier cosine and sine transforms are Hankel transforms of order $-1/2$ and $1/2$, respectively. In 1996, while trying to extend the result of [BI1] to that for Hankel transforms of arbitrary order $\nu \geq -1/2$, we found a useful method, which we call a *localization method*. Though it was simple, the output was large. Indeed, by the method, we could prove the desired Mercerian theorem for Hankel transforms ([BI3]). Moreover, the method enabled us to free the Mercerian theorem for absolutely convergent integral transforms, due to Drasin, Shea and Jordan, from unnecessary conditions ([BI4]). Also in [BI3], we introduced the notion of *Ratio Mercerian Theorem*, which we thought was worth formulating. All of these results are of Mercerian type.

In 1999, we took interest in some problems in analytic number theory. What we wanted to do was to prove Tauberian[3] theorems for some arithmetic sums. When we were trying to do so, we had a revelation that the Ratio Mercerian Theorem, if extended suitably to that for systems of kernels, would supply us with a new and powerful method to prove Tauberian theorems (not only for arithmetic sums). Indeed, by the method, we could prove a Tauberian theorem for a wide class of integral transforms with nonnegative kernels, under the same weak Tauberian conditions as in that for Laplace transforms ([BI5]). Tauberian conditions are assumptions in Tauberian theorems to be checked, whence the weaker, the better. We remark that Karamata's[4] Tauberian theorem for Laplace transforms is regarded as a prototype of Tauberian theorems. In [BI5], we could also extend de Haan's Tauberian theorem for Laplace transforms (which deals with a boundary case) to a class of integral transforms. Before our work, such theorems had been known only for sporadic examples of integral transforms. In [BI6], using the results of [BI5] as well as the idea to use a system of kernels itself, we could prove the desired Tauberian theorems for the arithmetic sums. See A. Ivić [Iv] for the review of this and related work in analytic number theory.

As stated above, the aim of this lecture is to review the work described above but, in the rest of this section, we explain what Abelian, Tauberian and Mercerian theorems are, in the framework of Hardy–Littlewood–Karamata or that of regular variation.

First we recall the notion of regular variation due to Karamata. A positive measurable function $f : [X, \infty) \to (0, \infty)$, defined on a neighborhood $[X, \infty)$ of infinity, is *regularly varying with index* ρ ($\in \mathbf{R}$), or $f \in R_\rho$, if the following holds[5]:

$$\forall \lambda > 0, \qquad \lim_{x \to \infty} \frac{f(\lambda x)}{f(x)} = \lambda^\rho. \tag{1.1}$$

In particular, a function belonging to R_0, that is, a regularly varying function with index 0, is said to be *slowly varying*. Usually, a slowly varying function is denoted such as $\ell(\cdot)$ or $L(\cdot)$[6]. Thus, in this paper, if we write $\ell(\cdot)$, then we mean that it is a slowly varying function. It is easy to see the following equivalence:

$$f \in R_\rho \quad \Longleftrightarrow \quad f(x) \sim x^\rho \ell(x) \quad (x \to \infty) \quad \text{for some } \ell \in R_0,$$

where $f(x) \sim g(x)$ implies $f(x)/g(x) \to 1$. Roughly speaking, a function $f \in R_\rho$ has an asymptotic behavior close to x^ρ.

Example. Positive constants, $\log x$ and $\log \log x$ are slowly varying functions, while x^2 and $x^2 \log x$ are regularly varying with index 2.

To explain basic notions, it would be perhaps the best to use the simplest integral transforms, that is, arithmetic means. Let f be a real-valued function in $L^1_{\mathrm{loc}}[0, \infty)$. In the case of arithmetic means, we assume that $\rho \in (-1, \infty)$. Then

$$f(x) \sim x^\rho \ell(x) \qquad (x \to \infty) \tag{1.2}$$

implies

$$\frac{1}{x} \int_0^x f(t)dt \sim x^\rho \ell(x) \frac{1}{\rho + 1} \qquad (x \to \infty). \tag{1.3}$$

The converse does not necessarily hold. However, if f satisfies a so-called Tauberian condition (such as monotonicity), then (1.3) implies (1.2). The assertion (1.2) $\Rightarrow$ (1.3) is called an *Abelian theorem*, while its partial converse (1.3) + (Tauber condition on f) $\Rightarrow$ (1.2) a *Tauberian theorem*.

Now, combining (1.2) and (1.3), we soon notice the following asymptotic relation:

$$\frac{1}{x} \int_0^x f(t)dt \sim C f(x) \qquad (x \to \infty) \tag{1.4}$$

with $C = 1/(1 + \rho)$. Interestingly, this happens only when f is regularly varying, that is, *f and its arithmetic mean have the same asymptotic behavior (if and) only if f satisfies* (1.2) for some $\rho > -1$ and $\ell(\cdot) \in R_0$. More precisely, if (1.4) holds for some $C \in (0, \infty)$, then we have $f \in R_\rho$ with $C = 1/(\rho + 1)$ (Karamata, 1930). This assertion is called a *Mercerian theorem*. We refer to Bingham et al. [BGT, Chapter 1] for the proofs of these Abelian, Tauberian and Mercerian theorems for arithmetic means.

It is easy to understand the importance of Tauberian theorems since they have many applications in various fields of mathematics[7]. On the other hand, at this stage, the importance of Mercerian theorems is not so clear; they look like only telling us that regular variation is not only sufficient but also necessary for Abel–Tauber theorems to hold. However, this lecture will lay emphasis on the usefulness of Mercerian viewpoints.

Abelian, Tauberian and Mercerian theorems similar to the above also hold for Laplace transforms[8]. From the viewpoint of application, it is desirable to be able to prove similar results for various integral transforms. To deal with general integral transforms, it is useful to use the Mellin convolution notation[9]. In fact, it has been more than useful to us since it led us to several unexpected discoveries.

