ASYMPTOTICS FOR PREDICTION ERRORS OF STATIONARYPROCESSES WITH REFLECTION POSITIVITY

AKIHIKO INOUE AND YUKIO KASAHARA

Abstract. We consider the stationary processes that have completely monotone autocovariance functions $R(\cdot)$. We prove that regular variation of $R(\cdot)$ implies an asymptotic formula for the prediction error.

1. Introduction

Let $(X_t) = (X_t : t \in \mathbb{R})$ be a real, centered, mean-continuous, weakly stationary process defined on a probability space $(\Omega, \mathcal{F}, P)$, which we shall simply call a *stationary process*. We write $R(\cdot)$ for the autocovariance function of (X_t):

$$R(t) := E[X_t X_0] \qquad (t \in \mathbb{R}).$$

This paper is concerned with the asymptotic behavior of the prediction error of (X_t). We are especially interested in the case in which $R(\cdot)$ is regularly varying with negative index, that is,

$$\forall\lambda > 0, \quad \lim_{t\to\infty} \frac{R(\lambda t)}{R(t)} = \lambda^{-p}$$

for some $p > 0$ (see Bingham *et al.* [1]). This implies that $R(t)$ is 'close' to the power function t^{-p} with negative index for large t.

Let H be the closed real linear hull of $\{X_s : s \in \mathbb{R}\}$ in $L^2(\Omega, \mathcal{F}, P)$. Then H is a real Hilbert space with inner product $(Y_1, Y_2) := E[Y_1 Y_2]$ and norm $\|Y\| := (Y, Y)^{1/2}$. For $I \subset \mathbb{R}$, put H_I for the closed subspace of H spanned by $\{X_s : s \in I\}$, and P_I for the orthogonal projection operator of H onto H_I. We write $P_I^\perp$ for the projection operator onto the orthogonal complement $H_I^\perp$ of H_I, i.e., $P_I^\perp Y = Y - P_I Y$ for $Y \in H$. Then, for $T > 0$ and $t > 0$, the projection $P_{[-t,0]} X_T$ may be regarded as the best linear predictor of X_T on the observations $\{X_s : -t \le s \le 0\}$, hence $P_{[-t,0]}^\perp X_T = X_T - P_{[-t,0]} X_T$ as its prediction error. Similarly, $P_{(-\infty,0]}^\perp X_T$ may be regarded as the prediction error for the prediction of X_T on the observations $\{X_s : -\infty < s \le 0\}$.

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We define

$$V_T(t) := \|P^{\perp}_{[-t,0]}X_T\|^2 - \|P^{\perp}_{(-\infty,0]}X_T\|^2 \qquad (T > 0,\ t > 0).$$

After preliminary investigations, we reached the working hypothesis that, for a wide class of (X_t), regular variation of $R(\cdot)$ would imply the following asymptotic formula:

$$(1.1) \qquad V_T(t) \sim \int_t^{\infty} \left\{ \frac{R(s)}{\int_{-s}^{s} R(u)du} \right\}^2 ds \cdot \left\{ \int_0^T C(s)ds \right\}^2 \qquad (t \to \infty),$$

where $C(\cdot)$ is the canonical representation kernel of (X_t) (see (2.1) below).

In [7], (1.1) is proved for a special stationary process that has a spectral density proportional to $|\xi|^{-q}/(1+\xi^2)$ $(0 < q < 1)$. The proof is by an explicit calculation of $\|P^{\perp}_{[-t,0]}X_T\|$ using techniques in Dym–McKean [2], and is not applicable to general stationary processes. Recently, one of the authors proved an analogue of (1.1) for *discrete-time* stationary processes in [6], under some assumptions (on the AR(∞)-coefficients and MA(∞)-coefficients). The present paper arose from an attempt to prove (1.1) by a continuous-time analogue of the method of [6]. Because of difficulties which are proper to continuous-time processes, it is still difficult to fully achieve this attempt. However we have at least one nice class of stationary processes to which we can apply the method. It is the class of stationary processes with *reflection positivity*, that is, those with completely monotone autocovariance functions.

A stationary process with autocovariance function $R(\cdot)$ is said to have reflection positivity, or simply (RP), if there exists a finite Borel measure σ on $(0, \infty)$ such that

$$(1.2) \qquad R(t) = \int_0^{\infty} e^{-|t|\lambda} \sigma(d\lambda) \quad (t \in \mathbb{R}).$$

For example, the stationary processes with $R(t) = (1 + |t|)^{-p}$, for $0 < p < \infty$, satisfy (RP) (see §4). We refer to, e.g., Okabe [9], [11] and [3, 4, 5] for earlier work on stationary processes with (RP). We may say that the class of stationary processes with (RP) has the advantage that it is especially suited for asymptotic analysis. The purpose of this paper is to establish the formula (1.1) for the class.

To state the results, we recall some notation from regular variation theory. We write $\mathcal{R}_0$ for the class of slowly varying functions at infinity: the class of positive, measurable ℓ, defined on some neighborhood $[A, \infty)$ of infinity, such that

$$\forall \lambda > 0, \quad \lim_{x \to \infty} \ell(\lambda x)/\ell(x) = 1.$$

Let $\ell \in \mathcal{R}_0$ and choose B so that $\ell(\cdot)$ is locally bounded on $[B, \infty)$ (cf. [1, Corollary 1.4.2]). When we say $\int^{\infty} \ell(s)ds/s = \infty$, it means that $\int_B^{\infty} \ell(s)ds/s = \infty$. If so, then we define another slowly varying function $\tilde{\ell}$ by

$$\tilde{\ell}(x) := \int_B^x \frac{\ell(s)}{s} ds \qquad (x \geq B)$$

(see [1, §1.5.6]). The asymptotic behavior of $\tilde{\ell}(x)$ as $x \to \infty$ does not depend on the choice of B because we have assumed $\int^\infty \ell(s)ds/s = \infty$.

Here is the main theorem.

