

# Note on martingales

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# 1 Conditional expectation

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space. We write  $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ .

In this note a random variable is an  $\mathcal{F}$ -measurable function  $X : \Omega \rightarrow \overline{\mathbb{R}}$  such that  $P(X = -\infty) = 0$ . Given two random variables  $X$  and  $Y$  we write

$$(X + Y)(\omega) := \begin{cases} X(\omega) + Y(\omega) & X(\omega) > -\infty, Y(\omega) > -\infty \\ -\infty & \text{otherwise} \end{cases}$$

Note that  $X + Y$  remains to be a random variable.

**1.1 Definition.** An  $\mathcal{F}$ -measurable function  $X : \Omega \rightarrow \overline{\mathbb{R}}$  is said to be lower semi-integrable if  $E[\max\{-X, 0\}] < +\infty$ . (If so then  $P(X = -\infty) = 0$ .)

*1.2 Remark.* If  $X$  and  $Y$  are lower semi-integrable random variables then so is  $X + Y$ .

**1.3 Lemma.** Suppose that  $\mathcal{G}$  is a sub  $\sigma$ -field of  $\mathcal{F}$ ,  $Y$  and  $Z$  are  $\mathcal{G}$ -measurable and lower semi-integrable random variables. If  $E[Y; A] \leq E[Z; A]$  for all  $A \in \mathcal{G}$  then  $Y \leq Z$  a.s.

*Proof.* We exploit the relation  $\{Z < Y\} = \bigcup_{n=1}^{\infty} \{Z \leq n, Z + 1/n \leq Y\}$ . Let  $n \in \mathbb{N}$  and set  $A := \{Z \leq n, Z + 1/n \leq Y\}$ . Then we have that  $A \in \mathcal{G}$ ,  $E[|Z|; A] < +\infty$  and  $E[Z + 1/n; A] \leq E[Y; A]$ . It therefore follows that

$$E[Z; A] + P(A)/n \leq E[Y; A] \leq E[Z; A].$$

This implies that  $P(A) = 0$ . Thus we get the claim. □

**1.4 Definition.** Given a lower semi-integrable random variable  $X$  and a sub  $\sigma$ -field  $\mathcal{G}$  of  $\mathcal{F}$ ,

$$\begin{aligned} E_{\leq}[X|\mathcal{G}] &:= \{Y : \mathcal{G}\text{-measurable, lower semi-integrable, } E[Y; A] \leq E[X; A] \forall A \in \mathcal{G}\}, \\ E[X|\mathcal{G}] &:= \{Y : \mathcal{G}\text{-measurable, lower semi-integrable, } E[Y; A] = E[X; A] \forall A \in \mathcal{G}\} \end{aligned}$$

We write  $E_{\geq}[X|\mathcal{G}] := -E_{\leq}[-X|\mathcal{G}]$  provided  $X$  is upper semi-integrable.

Suppose  $\xi$  is a signed measure on a measurable space  $(S, \mathcal{B})$  and  $\mu$  is a measure on  $(S, \mathcal{B})$ . The *Radon Nikodym* theorem asserts that

If  $\mu$  is  $\sigma$ -finite and  $\xi$  is  $\mu$ -absolutely continuous then  $D(\xi/\mu) \neq \emptyset$  where

$$D(\xi/\mu) := \{f : \mathcal{B}\text{-measurable, } \int_S \max\{-f, 0\} \mu < +\infty, \int_A f \mu = \xi(A) \forall A \in \mathcal{B}\}.$$

A lower semi-integrable random variable  $X$  defines a signed measure  $\mathcal{F} \rightarrow \overline{\mathbb{R}}$ ,  $A \mapsto E[X; A]$ , which we denote by  $XP$ . Let  $\mathcal{G}$  be a sub  $\sigma$ -field of  $\mathcal{F}$ . Then

$$E[X|\mathcal{G}] = D((XP)/P|\mathcal{G})$$

where we write  $D(\xi/\mu|\mathcal{C}) := D(\xi|_{\mathcal{C}}/\mu|_{\mathcal{C}})$  for a sub  $\sigma$ -field  $\mathcal{C}$  of  $\mathcal{B}$ .

In the rest of this section  $\mathcal{G}$  is a sub  $\sigma$ -field of  $\mathcal{F}$ .

**1.5 Theorem.** *Suppose  $X$  is a lower semi-integrable random variable. Then  $E[X|\mathcal{G}] \neq \emptyset$  and  $E_{\leq}[X|\mathcal{G}] = \{Y : \mathcal{G}\text{-measurable, lower semi-integrable and } Y \leq Z \text{ a.s.}\}$  for all  $Z \in E[X|\mathcal{G}]$ .*

**1.6 Lemma.** *Suppose that  $X$  is an integrable random variable. Then  $E[-X|\mathcal{G}] = -E[X|\mathcal{G}]$ , all  $Y \in E_{\leq}[X|\mathcal{G}]$  are integrable, and  $E[X|\mathcal{G}] = \{Y \in E_{\leq}[X|\mathcal{G}] : E[Y] = E[X]\}$ .*

**1.7 Lemma.** *Let  $X_1$  and  $X_2$  be lower semi-integrable random variables.*

(i) *If  $a \in \mathbb{R}_{>0}$  then  $aE[X_1|\mathcal{G}] = E[aX_1|\mathcal{G}]$  and  $aE_{\leq}[X_1|\mathcal{G}] = E_{\leq}[aX_1|\mathcal{G}]$ .*

(ii)  *$E[X_1|\mathcal{G}] + E[X_2|\mathcal{G}] \subset E[X_1 + X_2|\mathcal{G}]$  and  $E_{\leq}[X_1|\mathcal{G}] + E_{\leq}[X_2|\mathcal{G}] \subset E_{\leq}[X_1 + X_2|\mathcal{G}]$ . If either  $X_1$  or  $X_2$  is integrable then both of the inclusions  $\subset$  are replaced by the equality.*

*Proof.* (ii) Let  $Y_1 \in E[X_1|\mathcal{G}]$ ,  $Y_2 \in E[X_2|\mathcal{G}]$ ,  $Z_1 \in E[X_1|\mathcal{G}]$  and  $Z_2 \in E[X_2|\mathcal{G}]$ . We have that  $Y_1 + Y_2 \in E[X_1 + X_2|\mathcal{G}]$ . Indeed  $Y_1 + Y_2$  is lower semi-integrable and  $\mathcal{G}$ -measurable, and

$$E[Y_1 + Y_2; A] = E[Y_1; A] + E[Y_2; A] = E[X_1; A] + E[X_2; A] = E[X_1 + X_2; A] \forall A \in \mathcal{G}.$$

The second equality changes to  $\leq$  when  $Y_i$  are replaced by  $Z_i$ . Suppose that  $X_2$  is integrable and  $Z \in E_{\leq}[X_1 + X_2|\mathcal{G}]$ . Then, since  $Y_2$  is integrable and  $Z \leq Y_1 + Y_2$  a.s., it follows that  $Z - Y_2$  is lower semi-integrable and  $Z - Y_2 \leq Y_1$  a.s. This implies that  $Z - Y_2 \in E_{\leq}[X_1|\mathcal{G}]$ .  $\square$

**1.8 Lemma.** *Suppose  $X_1, X_2$  are lower semi-integrable random variables and  $X_1 \leq X_2$  a.s. Then  $E_{\leq}[X_1|\mathcal{G}] \subset E_{\leq}[X_2|\mathcal{G}]$ . If  $Y_1 \in E_{\leq}[X_1|\mathcal{G}]$  and  $Y_2 \in E[X_2|\mathcal{G}]$  then  $Y_1 \leq Y_2$  a.s.*

*Proof.* We have that  $E[Y_1; A] \leq E[X_1; A] \leq E[X_2; A] = E[Y_2; A]$  for all  $A \in \mathcal{G}$ .  $\square$

**1.9 Corollary.** *If  $X \geq 0$  a.s. then  $Y \geq 0$  a.s. for all  $Y \in E[X|\mathcal{G}]$ .*

**1.10 Theorem.** *Suppose that  $X_1$  is lower semi-integrable and  $X_n \leq X_{n+1}$  a.s. for all  $n \in \mathbb{N}$ .*

(i) *If  $Y_n \in E[X_n|\mathcal{G}]$  for all  $n \in \mathbb{N}$  then  $\sup_{n \in \mathbb{N}} Y_n \in E[\sup_{n \in \mathbb{N}} X_n|\mathcal{G}]$ .*

(ii) *If  $Z_n \in E_{\leq}[X_n|\mathcal{G}]$  for all  $n \in \mathbb{N}$  then  $\sup_{n \in \mathbb{N}} Z_n \in E_{\leq}[\sup_{n \in \mathbb{N}} X_n|\mathcal{G}]$ .*

*Proof.* Observe that  $\sup_{n \in \mathbb{N}} Y_n$  is  $\mathcal{G}$ -measurable. According to Lemma 1.8  $X_n \leq X_{n+1}$  a.s. implies that  $Y_n \leq Y_{n+1}$  a.s. On the other hand  $\sup_{n \in \mathbb{N}} Z_n \leq \sup_{n \in \mathbb{N}} Y_n$  a.s. because  $Z_n \leq Y_n$  a.s. Since  $E[X_1] = E[Y_1] > -\infty$ , invoking the monotone convergence theorem, we infer that

$$E[\sup_{n \in \mathbb{N}} Z_n; A] \leq E[\sup_{n \in \mathbb{N}} Y_n; A] = \sup_{n \in \mathbb{N}} E[Y_n; A] = \sup_{n \in \mathbb{N}} E[X_n; A] = E[\sup_{n \in \mathbb{N}} X_n; A] \forall A \in \mathcal{G}.$$

Consequently  $\sup_{n \in \mathbb{N}} Y_n \in E[\sup_{n \in \mathbb{N}} X_n|\mathcal{G}]$  and  $\sup_{n \in \mathbb{N}} Z_n \in E_{\leq}[\sup_{n \in \mathbb{N}} X_n|\mathcal{G}]$ .  $\square$

**1.11 Corollary.** *Suppose  $\inf_{n \in \mathbb{N}} X_n$  is lower semi-integrable and  $Y_n \in E[X_n|\mathcal{G}]$  for  $n \in \mathbb{N}$ .*

(i)  *$Z \leq \liminf_{n \rightarrow \infty} Y_n$  a.s. for all  $Z \in E[\liminf_{n \rightarrow \infty} X_n|\mathcal{G}]$ .*

(ii) *If  $\liminf_{n \rightarrow \infty} E[X_n] < +\infty$  then  $\liminf_{n \rightarrow \infty} X_n$  is integrable,  $\liminf_{n \rightarrow \infty} Y_n$  is integrable and  $\liminf_{n \rightarrow \infty} Y_n \in E_{\geq}[\liminf_{n \rightarrow \infty} X_n|\mathcal{G}]$ .*

*Proof.* (i) Let  $Z_n \in E[\inf_{k \geq n} X_k|\mathcal{G}]$ . Then we have that  $\sup_{n \in \mathbb{N}} Z_n \in E[\liminf_{n \rightarrow \infty} X_n|\mathcal{G}]$  by Theorem 1.10(i). On the other hand, since  $E_{\leq}[\inf_{k \geq n} X_k|\mathcal{G}] \subset E_{\leq}[X_k|\mathcal{G}]$  for all  $k \geq n$  by Lemma 1.8,  $Z_n \leq Y_k$  a.s. for all  $k \geq n$ . This implies that  $\sup_{n \in \mathbb{N}} Z_n \leq \liminf_{n \rightarrow \infty} Y_n$  a.s.

(ii) We see that  $\liminf_{n \rightarrow \infty} X_n$  is integrable since  $E[\liminf_{n \rightarrow \infty} X_n] \leq \liminf_{n \rightarrow \infty} E[X_n]$  by Fatou's lemma. It then follows that  $\inf_{n \in \mathbb{N}} X_n$  is integrable and so is  $Z_1$ . We have that  $Z_1 \leq Y_k$  a.s. for all  $k \geq 1$ . We get  $E[\liminf_{n \rightarrow \infty} Y_n] \leq \liminf_{n \rightarrow \infty} E[Y_n]$  by invoking Fatou's lemma. The right hand side equals  $\liminf_{n \rightarrow \infty} E[X_n] < +\infty$ .  $\square$

**1.12 Lemma.** (i) Suppose that  $f : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$  is convex. Then  $f(x) = \sup_{(a,b) \in F} (ax + b)$  for all  $x \in \mathbb{R}$  where  $F$  is a countable dense subset of  $\{(a, b) \in \mathbb{R}^2 : ax + b \leq f(x) \forall x \in \mathbb{R}\}$ .  
(ii) If  $f : \mathbb{R} \cup \{+\infty\} \rightarrow \mathbb{R} \cup \{+\infty\}$  is non-decreasing and convex then  $f(x) = \sup_{(a,b) \in F} (ax + b)$  for all  $x \in \mathbb{R}$  where  $F$  is a countable dense subset of  $\{(a, b) \in \mathbb{R}_{\geq 0} \times \mathbb{R} : ax + b \leq f(x) \forall x \in \mathbb{R}\}$ .

**1.13 Theorem.** (i) Suppose  $X$  is integrable and  $f : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$  is a convex function. Then  $f(X)$  is lower semi-integrable and  $f(Y) \in E_{\leq}[f(X)|\mathcal{G}]$  for all  $Y \in E[X|\mathcal{G}]$ .  
(ii) Suppose  $X$  is lower semi-integrable and  $f : \mathbb{R} \cup \{+\infty\} \rightarrow \mathbb{R} \cup \{+\infty\}$  is non-decreasing and convex. Then  $f(X)$  is lower semi-integrable and  $f(Y) \in E_{\leq}[f(X)|\mathcal{G}]$  for all  $Y \in E[X|\mathcal{G}]$ .

*Proof.* We choose a countable dense subset  $F$  of  $\{(a, b) \in \mathbb{R}^2 : ax + b \leq f(x) \forall x \in \mathbb{R}\}$ . Let  $(a, b) \in F$ . Then, since  $aX + b \leq f(X)$ , it follows that  $E_{\leq}[aX + b|\mathcal{G}] \subset E_{\leq}[f(X)|\mathcal{G}]$  by Lemma 1.8. On the other hand  $aY + b \in E[aX + b|\mathcal{G}]$  by Lemma 1.7 and Lemma 1.6. Consequently  $aY + b \in E_{\leq}[f(X)|\mathcal{G}]$  for all  $(a, b) \in F$ . The set  $F$  being countable, we infer that  $\sup_{(a,b) \in F} (aY + b) \in E_{\leq}[f(X)|\mathcal{G}]$  by Theorem 1.10(ii). Thus Lemma 1.12 claims that  $f(Y) \in E_{\leq}[f(X)|\mathcal{G}]$ .  $\square$

**1.14 Lemma.** Suppose that  $\mu$  is a measure on  $(\Omega, \mathcal{F})$  which is  $P$ -absolutely continuous on  $\mathcal{G}$  and  $\rho \in D(\mu/P|\mathcal{G})$ . Let  $A \in \mathcal{G}$  and  $f : \Omega \rightarrow \overline{\mathbb{R}}$  be a  $\mathcal{G}$ -measurable function.