For measurable functions $f, g : (0, \infty) \to \mathbf{R}$, we define the Mellin convolution $f * g$ by the following integral[10]:

$$f * g(x) := \int_0^\infty f(x/t)g(t)dt/t \qquad (x > 0).$$

Example. For the arithmetic mean, we have $x^{-1}\int_0^x f(t)dt = k*f(x)$ with $k(x) := x^{-1}I_{(1,\infty)}(x)$. For the Laplace transform, we have

$$x^{-1}\int_0^\infty e^{-t/x}f(t)dt = k * f(x)$$

with $k(x) := x^{-1}e^{-1/x}$. Notice that the integral on the left-hand side is the Laplace transform with x replaced by x^{-1}.

For given measurable function $k : (0, \infty) \to \mathbf{R}$, we define its Mellin transform $\check{k}(z)$ by

$$\check{k}(z) := \int_0^\infty t^{-z}k(t)dt/t$$

for $z \in \mathbf{C}$ for which the integral converges[11].

Example. For the integral kernel $k(x) = x^{-1}I_{(1,\infty)}(x)$ for arithmetic means, we have $\check{k}(z) = 1/(1 + z)$ ($\Re z > -1$). Notice that we have already seen the same function in (1.3). For the kernel $k(x) = x^{-1}e^{-1/x}$ for Laplace transforms, we have $\check{k}(z) = \Gamma(1 + z)$ ($\Re z > -1$).

The asymptotic behavior (1.2) implies, at least formally,

$$\frac{k * f(x)}{f(x)} = \int_0^\infty \frac{f(x/t)}{f(t)}k(t)dt/t \to \int_0^\infty t^{-\rho}k(t)dt/t = \check{k}(\rho) \quad (x \to \infty),$$

whence

$$k * f(x) \sim x^\rho \ell(x)\check{k}(\rho) \qquad (x \to \infty). \tag{1.5}$$

For the integral transform $k * f$, we may regard the implication (1.2) $\Rightarrow$ (1.5) as Abelian and (1.5) + (Tauber condition on f) $\Rightarrow$ (1.2) Tauberian. The Mercerian assertion may be stated in the following way: if

$$k * f(x) \sim Cf(x) \qquad (x \to \infty) \tag{1.6}$$

for some $C \neq 0$, then $f \in R_\rho$ for $\rho \in \mathbf{R}$ with $C = \check{k}(\rho)$. Of course, these assertions may turn out to be untrue as the case may be.

2. Ratio Mercerian Theorem

In this section, we explain the Ratio Mercerian Theorem introduced in [BI3]. Since the statement of [BI3, Theorem 3] contained an error (cf. Remark in [BI5, Section 3]), here we show the version corrected (and improved) in [BI5].

For a positive function $f : [X, \infty) \to (0, \infty)$, we define its *upper order* $\rho(f)$ by

$$\rho(f) := \limsup_{t\to\infty} \frac{\log f(t)}{\log t}.$$

Recall from Section 1 that the class R_ρ is the class of measurable f regularly varying with index ρ. If $f \in R_\rho$, then $\rho(f) = \rho$. So, for a function f for which we wish to show $f \in R_\rho$, we can use the upper order $\rho(f)$ as a tentative alternative to ρ.

The Ratio Mercerian Theorem is roughly the following assertion (see Theorem 5.1 below with $\#\Lambda = 1$ for precise statement):

Theorem 2.1 (Ratio Mercerian Theorem, [BI3, BI5])**.** *Under some suitable conditions on the integral kernels k^1 and k^2 as well as on the function f,*

$$\frac{k^2 * f(x)}{k^1 * f(x)} \to C \neq 0 \qquad (x \to \infty) \tag{2.1}$$

implies $C = \check{k}^2(\rho)/\check{k}^1(\rho)$ and $f \in R_\rho$, where ρ is the upper order of f.

We put

$$k^0(t) := \check{k}^1(\rho)k^2(t) - \check{k}^2(\rho)k^1(t) \qquad (0 < t < \infty). \tag{2.2}$$

Two key assumptions of Theorem 2.1 are as follows: for some $\sigma \in \mathbf{R}$ and $\epsilon \in (0, \infty)$ such that $\sigma - \epsilon < \rho < \sigma + \epsilon$,

$$\begin{cases} \rho \text{ is the unique zero of } \check{k}^0(z) = \check{k}^1(\rho)\check{k}^2(z) - \check{k}^2(\rho)\check{k}^1(z) \\ \text{in the strip } \sigma - \epsilon \le \Re z \le \sigma + \epsilon; \end{cases} \tag{2.3}$$

$$\exp\left(-\frac{\pi|t|}{2\epsilon}\right) \log|\check{k}^0(\sigma + it)| \to 0 \qquad (t \to \pm\infty). \tag{2.4}$$

Notice that, formally, $f \in R_\rho$ implies (2.1) with $C = \check{k}^2(\rho)/\check{k}^1(\rho)$ by Abelian theorem. Thus the Ratio Mercerian Theorem (Theorem 2.1) asserts the converse implication.

There are two keys to the proof of Theorem 2.1; one is the localization method and the other is Korenblum's theorem.

First we explain the localization method. We define two functions E_1 and E_2 by

$$E_1(x) := I_{(1,\infty)}(x)x^{\sigma-\epsilon}, \qquad E_2(x) := I_{(0,1)}(x)x^{\sigma+\epsilon} \qquad (0 < x < \infty),$$

and we set $h(x) := E_2 * E_1 * f(x)$. Then, from the two different representations

$$h(x) = x^{\sigma-\epsilon}\int_0^x (E_2 * f)(t)dt/t^{1+\sigma-\epsilon} = x^{\sigma+\epsilon}\int_x^\infty (E_1 * f)(t)dt/t^{1+\sigma+\epsilon}$$

for h, we see that $x^{-\sigma+\epsilon}h(x)$ is increasing and $x^{-\sigma-\epsilon}h(x)$ is decreasing (here f is assumed to be nonnegative). From this observation, we obtain the following key estimate:

$$h(ux)/h(x) \le \max(u^{\sigma-\epsilon}, u^{\sigma+\epsilon}) \qquad (0 < u < \infty,\ 0 < x < \infty). \tag{2.5}$$