Theorem 1.1. *Let $T > 0$, $p > 0$ and $\ell \in \mathcal{R}_0$. Let (X_t) be a stationary process with* (RP). *We assume*

$$R(t) \sim t^{-p}\ell(t) \qquad (t \to \infty). \tag{1.3}$$

(i) *If $0 < p < 1$, then*

$$V_T(t) \sim t^{-1}\frac{(1-p)^2}{4} \cdot \left\{\int_0^T C(s)ds\right\}^2 \qquad (t \to \infty).$$

(ii) *If $p = 1$ and $\int^\infty \ell(s)ds/s = \infty$, then*

$$V_T(t) \sim t^{-1}\left\{\frac{\ell(t)}{2\tilde{\ell}(t)}\right\}^2 \cdot \left\{\int_0^T C(s)ds\right\}^2 \qquad (t \to \infty).$$

(iii) *If either $1 < p < \infty$ or $p = 1$ and $\int^\infty \ell(s)ds/s < \infty$, then*

$$V_T(t) \sim \frac{t^{-(2p-1)}\left\{\ell(t)\right\}^2}{(2p-1)\{\int_{-\infty}^\infty R(s)ds\}^2} \cdot \left\{\int_0^T C(s)ds\right\}^2 \qquad (t \to \infty).$$

We remark that (i)–(iii) above correspond to the following cases, respectively: (i)' $\int_0^\infty R(t)dt = \infty$ and $0 < p < 1$, (ii)' $\int_0^\infty R(t)dt = \infty$ and $p = 1$, (iii)' $\int_0^\infty R(t)dt < \infty$. Stationary processes that satisfy (1.3) with $0 < p < 1$ are called *long-memory processes*. We see that the result (i) for long-memory processes is especially simple; in this case, the index over t is -1, and so does not depend on p, and the slowly varying function $\ell(\cdot)$ has even disappeared.

We can state the results (i)–(iii) above more simply as follows:

Theorem 1.1'. *Let $p > 0$, $T > 0$ and $\ell \in \mathcal{R}_0$, and let (X_t) be a stationary process with* (RP). *Then* (1.3) *implies* (1.1).

We comment on the method of the proof of Theorem 1.1. We define, for $t > 0$ and $n = 1, 2, \ldots,$

$$P_t^n := P_{(-\infty,0]} \quad \text{if } n \text{ is odd}, \quad := P_{[-t,\infty)} \quad \text{if } n \text{ is even}.$$

Then, for $t > 0$, $T > 0$ and $n \geq 2$, repeated use of the orthogonal decompositions

$$P_{[-t,0]}^\perp = P_{(-\infty,0]}^\perp + P_{[-t,0]}^\perp P_{(-\infty,0]} = P_{[-t,\infty)}^\perp + P_{[-t,0]}^\perp P_{[-t,\infty)}, \tag{1.4}$$

yields

$$\|P_{[-t,0]}^{\perp}X_T\|^2 = \|P_{(-\infty,0]}^{\perp}X_T\|^2 + \sum_{k=1}^{n-1}\|(P_t^{k+1})^{\perp}P_t^k\cdots P_t^1X_T\|^2 + \|P_{[-t,0]}^{\perp}P_t^n\cdots P_t^1X_T\|^2 \tag{1.5}$$

(compare [6, (4.2)]). If we set

$$U_T^n(t) := \|(P_t^{n+1})^{\perp}P_t^n\cdots P_t^1X_T\|^2 \qquad (n=1,2,\ldots),$$
$$Z_T^n(t) := \|P_{[-t,0]}^{\perp}P_t^n\cdots P_t^1X_T\|^2 \qquad (n=1,2,\ldots),$$

then, from (1.5), we have

$$V_T(t) = \sum_{k=1}^{n-1}U_T^k(t) + Z_T^n(t) \tag{1.6}$$

Eq. (1.6) suggests that the problem of $V_T(t)$ could be reduced to that of $U_T^k(t)$ if we could show that $Z_T^n(t)$ is small enough in a proper sense. The advantage here is that $U_T^n(t)$ $(n=1,2,\cdots)$ are easier to handle than $V_T(t)$. This is because $U_T^n(t)$ are defined only in terms of the projection operators $P_{(-\infty,0]}$ and $P_{[-t,\infty)}$ which are are much easier to handle than $P_{[-t,0]}$.

To prove (ii) and (iii) of Theorem 1.1, it is enough to use (1.6) with only $n=2$. Indeed, in this case, we can show that $V_T(t)\sim U_T^1(t)$ as $t\to\infty$. The proof of (i) is much harder; in this case, $U_T^n(t)$ for $n\geq 2$ are not negligible since they turn out to have the same order of asymptotics as that of $U_T^1(t)$. In order to prove (i), we must determine the asymptotic behavior of $U_T^n(t)$ for every $n\geq 1$.

To follow the line above, we need the representation of $P_{(-\infty,0]}X_t$ in terms of $\{X_s : -\infty < X_s \leq 0\}$. It is at this point where the difficulties proper to continuous-time processes arise. Formally, the representation would be of the form $\int_0^{\infty}k_t(s)X_{-s}ds$ with some kernel $k_t(\cdot)$. Wiener tackled this problem earlier in [13] and formally gave a formula identifying $k_t(\cdot)$ in terms of the outer function of the stationary process (see [2, §4.4]). However the derivation is purely formal ([2, p. 92]). In §2, we establish such a representation theorem for stationary processes with (RP) using the assumption (RP) essentially.