- (i)  $\mu(A) = 0$  if and only if  $P(A \cap \{\rho > 0\}) = 0$ .
- (ii)  $f \geq 0$   $\mu$ -a.e. if and only if  $f\rho \geq 0$   $P$ -a.s. If so then  $\int_A f \mu = E[f\rho; A]$ .
- (iii)  $f$  is  $\mu$ -integrable if and only if  $f\rho$  is  $P$ -integrable. If so then  $\int_A f \mu = E[f\rho; A]$ .

**1.15 Theorem.** (i) Suppose that  $X \geq 0$  a.s. and  $E[X; Z < 0] = 0$ . Then  $XZ \geq 0$   $P$ -a.s. If  $Y \in E[X|\mathcal{G}]$  and  $Z$  is  $\mathcal{G}$ -measurable then  $YZ \geq 0$   $P$ -a.s. and  $YZ \in E[XZ|\mathcal{G}]$ .  
(ii) Suppose that  $X$  is integrable and  $XZ$  is integrable. If  $Y \in E[X|\mathcal{G}]$  and  $Z$  is  $\mathcal{G}$ -measurable then  $YZ$  is integrable and  $YZ \in E[XZ|\mathcal{G}]$ .

**1.16 Lemma.** Let  $X$  be a lower semi-integrable random variable and  $\mathcal{N} := \text{Null}(P)$ . Then  $E[X|\mathcal{G}] = E[X|\mathcal{G} \vee \mathcal{N}] \cap \{\mathcal{G}\text{-measurable}\}$  and  $E_{\leq}[X|\mathcal{G}] = E_{\leq}[X|\mathcal{G} \vee \mathcal{N}] \cap \{\mathcal{G}\text{-measurable}\}$ .

**1.17 Theorem.** Suppose that  $X$  is lower semi-integrable and  $\mathcal{H}$  is a sub  $\sigma$ -field of  $\mathcal{G}$ .

- (i) If  $Y \in E_{\leq}[X|\mathcal{G}]$  and  $Z \in E_{\leq}[Y|\mathcal{H}]$  then  $Z \in E_{\leq}[X|\mathcal{H}]$ .
- (ii) If  $Y \in E[X|\mathcal{G}]$  and  $Z \in E[Y|\mathcal{H}]$  then  $Z \in E[X|\mathcal{H}]$ .

*Proof.* Let  $A \in \mathcal{H}$ . Then, since  $A \in \mathcal{G}$  as well, we have that  $E[X; A] \geq E[Y; A] \geq E[Z; A]$ . This means  $Z \in E_{\leq}[X|\mathcal{H}]$  because  $Z$  is  $\mathcal{H}$ -measurable.  $\square$

**1.18 Lemma.** Suppose that  $X$  is a non-negative random variable,  $A \in \mathcal{F}$  and  $c \in \mathbb{R}_{\geq 0}$ . Then  $E[X; A] \leq E[X; X > c] + cP(A)$ .

**1.19 Corollary.** (i) Let  $\Lambda$  be a family of random variables. Then the following holds:

$$\inf_{c>0} \sup_{X \in \Lambda} E[|X|; |X| > c] = 0 \Leftrightarrow \sup_{X \in \Lambda} E[|X|] < +\infty \text{ and } \inf_{\delta>0} \sup_{X \in \Lambda} \sup_{A \in \mathcal{F}: P(A) < \delta} E[|X|; A] = 0.$$

- (ii) If  $X$  is integrable then  $\inf_{\delta>0} \sup_{A \in \mathcal{F}: P(A) < \delta} E[|X|; A] = 0$ .

*Proof.* (i)  $\Rightarrow$  Lemma 1.18 shows that  $\sup_{X \in \Lambda} E[|X|] \leq \sup_{X \in \Lambda} E[|X|; |X| > c] + c$  and

$$\inf_{\delta > 0} \sup_{X \in \Lambda} \sup_{A \in \mathcal{F}: P(A) < \delta} E[|X|; A] \leq \sup_{X \in \Lambda} E[|X|; |X| > c] \text{ for all } c > 0.$$

$\Leftarrow$  Since  $P(|X| > c) \leq \sup_{X \in \Lambda} E[|X|]/c$  for  $X \in \Lambda$  and  $c > 0$ , it follows that

$$\sup_{X \in \Lambda} E[|X|; |X| > c] \leq \sup_{X \in \Lambda} \sup_{A \in \mathcal{F}: P(A) \leq M/c} E[|X|; A] \text{ for all } c \in \mathbb{R}_{>0}$$

where  $M := \sup_{X \in \Lambda} E[|X|] < +\infty$ . The infimum of the right hand side over  $c > 0$  is 0.

(ii) We see that  $E[|X|; |X| > c]$  converges to  $E[|X|; |X| = +\infty] = 0$  as  $c$  tends to  $+\infty$  by the monotone convergence theorem.  $\square$

*1.20 Remark.* Let  $P$  and  $Q$  be mutually absolutely continuous probability measures on  $(\Omega, \mathcal{F})$ . We infer by Corollary 1.19(ii) that the convergence in probability with respect to  $P$  and with respect to  $Q$  are equivalent.

**1.21 Definition.** Let  $\Lambda$  be a family of random variables. It is said to be uniformly integrable if  $\inf_{c > 0} \sup_{X \in \Lambda} E[|X|; |X| > c] = 0$ .

**1.22 Example.** Let  $\Lambda$  be a family of random variables. If there exists an integrable dominating random variable  $X$ , that is  $P(|Y| \leq |X|) = 1$  for all  $Y \in \Lambda$ , then  $\Lambda$  is uniformly integrable. Indeed  $E[|Y|; |Y| > c] \leq E[|X|; |X| > c]$  for all  $Y \in \Lambda$ .

**1.23 Theorem.** Let  $\Lambda$  be a uniformly integrable family of random variables and  $\Theta$  be the collection of all sub  $\sigma$ -fields of  $\mathcal{F}$ . Then the family  $\bigcup_{X \in \Lambda, \mathcal{G} \in \Theta} E[X|\mathcal{G}]$  is uniformly integrable.

*Proof.* Let  $X \in \Lambda$ ,  $\mathcal{G} \in \Theta$  and  $Y \in E[X|\mathcal{G}]$ . Since  $|Y| \in E_{\leq}[|X||\mathcal{G}]$  by Theorem 1.13,

$$P(|Y| > c) \leq E[|Y|]/c \leq E[|X|]/c \text{ and } E[|Y|; |Y| > c] \leq E[|X|; |Y| > c] \text{ for all } c > 0.$$

Here we used that  $\{|Y| > c\} \in \mathcal{G}$ . Invoking Lemma 1.18, we see that

$$E[|X|; |Y| > c] \leq E[|X|; |X| > \sqrt{c}] + \sqrt{c}P(|Y| > c).$$

The second term in the right hand side is dominated by  $E[|X|]/\sqrt{c}$ . It therefore follows that

$$E[|Y|; |Y| > c] \leq E[|X|; |Y| > c] \leq E[|X|; |X| > \sqrt{c}] + E[|X|]/\sqrt{c}.$$

The infimum of the right hand side over  $c \in \mathbb{R}_{>0}$  is 0.  $\square$

**1.24 Lemma.** Suppose that  $X$  is a random variable and  $\Lambda$  is a family of random variables such that  $P(|Y| = \infty) = 0 \forall Y \in \Lambda$  and  $\inf_{Y \in \Lambda} P(|X - Y| > \varepsilon) = 0$  for all  $\varepsilon \in \mathbb{R}_{>0}$ . Then

$$E[|X|; |X| > c] \leq \liminf_{\delta \downarrow 0} \inf_{Y \in \Lambda: P(|X - Y| > \delta) < \delta} E[|Y|; |Y| > c] \text{ for all } c \in \mathbb{R}_{\geq 0}.$$

*Proof.* Let  $c \in \mathbb{R}_{\geq 0}$ ,  $\delta \in \mathbb{R}_{>0}$ ,  $M \in \mathbb{R}_{>0}$  and  $Y \in \Lambda$ . It follows that

$$E[|X|; |X| > c + \delta, |X| \leq |Y| + \delta] \leq E[|Y| + \delta; |Y| > c] \leq E[|Y|; |Y| > c] + \delta$$

since  $\{|X| > c + \delta, |X| \leq |Y| + \delta\} \subset \{|Y| > c, |X| \leq |Y| + \delta\}$ . On the other hand we have that  $\{|X| > c + \delta, |X| > |Y| + \delta\} \subset \{|X| > |Y| + \delta\} \subset \{|X - Y| > \delta\}$ . Consequently

$$E[\min\{|X|, M\}; |X| > c + \delta] \leq E[|Y|; |Y| > c] + \delta + MP(|X - Y| > \delta).$$

Observe that  $\{Y \in \Lambda : P(|X - Y| > \delta) < \delta\} \neq \emptyset$  for all  $\delta \in \mathbb{R}_{>0}$ . We thus get

$$E[\min\{|X|, M\}; |X| > c + \delta] \leq \inf_{Y \in \Lambda: P(|X-Y|>\delta)<\delta} E[|Y|; |Y| > c] + (1 + M)\delta.$$

Tending  $\delta$  to 0 we see by the monotone convergence theorem that

$$E[\min\{|X|, M\}; |X| > c] \leq \liminf_{\delta \downarrow 0} \inf_{Y \in \Lambda: P(|X-Y|>\delta)<\delta} E[|Y|; |Y| > c].$$

The number  $M$  being arbitrary the monotone convergence theorem yields the claim.  $\square$

**1.25 Corollary.** *Suppose that  $\Lambda$  is a family of random variables. If there exists a uniformly integrable family of random variables  $\Xi$  such that  $\inf_{Y \in \Xi} P(|X - Y| > \varepsilon) = 0$  for all  $X \in \Lambda$  and  $\varepsilon \in \mathbb{R}_{>0}$  then  $\Lambda$  is uniformly integrable.*

*Proof.* Lemma 1.24 shows  $E[|X|; |X| > c] \leq \sup_{Y \in \Xi} E[|Y|; |Y| > c]$  for  $X \in \Lambda, c \in \mathbb{R}_{\geq 0}$ .  $\square$

**1.26 Lemma.** *Let  $\Lambda_1$  and  $\Lambda_2$  be families of random variables. If both  $\Lambda_1$  and  $\Lambda_2$  are uniformly integrable then so is  $\{X + Y; X \in \Lambda_1, Y \in \Lambda_2\}$ .*

*Proof.* Let  $X \in \Lambda_1$  and  $Y \in \Lambda_2$ . Since  $|X + Y| \leq 2 \max\{|X|, |Y|\}$ , we have that

$$E[|X + Y|; |X + Y| > c] \leq E[2 \max\{|X|, |Y|\}; 2 \max\{|X|, |Y|\} > c].$$

The right hand side reads  $E[2|X|; |X| \geq |Y|, 2|X| > c] + E[2|Y|; |X| < |Y|, 2|Y| > c]$ , which is dominated by  $2 \sup_{X \in \Lambda_1} E[|X|; |X| > c/2] + 2 \sup_{Y \in \Lambda_2} E[|Y|; |Y| > c/2]$ .  $\square$

**1.27 Corollary.** *Suppose that  $X_n$  is a sequence of uniformly integrable random variables. If  $X_n$  converges to a random variable  $X$  in probability then  $E[|X_n - X|]$  converges to 0.*

*Proof.* Since  $E[|X|; |X| > c] \leq \liminf_{n \rightarrow \infty} E[|X_n|; |X_n| > c]$  for all  $c \in \mathbb{R}_{\geq 0}$  by Lemma 1.24, we see that the sequence  $X_n - X$  is also uniformly integrable by Lemma 1.26. Suppose that  $c, \varepsilon \in \mathbb{R}_{>0}$  and  $\varepsilon < c$ . In general we have that

$$E[|Y|] \leq \varepsilon P(|Y| \leq \varepsilon) + cP(\varepsilon < |Y| \leq c) + E[|Y|; |Y| > c] \leq \varepsilon + cP(|Y| > \varepsilon) + E[|Y|; |Y| > c].$$

where  $Y$  is a random variable. It therefore follows that

$$E[|X_n - X|] \leq \varepsilon + cP(|X_n - X| > \varepsilon) + \sup_{k \in \mathbb{N}} E[|X_k - X|; |X_k - X| > c].$$

Tending  $n$  to  $\infty$  first and  $\varepsilon$  to 0 next we get

$$\limsup_{n \rightarrow \infty} E[|X_n - X|] \leq \sup_{k \in \mathbb{N}} E[|X_k - X|; |X_k - X| > c].$$

The number  $c$  being arbitrary, we get the claim due to the uniform integrability.  $\square$

## 2 Optional sampling

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space. In this section a stochastic process is simply a family of random variables with index set  $\mathbb{Z}_{\geq 0}$ . We shall discuss submartingales. To describe this notion we need to introduce a filtration.

$\mathcal{F}$  is a filtration of  $\mathcal{F}$  with index set  $\mathbb{Z}_{\geq 0}$ , i.e., it is family of sub  $\sigma$ -fields of  $\mathcal{F}$  with index set  $\mathbb{Z}_{\geq 0}$  such that  $\mathcal{F}_m \subset \mathcal{F}_n$  for all  $m, n \in \mathbb{Z}_{\geq 0}$  with  $m < n$ .

**2.1 Example.** Let  $\tau, \sigma$  be  $\mathbb{Z}_{\geq 0} \cup \{+\infty\}$ -valued random variables and  $X$  a stochastic process. We discuss the following random variable:

$$X_\tau := \begin{cases} X_n & \text{on } \{\tau = n\} \\ 0 & \text{on } \{\tau = +\infty\} \end{cases},$$

which coincides with  $X_0 + \sum_{n=0}^{\tau-1} (X_{n+1} - X_n)$  on  $\Lambda := \{\tau < +\infty, |X_n| < +\infty \forall n \in \mathbb{Z}_{\geq 0}\}$ . Suppose that  $E[|X_n|] < +\infty$  for all  $n \in \mathbb{Z}_{\geq 0}$  and there exists  $T \in \mathbb{Z}_{\geq 0}$  such that  $\tau \leq T$  a.s. Note that  $X_\tau$  is integrable. Suppose that  $\sigma \leq \tau$  a.s. Then we see that

$$X_\tau - X_\sigma \stackrel{\text{a.s.}}{=} \sum_{n:\sigma \leq n < \tau} (X_{n+1} - X_n) 1_\Lambda = \sum_{n=0}^{T-1} (X_{n+1} - X_n) 1_{\{\sigma \leq n\}} 1_{\{\tau > n\}} 1_\Lambda.$$

Let  $A \in \mathcal{F}$ . If  $E[X_n | \mathcal{F}_n] \subset E_{\leq}[X_{n+1} | \mathcal{F}_n]$  for all  $n \in \mathbb{Z}_{\geq 0}$ ,  $\{\tau > n\} \in \mathcal{F}_n \vee \text{Null}(P)$  for all  $n \in \mathbb{Z}_{\geq 0}$  and  $A \cap \{\sigma \leq n\} \in \mathcal{F}_n \vee \text{Null}(P)$  for all  $n \in \mathbb{Z}_{\geq 0}$  then  $E[X_\tau; A] \geq E[X_\sigma; A]$ .  $\square$

**2.2 Definition.** (i) A mapping  $\sigma : \Omega \rightarrow \mathbb{Z}_{\geq 0} \cup \{+\infty\}$  such that  $\{\sigma = n\} \in \mathcal{F}_n$  for all  $n \in \mathbb{Z}_{\geq 0}$  (equivalently  $\{\sigma \leq n\} \in \mathcal{F}_n$  for all  $n \in \mathbb{Z}_{\geq 0}$ ) is called an  $\mathcal{F}$ -stopping time.  
(ii) Given an  $\mathcal{F}$ -stopping time  $\sigma$ , set

$$\mathcal{F}_\sigma := \{A \in \mathcal{F} : A \cap \{\sigma = n\} \in \mathcal{F}_n \forall n \in \mathbb{Z}_{\geq 0}\} (= \{A \in \mathcal{F} : A \cap \{\sigma \leq n\} \in \mathcal{F}_n \forall n \in \mathbb{Z}_{\geq 0}\}).$$

We see that  $\mathcal{F}_\sigma$  is a sub  $\sigma$ -field of  $\mathcal{F}$  since  $(A^c \cap \{\sigma \leq n\})^c = (A \cap \{\sigma \leq n\}) \cup \{\sigma > n\}$ . As for a constant stopping time, say  $\sigma$ , we have that  $\mathcal{F}_\sigma = \mathcal{F}_m$  where  $\sigma(\Omega) = \{m\}$ .