Now, by (2.1), we can show that $E_2 * E_1 * k^2 * f(x)/E_2 * E_1 * k^1 * f(x) \to C$ $(x \to \infty)$ or

$$\frac{k^2 * h(x)}{k^1 * h(x)} \to C \neq 0 \qquad (x \to \infty). \tag{2.6}$$

Since $h = E_2 * E_1 * f \in R_\rho$ implies $f \in R_\rho$ by a simple Tauberian argument, we see from (2.6) that our problem is reduced to that for h from that for f. The advantage here is that h satisfies the good estimate (2.5) which we can never expect to hold for f. The localization method is the method that reduces the problem for f to that for $E_2 * E_1 * f$ in this way[12]. It actually localizes the domain, on which we must consider the behavior of the Mellin transform ($\check{k}^0$ in this case), to the strip $\sigma - \epsilon \le \Re z \le \sigma + \epsilon$.

Next we turn to Korenblum's theorem. To state it, we put, for $\alpha > 0$,

$$L(\alpha) := L^1(\mathbf{R}, e^{\alpha|x|}dx).$$

Then $L(\alpha)$ becomes a commutative Banach algebra with respect to the usual convolution. For $K \in L(\alpha)$, we define its Fourier transform $\hat{K}$ by

$$\hat{K}(z) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixz} K(x) dx \qquad (|\Im z| \le \alpha).$$

The function $\hat{K}(z)$ is holomorphic in $|\Im z| < \alpha$.

Here is Korenblum's theorem[13] that we use.

Theorem (Korenblum, [K1, K2]). *Let I be a closed ideal of $L(\alpha)$. We assume that z_0 is the unique common zero of $\hat{K}(z)$, $K \in I$, in $|\Im z| \le \alpha$ and that z_0 is an inner point, i.e., $|\Im z| < \alpha$. We further assume that*

$$\sup_{K \in I} \limsup_{x \to \infty} \frac{\log |\hat{K}(x)|}{\exp(\pi x/2\alpha)} = \sup_{K \in I} \limsup_{x \to -\infty} \frac{\log |\hat{K}(x)|}{\exp(-\pi x/2\alpha)} = 0.$$

Let n_0 be the minimum of the orders of the zero point z_0 for $\hat{K}(z)$, $K \in I$. Then I is described in the following way:

$$I = \{K \in L(\alpha) : z_0 \text{ is a zero of } \hat{K} \text{ of at least order } n_0.\}$$

We explain how Korenblum's theorem is used in our argument. As stated above, our problem is reduced to showing $h \in R_\rho$, that is, $h(ux)/h(x) \to u^\rho$ as $x \to \infty$ for all $u > 0$. To do that, we may show that any sequence $x_n \uparrow \infty$ has a subsequence $x_{n'}$ satisfying $h(ux_{n'})/h(x_{n'}) \to u^\rho$ $(\forall u > 0)$. However, by the Helly selection principle, we can choose a subsequence $(x_{n'})$ such that $h(ux_{n'})/h(x_{n'})$ converges to a function, say, $j(u)$. Therefore we may show that $j(u) = u^\rho$. From (2.5), (2.6) and the equality $C = \check{k}^2(\rho)/\check{k}^1(\rho)$ (which can be shown easily by a standard method), we see that $j(\cdot)$ is a solution to the following linear integral equation :

$$k^0 * j(x) = 0 \qquad (0 < x < \infty), \tag{2.7}$$

where k^0 is the integral kernel defined by (2.2). Now, from the definition of k^0, we see the equality $\check{k}^0(\rho) = 0$, which implies that u^ρ is a solution to (2.7). Moreover, we easily see that $j(1) = 1$. Thus the problem is reduced to showing that the general solution to (2.7) is of the form const. $\times\, u^\rho$. Korenblum's theorem does the job, and, in so doing, the conditions (2.3) and (2.4) are required.

As an application of the Ratio Mercerian Theorem (Theorem 2.1), we show the next Mercerian theorem for cosine transforms[14].

Theorem 2.2 ([BI1, BI3]). *Let $f \in L^1_{\mathrm{loc}}[0,\infty)$. We assume that $f(t)$ is eventually decreasing to 0 as $t \to \infty$. Let F_c it its Fourier cosine transform* :

$$F_c(x) := \int_0^{\infty-} f(t) \cos tx dt \qquad (0 < x < \infty).$$

If $\int_0^\infty f(t)dt \neq 0$ and

$$x^{-1} F_c(1/x) \sim C f(x) \qquad (x \to \infty) \tag{2.8}$$

for some constant C such that $C \neq 0, \sqrt{\pi/2}$, then we have $C > 0$ and $f \in R_\rho$, where ρ is the unique solution on $(-1, 0)$ to the equation $\Gamma(1+\rho)\sin(-\frac{1}{2}\pi\rho) = C$.

In the theorem above, $\int_0^{\infty-}$ denotes the improper integral $\lim_{M\to\infty} \int_0^M$. Since $f \downarrow 0$, the integral converges [15]. We cannot drop the assumption $C \neq \sqrt{\pi/2}$ (which corresponds to the case $\rho = -\frac{1}{2}$); see [BI1, §7].

We outline the proof of Theorem 2.2. To write the integral transform in the Mellin convolution form, we put $k(x) := x^{-1}\cos(1/x)$. Then we can write (2.8) as

$$k * f(x)/f(x) \to C \qquad (x \to \infty).$$

If we put $B(x) := e^{-x}$, then (2.9) implies $B * k * f(x)/B * f(x) \to C$, that is,

$$D * f(x)/B * f(x) \to C \qquad (x \to \infty), \tag{2.9}$$

where $D(x) := B * k(x) = x/(1+x^2)$. The advantage here is that the integrals $D * f$ and $B * f$ in (2.9) converge absolutely. Thus, by taking Laplace transforms, the original Mercerian problem for conditionally convergent integrals has been converted into a problem of ratio Mercerian type for absolutely convergent integrals. By applying the Ratio Mercerian Theorem, we obtain the desired result.