2. Projection

In what follows, throughout this paper, we assume that the autocovariance function $R(\cdot)$ of (X_t) is of the form (1.2) with a finite Borel measure σ on $(0,\infty)$. Then we know that (X_t) is purely nondeterministic: $\cap_t H_{(-\infty,t]} = \{0\}$. Let $\Delta(\cdot)$ be the spectral density of (X_t): $R(t) = \int_{-\infty}^{\infty}e^{-it\xi}\Delta(\xi)d\xi$ for $t\in\mathbb{R}$. It follows that

$\{\log \Delta(\xi)\}/(1+\xi^2) \in L^1(\mathbb{R})$. We write $h(\cdot)$ for the outer function of (X_t):

$$h(z) := \exp\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1+\xi z}{\xi - z}\cdot\frac{\log \Delta(\xi)}{1+\xi^2}d\xi\right\} \qquad (\Im z > 0).$$

The *canonical representation kernel* $C(\cdot)$ of (X_t) is defined by

$$C(t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\xi}h(\xi)d\xi \tag{2.1}$$

in $L^2(\mathbb{R})$-sense, where $h(\cdot) := \mathrm{l.i.m.}_{\eta\downarrow 0} h(\cdot + i\eta) \in L^2(\mathbb{R})$. The kernel $C(\cdot)$ vanishes on $(-\infty, 0)$ and satisfies the following equalities:

$$h(z) = \int_0^\infty e^{izt}C(t)dt \qquad (\Im z > 0), \tag{2.2}$$

$$R(t) = \int_0^\infty C(t+s)C(s)ds \qquad (t \geq 0). \tag{2.3}$$

See, e.g., Rozanov [12], [2] and [9, §2] for details.

By [10, Theorem A] and [4, Theorems 2.4 and 2.5], there exists a unique Borel measure ν on $(0,\infty)$ such that

$$h(z) = \int_0^\infty \frac{1}{\lambda - iz}\nu(d\lambda) \quad (\Im z > 0), \qquad \int_0^\infty\int_0^\infty \frac{1}{\lambda+\lambda'}\nu(d\lambda)\nu(d\lambda') < \infty. \tag{2.4}$$

It follows that the canonical representation kernel $C(\cdot)$ of (X_t) admits the following representation:

$$C(t) = I_{(0,\infty)}(t)\int_0^\infty e^{-t\lambda}\nu(d\lambda) \qquad (t \in \mathbb{R}).$$

By (2.4) as well as Theorems S1.5.1, S1.5.2 and Lemma S1.5.2 of Kac and Krein [8], $1/h(z)$ admits the representation

$$\frac{1}{h(z)} = b - iaz - iz\int_0^\infty \frac{1}{(\lambda - iz)\lambda}\tau(d\lambda) \qquad (\Im z > 0), \tag{2.5}$$

where $a \geq 0$, $b \geq 0$ and τ is a Borel measure on $(0,\infty)$ such that

$$\int_0^\infty \frac{1}{\lambda(1+\lambda)}\tau(d\lambda) < \infty.$$

We note that the triple (a, b, τ) is determined uniquely by $h(\cdot)$ hence by (X_t). We define

$$G(t) := \int_0^\infty e^{-t\lambda}\lambda^{-1}\tau(d\lambda) \qquad (t > 0).$$

Given a measurable function $f : (0,\infty) \to \mathbb{R}$, we define its Laplace transform

$$\hat{f}(x) := \int_0^\infty e^{-xt}f(t)dt$$

for $x > 0$ such that the integral converges absolutely. It follows from (2.2) and (2.5) that

$$\hat{C}(x)\left\{b + ax + x\hat{G}(x)\right\} = 1 \qquad (x > 0). \tag{2.6}$$

This implies

$$aC(t) + b\int_0^t C(s)ds + \int_0^t G(s)C(t-s)ds = 1 \qquad (t > 0) \tag{2.7}$$

since the Laplace transforms of both sides coincide by (2.6).

We set

$$A(t) := \int_0^\infty e^{-t\lambda}\tau(d\lambda) \qquad (t > 0).$$

Then $A(t) = -\dot{G}(t)$ for every $t > 0$. For $t > 0$, we define

$$K_t(s) := \int_0^t A(s+u)C(t-u)du \qquad (s > 0).$$

As (2.7) implies $\int_0^\infty K_t(s)ds = \int_0^t G(s)C(t-s)ds \le 1$, $K_t(\cdot)$ is integrable on $(0,\infty)$ for every $t > 0$.

Here is the representation of $P_{(-\infty,0]}X_t$ in terms of $\{X_s : -\infty < s \le 0\}$.

Theorem 2.1. *For $t > 0$, we have*

$$P_{(-\infty,0]}X_t = aC(t)X_0 + \int_0^\infty K_t(s)X_{-s}ds, \tag{2.8}$$

where the integral converges absolutely in H.

Proof. Let

$$X_u = \int_{-\infty}^\infty C(u-s)\xi(ds) = \int_{-\infty}^u C(u-s)\xi(ds) \qquad (u \in \mathbb{R}) \tag{2.9}$$

be the canonical representation of (X_u) (see, e.g., [12, Ch. III, §3]). Define the linear map $T : L^2(\mathbb{R}) \to H$ by

$$Tf := \int_{-\infty}^\infty f(-s)\xi(ds) \qquad (f \in L^2(\mathbb{R})). \tag{2.10}$$

Then T is a Hilbert space isomorphism of $L^2(\mathbb{R})$ onto H; for example, the representation (2.9) implies that T is onto. From (2.9) we see that $(T^{-1}X_u)(\cdot) = C(u+\cdot)$. Moreover, since $P_{(-\infty,0]}Tf = \int_{-\infty}^0 f(-s)\xi(ds)$, we have

$$(T^{-1}P_{(-\infty,0]}Tf)(u) = I_{(0,\infty)}(u)f(u) \qquad (f \in L^2(\mathbb{R})), \tag{2.11}$$

in particular, $(T^{-1}P_{(-\infty,0]}X_t)(\cdot) = I_{(0,\infty)}(u)C(t+\cdot)$. Thus, (2.8) follows if we prove

$$I_{(0,\infty)}(u)C(t+u) = aC(t)C(u) + \int_0^\infty K_t(s)C(u-s)ds \qquad (u \in \mathbb{R}).$$