**2.3 Definition.** (i) A stochastic process  $X$  such that  $X_n$  is integrable for all  $n \in \mathbb{Z}_{\geq 0}$  and  $X_n \in E_{\leq}[X_{n+1} | \mathcal{F}_n]$  for all  $n \in \mathbb{Z}_{\geq 0}$  is called an  $\mathcal{F}$ -submartingale.  
(ii) A stochastic process  $X$  such that  $-X$  is an  $\mathcal{F}$ -submartingale, i.e.,  $X_n$  is integrable for all  $n \in \mathbb{Z}_{\geq 0}$  and  $X_n \in E_{\geq}[X_{n+1} | \mathcal{F}_n]$  for all  $n \in \mathbb{Z}_{\geq 0}$ , is called an  $\mathcal{F}$ -supermartingale.  
(iii) A stochastic process  $X$  such that  $X_n$  is integrable for all  $n \in \mathbb{Z}_{\geq 0}$  and  $X_n \in E[X_{n+1} | \mathcal{F}_n]$  for all  $n \in \mathbb{Z}_{\geq 0}$  is called an  $\mathcal{F}$ -martingale.

**2.4 Definition.** A stochastic process  $X$  is said to be  $\mathcal{F}$ -adapted if the random variable  $X_n$  is  $\mathcal{F}_n$ -measurable for each  $n \in \mathbb{Z}_{\geq 0}$ .

*2.5 Remark.* All  $\mathcal{F}$ -submartingales are  $\mathcal{F}$ -adapted due to the condition  $X_n \in E_{\leq}[X_{n+1} | \mathcal{F}_n]$ .

**2.6 Theorem.** Suppose that  $X$  is an  $\mathcal{F}$ -submartingale and  $\sigma$  and  $\tau$  are  $\mathcal{F} \vee \text{Null}(P)$ -stopping times such that  $\sigma \leq \tau \leq T$  a.s. for some  $T \in \mathbb{R}_{>0}$ .

- (i)  $X_\sigma$  and  $X_\tau$  are integrable and  $X_\sigma \in E_{\leq}[X_\tau | (\mathcal{F} \vee \text{Null}(P))_\sigma]$ .
- (ii) If  $\sigma$  is an  $\mathcal{F}$ -stopping time then  $X_\sigma \in E_{\leq}[X_\tau | \mathcal{F}_\sigma]$ .

*Proof.* According to the discussion in Example 2.1, the random variables  $X_\sigma$  and  $X_\tau$  are integrable and  $E[X_\tau; A] \geq E[X_\sigma; A]$  for all  $A \in (\mathcal{F} \vee \text{Null}(P))_\sigma$ .  $\mathcal{F}_\sigma \subset (\mathcal{F} \vee \text{Null}(P))_\sigma$ . The  $\mathcal{F}_\sigma$ -measurability of  $X_\sigma$  for  $\mathcal{F}$ -stopping times  $\sigma$  shall be verified in Lemma 2.7.  $\square$

**2.7 Lemma.** *Let  $X$  be an  $\mathcal{F}$ -adapted stochastic process and  $\sigma$  be an  $\mathcal{F}$ -stopping time. Then  $X_\sigma$  is  $\mathcal{F}_\sigma$ -measurable. In particular  $\sigma$  is  $\mathcal{F}_\sigma$ -measurable.*

*Proof.* Let  $a \in \mathbb{R}$ . Then  $\{X_\sigma < a\} \cap \{\sigma = n\} = \{X_n < a\} \cap \{\sigma = n\} \in \mathcal{F}_n$  for all  $n \in \mathbb{Z}_{\geq 0}$ .  $\square$

**2.8 Lemma.** *If  $\sigma$  and  $\tau$  are  $\mathcal{F}$ -stopping times then so are  $\sigma \wedge \tau$ ,  $\sigma \vee \tau$  and  $\sigma + \tau$ .*

*Proof.*  $\{\sigma \wedge \tau \leq n\} = \{\sigma \leq n\} \cup \{\tau \leq n\}$ ,  $\{\sigma \vee \tau \leq n\} = \{\sigma \leq n\} \cap \{\tau \leq n\}$  and  $\{\sigma + \tau = n\} = \bigcup_{k=0}^n \{\sigma = k\} \cap \{\tau = n - k\}$ .  $\square$

**2.9 Remark.** Suppose that  $\sigma$  and  $\tau$  are  $\mathcal{F}$ -stopping times and  $\sigma \leq \tau$  a.s.

- (i) the  $\mathcal{F}$ -stopping times  $\sigma \wedge \tau$  and  $\sigma \vee \tau$  satisfy  $\sigma \wedge \tau = \sigma$  a.s. and  $\sigma \vee \tau = \tau$  a.s. respectively.
- (ii) Suppose in addition that there exists  $T \in \mathbb{Z}_{\geq 0}$  such that  $\tau \leq T$  a.s. Then the  $\mathcal{F}$ -stopping times  $\sigma \wedge \tau \wedge T$  and  $(\sigma \vee \tau) \wedge T$  satisfy  $\sigma \wedge \tau \wedge T = \sigma$  a.s. and  $(\sigma \vee \tau) \wedge T = \tau$  a.s. respectively.

**2.10 Lemma.** *Suppose that  $\sigma$  and  $\tau$  are  $\mathcal{F}$ -stopping times.*

- (i)  $\mathcal{F}_{\sigma \wedge \tau} = \mathcal{F}_\sigma \cap \mathcal{F}_\tau$ . If  $\sigma \leq \tau$  then  $\mathcal{F}_\sigma \subset \mathcal{F}_\tau$ .
- (ii)  $\{\tau \leq \sigma\}$ ,  $\{\sigma \leq \tau\}$ ,  $\{\tau < \sigma\}$ ,  $\{\sigma < \tau\}$  and  $\{\tau = \sigma\}$  belong to  $\mathcal{F}_{\sigma \wedge \tau}$
- (iii)  $A \cap \{\sigma \leq \tau\} \in \mathcal{F}_\tau$  and  $A \cap \{\sigma < \tau\} \in \mathcal{F}_\tau$  for all  $A \in \mathcal{F}_{\sigma \vee \tau}$ .  $\mathcal{F}_{\sigma \vee \tau} = \mathcal{F}_\sigma \vee \mathcal{F}_\tau$ .
- (iv)  $A \cap \{\sigma \leq \tau\} \in \mathcal{F}_{\sigma \wedge \tau}$  and  $A \cap \{\sigma < \tau\} \in \mathcal{F}_{\sigma \wedge \tau}$  for all  $A \in \mathcal{F}_\sigma$ .

*Proof.* Suppose  $B \in \mathcal{F}_{\sigma \wedge \tau}$ . Then it follows that  $B \in \mathcal{F}_\tau$ . Indeed

$$B \cap \{\tau \leq n\} = (B \cap \{\sigma \wedge \tau \leq n\}) \cap (\{\sigma > n\} \cup \{\sigma \leq n, \tau \leq n\}) \in \mathcal{F}_n \text{ for all } n \in \mathbb{Z}_{\geq 0}.$$

Similarly we get  $B \in \mathcal{F}_\sigma$ . Conversely if  $B \in \mathcal{F}_\sigma \cap \mathcal{F}_\tau$  then  $B \in \mathcal{F}_{\sigma \wedge \tau}$  since

$$B \cap \{\sigma \wedge \tau \leq n\} = (B \cap \{\sigma \leq n\}) \cup (B \cap \{\tau \leq n\}) \in \mathcal{F}_n \text{ for all } n \in \mathbb{Z}_{\geq 0}.$$

Let  $A \in \mathcal{F}_{\sigma \vee \tau}$ . We infer that  $A \cap \{\sigma \leq \tau\} \in \mathcal{F}_\tau$  by using

$$(A \cap \{\sigma \leq \tau\}) \cap \{\tau = n\} = (A \cap \{\sigma \vee \tau \leq n\}) \cap \{\tau = n\} \in \mathcal{F}_n \text{ for all } n \in \mathbb{Z}_{\geq 0}.$$

Since  $\sigma = \sigma \vee (\sigma \wedge \tau)$  and  $\{\sigma \leq \tau\} = \{\sigma \leq \sigma \wedge \tau\}$ , we get  $C \cap \{\sigma \leq \tau\} \in \mathcal{F}_{\sigma \wedge \tau}$  for all  $C \in \mathcal{F}_\sigma$ . It then follows that  $\{\sigma \leq \tau\} \in \mathcal{F}_{\sigma \wedge \tau}$  and  $\{\sigma < \tau\} = \{\tau \leq \sigma\}^c \in \mathcal{F}_{\sigma \wedge \tau}$ . We finally get  $A \cap \{\sigma < \tau\} = (A \cap \{\sigma \leq \tau\}) \cap \{\sigma < \tau\} \in \mathcal{F}_\tau$  because  $\mathcal{F}_{\sigma \wedge \tau} \subset \mathcal{F}_\tau$ .  $\square$

**2.11 Theorem.** *Suppose  $X$  is an  $\mathcal{F}$ -submartingale and  $\sigma(1)$ ,  $\sigma(2)$  and  $\tau$  are  $\mathcal{F} \vee \text{Null}(P)$ -stopping times such that  $\sigma(1) \leq \sigma(2)$  a.s. and  $\sigma(2) \wedge \tau \leq T$  a.s. for some  $T \in \mathbb{R}_{>0}$ .*

- (i)  $X_{\sigma(1) \wedge \tau} \in E_{\leq}[X_{\sigma(2) \wedge \tau} | (\mathcal{F} \vee \text{Null}(P))_{\sigma(1)}]$ .
- (ii) If  $\sigma(1)$  and  $\tau$  are  $\mathcal{F}$ -stopping times then  $X_{\sigma(1) \wedge \tau} \in E_{\leq}[X_{\sigma(2) \wedge \tau} | \mathcal{F}_{\sigma(1)}]$ .

*Proof.* We write  $\tilde{\mathcal{F}}_t := \mathcal{F}_t \vee \text{Null}(P)$ . Let  $A \in \tilde{\mathcal{F}}_{\sigma(1)}$ . Then  $A \cap \{\sigma(1) \leq \tau\} \in \tilde{\mathcal{F}}_{\sigma(1) \wedge \tau}$  by Lemma 2.10(iv). Therefore, invoking Theorem 2.6, we see that

$$E[X_{\sigma(1) \wedge \tau}; A \cap \{\sigma(1) \leq \tau\}] \leq E[X_{\sigma(2) \wedge \tau}; A \cap \{\sigma(1) \leq \tau\}].$$

Adding  $E[X_\tau; A \cap \{\tau < \sigma(1)\}]$  we get

$$E[X_{\sigma(1) \wedge \tau}; A] \leq E[X_{\sigma(2) \wedge \tau}; A].$$

On the other hand  $\tilde{\mathcal{F}}_{\sigma(1) \wedge \tau} \subset \tilde{\mathcal{F}}_{\sigma(1)}$  according to Lemma 2.10(i). Therefore  $X_{\sigma(1) \wedge \tau}$  is  $\tilde{\mathcal{F}}_{\sigma(1)}$ -measurable by Lemma 2.7. Consequently we get  $X_{\sigma(1) \wedge \tau} \in E_{\leq}[X_{\sigma(2) \wedge \tau} | \tilde{\mathcal{F}}_{\sigma(1)}]$ .  $\square$

**2.12 Corollary.** *If  $X$  is an  $\mathcal{F}$ -submartingale and  $\tau$  is an  $\mathcal{F}$ -stopping time then the stopped process  $X_{\cdot \wedge \tau}$  is an  $\mathcal{F}$ -submartingale.*

**2.13 Lemma.** *Let  $X$  be an  $\mathcal{F}$ -submartingale with index set  $\mathbb{Z}_{\leq 0}$ . If  $\inf E[X] > -\infty$  then  $X$  is uniformly integrable.*

*Proof.* Suppose that  $\lambda \in \mathbb{R}_{>0}$ ,  $k, m \in \mathbb{Z}_{\leq 0}$  and  $k < m$ . We see that

$$E[|X_k|; |X_k| \geq \lambda] = E[X_k; X_k \geq \lambda] + E[X_k; X_k > -\lambda] - E[X_k]$$

Due to the submartingale property, the first term and the second term in the right hand side are dominated by  $E[X_m; X_k \geq \lambda]$  and  $E[X_m; X_k > -\lambda]$  respectively. Therefore

$$\begin{aligned} E[|X_k|; |X_k| \geq \lambda] &\leq E[X_m; X_k \geq \lambda] - E[X_m; X_k \leq -\lambda] + E[X_m] - E[X_k] \\ &\leq E[|X_m|; |X_k| \geq \lambda] + E[X_m] - \inf E[X]. \end{aligned}$$

Since  $n \mapsto E[X_n]$  is non-decreasing, it follows that  $\limsup_{n \rightarrow -\infty} E[X_n] = \inf E[X] > -\infty$ . Given  $\varepsilon > 0$ . There exists  $m \in \mathbb{Z}_{\leq 0}$  such that

$$E[X_m] - \inf E[X] < \varepsilon.$$

By Corollary 1.19(i) we can choose  $\delta > 0$  such that

$$\text{if } A \in \mathcal{F} \text{ and } P(A) < \delta \text{ then } E[|X_k|; A] < \varepsilon \text{ for all } k \in \mathbb{Z}_{[m,0]}.$$

Using Theorem 1.13 we observe that

$$E[|X_k|] = 2E[\max\{X_k, 0\}] - E[X_k] \leq 2E[\max\{X_0, 0\}] - \inf E[X].$$

This implies that there exists  $\lambda \in \mathbb{R}_{>0}$  such that

$$P(|X_k| \geq \lambda) < \delta \text{ for all } k \in \mathbb{Z}_{\leq 0}.$$

It follows that

$$E[|X_k|; |X_k| \geq \lambda] < \varepsilon \text{ for all } k \in \mathbb{Z}_{[m,0]} \text{ and } E[|X_m|; |X_k| \geq \lambda] < \varepsilon \text{ for all } k \in \mathbb{Z}_{< m}.$$

Thus we get  $E[|X_k|; |X_k| \geq \lambda] < 2\varepsilon$  for all  $k \in \mathbb{Z}_{\leq 0}$ . □

We call  $\mathcal{N}$  an exceptional family if  $\emptyset \in \mathcal{N}$  and  $\{A : A \subset B\} \subset \mathcal{N}$  for all  $B \in \mathcal{N}$ . An exceptional family  $\mathcal{N}$  is said to be  $\sigma$ -consistent if  $\bigcup_n A(n) \in \mathcal{N}$  for all  $A \in \text{Seq}(\mathcal{N})$ .