3. Ratio Mercerian Theorem for systems of kernels with application to Tauberian theorems

In this section, we show a new method to prove Tauberian theorems. It is based on the Ratio Mercerian Theorem for systems ([BI5]).

The Tauberian theorem that we want to prove now is the implication (1.5) ⇒ (1.2) under suitable conditions on k and f. Now we easily see that, for $\lambda > 1$, (1.5) implies

$$\frac{k * f(\lambda x)}{k * f(x)} \to \lambda^\rho \qquad (x \to \infty),$$

or

$$\frac{k^2_\lambda * f(x)}{k^1 * f(x)} \to \lambda^\rho \qquad (x \to \infty), \tag{3.1}$$

where

$$k^1(x) := k(x), \quad k^2_\lambda(x) := k(\lambda x) \qquad (0 < x < \infty).$$

This is the same setting as (2.1), and it seems to be a very attractive idea to apply the Ratio Mercerian Theorem to show $f \in R_\rho$. If we could do this, then we obtain

(1.2) easily by Abelian theorem. However, we soon realize that there is a problem. Since we have

$$\check{k}^1(\rho)\check{k}^2_\lambda(z) - \check{k}^2_\lambda(\rho)\check{k}^1(z) = (\lambda^z - \lambda^\rho)\check{k}(\rho)\check{k}(z), \tag{3.2}$$

the function on the right-hand side, whence on the left-hand side, has infinitely many zeros $z = \rho + i(2n\pi/\log\lambda)$ $(n = \pm1, \pm2, \cdots)$, other than $z = \rho$, on the vertical line $\Re z = \rho$, whence in any vertical strip $\sigma - \epsilon \le \Re z \le \sigma + \epsilon$ such that $\rho \in (\sigma - \epsilon, \sigma + \epsilon)$. Thus the key condition (2.3) does not hold at all.

In [BI5], we introduced a technique to overcome this trouble. It is to consider not a single λ but more than one λ's. To see this roughly, choose $\lambda_1 > 1$ and $\lambda_2 > 1$ so that $\log\lambda_2/\log\lambda_1$ is irrational, e.g., $\lambda_1 = 2$ and $\lambda_2 = 3$. Then we have

$$\{\rho + i(2\pi n/\log\lambda_1) : z \in \mathbf{Z}\} \cap \{\rho + i(2\pi n/\log\lambda_2) : z \in \mathbf{Z}\} = \{\rho\}.$$

This suggests the following: *if we could extend the Ratio Mercerian Theorem (Theorem 2.1) to that for systems of integral transforms, then we would be able to prove the desired Tauberian implication* (1.5) $\Rightarrow$ (1.2) *by following the plan above.*

In effect, we could prove the next Ratio Tauberian Theorem for systems almost in parallel with the proof of Theorem 2.1.

Theorem 3.1 (Ratio Mercerian Theorem for Systems, [BI5])**.** *Under suitable conditions on the family of integral kernels k^i_λ $(i = 1, 2,\ \lambda \in \Lambda)$ as well as on the function f,*

$$\forall\lambda \in \Lambda, \qquad \frac{k^2_\lambda * f(x)}{k^1_\lambda * f(x)} \to C_\lambda \neq 0 \qquad (x \to \infty) \tag{3.3}$$

implies $C_\lambda = \check{k}^2_\lambda(\rho)/\check{k}^1_\lambda(\rho)$ $(\lambda \in \Lambda)$ and $f \in R_\rho$, where ρ is the upper order of f.

We put

$$k^0_\lambda(t) := \check{k}^1_\lambda(\rho)k^2_\lambda(t) - \check{k}^2_\lambda(\rho)k^1_\lambda(t) \qquad (0 < t < \infty,\ \lambda \in \Lambda). \tag{3.4}$$

The assumptions (2.3) and (2.4) of Theorem 2.1 must be replaced in Theorem 3.1 by the following ones, respectively: for some $\sigma \in \mathbf{R}$ and $\epsilon \in (0, \infty)$ such that $\sigma - \epsilon < \rho < \sigma + \epsilon$,

$$\begin{cases} \rho \text{ is the unique common zero of } \check{k}^0_\lambda(z) = \check{k}^1_\lambda(\rho)\check{k}^2_\lambda(z) - \check{k}^2_\lambda(\rho)\check{k}^1_\lambda(z) \\ (\lambda \in \Lambda) \text{ in the strip } \sigma - \epsilon \le \Re z \le \sigma + \epsilon; \end{cases} \tag{3.5}$$

$$\forall\lambda \in \Lambda, \qquad \exp\left(-\frac{\pi|t|}{2\epsilon}\right)\log|\check{k}^0_\lambda(\sigma + it)| \to 0 \qquad (t \to \pm\infty). \tag{3.6}$$

Using Theorem 3.1, we can prove the following Tauberian theorem (see Theorem 5.2 below with $\#\Lambda = 1$ for precise statement):

Theorem 3.2 (Tauberian theorem for nonnegative kernels, [BI5])**.** *If the kernel k satisfies Korenblum's conditions, then* (1.5) *implies* (1.2) *under the same weak Tauberian conditions on f as in the Tauberian theorem for arithmetic means.*

The point of Theorem 3.2 is that it holds under weak Tauberian conditions. This is useful in applications. We remark that the conditions on k in Theorem 3.2 is of the type that usually holds. In particular, Theorem 3.2 includes Karamata's Tauberian theorem for Laplace transforms.

As an application of Theorem 3.2, we have the next theorem.

Theorem 3.3 (Tauberian theorem for arithmetic sums (I), [BI6]). *Let* $\rho > 0$. *For* $f : [2, \infty) \to [0, \infty)$, (1.2) *implies*

$$\sum_{n \le x} \sum_{p|n} f(p) \sim \frac{x^\rho \ell(x)}{\log x} \cdot \frac{\zeta(1+\rho)}{1+\rho} \qquad (x \to \infty). \tag{3.7}$$

Conversely, if f *is increasing, then* (3.7) *implies* (1.2).