However, since both sides of the equality above vanish for $u \le 0$, it is enough to prove

$$C(t+u) = aC(t)C(u) + \int_0^u K_t(s)C(u-s)ds \qquad (u>0). \tag{2.12}$$

Let $x>0$. Then by (2.6) and the equalities

$$\int_0^\infty e^{-xu}C(t+u)du = x\int_0^\infty e^{-xs}\left\{\int_t^{t+s} C(u)du\right\}ds,$$
$$\hat{G}(x)\int_0^\infty e^{-xu}C(t+u)du = \int_0^\infty e^{-xs}\left\{\int_0^s G(u)C(t+s-u)du\right\}ds,$$

the Laplace transform $\int_0^\infty e^{-xu}C(t+u)du$ is equal to

$$\hat{C}(x)\left\{b+ax+x\hat{G}(x)\right\}\int_0^\infty e^{-xu}C(t+u)du = x\hat{C}(x)\hat{F}_t(x)$$

where

$$F_t(s) := bC(t+s) + a\int_t^{t+s} C(u)du + \int_0^s G(u)C(t+s-u)du.$$

On the other hand, it follows from integration by parts that

$$\begin{aligned}\hat{K}_t(x) &= \int_0^t G(u)C(t-u)du - x\int_0^\infty e^{-xs}\left\{\int_0^t G(s+u)C(t-u)du\right\}ds\\ &= x\int_0^\infty e^{-xs}\left\{\int_0^t G(u)C(t-u)du - \int_s^{t+s} G(u)C(t+s-u)du\right\}ds,\end{aligned}$$

so that the Laplace transform of the right-hand side of (2.12) is equal to

$$x\hat{C}(x)\hat{H}_t(x),$$

where

$$H_t(s) := bC(t) + \int_0^t G(u)C(t-u)du - \int_s^{t+s} G(u)C(t+s-u)du.$$

However, we have from (2.7) that $F_t(s) = H_t(s)$ for every $s>0$, and so the uniqueness of Laplace transforms implies (2.12). $\square$

Corollary 2.2. *Let $t>0$, $T>0$ and $n\in\mathbb{N}$.*

(i) *If n is odd, then we have*

$$\begin{aligned}P_t^n\cdots P_t^1 X_T = &\int_0^\infty ds_1 K_T(t+s_1)\int_0^\infty ds_2 K_{s_1}(t+s_2)\\ &\cdots\int_0^\infty K_{s_{n-1}}(t+s_n)X_{-t-s_n}ds_n \qquad (\mathrm{mod}\ H_{[-t,0]}).\end{aligned}$$

(ii) *If n is even, then we have*

$$P_t^n \cdots P_t^1 X_T = \int_0^\infty ds_1 K_T(t+s_1) \int_0^\infty ds_2 K_{s_1}(t+s_2) \cdots \int_0^\infty K_{s_{n-1}}(t+s_n) X_{s_n} ds_n \qquad (\mathrm{mod}\ H_{[-t,0]}).$$

Proof. It follows from Theorem 2.1 that, for $u>0$,

$$P_{(-\infty,0]} X_u = \int_0^\infty K_u(t+s) X_{-t-s} ds \qquad (\mathrm{mod}\ H_{[-t,0]}). \tag{2.13}$$

Let S_v ($v \in \mathbb{R}$) and θ be the Hilbert space automorphisms of H such that $S_v(X_s) = X_{v+s}$, $\theta(X_s) = X_{-s}$ for every $s \in \mathbb{R}$. Then we have

$$S_v^{-1} = S_{-v}, \quad \theta^{-1} = \theta, \quad P_{[-t,\infty)} = (\theta S_t)^{-1} P_{(-\infty,0]} (\theta S_t). \tag{2.14}$$

Therefore, we have from (2.13) that, for $u>0$,

$$P_{[-t,\infty)} X_{-t-u} = S_{-t} \theta P_{(-\infty,0]} X_u = \int_0^\infty K_u(t+s) X_s ds \quad (\mathrm{mod}\ H_{[-t,0]}). \tag{2.15}$$

The absolute convergence of the integrals in (2.13) and (2.15) allow us to use them repeatedly, and so we obtain the corollary. □

For $T>0$, $t>0$, $u>0$ and $k \in \mathbb{N}$, we define $D_T^k(t,u)$ by

$$\begin{aligned}
D_T^1(t,u) &:= \int_0^\infty C(v_1) K_T(t+v_1+u) dv_1,\\
D_T^2(t,u) &:= \int_0^\infty dv_2 C(v_2) \int_0^\infty dv_1 C(v_1)\\
&\qquad \int_0^\infty A(t+s_1+v_2+u) K_T(t+s_1+v_1) ds_1,\\
D_T^k(t,u) &:= \int_0^\infty dv_k C(v_k) \cdots \int_0^\infty dv_1 C(v_1) \int_0^\infty ds_{k-1} A(t+s_{k-1}+v_k+u)\\
&\quad \int_0^\infty ds_{k-2} A(t+s_{k-1}+s_{k-2}+v_{k-1}) \cdots \int_0^\infty ds_2 A(t+s_3+s_2+v_3)\\
&\qquad \int_0^\infty A(t+s_2+s_1+v_2) K_T(t+s_1+v_1) ds_1 \qquad (k \geq 3).
\end{aligned}$$

We note that the integrals above are well-defined since all the integrands are non-negative.

Recall $U_T^n(t)$ from §1. Here is the representation of $U_T^n(t)$ in terms of $A(\cdot)$ and $C(\cdot)$.

Proposition 2.3. *For $T>0$, $t>0$ and $n \in \mathbb{N}$, we have $U_T^n(t) = \int_0^\infty D_T^n(t,u)^2 du$.*

Proof. By Corollary 2.2 and (2.14), $U_T^n(t)$ is equal to

$$\left\| P^{\perp}_{(-\infty,0]} \int_0^\infty ds_1 K_T(t+s_1) \int_0^\infty ds_2 K_{s_1}(t+s_2) \cdots \int_0^\infty K_{s_{n-1}}(t+s_n) X_{s_n} ds_n \right\|^2$$