Let  $\mathcal{N}$  be a  $\sigma$ -consistent exceptional family of  $\text{Sbset}(\Omega)$ . For example  $\mathcal{N} = \text{Null}(P)$ .

We denote the symmetric difference of two sets  $A$  and  $B$  by  $A \bowtie B$ .

**2.14 Lemma.** *If  $\sigma$  is an  $\mathcal{F}$ -stopping time then  $(\mathcal{F} \vee \mathcal{N})_\sigma = \mathcal{F}_\sigma \vee \mathcal{N}$ .*

*Proof.* Clearly we have that  $\mathcal{F}_\sigma \subset (\mathcal{F} \vee \mathcal{N})_\sigma$ . On the other hand  $\text{Sbset}(A) \subset \mathcal{N}$  for  $A \in \mathcal{N}$ , which implies that  $\mathcal{N} \subset (\mathcal{F} \vee \mathcal{N})_\sigma$ . Conversely suppose that  $A \in (\mathcal{F} \vee \mathcal{N})_\sigma$ . There exists a sequence of sets  $B_n$  such that  $B_n \in \mathcal{F}_n$  and  $(A \cap \{\sigma = n\}) \bowtie B_n \in \mathcal{N}$  for all  $n \in \mathbb{Z}_{\geq 0}$ . We see that  $B_n \cap \{\sigma = n\} \in \mathcal{F}_n$ . It therefore follows that

$$B := (A \cap \{\sigma = +\infty\}) \cup \bigcup_{n=0}^{\infty} (B_n \cap \{\sigma = n\}) \in \mathcal{F}_\sigma.$$

Since  $A \bowtie B \subset \bigcup_{n=0}^{\infty} ((A \cap \{\sigma = n\}) \bowtie B_n) \in \mathcal{N}$ , we get  $A \in \mathcal{F}_\sigma \vee \mathcal{N}$ . □

**2.15 Lemma.** *Suppose  $\sigma$  and  $\tau$  are  $\mathcal{F} \vee \mathcal{N}$ -stopping times.*

- (i) *If  $\{\tau < \sigma\} \in \mathcal{N}$  then  $(\mathcal{F} \vee \mathcal{N})_\sigma \subset (\mathcal{F} \vee \mathcal{N})_\tau$ . If  $\{\tau \neq \sigma\} \in \mathcal{N}$  then  $(\mathcal{F} \vee \mathcal{N})_\sigma = (\mathcal{F} \vee \mathcal{N})_\tau$ .*  
(ii) *There exists an  $\mathcal{F}$ -stopping time  $\tilde{\sigma}$  such that  $\{\tilde{\sigma} \neq \sigma\} \in \mathcal{N}$ .*

*Proof.* (i) We see by Lemma 2.10 that  $A \cap \{\sigma \leq \tau\} \in (\mathcal{F} \vee \mathcal{N})_\tau$  for all  $A \in (\mathcal{F} \vee \mathcal{N})_\sigma$ .

(ii) For each  $n \in \mathbb{Z}_{\geq 0}$  there exists  $A_n \in \mathcal{F}_n$  such that

$$\{\sigma = n\} \times A_n \in \mathcal{N}.$$

We set  $N := \{\sigma \notin \mathbb{Z}_{\geq 0} \cup \{+\infty\}\} \cup \bigcup_{n=0}^{\infty} (\{\sigma = n\} \times A_n)$ . Then we have that

$$A_n \subset \{\sigma = n\} \cup N, \{\sigma = n\} \subset A_n \cup N \text{ for all } n \in \mathbb{Z}_{\geq 0} \text{ and } N \in \mathcal{N}.$$

It follows that

$$A_n \cap \left(\bigcup_{k=0}^{n-1} A_k\right) \subset N \text{ and } \left(\bigcup_{n=0}^{\infty} A_n\right)^c \times \{\sigma = +\infty\} \subset N.$$

Define  $\tilde{\sigma}(\omega) := \inf\{n \in \mathbb{Z}_{\geq 0} : \omega \in \bigcup_{k=0}^n A_k\}$ . Then, since

$$\tilde{\sigma} = n \text{ on } A_n \cap \left(\bigcup_{k=0}^{n-1} A_k\right)^c \text{ and } \tilde{\sigma} = +\infty \text{ on } \left(\bigcup_{n=0}^{\infty} A_n\right)^c,$$

we see that  $\{\tilde{\sigma} = n\} \in \mathcal{F}_n$  for all  $n \in \mathbb{Z}_{\geq 0}$  and  $\{\tilde{\sigma} \neq \sigma\} \in \mathcal{N}$  a.s.  $\square$

*2.16 Remark.* Let  $\sigma$  be as above. Suppose in addition that there exists  $T \in \mathbb{Z}_{\geq 0}$  such that  $\sigma \leq T$  a.s. Then the  $\mathcal{F}$ -stopping time  $\tilde{\sigma} \wedge T$  satisfies  $\tilde{\sigma} \wedge T = \sigma$  a.s.

We slightly extend the class of submartingales.

**2.17 Definition.** A stochastic process  $X$  such that  $X_n$  is lower semi-integrable for all  $n \in \mathbb{Z}_{\geq 0}$  and  $X_n \in E_{\leq}[X_{n+1}|\mathcal{F}_n]$  for all  $n \in \mathbb{Z}_{\geq 0}$  is called an  $\mathcal{F}$ -submartingale.

**2.18 Lemma.** *Suppose that  $X$  is an  $\mathcal{F}$ -submartingale and  $f : \mathbb{R} \cup \{+\infty\} \rightarrow \mathbb{R} \cup \{+\infty\}$  is a non-decreasing convex function. Then  $f(X)$  is an  $\mathcal{F}$ -submartingale.*

*Proof.* Let  $Y \in E[X_{n+1}|\mathcal{F}_n]$ . Then it follows by Theorem 1.13 that  $f(X_{n+1})$  is lower semi-integrable and  $f(Y) \in E_{\leq}[f(X_{n+1})|\mathcal{F}_n]$ . Since  $X_n \leq Y$  a.s. and the function  $f$  is non-decreasing, we have that  $f(X_n) \leq f(Y)$  a.s. Thus  $f(X_n) \in E_{\leq}[f(X_{n+1})|\mathcal{F}_n]$ .  $\square$

*2.19 Remark.* If  $X$  is an  $\mathcal{F}$ -martingale then  $f(X)$  is an  $\mathcal{F}$ -submartingale without  $f$  being non-decreasing.

**2.20 Lemma.** *Let  $X$  be an  $\mathcal{F}$ -submartingale. If  $\sigma$  and  $\tau$  are bounded  $\mathcal{F}$ -stopping times such that  $P(\{\tau = \sigma\} \cup \{\tau = \sigma + 1\}) = 1$  then  $X_\tau$  is lower semi-integrable and  $X_\sigma \in E_{\leq}[X_\tau|\mathcal{F}_\sigma]$ .*

*Proof.* There exists  $T \in \mathbb{Z}_{\geq 0}$  such that  $\sigma \vee \tau \leq T$ . Let  $A \in \mathcal{F}_\sigma$ . Then

$$E[X_\tau; A] = \sum_{n=0}^T E[X_\tau; A, \sigma = n],$$

where  $A \cap \{\sigma = n\} \in \mathcal{F}_n$  for each  $n$ . Observe that  $\{\sigma = n, \tau \neq n\} = \{\sigma = n, \tau = n + 1\}$  up to  $\text{Null}(P)$  and the left hand side belongs to  $\mathcal{F}_n$ . It then follows that

$$E[X_\tau; A, \sigma = n, \tau \neq n] = E[X_{n+1}; A, \sigma = n, \tau \neq n] \geq E[X_n; A, \sigma = n, \tau \neq n].$$

The right hand side reads  $E[X_\sigma; A, \sigma = n, \tau \neq n]$ . We thus get  $E[X_\tau; A] \geq E[X_\sigma; A]$  for all  $A \in \mathcal{F}_\sigma$  by adding  $E[X_\tau; A, \sigma = n, \tau = n] = E[X_\sigma; A, \sigma = n, \tau = n]$ . Since  $X_\sigma$  is  $\mathcal{F}_\sigma$ -measurable by Lemma 2.7, we reach the statement.  $\square$

**2.21 Corollary.** *Let  $X$  be an  $\mathcal{F}$ -submartingale. If  $\sigma$  and  $\tau$  are bounded  $\mathcal{F}$ -stopping times with  $\sigma \leq \tau$  then  $X_\tau$  is lower semi-integrable and  $X_\sigma \in E_{\leq}[X_\tau|\mathcal{F}_\sigma]$ .*

*Proof.* By applying Lemma 2.20 to the pair of stopping times  $\tau \wedge (\sigma + n)$  and  $\tau \wedge (\sigma + n + 1)$  we get  $X_{\tau \wedge (\sigma + n)} \in E_{\leq}[X_{\tau \wedge (\sigma + n + 1)}|\mathcal{F}_{\tau \wedge (\sigma + n)}]$  for all  $n \in \mathbb{Z}_{\geq 0}$ . There exists  $T \in \mathbb{Z}_{\geq 0}$  such that  $\tau \leq T$ . Since  $\tau \wedge \sigma = \sigma$  and  $\tau \wedge (\sigma + T) = \tau$ , we get the conclusion by Theorem 1.17.  $\square$

### 3 Optional sampling–continuous time

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\mathcal{F}$  be a filtration of sub  $\sigma$ -fields. We switch to discuss submartingales with continuous parameter. So the index set is  $\mathbb{R}_{\geq 0}$ .

**3.1 Lemma.** *Given  $A \in \text{Sbset}(E)$  define  $\tau : \text{Map}(\mathbb{R}_{\geq 0}, E) \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$  as follows:  $\tau(\omega) := \inf\{s \in \mathbb{R}_{>0} : \omega(s) \in A\}$  the hitting time of path  $\omega$  to the set  $A$ . Let  $t \in \mathbb{R}_{>0}$ .*

(i)  $\{\omega : \tau(\omega) < t\} = \bigcup_{0 < s < t} \{\omega : \omega(s) \in A\} = \text{proj}_\Omega\{(s, \omega) : 0 < s < t, \omega(s) \in A\}$ .

(ii)  $A$  open  $\Rightarrow \{\omega \in \text{RC}(\mathbb{R}_{\geq 0}, E) : \tau(\omega) < t\} = \bigcup_{s \in \mathbb{Q}: 0 < s < t} \{\omega \in \text{RC}(\mathbb{R}_{\geq 0}, E) : \omega(s) \in A\}$ .

**3.2 Definition.** A mapping  $\sigma : \Omega \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$  such that  $\{\sigma < t\} \in \mathcal{F}_t$  for all  $t \in \mathbb{R}_{\geq 0}$  is called an  $\mathcal{F}$ -optional time. Given an  $\mathcal{F}$ -optional time  $\sigma$ , set

$$\mathcal{F}_{\sigma+} := \{A \in \mathcal{F} : A \cap \{\sigma < t\} \in \mathcal{F}_t \forall t \in \mathbb{R}_{\geq 0}\}.$$

*3.3 Remark.* If  $\sigma$  is a constant optional time then  $\mathcal{F}_{\sigma+} = \bigcap\{\mathcal{F}_t; t \in \mathbb{R}_{>s}\}$  where  $\sigma(\Omega) = \{s\}$ . Thus we shall write  $\mathcal{F}_{s+} := \bigcap\{\mathcal{F}_t; t \in \mathbb{R}_{>s}\}$  where there is no fear of confusion. Note that  $s \mapsto \mathcal{F}_{s+}$  is a right continuous filtration of sub  $\sigma$ -fields with index space  $\mathbb{R}_{\geq 0}$ .

**3.4 Example.** Let  $X$  be an  $\mathcal{F}$ -adapted process and  $D$  be an open set. If every sample path (almost every sample path) is right continuous then  $\inf\{s \in \mathbb{R}_{>0} : X_s \in D\}$  as well as  $\inf\{s \in \mathbb{R}_{\geq 0} : X_s \in D\}$ , called the entry time of sample path  $t \mapsto X_t$  to the set  $D$ , are  $\mathcal{F}$ -optional times ( $\mathcal{F} \vee \text{Null}(P)$ -optional times) by Lemma 3.1(ii).

**3.5 Definition.** A mapping  $\sigma : \Omega \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$  such that  $\{\sigma \leq t\} \in \mathcal{F}_t$  for all  $t \in \mathbb{R}_{\geq 0}$  is called an  $\mathcal{F}$ -stopping time. Given an  $\mathcal{F}$ -stopping time  $\sigma$ , set

$$\mathcal{F}_\sigma := \{A \in \mathcal{F} : A \cap \{\sigma \leq t\} \in \mathcal{F}_t \forall t \in \mathbb{R}_{\geq 0}\}.$$

**3.6 Lemma.** (i) *Any  $\mathcal{F}$ -stopping time is an  $\mathcal{F}$ -optional time. Both  $\mathcal{F}_\sigma$  and  $\mathcal{F}_{\sigma+}$  are defined for an  $\mathcal{F}$ -stopping time  $\sigma$  and  $\mathcal{F}_\sigma \subset \mathcal{F}_{\sigma+}$ .*

(ii)  *$\sigma$  is an  $\mathcal{F}$ -optional time if and only if it is an  $\mathcal{F}_{\cdot+}$ -stopping time. If the filtration  $\mathcal{F}$  is right continuous then any  $\mathcal{F}$ -optional time  $\sigma$  is an  $\mathcal{F}$ -stopping time and  $\mathcal{F}_\sigma = \mathcal{F}_{\sigma+}$  holds.*

(iii) *The notions  $\mathcal{F}$ -optional time,  $\mathcal{F}_{\cdot+}$ -stopping time and  $\mathcal{F}_{\cdot+}$ -optional time are equivalent. Moreover  $\mathcal{F}_{\sigma+} = (\mathcal{F}_{\cdot+})_\sigma = (\mathcal{F}_{\cdot+})_{\sigma+}$  for an  $\mathcal{F}$ -optional time  $\sigma$ .*

We shall later show that the hitting time of a continuous process to a closed set is a stopping time under some mild assumption on the state space, see Theorem 4.4.

**3.7 Lemma.** *Let  $X_n$  be a uniformly integrable sequence of random variables.*

(i)  *$\liminf_{n \rightarrow \infty} |X_n|$  is integrable.  $\limsup_{n \rightarrow \infty} E[|X_n|] \leq E[\limsup_{n \rightarrow \infty} |X_n|]$ .*

(ii) *If  $\liminf_{n \rightarrow \infty} X_n = \limsup_{n \rightarrow \infty} X_n$  a.s then  $\liminf_{n \rightarrow \infty} X_n$  is integrable and  $X$  converges to  $\liminf_{n \rightarrow \infty} X_n$  in  $L^1$ -sense.*

*Proof.* (i) Invoking Fatou's lemma, we get  $E[\liminf_{n \rightarrow \infty} |X_n|] \leq \liminf_{n \rightarrow \infty} E[|X_n|]$ . Therefore  $\liminf_{n \rightarrow \infty} |X_n|$  is integrable. On the other hand

$$E[|X_k|] \leq E[\min\{|X_k|, c\}] + E[|X_k|; |X_k| > c] \text{ for all } c > 0.$$

Since  $\limsup_{k \rightarrow \infty} \min\{|X_k|, c\} = \min\{\limsup_{k \rightarrow \infty} |X_k|, c\}$ , we see by Fatou's lemma that

$$\limsup_{k \rightarrow \infty} E[|X_k|] \leq E[\min\{\limsup_{k \rightarrow \infty} |X_k|, c\}] + \sup_{n \in \mathbb{N}} E[|X_n|; |X_n| > c].$$

Tending  $c \rightarrow +\infty$ , we get (i).