In Theorem 3.3, p denotes a prime. The Abelian implication (1.2) $\Rightarrow$ (3.7) is due to De Koninck and Ivić [DI]. We explain the motivation to consider the arithmetic sum

$$\sum_{n \le x} \sum_{p|n} f(p).$$

In analytic number theory, we are sometimes concerned with the asymptotic behavior of a function of the form $g(n) := \sum_{p|n} f(p)$. For example, if $f(x) \equiv 1$, then $g(n)$ is the number of prime factors of n (see the remark just after Theorem 4.2 below). In Theorem 3.3, we consider the asymptotic behavior of its arithmetic mean, rather than $g(n)$ itself, which is easier to handle. In the proof of Theorem 3.3, we take $I_{(1,\infty)}(x)[x]/x$ as the integral kernel k^{17}.

4. Tauberian theorems of de Haan type

As we have seen in Section 1, if $\rho > -1$, then we can characterize the asymptotic behavior (1.2) of a monotone function f in terms of its arithmetic mean. It would be natural to ask if we can do the same thing in the boundary case $\rho = -1$. The answer is yes but, to do that, we need the notion of π-variation due to de Haan (cf. [BGT, Chapter 3]).

Definition (π-variation). Let $\ell \in R_0$ and $c \in \mathbf{R}$, and let $g : [X, \infty) \to \mathbf{R}$ be a measurable function. Then g is said to be in Π_ℓ with ℓ-index c if

$$\forall \lambda > 1, \qquad \lim_{x \to \infty} \frac{g(\lambda x) - g(x)}{\ell(x)} = c \log \lambda.$$

Example. For $\ell(x) \equiv 1$, $\log x \in \Pi_\ell$ with ℓ-index 1. For, $\log(\lambda x) - \log x = \log \lambda$. For general ℓ, we can produce many examples by the assertion (4.1) $\Rightarrow$ (4.2) below.

Here is the characterization of (1.2) with $\rho = -1$ in terms of the arithmetic mean of f.

Theorem (de Haan [H]). *Let* $f \in L^1_{\text{loc}}[0, \infty)$. *Then*

$$f(x) \sim x^{-1} \ell(x) \qquad (x \to \infty) \tag{4.1}$$

implies

$$\textstyle\int_0^x f(t)dt \in \Pi_\ell \text{ with } \ell\text{-index } 1. \tag{4.2}$$

Conversely, if f *satisfies a suitable Tauberian condition (such as monotonicity), then* (4.2) *implies* (4.1).

De Haan [H] proved a similar result for Laplace transforms (cf. [BGT, Theorem 3.9.1]). Similar results also hold for Fourier series and integrals as well as for Hankel transforms[18]. Thus we may think that, for at least basic integral transforms, π-variation naturally appears in Tauberian theorems in the boundary case. The proofs of these results, however, have been based on special properties of the special integral transforms, whence the mechanism behind them was unclear. Also it was difficult to prove similar results for general integral transforms.

The Ratio Mercerian Theorem for Systems (Theorem 3.1) enables us to prove such general results. It turns out that the key condition is described in terms of a pole of the Mellin transform $\check{k}(z)$.

Theorem 4.1 (Tauberian theorem of de Haan type, [BI5]). *Let k be a kernel satisfying Korenblum's conditions. We assume suitable conditions on k and f. In particular, we assume that $z = \rho$ ($\in \mathbf{R}$) is a simple pole of the analytic continuation of the Mellin transform $\check{k}(z)$. We also assume that f satisfies a suitable Tauberian condition. Then, in the below,* (4.4) *implies* (4.3)*:*

$$f(x) \sim x^{\rho}\ell(x) \qquad (x \to \infty), \tag{4.3}$$

$$x^{-\rho}(k * f)(x) \in \Pi_{\ell} \text{ with } \ell\text{-index } c. \tag{4.4}$$

See Theorem 5.3 below for the precise statement of Theorem 4.1. We remark that, under weak conditions, the Abelian implication (4.3) $\Rightarrow$ (4.4) also holds. Theorem 4.1 includes the known results for arithmetic means as well as for Laplace transforms. In the case of arithmetic means, we have $\check{k}(z) = 1/(1+z)$ as we have seen in Section 1. Thus this is the case $\rho = -1$, $c = 1$. For Laplace transforms, we have $\check{k}(z) = \Gamma(1+z)$, whence this also corresponds to the case $\rho = -1$, $c = 1$.

Korenblum's conditions on k in Theorem 4.1 is of the type that usually holds. However, we encountered a case in which it is impossible to check them. More precisely, we were trying to prove a Tauberian theorem for arithmetic sums in the boundary case. The kernel $k(x) = I_{(1,\infty)}(x)[x]/x$ has the Mellin transform

$$\check{k}(z+1) = \frac{\zeta(1+z)}{1+z},$$

where $\zeta(\cdot)$ is Riemann's zeta function. Thus the Mellin transform $\check{k}(z+1)$ has a simple pole at $z = 0$, so that this corresponds to the case $\rho = 0$, $c = 1$. However, to check Korenblum's conditions in this case, we need to know the behavior of $\zeta(z)$ in the strip $1 - \epsilon < \Re z < 1 + \epsilon$, in particular, nonexistence of zeros there. Of course, these are completely inaccessible.