C (use the maps S_t and θ in the proof of Corollary 2.2 if n is even). Recall the map $T : L^2(\mathbb{R}) \to H$ from (2.10). Since (2.11) implies

$$(T^{-1} P^{\perp}_{(-\infty,0]} T f)(u) = I_{(-\infty,0)}(u) f(u) \qquad (f \in L^2(\mathbb{R})),$$

we see that $U_T^n(t) = \int_0^\infty d_T^n(t,u)^2 du$, where we write $d_T^n(t,u)$ for the integral

$$\int_0^\infty ds_1 K_T(t+s_1) \int_0^\infty ds_2 K_{s_1}(t+s_2) \cdots \int_u^\infty K_{s_{n-1}}(t+s_n) C(s_n - u) ds_n.$$

It is enough to prove $d_T^n(t,u) = D_T^n(t,u)$. By the Fubini–Tonelli theorem, $d_T^n(t,u)$ is equal to

$$\int_0^\infty dv_n C(v_n) \int_0^\infty ds_{n-1} K_{s_{n-1}}(t+s_n+u) \int_0^\infty ds_{n-2} K_{s_{n-2}}(t+s_{n-1})$$
$$\cdots \int_0^\infty ds_2 K_{s_2}(t+s_3) \int_0^\infty K_{s_1}(t+s_2) K_T(t+s_1) ds_1.$$

First, $\int_0^\infty K_{s_1}(t+s_2) K_T(t+s_1) ds_1$ is equal to

$$\begin{aligned}&\int_0^\infty \left\{ \int_0^{s_1} C(v_1) A(t+s_2+s_1-v_1) dv_1 \right\} K_T(t+s_1) ds_1 \\ &\qquad = \int_0^\infty dv_1 C(v_1) \int_0^\infty A(t+s_2+s_1) K_T(t+s_1+v_1) ds_1.\end{aligned}$$

Second, $\int_0^\infty K_{s_2}(t+s_3) A(t+s_2+s_1) ds_2$ is equal to

$$\begin{aligned}&\int_0^\infty \left\{ \int_0^{s_2} C(v_2) A(t+s_3+s_2-v_2) dv_2 \right\} A(t+s_2+s_1) ds_2 \\ &\qquad = \int_0^\infty dv_2 C(v_2) \int_0^\infty A(t+s_3+s_2) A(t+s_2+s_1+v_2) ds_2.\end{aligned}$$

Repeating this argument, we finally obtain $d_T^n(t,u) = D_T^n(t,u)$, as desired. □

3. Proof of Theorem 1.1

For simplicity, we set

$$J(t) := \int_0^t C(s) ds \qquad (t > 0).$$

Proof of Theorem 1.1(iii). It follows from (2.3) that

$$\left\{ \int_0^\infty C(s) ds \right\}^2 = 2 \int_0^\infty R(t) dt. \tag{3.1}$$

Since the assumptions on p and $\ell(\cdot)$ imply $\int_0^\infty R(t)dt < \infty$, we have $\int_0^\infty C(s)ds < \infty$. Therefore, applying the monotone convergence theorem to (2.6), we obtain

$$b = \left\{ \int_0^\infty C(s)ds \right\}^{-1}$$

(cf. [9, Lemma 2.7]). Hence we have from [3, Lemma 3.8] that

$$C(t) \sim bt^{-p}\ell(t) \qquad (t \to \infty).$$

If $1 < p < \infty$, then it follows from [3, Theorem 4.1] that

$$G(t) \sim t^{1-p}\ell(t) \cdot \frac{b^3}{p-1} \qquad (t \to \infty).$$

Since $G(t) = \int_t^\infty A(s)ds$ and $A(\cdot)$ is decreasing, this implies

$$A(t) \sim t^{-p}\ell(t)b^3 \qquad (t \to \infty) \tag{3.2}$$

(cf. [1, §1.7.3]). On the other hand, if $p = 1$ and $\int^\infty \ell(s)ds/s < \infty$, then as in the proof of [5, Theorem 1.2] we see that $G(\cdot) \in \Pi_\ell$ with ℓ-index $-b^3$. However, by de Haan's monotone convergence theorem (cf. [1, Theorem 3.6.8]), this also implies (3.2).

By (1.6) with $n = 2$, we have the lower bound for $V_T(t)$: $V_T(t) \geq U_T^1(t)$. On the other hand, by (1.4) and Theorem 2.1, we have the upper bound for $V_T(t)$:

$$\begin{aligned} V_T(t) &= \left\| P_{[-t,0]}^\perp P_{(-\infty,0]} X_T \right\|^2 = \left\| P_{[-t,0]}^\perp \int_t^\infty K_T(s) X_{-s} ds \right\|^2 \\ &\leq \left\| \int_t^\infty K_T(s) X_{-s} ds \right\|^2 = 2 \int_t^\infty ds K_T(s) \int_0^\infty K_T(s+u) R(u) du. \end{aligned} \tag{3.3}$$

First, we consider the lower bound. Since

$$A(t+T)J(T) \leq K_T(t) \leq A(t)J(T), \tag{3.4}$$

it follows from (3.2) that

$$K_T(t) \sim t^{-p}\ell(t)b^3 J(T) \qquad (t \to \infty). \tag{3.5}$$

As in [3, Lemma 3.8], this implies

$$\int_0^\infty C(v) K_T(t+v) dv \sim t^{-p}\ell(t)b^2 J(T) \qquad (t \to \infty),$$

which by (3.1) yields

$$U_T^1(t) = \int_t^\infty \left\{ \int_0^\infty C(v) K_T(u+v) dv \right\}^2 du \sim \frac{t^{-(2p-1)} \left\{ \ell(t) \right\}^2 J(T)^2}{(2p-1)\{\int_{-\infty}^\infty R(s)ds\}^2} \quad (t \to \infty).$$

Next we consider the upper bound. From (3.4), it follows, as above, that

$$\int_0^\infty K_T(t+u) R(u) du \sim K_T(t) \int_0^\infty R(u) du \qquad (t \to \infty),$$

hence

$$2\int_t^\infty ds K_T(s)\int_0^\infty K_T(s+u)R(u)du \sim \frac{t^{-(2p-1)}\left\{\ell(t)\right\}^2 J(T)^2}{(2p-1)\{\int_{-\infty}^\infty R(s)ds\}^2} \qquad (t\to\infty).$$

Since the asymptotics for the upper and lower bounds coincide, (iii) follows. □

Proof of Theorem 1.1(ii). We have from [5, Theorem 5.2] that

$$C(t)\sim t^{-1}\frac{\ell(t)}{\{2\tilde{\ell}(t)\}^{1/2}} \qquad (t\to\infty),$$

so that

$$J(t)\sim \{2\tilde{\ell}(t)\}^{1/2} \qquad (t\to\infty) \tag{3.6}$$

(cf. [6, (5.29)]).