(ii) It follows that  $|\liminf_{n \rightarrow \infty} X_n| = \liminf_{n \rightarrow \infty} |X_n|$  a.s. Hence  $Y := \liminf_{n \rightarrow \infty} X_n$  is integrable by (i). Since  $-\infty < Y < +\infty$  a.s., we have that  $\limsup_{n \rightarrow \infty} |X_n - Y| = 0$  a.s. The sequence  $X_n - Y$  is uniformly integrable by Lemma 1.26. Applying (i) we infer that  $\limsup_{n \rightarrow \infty} E[|X_n - Y|] \leq E[\limsup_{n \rightarrow \infty} |X_n - Y|] = 0$ .  $\square$

**3.8 Lemma.** (i) *Suppose that  $X$  is an  $\mathcal{F}$ -adapted process. If almost every sample path (every sample path) is right continuous and  $\sigma$  is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time ( $\mathcal{F}$ -optional time) then  $X_\sigma 1_{\{\sigma < +\infty\}}$  is  $(\mathcal{F} \vee \text{Null}(P))_{\sigma+}$ -measurable ( $\mathcal{F}_{\sigma+}$ -measurable).*

(ii) *Suppose that  $X$  is an  $\mathcal{F}$ -submartingale. If almost every sample path is right continuous and  $\sigma$  is a bounded  $\mathcal{F} \vee \text{Null}(P)$ -optional time then  $X_\sigma$  is integrable.*

*Proof.* (i) To save the space we write the filtration  $\mathcal{F} \vee \text{Null}(P)$  by  $\mathcal{F}$ . Let  $n \in \mathbb{Z}_{\leq 0}$ . Denote by  $\mathcal{F}^n$  the filtration  $q \mapsto \mathcal{F}_q$  with index set  $2^n \mathbb{Z}_{\geq 0}$ . We set

$$\sigma(n) := \min\{q \in 2^n \mathbb{Z}_{\geq 0} : q > \sigma\}.$$

Then  $\sigma(n)$  is an  $\mathcal{F}^n$ -stopping time since  $\sigma(n) \in 2^n \mathbb{Z}_{\geq 0} \cup \{+\infty\}$  and

$$\{\sigma(n) = q\} = \{\sigma < q\} \cap \{\sigma \geq q - 2^n\} \in \mathcal{F}_q \text{ for all } q \in 2^n \mathbb{Z}_{\geq 0}.$$

Since  $\sigma(n) \downarrow \sigma$  as  $n$  tends to  $-\infty$  and  $X$  is right continuous a.s., it follows that

$$\liminf_{n \rightarrow -\infty} X_{\sigma(n)} \stackrel{\text{a.s.}}{=} X_\sigma \text{ and } \limsup_{n \rightarrow -\infty} X_{\sigma(n)} \stackrel{\text{a.s.}}{=} X_\sigma \text{ on } \{\sigma < +\infty\}.$$

The random variable  $X_{\sigma(n)} 1_{\{\sigma(n) < +\infty\}}$  is  $\mathcal{F}_{\sigma(n)}^n$ -measurable by Lemma 2.7. We have that

$$(\star) \quad \{\sigma(n) \leq t\} = \bigcup \{\{\sigma(n) \leq q\}; q \in 2^n \mathbb{Z}_{\geq 0} : q \leq t\} = \bigcup \{\{\sigma < q\}; q \in 2^n \mathbb{Z}_{\geq 0} : q \leq t\}.$$

for all  $t \in \mathbb{R}_{\geq 0}$ . This implies that  $\sigma(n)$  is an  $\mathcal{F}$ -stopping time and  $\mathcal{F}_{\sigma(n)} = \mathcal{F}_{\sigma(n)}^n$ . In particular  $X_{\sigma(n)} 1_{\{\sigma(n) < +\infty\}}$  is  $\mathcal{F}_{\sigma(n)}$ -measurable. We show that

$$\mathcal{F}_{\sigma+} = \bigcap_{n \in \mathbb{Z}; n \leq 0} \mathcal{F}_{\sigma(n)}$$

Indeed let  $A \in \mathcal{F}_{\sigma+}$ . The relation  $(\star)$  implies that

$$A \cap \{\sigma(n) \leq q\} = A \cap \{\sigma < q\} \in \mathcal{F}_q \text{ for } q \in 2^n \mathbb{Z}_{\geq 0},$$

which means  $A \in \mathcal{F}_{\sigma(n)}^n = \mathcal{F}_{\sigma(n)}$ . Conversely, since  $\sigma = \inf_{n \in \mathbb{Z}; n \leq 0} \sigma(n)$ , it follows that

$$A \cap \{\sigma < t\} = \bigcup_{n \in \mathbb{Z}; n \leq 0} (A \cap \{\sigma(n) < t\}) \in \mathcal{F}_t \text{ for all } A \in \bigcap_{n \in \mathbb{Z}; n \leq 0} \mathcal{F}_{\sigma(n)} \text{ and } t \in \mathbb{R}_{>0}.$$

On the other hand, taking  $\sigma(n-1) \leq \sigma(n)$  into account, we see that

$$\mathcal{F}_{\sigma(n-1)} \subset \mathcal{F}_{\sigma(n)} \text{ for all } n \in \mathbb{Z}_{\leq 0}.$$

Consequently  $\liminf_{n \rightarrow -\infty} X_{\sigma(n)} 1_{\{\sigma(n) < +\infty\}}$  is  $\mathcal{F}_{\sigma+}$ -measurable and so is  $X_{\sigma} 1_{\{\sigma < +\infty\}}$  because  $\{\sigma(n) < +\infty\} = \{\sigma < +\infty\}$  for all  $n \in \mathbb{Z}_{\leq 0}$  and  $\text{Null}(P) \subset \mathcal{F}_{\sigma+}$ .

(ii) We shall discuss the uniform integrability of  $X_{\sigma(\cdot)}$ . Observe that  $q \mapsto X_q$  with index set  $2^{n-1}\mathbb{Z}_{\geq 0}$  is an  $\mathcal{F}^{n-1}$ -submartingale. On the other hand  $\sigma(n)$  is an  $\mathcal{F}^{n-1}$ -stopping time as well because  $\{\sigma(n) = q\} = \emptyset$  if  $q \notin 2^n\mathbb{Z}_{\geq 0}$ . Moreover  $\sigma(n)$  is bounded and

$$\sigma(n-1) = \sigma(n) \text{ or } \sigma(n-1) = \sigma(n) - 2^{n-1}.$$

Therefore we see by Theorem 2.6 that both  $X_{\sigma(n)}$  and  $X_{\sigma(n-1)}$  are integrable and

$$X_{\sigma(n-1)} \in E_{\leq}[X_{\sigma(n)} | \mathcal{F}_{\sigma(n-1)}^{n-1}] \text{ and } X_0 \in E_{\leq}[X_{\sigma(n)} | \mathcal{F}_0^{n-1}]$$

The latter implies that  $E[X_{\sigma(n)}] \geq E[X_0] > -\infty$  for all  $n \in \mathbb{Z}_{\leq 0}$ . Thus we get an  $\mathcal{F}_{\sigma(\cdot)}$ -submartingale  $X_{\sigma(\cdot)}$  with index set  $\mathbb{Z}_{\leq 0}$ . Since  $\inf E[X_{\sigma(\cdot)}] > -\infty$ , it follows by Lemma 2.13 that  $X_{\sigma(\cdot)}$  is uniformly integrable. We infer by Lemma 3.7 that  $\liminf_{n \rightarrow -\infty} X_{\sigma(n)}$  is integrable and  $X_{\sigma(\cdot)}$  converges to  $\liminf_{n \rightarrow -\infty} X_{\sigma(n)}$  in  $L^1$ -sense.  $\square$

**3.9 Theorem.** *Suppose that  $X$  is an  $\mathcal{F}$ -submartingale which is right continuous a.s. and  $\sigma$  and  $\tau$  are  $\mathcal{F} \vee \text{Null}(P)$ -optional times such that  $\sigma \leq \tau \leq T$  a.s. for some  $T \in \mathbb{R}_{>0}$ .*

(i)  $X_{\tau}$  is  $\mathcal{F}$ -measurable and integrable and  $X_{\sigma} \in E_{\leq}[X_{\tau} | (\mathcal{F} \vee \text{Null}(P))_{\sigma+}]$ .

(ii) *If every sample path is right continuous and  $\sigma$  is an  $\mathcal{F}$ -optional time (respectively  $\mathcal{F}$ -stopping time) then  $X_{\sigma} 1_{\{\sigma < +\infty\}} \in E_{\leq}[X_{\tau} | \mathcal{F}_{\sigma+}]$  ( $X_{\sigma} 1_{\{\sigma < +\infty\}} \in E_{\leq}[X_{\tau} | \mathcal{F}_{\sigma}]$ ).*

*Proof.* To save the space we write the filtration  $\mathcal{F} \vee \text{Null}(P)$  by  $\mathcal{F}$ . Let  $n \in \mathbb{Z}_{\leq 0}$ . Denote by  $\mathcal{F}^n$  the filtration  $q \mapsto \mathcal{F}_q$  with index set  $2^n\mathbb{Z}_{\geq 0}$ . We set

$$\sigma(n) := \min\{q \in 2^n\mathbb{Z}_{\geq 0} : q > \sigma\}, \tau(n) := \min\{q \in 2^n\mathbb{Z}_{\geq 0} : q > \tau\}$$

As we have seen in the proof of Lemma 3.8 both  $\sigma(n)$  and  $\tau(n)$  are bounded  $\mathcal{F}^n$ -stopping times. Since  $q \mapsto X_q$  with index set  $2^n\mathbb{Z}_{\geq 0}$  is an  $\mathcal{F}^n$ -submartingale and  $\sigma(n) \leq \tau(n)$  a.s., we see by Theorem 2.6 that both  $X_{\sigma(n)}$  and  $X_{\tau(n)}$  are integrable and

$$X_{\sigma(n)} \in E_{\leq}[X_{\tau(n)} | \mathcal{F}_{\sigma(n)}^n].$$

We verified in the proof of Lemma 3.8 that  $\mathcal{F}_{\sigma+} \subset \mathcal{F}_{\sigma(n)}^n$ . It therefore follows that

$$E[X_{\sigma(n)}; A] \leq E[X_{\tau(n)}; A] \text{ for all } A \in \mathcal{F}_{\sigma+}.$$

We also verified in the proof of Lemma 3.8 that  $X_{\sigma(\cdot)}$  converges to  $X_{\sigma}$  in  $L^1$ -sense. The same claim holds for the pair  $X_{\tau(\cdot)}$  and  $X_{\tau}$ . Tending  $n$  to  $-\infty$ , we obtain that

$$E[X_{\sigma}; A] \leq E[X_{\tau}; A] \text{ for all } A \in \mathcal{F}_{\sigma+}.$$

According to Lemma 3.8,  $X_{\sigma}$  is  $\mathcal{F}_{\sigma+}$ -measurable. Thus we get (i). Finally Lemma 3.12 and Lemma 3.13 show the parenthesis part of (ii).  $\square$

*3.10 Remark.* Suppose that  $\sigma$  and  $\tau$  are  $\mathcal{F}$ -stopping times and  $\sigma \leq \tau$ . Given  $A \in \mathcal{F}_{\sigma}$ , the function  $S$  such that  $S = \sigma$  on  $A$  and  $S = \tau$  on  $A^c$  is an  $\mathcal{F}$ -stopping time. Clearly  $S \leq \tau$ . Let  $X$  be an  $\mathcal{F}$ -adapted process with a.s. right continuous sample path. Then  $E[X_S] \leq E[X_{\tau}]$  implies  $E[X_{\sigma}; A] \leq E[X_{\tau}; A]$  provided the integrals make sense.

In what follows we are concerned with optional stopping theorem.

**3.11 Definition.** A stochastic process  $X$ . with state space  $E$  is said to be  $\mathcal{F}$ .-progressively measurable if  $\mathbb{R}_{[0,t]} \times \Omega \rightarrow E$ ,  $(s, \omega) \mapsto X_s(\omega)$  is  $\text{Borel}(\mathbb{R}_{[0,t]}) \otimes \mathcal{F}_t$ -measurable for each  $t \geq 0$ .

**3.12 Lemma.** Let  $X$ . be an  $\mathcal{F}$ .-adapted process. If every sample path is right continuous (every sample path is left continuous) then  $X$ . is  $\mathcal{F}$ .-progressively measurable.

*Proof.* Suppose that  $f : I \times \Omega \rightarrow \mathbb{R}$  where  $I$  is an interval of  $\mathbb{R}$ , and  $\mathcal{G}$  is a sub  $\sigma$ -field of  $\mathcal{F}$ . If  $f(\cdot, \omega)$  is right continuous for every  $\omega \in \Omega$  (or left continuous for every  $\omega \in \Omega$ ) and  $f(t, \cdot)$  is  $\mathcal{G}$ -measurable for every  $t \in I$  then  $f$  is  $\text{Borel}(I) \otimes \mathcal{G}$ -measurable.  $\square$

**3.13 Lemma.** Let  $X$ . be an  $\mathcal{F}$ .-progressively measurable process and  $\tau$  an  $\mathcal{F}$ .-optional time

(i) The stopped process  $X_{\cdot \wedge \tau}$  is  $\mathcal{F} \cap \mathcal{F}_{\tau+}$ -progressively measurable. If  $\tau$  is an  $\mathcal{F}$ .-stopping time then  $X_{\cdot \wedge \tau}$  is  $\mathcal{F} \cap \mathcal{F}_{\tau}$ -progressively measurable.

(ii)  $1_{\{\tau < +\infty\}} X_{\tau}$  is  $\mathcal{F}_{\tau+}$ -measurable while if  $\tau$  is an  $\mathcal{F}$ .-stopping time then it is  $\mathcal{F}_{\tau}$ -measurable.

(iii) If  $\sigma$  is an  $\mathcal{F}$ .-stopping (optional) time then  $1_{\{\sigma < +\infty\}} X_{\sigma \wedge \tau}$  is  $\mathcal{F}_{\sigma}$ -( $\mathcal{F}_{\sigma+}$ -)measurable.