Fortunately, we could prove an analogue of Theorem 4.1 without Korenblum's conditions ([BI6]). In the proof, we use Wiener's $L^1(\mathbf{R})$ theory instead of Korenblum's $L(\alpha)$ theory. Also we use the idea to consider systems itself rather than the Ratio Mercerian Theorem. The result thus obtained (Theorem 5.4 below) also asserts the implication (4.4) $\Rightarrow$ (4.3) but the conditions on k are weakened at the cost of loss of flexibility about the Tauberian condition on f. As for the application to arithmetic sums, we can check both conditions on k and f, whence this is enough. The result thus obtained is the following:

Theorem 4.2 (Tauberian theorem for arithmetic sums (II), [BI6])**.** *For $\ell \in R_0$, we put $\tilde{\ell}(x) := \ell(x)/\log x$. Then, for $f : [2,\infty) \to [0,\infty)$,*

$$f(x) \sim \ell(x) \qquad (x \to \infty) \tag{4.5}$$

implies

$$\frac{1}{x}\sum_{n\le x}\sum_{p|n} f(p) \in \Pi_{\tilde{\ell}} \quad \text{with } \tilde{\ell}\text{-index } 1. \tag{4.6}$$

Conversely, if f is increasing and ℓ satisfies

$$\int^{\infty} \frac{\ell(t)e^{-\sqrt{\log t}}}{t}dt < \infty \quad \text{and} \quad \log x = O(\ell(x)), \tag{4.7}$$

then (4.6) *implies* (4.5).

For example, $\ell(x) = \log x$ satisfies (4.7) but $\ell(x) \equiv 1$ does not. It is an open problem to prove the theorem without (4.7) so that the case $\ell(x) \equiv 1$ or, more specifically, the case of $\sum_{p|n} f(p)$ being the number of prime factors, is covered.

5. Precise statements

In this section, we shall give the precise statements of Theorems 2.1, 3.1, 3.2, 4.1 etc. We follow the notation of [BGT]. In particular, we recall from [BGT, Section 2.1.2] the Matuszewska indices of a positive function f. The *upper Matuszewska index* $\alpha(f)$ is the infimum of those α for which there exists a constant $C = C(\alpha)$ such that for each $\Lambda > 1$,

$$f(\lambda x)/f(x) \le C\{1 + o(1)\}\lambda^{\alpha} \qquad (x \to \infty) \text{ uniformly in } \lambda \in [1, \Lambda];$$

the *lower Matuszewska index* $\beta(f)$ is the supremum of those β for which, for some constant $D = D(\beta) > 0$ and all $\Lambda > 1$,

$$f(\lambda x)/f(x) \ge D\{1 + o(1)\}\lambda^{\beta} \qquad (x \to \infty) \text{ uniformly in } \lambda \in [1, \Lambda].$$

One says that f has *bounded increase*, written $f \in BI$, if $\alpha(f) < \infty$, *bounded decrease*, written $f \in BD$, if $\beta(f) > -\infty$.

Theorem 5.1 ([BI5])**.** *Let $\sigma \in \mathbf{R}$, $\epsilon > 0$, and $\rho \in (\sigma - \epsilon, \sigma + \epsilon)$. Let $k^1_\lambda : (0,\infty) \to [0,\infty)$ $(\lambda \in \Lambda)$ and $k^2_\lambda : (0,\infty) \to \mathbf{R}$ $(\lambda \in \Lambda)$ be measurable kernels such that the Mellin transforms $\check{k}^i_\lambda$ $(i = 1, 2,\ \lambda \in \Lambda)$ converge absolutely in the strip $\sigma - \epsilon \le \Re \le \sigma + \epsilon$. We define k^0_λ $(\lambda \in \Lambda)$ by* (3.4)*. We assume* (3.5)*,* (3.6)*, and that $|\check{k}^{0\prime}_\lambda(\rho)| + |\check{k}^{0\prime\prime}_\lambda(\rho)| > 0$ for some $\lambda \in \Lambda$. Let f be non-negative and locally bounded on $[0,\infty)$, vanish in a neighborhood of zero, have upper order ρ, and $f \in BD \cup BI$. Then* (3.3) *implies $C_\lambda = \check{k}^2_\lambda(\rho)/\check{k}^1_\lambda(\rho)$ $(\lambda \in \Lambda)$ and $E_1 * f \in R_\rho$ with $E_1(x) := I_{(1,\infty)}(x)x^{\sigma-\epsilon}$.*

Note that $E_1 * f \in R_\rho$ implies $f \in R_\rho$ under an adequate Tauberian condition on f.

Theorem 5.2 ([BI5])**.** *Let $\ell \in R_0$, $\sigma \in \mathbf{R}$, $\epsilon > 0$, and $\rho \in (\sigma - \epsilon, \sigma + \epsilon) \setminus \{\sigma\}$. Let k_λ $(\lambda \in \Lambda)$ be a system of non-negative measurable kernels on $(0,\infty)$ such that all*

$\check{k}_\lambda(z)$ $(\lambda \in \Lambda)$ converge absolutely in the strip $\sigma - \epsilon \le \Re z \le \sigma + \epsilon$. We assume that $\check{k}_\lambda(z)$ $(\lambda \in \Lambda)$ have no common zeros in $\sigma - \epsilon \le \Re z \le \sigma + \epsilon$ and that

$$\forall \lambda \in \Lambda, \qquad \exp\left(-\frac{\pi|t|}{2\epsilon}\right) \log|\check{k}_\lambda(\sigma + it)| \to 0 \qquad (t \to \pm\infty). \tag{5.1}$$

Let f be non-negative, measurable and locally bounded on $[0,\infty)$, and vanish in a neighborhood of zero. Then (1.2) implies

$$\forall \lambda \in \Lambda, \qquad k_\lambda * f(x) \sim x^\rho \ell(x) \check{k}_\lambda(\rho) \qquad (x \to \infty). \tag{5.2}$$

Conversely, (5.2) implies (1.2) if f satisfies one of the following:

$$f \text{ is eventually positive and } \log f \text{ is slowly decreasing} \tag{5.3}$$

$$f(x)/\{x^\rho \ell(x)\} \text{ is slowly decreasing} \tag{5.4}$$

$$\lim_{t \downarrow 1} \liminf_{x \to \infty} \inf_{y \in [x,tx]} \frac{y^{-\tau} f(y) - x^{-\tau} f(x)}{x^{\rho - \tau} \ell(x)} \ge 0 \quad (\text{hence} = 0) \text{ for some } \tau \in \mathbf{R}, \tag{5.5}$$