Since $\int_0^\infty C(t)dt=\infty$, we see that $b=0$, by letting $x\downarrow 0$ in (2.6). Hence, comparing (2.6) with [5, (6.1)], we find, as in [5, Theorem 1.1(3)'], that $G(\cdot)\in \Pi_{\ell_1}$ with ℓ_1-index -1, where $\ell_1(t) := \ell(t)\{2\tilde{\ell}(t)\}^{-3/2}$. Therefore, by de Haan's monotone density theorem, we have $A(t)\sim t^{-1}\ell_1(t)$ as $t\to\infty$, and so

$$K_T(t)\sim t^{-1}\ell_1(t)J(T) \qquad (t\to\infty). \tag{3.7}$$

Since $-\dot{K}_T(t) = -\int_0^T \dot{A}(t+v)C(T-v)dv$ is decreasing in t, this implies

$$-\dot{K}_T(t)\sim t^{-2}\ell_1(t)J(T) \qquad (t\to\infty). \tag{3.8}$$

By the Fubini–Tonelli theorem, we have

$$\int_0^\infty K_T(t+v)C(v)dv = -\int_0^\infty \dot{K}_T(t+s)J(s)ds.$$

Using (3.6) and (3.8), we apply [5, Proposition 4.3] to the above to obtain

$$\int_0^\infty K_T(t+v)C(v)dv\sim t^{-1}\frac{\ell(t)}{2\tilde{\ell}(t)}J(T) \qquad (t\to\infty), \tag{3.9}$$

and so

$$U_T^1(t)=\int_t^\infty\left\{\int_0^\infty C(v)K_T(u+v)dv\right\}^2du\sim t^{-1}\left\{\frac{\ell(t)}{2\tilde{\ell}(t)}\right\}^2 J(T)^2 \qquad (t\to\infty).$$

On the other hand, in the same way as above, it follows that

$$\int_0^\infty K_T(t+u)R(u)du\sim t^{-1}\frac{\ell(t)}{2^{3/2}\tilde{\ell}(t)^{1/2}}J(T) \qquad (t\to\infty),$$

which with (3.7) implies

$$2\int_t^\infty ds K_T(s)\int_0^\infty K_T(s+u)R(u)du\sim t^{-1}\left\{\frac{\ell(t)}{2\tilde{\ell}(t)}\right\}^2 J(T)^2 \qquad (t\to\infty).$$

Thus the asymptotic behavior of the upper bound in (3.3) is equal to that of the lower bound $U_T^1(t)$, and so (ii) follows. $\square$

Proof of Theorem 1.1(i). Step 1. For simplicity, we use the following number $d \in (0, 1/2)$ rather than p:

$$d := \frac{1-p}{2}.$$

Then it follows from [5, Theorem 4.1] that

$$C(t) \sim t^{-(1-d)} \left\{ \frac{\ell(t)}{B(d, 1-2d)} \right\}^{1/2} \qquad (t \to \infty), \tag{3.10}$$

where $B(\cdot, \cdot)$ is the beta function (we note that the index $-1/2$ in [5, (4.3)] is mistaken; it should be $1/2$). As in the previous proof, we have $b = 0$. Therefore, comparing [5, (3.5)] with (2.6), we obtain, as in [5, Theorem 1.1(i)],

$$G(t) \sim t^{-d} \left\{ \frac{\ell(t)}{B(d, 1-2d)} \right\}^{-1/2} \frac{\sin(\pi d)}{\pi} \qquad (t \to \infty).$$

By the monotone convergence theorem, this implies

$$A(t) \sim t^{-(d+1)} \left\{ \frac{\ell(t)}{B(d, 1-2d)} \right\}^{-1/2} \frac{d \sin(\pi d)}{\pi} \qquad (t \to \infty). \tag{3.11}$$

Step 2. We prove, for $T > 0$ and $k \in \mathbb{N}$,

$$U_T^k(t) \sim t^{-1} a_k \sin^{2k}(\pi d) J(T)^2 \qquad (t \to \infty), \tag{3.12}$$

where

$$a_k := \frac{(2k-2)!!}{\pi^2 (2k-1)!! k} \qquad (k \in \mathbb{N}).$$

Choose δ from the interval $(0, \min(d, 1/(8k-2)))$. By (3.10) and (3.11), there exists $M > 0$ such that

$$0 \le \frac{A(t)}{A(s)} \le 2(t/s)^{-(1+d-\delta)} \qquad (t \ge s \ge M), \tag{3.13}$$

$$0 \le \frac{C(t)}{C(s)} \le 2 \max\left\{ (t/s)^{-(1-d+\delta)}, (t/s)^{-(1-d-\delta)} \right\} \quad (t \ge M,\ s \ge M) \tag{3.14}$$

(see [1, Theorem 1.5.6]). We set, for $t > 0$,

$$C_0(t) := I_{(0,M)}(t) C(t), \quad C_1(t) := I_{[M,\infty)}(t) C(t). \tag{3.15}$$

For $k \geq 3$ and $(i_1, \cdots, i_k) \in \{0,1\}^k$, we write $M_T(t, u; i_1, \cdots, i_k)$ for the integral

$$\int_0^\infty dv_k \frac{C_{i_k}(tv_k)}{C(t)} \cdots \int_0^\infty dv_1 \frac{C_{i_1}(tv_1)}{C(t)} \int_0^\infty ds_{k-1} \frac{A(t(1+s_{k-1}+v_k+u))}{A(t)}$$
$$\int_0^\infty ds_{k-2} \frac{A(t(1+s_{k-1}+s_{k-2}+v_{k-1}))}{A(t)} \cdots \int_0^\infty ds_2 \frac{A(t(1+s_3+s_2+v_3))}{A(t)}$$
$$\int_0^\infty \frac{A(t(1+s_2+s_1+v_2))}{A(t)} \cdot \frac{K_T(t(1+s_1+v_1))}{A(t)} ds_1.$$