*Proof.* Let  $t \in \mathbb{R}_{>0}$ . Since  $\{t \wedge \tau \leq a\} = \{\tau \leq a\}$  for all  $a \in \mathbb{R}_{[0,t]}$ , the random variable  $t \wedge \tau$  is  $\mathcal{F}_t$ -measurable. It is  $\mathcal{F}_{\tau+}$ -measurable as well for  $\{\tau \leq a\} \cap \{\tau < s\}$  reads  $\{\tau < s\}$  if  $s \leq a$  and  $\{\tau \leq a\}$  if  $s > a$ . It follows that the mapping  $\mathbb{R}_{[0,t]} \times \Omega \rightarrow \mathbb{R}_{[0,t]}$ ,  $(s, \omega) \mapsto \min\{s, t \wedge \tau(\omega)\}$  is  $\text{Borel}(\mathbb{R}_{[0,t]}) \otimes (\mathcal{F}_t \cap \mathcal{F}_{\tau+})$ -measurable. Consequently the self mapping

$$\mathbb{R}_{[0,t]} \times \Omega \rightarrow \mathbb{R}_{[0,t]} \times \Omega, (s, \omega) \mapsto (s \wedge \tau(\omega), \omega)$$

is measurable with respect to  $\text{Borel}(\mathbb{R}_{[0,t]}) \otimes (\mathcal{F}_t \cap \mathcal{F}_{\tau+})$ . We thus infer that the mapping

$$\mathbb{R}_{[0,t]} \times \Omega \rightarrow E, (s, \omega) \mapsto X_{s \wedge \tau(\omega)}(\omega)$$

is  $\text{Borel}(\mathbb{R}_{[0,t]}) \otimes (\mathcal{F}_t \cap \mathcal{F}_{\tau+})$ -measurable. In particular  $X_{t \wedge \tau}$  is  $(\mathcal{F}_t \cap \mathcal{F}_{\tau+})$ -measurable.  $\square$

**3.14 Corollary.** Suppose that  $X$ . is an  $\mathcal{F}$ .-adapted process whose almost every sample path is right continuous and  $\tau$  is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time. If  $\sigma$  is an  $\mathcal{F} \vee \text{Null}(P)$ -stopping (optional) time then  $1_{\{\sigma < +\infty\}} X_{\sigma \wedge \tau}$  is  $(\mathcal{F} \vee \text{Null}(P))_{\sigma}$ -measurable ( $(\mathcal{F} \vee \text{Null}(P))_{\sigma+}$ -measurable).

*Proof.* There exists  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and  $X_{\cdot}(\omega)$  is right continuous for all  $\omega \in \Omega_0$ . Being  $\mathcal{F} \vee \text{Null}(P)$ -adapted and right continuous, the modification  $1_{\Omega_0} X_{\cdot}$  is  $\mathcal{F} \vee \text{Null}(P)$ -progressively measurable. (Nevertheless  $X_{\cdot}$  itself may fail to be  $\mathcal{F} \vee \text{Null}(P)$ -progressively measurable.) Since  $\text{Null}(P) \subset (\mathcal{F} \vee \text{Null}(P))_{\sigma}$ , we get the claim by Lemma 3.13.  $\square$

**3.15 Lemma.** Suppose that  $\sigma$  and  $\tau$  are  $\mathcal{F}$ .-stopping times.

(i)  $\sigma \wedge \tau$  and  $\sigma \vee \tau$  are  $\mathcal{F}$ .-stopping times.

(ii)  $\mathcal{F}_{\sigma \wedge \tau} = \mathcal{F}_{\sigma} \cap \mathcal{F}_{\tau}$ . If  $\sigma \leq \tau$  then  $\mathcal{F}_{\sigma} \subset \mathcal{F}_{\tau}$ .

(iii)  $\{\tau \leq \sigma\}$ ,  $\{\sigma \leq \tau\}$ ,  $\{\tau < \sigma\}$ ,  $\{\sigma < \tau\}$  and  $\{\tau = \sigma\}$  belong to  $\mathcal{F}_{\sigma \wedge \tau}$

(iv)  $A \cap \{\sigma \leq \tau\} \in \mathcal{F}_{\tau}$  and  $A \cap \{\sigma < \tau\} \in \mathcal{F}_{\tau}$  for all  $A \in \mathcal{F}_{\sigma \vee \tau}$ .  $\mathcal{F}_{\sigma \vee \tau} = \mathcal{F}_{\sigma} \vee \mathcal{F}_{\tau}$ .

(v)  $A \cap \{\sigma \leq \tau\} \in \mathcal{F}_{\sigma \wedge \tau}$ ,  $A \cap \{\sigma < \tau\} \in \mathcal{F}_{\sigma \wedge \tau}$  for all  $A \in \mathcal{F}_{\sigma}$ .

*Proof.* The random variable  $\tau \wedge t$  is  $\mathcal{F}_t$ -measurable for all  $t \in \mathbb{R}_{\geq 0}$ . It then follows that  $A \cap \{\sigma \leq \tau\} \in \mathcal{F}_{\tau}$  for all  $A \in \mathcal{F}_{\sigma \vee \tau}$ . Indeed

$$(A \cap \{\sigma \leq \tau\}) \cap \{\tau \leq t\} = (A \cap \{\sigma \vee \tau \leq t\}) \cap \{\sigma \wedge \tau \leq \tau \wedge t\} \in \mathcal{F}_t \text{ for all } t \in \mathbb{R}_{\geq 0}.$$

The rest of the discussion is the same as in the proof of Lemma 2.10.  $\square$

*3.16 Remark.* (i) Suppose that  $\sigma$  and  $\tau$  are  $\mathcal{F}$ -optional times, equivalently  $\mathcal{F}_+$ -stopping times. Recall that  $(\mathcal{F}_+)_\sigma = \mathcal{F}_{\sigma+}$  etc. Thus Lemma 3.15 reads as follows:  $\sigma \wedge \tau$  and  $\sigma \vee \tau$  are  $\mathcal{F}$ -optional times.  $\mathcal{F}_{(\sigma \wedge \tau)+} = \mathcal{F}_{\sigma+} \cap \mathcal{F}_{\tau+}$ . If  $\sigma \leq \tau$  then  $\mathcal{F}_{\sigma+} \subset \mathcal{F}_{\tau+}$ . And so on.

(ii) If  $\sigma$  is an  $\mathcal{F}$ -optional time and  $\tau$  is an  $\mathcal{F}$ -stopping time then  $A \cap \{\sigma < \tau\} \in \mathcal{F}_\tau \forall A \in \mathcal{F}_{\sigma+}$ . Indeed  $(A \cap \{\sigma < \tau\}) \cap \{\tau \leq t\} = (A \cap \{\sigma < t\}) \cap \{\tau \leq t\} \cap \{\sigma \wedge t < \tau \wedge t\} \in \mathcal{F}_t \forall t \geq 0$ .

**3.17 Corollary.** *Suppose that  $X$  is an  $\mathcal{F}$ -submartingale such that almost every sample path (every sample path) is right continuous, and  $\sigma(1)$ ,  $\sigma(2)$  and  $\tau$  are  $\mathcal{F} \vee \text{Null}(P)$ -optional times ( $\mathcal{F}$ -optional times) such that  $\sigma(1) \leq \sigma(2)$  a.s. and  $\sigma(2) \wedge \tau \leq T$  a.s. for some  $T \in \mathbb{R}_{>0}$ .*

(i)  $X_{\sigma(1) \wedge \tau} \in E_{\leq} [X_{\sigma(2) \wedge \tau} | (\mathcal{F} \vee \text{Null}(P))_{\sigma(1)+}]$  ( $1_{\{\sigma(1) \wedge \tau < +\infty\}} X_{\sigma(1) \wedge \tau} \in E_{\leq} [X_{\sigma(2) \wedge \tau} | \mathcal{F}_{\sigma(1)+}]$ ).

(ii) *The stopped process  $X_{\cdot \wedge \tau}$  is an  $\mathcal{F} \vee \text{Null}(P)$ -submartingale ( $\mathcal{F}$ -submartingale).*

(iii) *If every sample path is right continuous,  $\sigma(1)$  is an  $\mathcal{F}$ -stopping time with  $\sigma(1) < +\infty$  a.s. and  $\tau$  is an  $\mathcal{F}$ -optional time then  $1_{\{\sigma(1) < +\infty\}} X_{\sigma(1) \wedge \tau} \in E_{\leq} [X_{\sigma(2) \wedge \tau} | \mathcal{F}_{\sigma(1)}]$ .*

*Proof.* (i) Invoking Corollary 3.14 and Lemma 3.15(ii), we see that  $1_{\{\sigma(1) \wedge \tau < +\infty\}} X_{(\sigma(1) \wedge \tau) \wedge \infty}$  is  $(\mathcal{F} \vee \text{Null}(P))_{\sigma(1)+}$ -measurable. On the other hand  $A \cap \{\sigma(1) \leq \tau\} \in (\mathcal{F} \vee \text{Null}(P))_{(\sigma(1) \wedge \tau)+}$  for all  $A \in (\mathcal{F} \vee \text{Null}(P))_{\sigma(1)+}$  by Lemma 3.15(v). The rest of the discussion is the same as in the proof of Corollary 2.11. (ii)  $X_{\cdot \wedge \tau}$  is  $\mathcal{F} \vee \text{Null}(P)$ -adapted by Corollary 3.14.  $\square$

*3.18 Question.* Suppose that  $X$  is a measurable process and  $\sigma$  is an  $\sigma\{X\}_{\leq}$ -stopping time. When  $X$  is  $\sigma\{X\}_{\leq}$ -progressively measurable? Is the relation  $\sigma\{X_{\cdot \wedge \sigma}\}_{\leq t} = \sigma\{X_{\cdot}\}_{\leq t \wedge \sigma}$  holds?

## 4 More on stopping times

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\mathcal{F}$  be a filtration of sub  $\sigma$ -fields. We discuss hitting times of continuous process to closed subsets. So the index space is  $\mathbb{R}_{\geq 0}$ .

**4.1 Lemma.** *Suppose that  $E$  is a topological space,  $D_n$  is sequence of open subsets with  $\overline{D_{n+1}} \subset D_n$  for all  $n \in \mathbb{N}$ ,  $F$  is a closed subset and  $\bigcap_{n \in \mathbb{N}} D_n = F$ . Given  $\omega \in C(\mathbb{R}_{\geq 0} \rightarrow E)$  define  $\sigma_n := \inf\{s \in \mathbb{R}_{\geq 0} : \omega(s) \in D_n\}$ . Then  $\sup_{n \in \mathbb{N}} \sigma_n = \inf\{s \in \mathbb{R}_{\geq 0} : \omega(s) \in F\}$ .*

*Proof.* Since  $F \subset D_n$ , we have that  $\sigma_n \leq \tau := \inf\{s \in \mathbb{R}_{\geq 0} : \omega(s) \in F\}$ . In particular

$$\sup_{n \in \mathbb{N}} \sigma_n \leq \tau.$$

If  $\sup_{n \in \mathbb{N}} \sigma_n = +\infty$  then it follows that  $\tau = +\infty$ . Thus we assume that  $\sup_{n \in \mathbb{N}} \sigma_n < +\infty$  in addition. Since  $\omega$  is right continuous,

$$\omega(\sigma_n) \in \overline{D_n} \subset \overline{D_{m+1}} \text{ for all } n \in \mathbb{Z}_{> m}.$$

(Note that  $\sigma_n$  is an optional time.) The (quasi-) left continuity implies that

$$\omega(\sup_{n \in \mathbb{N}} \sigma_n) \in \overline{D_{m+1}} \subset D_m \text{ for all } m \in \mathbb{N} \text{ and hence } \omega(\sup_{n \in \mathbb{N}} \sigma_n) \in F.$$

The latter means  $\sup_{n \in \mathbb{N}} \sigma_n \geq \tau$ . Thus we get  $\sup_{n \in \mathbb{N}} \sigma_n = \tau$ .  $\square$

**4.2 Lemma.** *Suppose that  $E$  is a topological space,  $D$  is an open subset,  $F$  is a closed subset and  $F \subset D$ . Given  $\omega \in C(\mathbb{R}_{\geq 0} \rightarrow E)$  define*

$$S := \inf\{s \in \mathbb{R}_{\geq 0} : \omega(s) \in D\} \text{ and } T := \inf\{s \in \mathbb{R}_{\geq 0} : \omega(s) \in F\}.$$

*If  $S = +\infty$  then  $T = +\infty$ . If  $T = 0$  then  $S = 0$ . If  $S < +\infty$  and  $T > 0$  then  $S < T$ .*

*Proof.* Since  $F \subset D$ , we have that  $S \leq T$ . Hence the first two statements are obvious. Suppose that  $S < +\infty$  and  $T > 0$ . If  $T = +\infty$  or  $S = 0$  then it follows that  $S < T$ . Thus we assume that  $T < +\infty$  and  $S > 0$  in addition. We see that

$$\omega(T) \in F \text{ and } \omega(S) \in D^c.$$

The former is because  $\omega$  is right continuous and  $F$  is closed while the latter is because  $\omega$  is left continuous and  $D^c$  is closed. On the other hand  $F \cap D^c = \emptyset$ . Therefore we get  $T \neq S$ .  $\square$

**4.3 Lemma.** *Suppose that  $\sigma_n \in \text{Map}(\Omega, \mathbb{R}_{\geq 0} \cup \{+\infty\})$  for  $n \in \mathbb{N}$ . If  $\sigma_k < \sup_{n \in \mathbb{N}} \sigma_n$  on  $\{\sigma_k < +\infty, \sup_{n \in \mathbb{N}} \sigma_n > 0\}$  for all  $k \in \mathbb{N}$  then  $\{\sup_{n \in \mathbb{N}} \sigma_n \leq t\} = \bigcap_{n \in \mathbb{N}} \{\sigma_n < t\}$  for  $t \in \mathbb{R}_{>0}$ .*

**4.4 Theorem.** *Suppose that  $E$  is metrizable,  $X$  is an  $\mathcal{F}$ -adapted process and  $F$  is a closed set. If every sample path is continuous then  $\inf\{s \in \mathbb{R}_{\geq 0} : X_s \in F\}$  is an  $\mathcal{F}$ -stopping time.*

*Proof.* Example 3.4, Lemma 4.1, Lemma 4.2 and Lemma 4.3.  $\square$

Since  $\inf\{s \in \mathbb{R}_{\geq 1/n} : X_s \in F\} \searrow \inf\{s \in \mathbb{R}_{>0} : X_s \in F\}$  as  $n$  tends to  $\infty$ , the hitting time  $\inf\{s \in \mathbb{R}_{>0} : X_s \in F\}$  is an  $\mathcal{F}$ -optional time.