Theorem 5.3 ([BI5]). *Let $c \in \mathbf{R} \setminus \{0\}$. Let $\ell \in R_0$, $\sigma \in \mathbf{R}$, $\epsilon > 0$, and $\rho \in (\sigma - \epsilon, \sigma + \epsilon) \setminus \{\sigma\}$. Let $k : (0,\infty) \to \mathbf{R}$ be a measurable kernel such that the Mellin transform $\check{k}(z)$ converges absolutely in the strip $\rho < \Re z \le \sigma + \epsilon$. We assume the following:*

$$\begin{cases} \text{there exist } \lambda_1, \lambda_2 \in (0,\infty) \setminus \{1\} \text{ such that } \log \lambda_2 / \log \lambda_1 \text{ is irrational} \\ \text{and that } (\lambda_j x)^{-\rho} k(\lambda_j x) - x^{-\rho} k(x) \ge 0 \text{ for } 0 < x < \infty \text{ and } j = 1, 2; \end{cases} \tag{5.6}$$

$$\begin{cases} \text{the analytic continuation of } \check{k}(z) \text{ is holomorphic in } \sigma - \epsilon \le \Re z \le \sigma + \epsilon \\ \text{except for a simple pole, with residue } c, \text{ at } z = \rho; \end{cases} \tag{5.7}$$

$$\exp\left(-\frac{\pi|t|}{2\epsilon}\right) \log|\check{k}(\sigma + it)| \to 0 \qquad (t \to \pm\infty); \tag{5.8}$$

$$\check{k}(z) \text{ has no zeros in } \sigma - \epsilon \le \Re z \le \sigma + \epsilon. \tag{5.9}$$

Let f be non-negative, measurable and locally bounded on $[0,\infty)$, and vanish in a neighborhood of zero. We also assume that f satisfies either (5.3) or (5.4) or (5.5). Then (4.4) implies (4.3).

Theorem 5.4 ([BI6]). *Let $c \in \mathbf{R} \setminus \{0\}$ and $\ell \in R_0$. Let $-\infty < \sigma_1 < \rho < \sigma_2 < \infty$. Let $k : (0,\infty) \to \mathbf{R}$ be a measurable kernel such that the Mellin transform $\check{k}(z)$ converges absolutely in the strip $\rho < \Re z < \sigma_2$. We assume (5.6) and the following:*

$$\begin{cases} \text{the analytic continuation of } \check{k}(z) \text{ is holomorphic in } \sigma_1 < \Re z < \sigma_2 \\ \text{except for a simple pole, with residue } c, \text{ at } z = \rho; \end{cases} \tag{5.10}$$

$$\check{k}(z) \text{ has no zeros on } \Re z = \rho. \tag{5.11}$$

Let f be non-negative, measurable and locally bounded on $[0,\infty)$, and vanish in a neighborhood of zero. We also assume

$$\lim_{t \downarrow 1} \liminf_{x \to \infty} \inf_{y \in [x,tx]} \frac{y^{-\rho} f(y) - x^{-\rho} f(x)}{\ell(x)} \ge 0 \quad (\text{hence} = 0). \tag{5.12}$$

Then (4.4) implies (4.3).

Comments

1. This lecture was given at the meeting of the Mathematical Society of Japan, Tokyo, 2000. This is a translation from the one originally written in Japanese.
2. The term *Mercerian* comes from the fact that J. Mercer proved the following theorem in 1907 : if $t > 0$, then $a_n t + n^{-1}(a_1 + a_2 + \cdots + a_n)(1-t) \to A$ $(n \to \infty)$ implies $a_n \to A$ $(n \to \infty)$ (cf. Pitt [P]). Mercer's theorem is not of the same type as Mercerian theorems in this lecture in the narrow sense. Many people would recognize his name via his expansion theorem for positive semidefinite kernels.
3. The term *Tauberian* comes from the fact that A. Tauber proved the converse of Abel's theorem on power series, in 1897, under the condition $a_n = o(1/n)$ on coefficients. This naming is due to Hardy–Littlewood. Carleson [C], after completely proving the theorem in only 4 lines, wrote: "It is almost unique in the history of mathematics that a result of the simplicity of Tauber's gave rise to so many deep and important results that the author was immortalized by having his name become an adjective." Full investigation of Tauberian theorems dates from the Hardy–Littlewood period (1910–1920). Inspired by their work, Wiener created his theory of General Tauberian Theorem (around 1930). Wiener's idea was novel. Gelfand found that it was well described in terms of Banach algebras (1940).
4. Inspired by the work of Hardy–Littlewood as Wiener and also by the work of Pólya (1917) and others, Yugoslav mathematician Karamata initiated the theory of regular variation. Through his efforts to extend the Tauberian theorem of Hardy–Littlewood, he was led to the notion of slowly varying function. See Tomić and Aljančić [TA] for the biography of Karamata.
5. To see the potential of definition (1.1), one may just look at the Uniform Convergence Theorem of Karamata (cf. [BGT, Theorems 1.2.1 and 1.5.2]) which asserts that, under the measurability of f, the convergence of (1.1) is automatically uniform in λ on each $[a, b]$ $(0 < a \le b < \infty)$.
6. The letter ℓ or L seems to be intended to mean something like logarithm. Notice that $\log x$ is a slowly varying function.
7. For example, Wiener's Tauberian theorem seems to have been striking to the people of his time primarily because it gave a novel proof of the Prime Number Theorem.
8. See [BGT, Theorem 1.7.6] for the Able–Tauber theorem for Laplace transforms (due to Karamata) and [BGT, Theorem 5.2.4] for the Mercerian counterpart (due to Drasin).
9. In other words, in terms of the locally compact Abelian group $(0, \infty)$ with respect to the multiplication of real numbers, which has invariant measure dt/t.
10. Unless stated otherwise, the integral is supposed to converge absolutely. However, when we consider, e.g., Fourier transforms, it may be in the sense of improper integrals.
11. The same remark as Comment 10 applies. In general, the maximal domain of absolute convergence for $\check{k}(z)$ is a vertical strip in $\mathbf{C}$.
12. Integral transforms of the form $E_1 * f$ or $E_2 * f$ had already been used in the proofs of Mercerian theorems by Drasin–Shea [DS] and Jordan [J] (cf. [BGT,

Chapter 5]). However, in them, the integral transforms are written explicitly rather than in Mellin convolution form. In the localization method, we just write them in Mellin convolution form and apply both simultaneously to consider $E_2 * E_1 * f$. Though it may seem nothing significant , it in fact enables us to simplify the long and complicated proof of the Drasin–Shea–Jordan theorem profoundly. More importantly, it enables us to remove unnecessary conditions, whereby broaden the applicability of the theorem ([BI4]; see Comment 15 below).