We also write

$$M_T(t,u;i_1) := \int_0^\infty \frac{C_{i_1}(tv_1)}{C(t)} \cdot \frac{K_T(t(1+v_1+u))}{A(t)} dv_1,$$
$$M_T(t,u;i_1,i_2) := \int_0^\infty dv_2 \frac{C_{i_2}(tv_2)}{C(t)} \int_0^\infty dv_1 \frac{C_{i_1}(tv_1)}{C(t)}$$
$$\int_0^\infty \frac{A(t(1+s_1+v_2+u))}{A(t)} \cdot \frac{K_T(t(1+s_1+v_1))}{A(t)} ds_1.$$

Then, from Proposition 2.3, we have

$$U_T^k(t) = t^{4k-1} \{A(t)C(t)\}^{2k} \int_0^\infty \left\{ \sum_i M_T(t,u;i_1,\cdots,i_k) \right\}^2 du,$$

where $\sum_i = \sum_{(i_1,\cdot,i_k)\in\{0,1\}^k}$. As in the proof of [6, Proposition 6.1], it follows that

$$\lim_{t\to\infty} \int_0^\infty M_T(t,u;i_1,\cdots,i_k)^2 du = \begin{cases} \dfrac{\pi^{2k} a_k J(T)^2}{d^{2k}} & (i_1 = 1, \cdots, i_k = 1), \\ 0 & ((i_1,\cdots,i_k) \neq (1,\cdots,1)) \end{cases}$$

(use (3.4) and [6, (6.15)]), hence

$$\lim_{t\to\infty} \int_0^\infty \left\{ \sum_i M_T(t,u;i_1,\cdots,i_k) \right\}^2 du = \frac{\pi^{2k} a_k J(T)^2}{d^{2k}}.$$

Since (3.10) and (3.11) imply

$$t^{4k-1} \{A(t)C(t)\}^{2k} \sim t^{-1} \left\{ \frac{d\sin(d\pi)}{\pi} \right\}^{2k} \qquad (t \to \infty), \tag{3.16}$$

(3.12) follows.

Step 3. Recall $Z_T^n(t)$ from §1. By Corollary 2.2, we see that $Z_T^n(t)$ is at most

$$\left\| \int_0^\infty ds_1 K_T(t+s_1) \int_0^\infty ds_2 K_{s_1}(t+s_2) \cdots \int_0^\infty K_{s_{n-1}}(t+s_n) X_{s_n} ds_n \right\|^2$$
$$= \int_{-\infty}^\infty \left\{ \int_0^\infty ds_1 K_T(t+s_1) \int_0^\infty ds_2 K_{s_1}(t+s_2) \right.$$
$$\left. \cdots \int_0^\infty K_{s_{n-1}}(t+s_n) C(s_n+u) ds_n \right\}^2 du$$
$$= U_T^n(t) + W_T^n(t),$$

where

$$W_T^n(t) := \int_0^\infty \left\{ \int_0^\infty ds_1 K_T(t+s_1) \int_0^\infty ds_2 K_{s_1}(t+s_2) \cdots \int_0^\infty K_{s_{n-1}}(t+s_n) C(s_n+u) ds_n \right\}^2 du.$$

Therefore, we have from (1.6) that

$$\sum_{k=1}^{n-1} U_T^k(t) \le V_T(t) \le \sum_{k=1}^{n} U_T^k(t) + W_T^n(t) \qquad (n = 2, 3, \cdots). \tag{3.17}$$

As in the proof of Proposition 2.3, we see that $W_T^n(t)$ is equal to

$$\int_0^\infty \left\{ \int_0^\infty dv_n C(v_n+u) \cdots \int_0^\infty dv_1 C(v_1) \int_0^\infty ds_{n-1} A(t+s_{n-1}+v_n) \int_0^\infty ds_{n-2} A(t+s_{n-1}+s_{n-2}+v_{n-1}) \cdots \int_0^\infty ds_2 A(t+s_3+s_2+v_3) \int_0^\infty A(t+s_2+s_1+v_2) K_T(t+s_1+v_1) ds_1 \right\}^2 du.$$

We prove, for $T > 0$ and $n \ge 3$,

$$W_T^n(t) \sim t^{-1} b_n(d) \sin^{2n}(\pi d) J(T)^2 \qquad (t \to \infty), \tag{3.18}$$

where we write $b_n(d)$ for the integral

$$\left(\frac{d}{\pi}\right)^{2n} \int_0^\infty du \left\{ \int_0^\infty \frac{dv_n}{(v_n+u)^{1-d}} \int_0^\infty \frac{dv_{n-1}}{(v_{n-1})^{1-d}} \cdots \int_0^\infty \frac{dv_1}{(v_1)^{1-d}} \int_0^\infty \frac{ds_{n-1}}{(v_n+s_{n-1}+1)^{1+d}} \int_0^\infty \frac{ds_{n-2}}{(v_{n-1}+s_{n-1}+s_{n-2}+1)^{1+d}} \cdots \int_0^\infty \frac{ds_2}{(v_3+s_3+s_2+1)^{1+d}} \int_0^\infty \frac{ds_1}{(v_2+s_2+s_1+1)^{1+d}(v_1+s_1+1)^{1+d}} \right\}^2.$$

Choose δ from the interval $(0, \min\{(1-2d)/2, d/(4n-3)\})$, and take $M > 0$ so large that both (3.13) and (3.14) hold. We define $C_0(\cdot)$ and $C_1(\cdot)$ by (3.15), and $\varphi_0(\cdot)$ and $\varphi_1(\cdot)$ by

$$\varphi_0(t) := I_{(0,M)}(t), \quad \varphi_1(t) := I_{[M,\infty)}(t) \qquad (t > 0).$$