**4.5 Lemma.** *Suppose that  $E$  is metrizable,  $\rho$  is a compatible metric on  $E$ ,  $\omega : \mathbb{R}_{\geq 0} \rightarrow E$  is right continuous and admits left-hand limits everywhere, and  $F$  is a closed set. Set  $\tau(n) := \inf\{t \in \mathbb{R}_{\geq 0} : \rho(\omega(t), F) < 1/n\}$  for  $n \in \mathbb{N}$ . Then  $\tau(n) \leq \tau(n+1)$  and*

$$\sup_{n \in \mathbb{N}} \tau(n) = \min\{\inf\{t \in \mathbb{R}_{\geq 0} : \omega(t) \in F\}, \inf\{t \in \mathbb{R}_{>0} : \omega(t-o) \in F\}\}.$$

*Proof.* We may assume that  $F$  is not void. We denote the right hand side by  $\zeta$ . Let  $n \in \mathbb{N}$ . Clearly  $\tau(n) \leq \tau(n+1)$ . If  $t \in \mathbb{R}_{>0}$  and  $\omega(t-o) \in F$  then  $\rho(\omega(t-o), F) < 1/n$  and,  $x \mapsto \rho(x, F)$  being continuous, there exists  $s \in \mathbb{R}_{[0,t]}$  such that  $\rho(\omega(s), F) < 1/n$ . This implies  $\tau(n) \leq \zeta$  for all  $n \in \mathbb{N}$ . Suppose that  $t \in \mathbb{R}$ ,  $\delta \in \mathbb{R}_{>0}$  and  $\sup_{n \in \mathbb{N}} \tau(n) < t - \delta$ . Then

$$\{s \in \mathbb{Q}_{>0} : s < t - \delta, \rho(\omega(s), F) < 1/n\} \neq \emptyset \text{ for each } n \in \mathbb{N}$$

due to the right continuity of  $\omega$ . Therefore we can find a sequence  $s(\cdot)$  such that  $s(n) < t - \delta$  and  $\rho(\omega(s(n)), F) < 1/n$  for all  $n \in \mathbb{N}$  (the countability of  $\mathbb{Q}_{>0}$  prevents us from using the axiom of choice). Recall that  $\omega$  is right continuous and admits left hand limits everywhere. On the other hand  $\rho(\cdot, F)$  is continuous and,  $F$  being closed,  $\{x \in E : \rho(x, F) = 0\} = F$ . If  $s \in \text{clust}s(\cdot)$  then, since  $s \leq t - \delta$  and either  $\rho(\omega(s), F) \leq 0$  or  $\rho(\omega(s-o), F) \leq 0$  holds,  $\zeta \leq s \leq t - \delta < t$ . The sequence  $s(\cdot)$  being bounded, we have that  $\text{clust}s(\cdot) \neq \emptyset$ , which implies that  $\zeta < t$ . This holds whenever  $\sup_{n \in \mathbb{N}} \tau(n) < t$ .  $\square$

**4.6 Lemma.** *Suppose that  $\sigma$  and  $\tau$  are  $\mathcal{F}$ -optional times.*

- (i)  $\{\tau > 0, \sigma > 0, \tau + \sigma \leq t\} \in \mathcal{F}_t$  for all  $t \in \mathbb{R}_{\geq 0}$ .
- (ii) If  $\sigma$  is an  $\mathcal{F}$ -stopping time then  $\{\sigma > 0, \tau + \sigma \leq t\} \in \mathcal{F}_t$  for all  $t \in \mathbb{R}_{\geq 0}$ .
- (iii)  $\tau + s$  is an  $\mathcal{F}$ -stopping time for all  $s \in \mathbb{R}_{>0}$ .
- (iv)  $\tau + \sigma$  is an  $\mathcal{F}$ -optional time. If  $\sigma$  and  $\tau$  are  $\mathcal{F}$ -stopping times then so is  $\tau + \sigma$ .

*Proof.* If  $0 < \tau \leq t < \tau + \sigma$  then  $\max\{t - \sigma, 0\} < \min\{\tau, t\}$ . It follows that

$$\{0 < \tau < t, \tau + \sigma > t\} \subset \bigcup_{s \in \mathbb{Q}: 0 < s < t} (\{0 < \tau < t\} \cap \{\tau > s, \sigma > t - s\}).$$

The converse inclusion clearly holds. Consequently

$$\{0 < \tau < t, \tau + \sigma \leq t\} = \{0 < \tau < t\} \cap \bigcap_{s \in \mathbb{Q}: 0 < s < t} (\{\tau \leq s\} \cup \{\sigma \leq t - s\}) \text{ for all } t \in \mathbb{R}_{\geq 0}.$$

We have that  $\{\tau > 0, \sigma > 0, \tau + \sigma \leq t\} = \{0 < \tau < t, \tau + \sigma \leq t\} \cap \{\sigma > 0\}$ . Since the right hand is  $\mathcal{F}_t$ -measurable when  $t \in \mathbb{R}_{>0}$ , we get (i). On the other hand  $\{\sigma > 0, \tau + \sigma \leq t\}$  equals  $\{\tau > 0, \sigma > 0, \tau + \sigma \leq t\} \cup \{\tau = 0, 0 < \sigma \leq t\}$ , which is  $\mathcal{F}_t$ -measurable by (i) provided  $t \in \mathbb{R}_{>0}$  and  $\sigma$  is an  $\mathcal{F}$ -stopping time. This shows (ii). Finally  $\{\tau + \sigma \leq t\}$  coincides with  $\{\sigma > 0, \tau + \sigma \leq t\} \cup \{\sigma = 0, \tau \leq t\}$ , which is  $\mathcal{F}_t$ -measurable by (ii) provided both  $\sigma$  and  $\tau$  are  $\mathcal{F}$ -stopping times. Thus we reach the second half of (iv). Since an  $\mathcal{F}$ -optional time is an  $\mathcal{F}_+$ -stopping time and vice versa, we obtain the first half of (iv).  $\square$

Let  $\mathcal{N}$  be a  $\sigma$ -consistent exceptional family of  $\text{Sbset}(\Omega)$ . For example  $\mathcal{N} = \text{Null}(P)$ .

**4.7 Lemma.**  $\inf\{r1_{[0,r)}(t); r \in \mathbb{Q}_{>0}\} = \max\{t, 0\}$  for all  $t \in \mathbb{R}$ .

**4.8 Lemma.** (i) If  $\sigma$  is an  $\mathcal{F}$ -optional time then  $(\mathcal{F} \vee \mathcal{N})_{\sigma+} = \mathcal{F}_{\sigma+} \vee \mathcal{N}$ .  
(ii) If  $\mathcal{N} \subset \mathcal{F}_0$ ,  $\sigma$  and  $\tau$  are  $\mathcal{F}$ -stopping times and  $\{\tau < \sigma\} \in \mathcal{N}$  then  $\mathcal{F}_\sigma \subset \mathcal{F}_\tau$ .  
(iii) Every  $\mathcal{F} \vee \mathcal{N}$ -optional time  $\tau$  has an  $\mathcal{F}$ -optional time modification  $\tilde{\tau}$ .

*Proof.* (i) Clearly we have that  $\mathcal{F}_{\sigma+} \subset (\mathcal{F} \vee \mathcal{N})_{\sigma+}$ . On the other hand  $\text{Sbset}(A) \subset \mathcal{N}$  for  $A \in \mathcal{N}$ , which implies that  $\mathcal{N} \subset (\mathcal{F} \vee \mathcal{N})_{\sigma+}$ . Conversely suppose that  $A \in (\mathcal{F} \vee \mathcal{N})_{\sigma+}$ . There exists a mapping  $B_0 : \mathbb{Q}_{>0} \rightarrow \mathcal{F}$  such that

$$B_0(q) \in \mathcal{F}_q \text{ and } (A \cap \{\sigma < q\}) \times B_0(q) \in \mathcal{N} \text{ for all } q \in \mathbb{Q}_{>0}.$$

Since  $B_0(r) \cap (A \cap \{\sigma < q\})^c \subset B_0(r) \cap (A \cap \{\sigma < r\})^c \in \mathcal{N}$  for  $r \in \mathbb{Q}_{(0,q]}$ , it follows that

$$(A \cap \{\sigma < q\}) \times B_1(q) \in \mathcal{N} \text{ for all } q \in \mathbb{Q}_{>0}$$

where  $B_1(q) := \bigcup_{r \in \mathbb{Q}: 0 < r \leq q} B_0(r) \in \mathcal{F}_q$ . We set

$$B := (A \cap \{\sigma = +\infty\}) \cup \bigcap_{r \in \mathbb{Q}: r > 0} ((B_1(r) \cup \{\sigma \geq r\}) \cap \{\sigma < +\infty\})$$

Let  $q \in \mathbb{Q}_{>0}$ . Since  $\{\sigma \geq r\} \cap \{\sigma < q\} = \emptyset$  and  $B_1(q) \subset B_1(r)$  for all  $r \in \mathbb{Q}_{\geq q}$ , we have that

$$B \cap \{\sigma < q\} = \bigcap_{r \in \mathbb{Q}: 0 < r < q} ((B_1(r) \cup \{\sigma \geq r\}) \cap \{\sigma < q\}) \cap (B_1(q) \cap \{\sigma < q\}).$$

(This relation may fail off  $\mathbb{Q}_{>0}$ , which means the impossibility of adjusting this claim for  $\mathcal{F}$ -stopping times.) Consequently  $B \cap \{\sigma < q\} \in \mathcal{F}_q$  for all  $q \in \mathbb{Q}_{>0}$  and hence  $B \in \mathcal{F}_{\sigma+}$ . The above relation also implies that  $(A \cap B^c) \cap \{\sigma < n\} \in \mathcal{N}$  for  $n \in \mathbb{N}$ . Indeed

$$(A \cap (B_1(r) \cup \{\sigma \geq r\})^c) \cap \{\sigma < n\} \subset (A \cap \{\sigma < r\}) \cap B_1(r)^c \in \mathcal{N} \text{ for all } r \in \mathbb{Q}_{(0,n]}.$$

On the other hand  $(B \cap A^c) \cap \{\sigma < n\} \in \mathcal{N}$  for  $n \in \mathbb{N}$  derives from

$$((B_1(n) \cup \{\sigma \geq n\}) \cap A^c) \cap \{\sigma < n\} \subset B_1(n) \cap (A \cap \{\sigma < n\})^c \in \mathcal{N}.$$

Since  $(A \bowtie B) \cap \{\sigma = +\infty\} = \emptyset$ , it follows that  $A \bowtie B = \bigcup_{n=1}^{\infty} (A \bowtie B) \cap \{\sigma < n\} \in \mathcal{N}$ .

(ii) According to Lemma 3.15  $\sigma \wedge \tau$  is an  $\mathcal{F}$ -stopping time and  $\mathcal{F}_{\sigma \wedge \tau} = \mathcal{F}_{\sigma} \cap \mathcal{F}_{\tau}$ . We show that  $\mathcal{F}_{\sigma \wedge \tau} = \mathcal{F}_{\sigma}$ . Let  $A \in \mathcal{F}_{\sigma}$  and  $t \in \mathbb{R}_{\geq 0}$ . We see that

$$A \cap \{\sigma \wedge \tau \leq t\} = (A \cap \{\sigma \leq t\}) \cup (A \cap \{\tau \leq t, \sigma > t\}).$$

Since  $A \cap \{\tau \leq t, \sigma > t\} \subset \{\tau < \sigma\}$  and  $\mathcal{N}$  is an exceptional family,  $A \cap \{\tau \leq t, \sigma > t\} \in \mathcal{N}$ . It follows that  $A \cap \{\sigma \wedge \tau \leq t\} \in \mathcal{F}_t \vee \mathcal{N} = \mathcal{F}_t$ . Consequently we get  $\mathcal{F}_{\sigma} \subset \mathcal{F}_{\sigma \wedge \tau}$ . Incidentally we have also verified that  $\mathcal{F}_{\sigma \vee \tau} = \mathcal{F}_{\tau}$ .

(iii) There exists a mapping  $B : \mathbb{Q}_{>0} \rightarrow \mathcal{F}$  such that

$$B(q) \in \mathcal{F}_q, \{\tau < q\} \bowtie B(q) \in \mathcal{N} \text{ for all } q \in \mathbb{Q}_{>0} \text{ and } B(q) \subset B(r) \text{ for all } r \in \mathbb{Q}_{\geq q}.$$

We set  $\tilde{\tau} := \inf\{q1_{B(q)}; q \in \mathbb{Q}_{>0}\}$  on  $B(\infty) := \bigcup_{q \in \mathbb{Q}_{>0}} B(q)$  and  $\tilde{\tau} := +\infty$  off  $B(\infty)$ . Then

$$\{\tau \neq \tilde{\tau}, \tau < +\infty\} \cup (\{\tau < +\infty\} \bowtie B(\infty)) \subset \bigcup_{q \in \mathbb{Q}_{>0}} (\{\tau < q\} \bowtie B(q)) \in \mathcal{N}$$

and  $\{\tilde{\tau} < t\} = \bigcup_{q \in \mathbb{Q}: 0 < q < t} B(q) \in \mathcal{F}_t$  for all  $t \in \mathbb{R}_{>0}$ .  $\square$

**4.9 Corollary.** (i) Suppose that  $\mathcal{F}_{t+} \subset \mathcal{F}_t \vee \mathcal{N}$  for all  $t \in \mathbb{R}_{\geq 0}$  and  $\sigma$  is an  $\mathcal{F}$ -optional time. Then  $\mathcal{F} \vee \mathcal{N}$  is a right continuous filtration and  $(\mathcal{F} \vee \mathcal{N})_{\sigma} = (\mathcal{F} \vee \mathcal{N})_{\sigma+} = \mathcal{F}_{\sigma+} \vee \mathcal{N}$ .

(ii) If  $\sigma$  is an  $\mathcal{F}$ -optional time then  $(\mathcal{F}_{\cdot+} \vee \mathcal{N})_{\sigma} = (\mathcal{F}_{\cdot+} \vee \mathcal{N})_{\sigma+} = \mathcal{F}_{\sigma+} \vee \mathcal{N} = (\mathcal{F} \vee \mathcal{N})_{\sigma+}$ .

*Proof.* (i) Let  $t \in \mathbb{R}_{\geq 0}$ . Since  $\mathcal{F}_t \subset \mathcal{F}_{t+}$ , the given condition implies that  $\mathcal{F}_{t+} \vee \mathcal{N} = \mathcal{F}_t \vee \mathcal{N}$ . It then follows by Lemma 4.8(i) that  $(\mathcal{F} \vee \mathcal{N})_{t+} = \mathcal{F}_{t+} \vee \mathcal{N} = \mathcal{F}_t \vee \mathcal{N}$ , which means the right continuity of  $\mathcal{F} \vee \mathcal{N}$ . Invoking Lemma 3.6(ii), we get  $(\mathcal{F} \vee \mathcal{N})_{\sigma} = (\mathcal{F} \vee \mathcal{N})_{\sigma+}$ . The right hand side coincides with  $\mathcal{F}_{\sigma+} \vee \mathcal{N}$  by Lemma 4.8(i).

(ii) Since  $(\mathcal{F}_{\cdot+})_{t+} = \mathcal{F}_{t+}$ , we see by (i) that  $\mathcal{F}_{\cdot+} \vee \mathcal{N}$  is right continuous. Finally we have that  $(\mathcal{F}_{\cdot+})_{\sigma+} = \mathcal{F}_{\sigma+}$  by Lemma 3.6(iii).  $\square$

## 5 Martingale convergence theorem

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\mathcal{F}$  be a filtration of sub  $\sigma$ -fields. We discuss submartingales with discrete parameter. So the index space is  $\mathbb{Z}_{\geq 0}$ .