13. The Nyman–Korenblum theory [N, K1, K2] may be regarded as an analogue of Wiener's $L^1(\mathbf{R})$ theory for $L(\alpha)$. It describes closed ideals of the commutative Banach algebra $L(\alpha)$. In particular, Korenblum determined all the primary ideals associated with the point ∞ of the maximal ideal space $\{|\Im z| \leq \alpha\} \cup \{\infty\}$ (cf. [C], [Bor]). The Banach algebra $L(\alpha)$ is the so-called *analytic case* of Beurling algebras. The function u^ρ is transformed into the exponential function $e^{\rho t}$ by the change of variable $u = e^t$. The dual space of $L^1(\mathbf{R})$ consists only of bounded functions but that of $L(\alpha)$ contains exponential functions. This is the reason why we need $L(\alpha)$ rather than $L^1(\mathbf{R})$. I am sorry that no English translation of [K2] has been published.

14. More generally, we proved, in [BI1], a Mercerian theorem for Hankel transforms of order ν such that $-1/2 \leq \nu \leq \nu_0 = 0.8660252\ldots$. This result was extended to Hankel transforms of arbitrary order $\nu \geq -1/2$ in [BI3]. Recall that Fourier cosine and sine transforms are Hankel transforms of order $-1/2$ and $1/2$, respectively. The difference between the two cases $-1/2 \leq \nu \leq \nu_0$ and $\nu > \nu_0$ is that, for the integral kernel $k(x) := x^{-3/2} J_\nu(1/x)$ of the Hankel transform of order ν (in Mellin convolution form), the following analytic continuation of Mellin transform $\check{k}(z)$ is monotone on the interval $(-\nu - \frac{3}{2}, \nu + \frac{1}{2})$ in the former case but not so in the latter case:

$$\check{k}(z) = 2^{z+(1/2)} \frac{\Gamma\left(\frac{3}{4} + \frac{1}{2}\nu + \frac{1}{2}z\right)}{\Gamma\left(\frac{1}{4} + \frac{1}{2}\nu - \frac{1}{2}z\right)}.$$

In [BI3], the localization method made such a global property irrelevant.

15. In the case $\nu > -1/2$, we can prove a Mercerian theorem for the Hankel transform of a nonmonotone function f (but assuming a weak Tauberian condition on f) if the integral converges absolutely ([BI4]). In this case, we can apply the Drasin–Shea–Jordan theorem, in the from extended in [BI4], for absolutely convergent integral transforms. However, in the case $\nu = -1/2$, i.e., that of Fourier cosine transforms, it is essential to consider conditionally convergent integrals. For, in this case, the absolute convergence strip of the Mellin transform is empty.

16. What is called a Tauberian theorem in [DI] is in fact a Mercerian one in our notation. The result [DI, Theorem 4] can be improved if we use the general theory of Mercerian theorems or, in effect, the Drasin–Shea theorem in this case (cf. [BI6]).

17. Here is an outline of the proof of the Tauberian implication (3.7) $\Rightarrow$ (1.2). By using the Prime Number Theorem of the form

$$\sum_{p \leq x} 1 = \int_2^x \frac{dt}{\log t} + R(x), \qquad R(x) = O(xe^{-\sqrt{\log x}}),$$

we write

$$\sum_{n\le x}\sum_{p|n} f(p) = \sum_{p\le x} f(p)\left[\frac{x}{p}\right] = \int_2^x \frac{f(t)}{\log t}\left[\frac{x}{t}\right]dt + \int_{[2,x]} f(t)\left[\frac{x}{t}\right]dR(t).$$

We can neglect the second term on the right-hand side. As for the first term, we write

$$\frac{1}{x}\int_2^x \frac{f(t)}{\log t}\left[\frac{x}{t}\right]dt = k * \tilde{f}(x),$$

with

$$k(x) := \frac{[x]}{x} I_{(1,\infty)}(x), \qquad \tilde{f}(x) := \frac{f(x)}{\log x} I_{(2,\infty)}(x).$$

We have $\check{k}(z) = \zeta(1+z)/(1+z)$ for $\Re z > 0$. Applying Theorem 3.2 to the integral transform $k * \tilde{f}$, we obtain (1.2).

18. The next theorem answered the Question 7.18 of Boas [Bo].

Theorem ([I1])**.** *Let* $a_n \downarrow 0$ *and put* $F(\theta) := \sum_1^\infty a_n \cos n\theta$ $(0 < \theta < 2\pi)$*. Then the following are equivalent:*

$$a_n \sim n^{-1}\ell(n) \qquad (n \to \infty),$$
$$F(1/\cdot) \in \Pi_\ell \text{ with } \ell\text{-index } 1.$$

In view of the assertion (4.1) ⇔ (4.2) for arithmetic means, this theorem does not seem so unexpected. However, strangely enough, no such results for Fourier series and integrals had been known until [I1]. See [I2, I3, BI2, IK] for subsequent work. The key to the proof of the above theorem is to use Laplace transforms to bypass the difficulty arising from conditional convergence; similar idea has been stated in the outline of the proof of Theorem 2.2 above. When I explained this idea to Nick Bingham in the spring of 1995 in London, he suggested possible use of it in the proof of yet unproven Mercerian theorems for Fourier transforms. The project described in this lecture started in this way.

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Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo 060-0810, Japan

E-mail address: `inoue@math.sci.hokudai.ac.jp`