For $(i_1, \cdots, i_n) \in \{0,1\}^n$, we define $N_T(t,u;i_1,\cdots,i_n)$ by

$$\begin{aligned} N_T(t,u;i_1,\cdots,i_n) := \int_0^\infty dv_n \varphi_{i_n}(tv_n) \frac{C(t(v_n+u))}{C(t)} \int_0^\infty dv_{n-1} \frac{C_{i_{n-1}}(tv_{n-1})}{C(t)} \\ \cdots \int_0^\infty dv_1 \frac{C_{i_1}(tv_1)}{C(t)} \int_0^\infty ds_{k-1} \frac{A(t(1+s_{k-1}+v_k))}{A(t)} \\ \int_0^\infty ds_{k-2} \frac{A(t(1+s_{k-1}+s_{k-2}+v_{k-1}))}{A(t)} \cdots \int_0^\infty ds_2 \frac{A(t(1+s_3+s_2+v_3))}{A(t)} \\ \int_0^\infty \frac{A(t(1+s_2+s_1+v_2))}{A(t)} \cdot \frac{K_T(t(1+s_1+v_1))}{A(t)} ds_1. \end{aligned}$$

Then, it follows that

$$W_T^n(t) = t^{4n-1} \{A(t)C(t)\}^{2n} \int_0^\infty \left\{ \sum_i N_T(t,u;i_1,\cdots,i_n) \right\}^2 du,$$

where $\sum_i = \sum_{(i_1,\cdot,i_n)\in\{0,1\}^n}$ as above. As in the proof of [6, Proposition 6.2], we have

$$\lim_{t\to\infty} \int_0^\infty N_T(t,u;i_1,\cdots,i_n)^2 du = \begin{cases} \dfrac{\pi^{2n} b_n(d) J(T)^2}{d^{2n}} & (i_1 = 1, \cdots, i_n = 1), \\ 0 & ((i_1,\cdots,i_n) \neq (1,\cdots,1)) \end{cases}$$

(use (3.4)), hence

$$\lim_{t\to\infty} \int_0^\infty \left\{ \sum_i N_T(t,u;i_1,\cdots,i_n) \right\}^2 du = \frac{\pi^{2n} b_n(d) J(T)^2}{d^{2n}}.$$

This and (3.16) yield (3.18).

Step 4. It follows from (3.12), (3.17) and (3.18) that

$$\begin{aligned} \sum_{k=1}^{n-1} a_k \sin^{2k}(\pi d) \le J(T)^{-2} \liminf_{t\to\infty} V_T(t)t \le J(T)^{-2} \limsup_{n\to\infty} V_T(t)t \\ \le \sum_{k=1}^{n} a_k \sin^{2k}(\pi d) + b_n(d) \sin^{2n}(\pi d). \end{aligned}$$

However, we have $b_n(d) \le \pi^{-2}\tan^2(\pi d)$ for $n \ge 3$ (see the proof of [6, Theorem 6.4]), hence, letting $n \to \infty$, we obtain

$$\lim_{t\to\infty} V_T(t)t = J(T)^2 \sum_{k=1}^{\infty} a_k \sin^{2k}(\pi d).$$

Since $\sum_1^\infty a_k x^{2k} = \pi^{-2} \arcsin^2 x$ for $|x| < 1$, this yields the desired result. $\square$

4. Examples

Let (X_t) be a stationary process with autocovariance function $R(\cdot)$ and canonical representation kernel $C(\cdot)$. Let $T > 0$.

Example 1. If $R(t) = (1 + |t|)^{-p}$ for $0 < p < \infty$, then (X_t) has (RP). For,

$$\frac{1}{(1+|t|)^p} = \int_0^\infty e^{-|t|\lambda} \frac{e^{-\lambda}\lambda^{p-1}}{\Gamma(p)} d\lambda \qquad (t \in \mathbb{R}).$$

In this case, $C(\cdot)$ seems to have no simple representation. By Theorem 1.1, we have the following results (we omit the case (i) $0 < p < 1$):

(ii) if $p = 1$, then

$$V_T(t) \sim \frac{1}{4t(\log t)^2} \cdot \left\{ \int_0^T C(s)ds \right\}^2 \qquad (t \to \infty);$$

(iii) if $1 < p < \infty$, then

$$V_T(t) \sim t^{-(2p-1)} \cdot \frac{4}{(2p-1)(p-1)^2} \left\{ \int_0^T C(s)ds \right\}^2 \qquad (t \to \infty).$$

Example 2. Let $1/2 < q < \infty$. If $C(t) = (1 + t)^{-q}$ $(t > 0)$, then (X_t) has (RP) (see [4, Theorem 2.6]).

(i) If $1/2 < q < 1$, then

$$R(t) \sim t^{-(2q-1)} \int_0^\infty \frac{1}{s^q(1+s)^q} ds \qquad (t \to \infty)$$

(cf. [5, Theorem 4.1]). Hence, by Theorem 1.1(i), we have

$$V_T(t) \sim t^{-1} \cdot \{(1+T)^{1-q} - 1\}^2 \qquad (t \to \infty).$$

(ii) If $q = 1$, then $R(t) \sim t^{-1} \log t$ as $t \to \infty$ by [5, Theorem 5.2]. Hence, by Theorem 1.1(ii), we have

$$V_T(t) \sim \frac{\{\log(1+T)\}^2}{t(\log t)^2} \qquad (t \to \infty).$$

(iii) If $q > 1$, then $R(t) \sim t^{-q} \int_0^\infty C(s)ds$ as $t \to \infty$ (cf. [3, Lemma 3.8]). Therefore, by Theorem 1.1(iii) and (3.1), we have

$$V_T(t) \sim t^{-(2q-1)} \cdot \frac{\{1 - (1+T)^{1-q}\}^2}{2q - 1} \qquad (t \to \infty).$$

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Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo 060-0810, Japan

E-mail address: `inoue@math.sci.hokudai.ac.jp, y-kasa@math.sci.hokudai.ac.jp`