**5.1 Lemma.** Suppose that  $f \in \text{map}(\mathbb{Z}_{\geq 0}, \mathbb{R})$  and  $a, b$  be a pair of real numbers with  $a < b$ . Then  $\liminf_{n \rightarrow \infty} f(n) < a$  and  $\limsup_{n \rightarrow \infty} f(n) > b$  if and only if

$$\sup\{\#\Delta; \Delta \text{ family of disjoint intervals s.t. } f(\min J) < a, f(\max J) > b \forall J \in \Delta\} = +\infty.$$

**5.2 Lemma.** Let  $X$  be an  $\mathcal{F}$ -submartingale and  $a, b$  be a pair of real numbers with  $a < b$ .

$$N := \sup\{\#\Delta; \Delta \text{ family of disjoint intervals s.t. } X_{\min J} < a, X_{\max J} > b \forall J \in \Delta\}$$

Then there exists a random variable  $N^*$  such that  $N \leq N^*$  and

$$E[N^*] \leq (\sup_{k \in \mathbb{N}} E[\max\{X_k, a\}] - E[\max\{X_0, a\}]) / (b - a).$$

*Proof.* We set  $S(0) := \inf\{i \in \mathbb{Z}_{\geq 0} : X_i < a\}$  with the convention  $\inf \emptyset = +\infty$ . For  $n \in \mathbb{N}$  we successively define

$$T(n) := \inf\{i \in \mathbb{Z}_{\geq 0} : i > S(n-1), X_i > b\}, S(n) := \inf\{i \in \mathbb{Z}_{\geq 0} : i > T(n), X_i < a\}.$$

Clearly  $S(n-1) < T(n)$  provided  $S(n-1) < +\infty$  and  $T(n) < S(n)$  provided  $T(n) < +\infty$ . Introduce the following random variable so that we have  $N = \sup_{k \in \mathbb{N}} N(k)$ :

$$N(k) := \#\{n \in \mathbb{N} : T(n) \leq k\} \quad k \in \mathbb{Z}_{\geq 0}.$$

Since  $N(k) \leq k$  for all  $k \in \mathbb{N}$ , we see that  $S(n) \geq T(n) \geq k$  when  $n > k$ . It follows that

$$\begin{aligned} \max\{X_k, a\} &= \sum_{n=1}^k (\max\{X_{T(n) \wedge k}, a\} - \max\{X_{S(n-1) \wedge k}, a\}) + \max\{X_{S(0) \wedge k}, a\} \\ &\quad + \sum_{n=1}^k (\max\{X_{S(n) \wedge k}, a\} - \max\{X_{T(n) \wedge k}, a\}) \end{aligned}$$

Observe that  $T(N(k)) \leq k < T(N(k)+1)$ . The first summation in the right hand side reads

$$\begin{aligned} &\sum_{n=1}^{N(k)} (\max\{X_{T(n)}, a\} - \max\{X_{S(n-1)}, a\}) \\ &\quad + (\max\{X_k, a\} - \max\{X_{S(N(k))}, a\}) \mathbf{1}_{\{S(N(k)) \leq k\}} \end{aligned}$$

Since  $X_{S(n-1)} < a$  provided  $S(n-1) < +\infty$  and  $X_{T(n)} > b$  provided  $X_{T(n)} < +\infty$ ,

$$\begin{aligned} \max\{X_{T(n)}, a\} - \max\{X_{S(n-1)}, a\} &\geq b - a \text{ for all } n \in \mathbb{N} \text{ with } n \leq N(k) \text{ and} \\ (\max\{X_k, a\} - \max\{X_{S(N(k))}, a\}) \mathbf{1}_{\{S(N(k)) \leq k\}} &= (\max\{X_k, a\} - a) \mathbf{1}_{\{S(N(k)) \leq k\}} \geq 0. \end{aligned}$$

Consequently we have that

$$\max\{X_k, a\} \geq (b-a)N(k) + \max\{X_{S(0) \wedge k}, a\} + \sum_{n=1}^k (\max\{X_{S(n) \wedge k}, a\} - \max\{X_{T(n) \wedge k}, a\}).$$

The random variables  $T(n)$  and  $S(n)$  are  $\mathcal{F}$ -stopping times. Indeed  $S(0)$  is an  $\mathcal{F}$ -stopping time since  $\{S(0) \leq i\} = \bigcap_{j \leq i} \{X_j < a\}$ . We then see that

$$\{T(n) \leq i\} = \bigcup_{j \leq i} \{S(n-1) < j, X_j > b\}, \{S(n) \leq i\} = \bigcup_{j \leq i} \{T(n) < j, X_j < b\}.$$

Recall that  $T(n) \leq S(n)$ . On the other hand, the function  $x \mapsto \max\{x, a\}$  being convex and non-decreasing, the process  $\max\{X, a\}$  is an  $\mathcal{F}$ -submartingale by Lemma 2.18. Invoking Theorem 2.6 we infer that

$$E[\max\{X_0, a\}] \leq E[\max\{X_{S(0) \wedge k}, a\}], E[\max\{X_{T(n) \wedge k}, a\}] \leq E[\max\{X_{S(n) \wedge k}, a\}]$$

for all  $n \in \mathbb{N}, k \in \mathbb{Z}_{\geq 0}$ . We thus obtain that

$$E[\max\{X_k, a\}] \geq (b-a)E[N(k)] + E[\max\{X_0, a\}].$$

Since  $N(k) \leq N(k+1)$  for all  $k \in \mathbb{N}$  and  $N = \sup_{k \in \mathbb{N}} N(k)$ , we get the conclusion by the monotone convergence theorem.  $\square$

**5.3 Lemma.** Let  $X$  be an  $\mathcal{F}$ -supermartingale and  $a, b$  be a pair of real numbers with  $a < b$ .

$$N := \sup\{\#\Delta; \Delta \text{ family of disjoint intervals s.t. } X_{\min J} < a, X_{\max J} > b \forall J \in \Delta\}$$

Then there exists a random variable  $N^*$  such that  $N \leq N^*$  and

$$E[N^*] \leq \sup_{k \in \mathbb{N}} E[\max\{a - X_k, 0\}] / (b - a).$$

*Proof.* We use the same notation as in the proof of Lemma 5.2. It follows that

$$\begin{aligned} \sum_{n=1}^k (X_{T(n) \wedge k} - X_{S(n-1) \wedge k}) &= \sum_{n=1}^{N(k)} (X_{T(n)} - X_{S(n-1)}) + (X_k - X_{S(N(k))}) 1_{\{S(N(k)) \leq k\}} \\ &\geq (b - a)N(k) + (X_k - a) 1_{\{S(N(k)) \leq k\}} \text{ for all } k \in \mathbb{Z}_{>0}. \end{aligned}$$

According to Theorem 2.6 the expectation of the left hand side is non-positive.  $\square$

**5.4 Theorem.** Suppose that  $X$  is an  $\mathcal{F}$ -submartingale,  $\sup_{k \in \mathbb{N}} E[\max\{X_k, 0\}] < +\infty$  and  $\sigma$  and  $\tau$  is an pair of  $\mathcal{F}$ -stopping times with  $\sigma \leq \tau$  a.s. Set  $X_\infty := \liminf_{k \rightarrow \infty} X_k$ .

(i)  $\liminf_{k \rightarrow \infty} X_k = \limsup_{k \rightarrow \infty} X_k$  a.s. and moreover  $E[|X_\sigma|] < +\infty$ .

(ii) If  $X$  is uniformly integrable then  $X$  converges to  $X_\infty$  in  $L^1$ -sense and  $X_\sigma \in E_{\leq}[X_\tau | \mathcal{F}_\sigma]$ .

*Proof.* (i) We infer by Lemma 5.1 and Lemma 5.2 that

$$P(\liminf_{k \rightarrow \infty} X_k < a, \limsup_{k \rightarrow \infty} X_k > b) = 0 \text{ for all pair } a, b \in \mathbb{Q} \text{ with } a < b.$$

This implies that  $P(\Omega_0) = 1$  where  $\Omega_0 := \{\omega : \liminf_{k \rightarrow \infty} X_k(\omega) = \limsup_{k \rightarrow \infty} X_k(\omega)\}$ . It follows by Lemma 2.18 and Theorem 2.11 that

$$E[|X_{\sigma \wedge k}|] = 2E[\max\{X_{\sigma \wedge k}, 0\}] - E[X_{\sigma \wedge k}] \leq 2E[\max\{X_k, 0\}] - E[X_0].$$

Since  $X_{\sigma \wedge k}$  converges to  $X_\sigma$  on  $\Omega_0$ , we see by Fatou's lemma that

$$E[|X_\sigma|] = E[\liminf_{k \rightarrow \infty} |X_{\sigma \wedge k}|] \leq 2 \sup_{k \in \mathbb{N}} E[\max\{X_k, 0\}] - E[X_0].$$

(ii) We infer by (i) and Lemma 3.7 that  $X$  converges to  $\liminf_{n \rightarrow \infty} X_n$  in  $L^1$ -sense. Let  $A \in \mathcal{F}_\sigma$ . Since  $\{\sigma \leq n\} \in \mathcal{F}_\sigma$  by Lemma 2.7, it follows by Theorem 2.11 that

$$E[X_\sigma; A \cap \{\sigma \leq n\}] = E[X_{\sigma \wedge n}; A \cap \{\sigma \leq n\}] \leq E[X_{\tau \wedge n}; A \cap \{\sigma \leq n\}] \text{ for all } n \in \mathbb{N}.$$

The left hand side converges to  $E[X_\sigma; A \cap \{\sigma < +\infty\}]$ . The right hand side reads as follows:

$$E[X_n; A \cap \{\sigma \leq n < \tau\}] + E[X_\tau; A \cap \{\tau \leq n\}]$$

We see that  $1_{\{\sigma \leq n < \tau\}}$  converges to  $1_{\{\sigma < +\infty, \tau = +\infty\}}$ . Due to the  $L^1$ -convergence of  $X_n$  to  $X_\infty$ , the first term of the above converges to  $E[X_\infty; A \cap \{\sigma < +\infty, \tau = +\infty\}]$ . Consequently

$$E[X_\sigma; A \cap \{\sigma < +\infty\}] \leq E[X_\infty; A \cap \{\sigma < +\infty, \tau = +\infty\}] + E[X_\tau; A \cap \{\tau < +\infty\}].$$

Since  $E[X_\sigma; A \cap \{\sigma = +\infty\}]$  equals  $E[X_\infty; A \cap \{\sigma = +\infty\}]$  and  $E[X_\tau; A \cap \{\tau = +\infty\}]$  equals  $E[X_\infty; A \cap \{\tau = +\infty\}]$ , adding the missing term  $E[X_\infty; A \cap \{\sigma = +\infty\}]$ , we reach that  $E[X_\sigma; A] \leq E[X_\tau; A]$  for all  $A \in \mathcal{F}_\sigma$ .  $\square$

**5.5 Corollary.** *Suppose that  $X$  is an  $\mathcal{F}$ -submartingale. If  $X$  is uniformly integrable then the family  $\{X_\sigma; \sigma \text{ } \mathcal{F}\text{-stopping times}\}$  is uniformly integrable where  $X_\infty := \liminf_{k \rightarrow \infty} X_k$ .*

*Proof.* Let  $\sigma$  be an  $\mathcal{F}$ -stopping time. Choose  $Y_n \in E[X_{n+1}|\mathcal{F}_n]$  for each  $n \in \mathbb{N}$ . We may assume that  $X_n(\omega) \leq Y_n(\omega)$  for all  $\omega \in \Omega$  else replace  $Y_n$  by  $\min\{X_n, Y_n\}$ . Let  $n \in \mathbb{N}$  and  $A \in \mathcal{F}_\sigma$ . Since  $E[Y_k; A \cap \{\sigma = n\}] = E[X_{k+1}; A \cap \{\sigma = n\}]$  for all  $k \in \mathbb{N}_{\geq n}$ , it follows that

$$\begin{aligned} E\left[\sum_{k=n}^{\infty} (Y_k - X_k); A \cap \{\sigma = n\}\right] &= \sum_{k=n}^{\infty} E[Y_k - X_k; A \cap \{\sigma = n\}] \\ &= \sum_{k=n}^{\infty} (E[X_{k+1}; A \cap \{\sigma = n\}] - E[X_k; A \cap \{\sigma = n\}]). \end{aligned}$$

The summation in the right hand side equals  $E[X_\infty; A \cap \{\sigma = n\}] - E[X_n; A \cap \{\sigma = n\}]$  according to Theorem 5.4(ii). We set  $M := X_\infty - \sum_{k=0}^{\infty} (Y_k - X_k)$ , which is integrable. Then

$$E[-M; A \cap \{\sigma = n\}] = E\left[\sum_{k=0}^{n-1} (Y_k - X_k); A \cap \{\sigma = n\}\right] - E[X_n; A \cap \{\sigma = n\}].$$

The right hand side equals  $E[\sum_{k < \sigma} (Y_k - X_k); A \cap \{\sigma = n\}] - E[X_\sigma; A \cap \{\sigma = n\}]$ . Therefore

$$E[-M; A \cap \{\sigma < +\infty\}] = E\left[\sum_{k < \sigma} (Y_k - X_k); A \cap \{\sigma < +\infty\}\right] - E[X_\sigma; A \cap \{\sigma < +\infty\}].$$

Note that  $-M = \sum_{k < \infty} (Y_k - X_k) - X_\infty$ . Adding the missing term  $E[-M; A \cap \{\sigma = +\infty\}]$ , we get

$$E[-M; A] = E\left[\sum_{k < \sigma} (Y_k - X_k); A\right] - E[X_\sigma; A] \text{ for all } A \in \mathcal{F}_\sigma.$$

This reads  $X_\sigma \in E[M|\mathcal{F}_\sigma] + \sum_{k < \sigma} (Y_k - X_k)$ . Since  $\sum_{k < \sigma} (Y_k - X_k)$  is non-negative and dominated by the integrable random variable  $\sum_{k=0}^{\infty} (Y_k - X_k)$ , we infer by Theorem 1.23 that  $\{X_\sigma; \sigma \text{ } \mathcal{F}\text{-stopping times}\}$  is uniformly integrable.  $\square$

**5.6 Theorem.** *Suppose  $X$  is an integrable random variable and  $X_n \in E[X|\mathcal{F}_n]$  for all  $n \in \mathbb{N}$ .*

- (i)  $X_\infty := \liminf_{n \rightarrow \infty} X_n = \limsup_{n \rightarrow \infty} X_n$  a.s. and  $X$  converges to  $X_\infty$  in  $L^1$ -sense.
- (ii)  $X_\infty \in E[X|\sigma(\mathcal{F})]$ . If  $X$  is  $\sigma(\mathcal{F}) \vee \text{Null}(P)$ -measurable then  $X_\infty = X$  a.s. and moreover  $X_\sigma \in E[X|\mathcal{F}_\sigma]$  for all  $\mathcal{F}$ -stopping times  $\sigma$ .

*Proof.* Applying Lemma 1.17 we see that  $X$  is an  $\mathcal{F}$ -martingale. According to Theorem 1.23  $\bigcup_{n=1}^{\infty} E[X|\mathcal{F}_n]$  is uniformly integrable. Hence Theorem 5.4 shows (i). Let  $A \in \mathcal{F}_n$ . Then

$$E[X_\infty; A] = E[X_n; A] = E[X; A].$$

by Corollary 5.5. Therefore the following Dynkin system contains the pre $\sigma$ -field  $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ :

$$\{A \in \mathcal{F} : E[X_\infty; A] = E[X; A]\}$$

The Dynkin system theorem shows that  $E[X_\infty; A] = E[X; A]$  for all  $A \in \sigma(\mathcal{F})$ . Since  $X_\infty$  is  $\sigma(\mathcal{F})$ -measurable, we get the first statement of (ii). The rest follows by Theorem 5.4(ii).  $\square$

**5.7 Definition.** An  $\mathcal{F}$ -supermartingale  $X$  such that  $X_k \geq 0$  a.s. for all  $k \in \mathbb{Z}_{\geq 0}$  and  $\limsup_{k \rightarrow \infty} X_k = 0$  a.s. is called an  $\mathcal{F}$ -potential.

**5.8 Lemma.** Suppose that  $X$  is an  $\mathcal{F}$ -supermartingale and  $X_k \geq 0$  a.s. for all  $k \in \mathbb{Z}_{\geq 0}$ .

(i)  $\liminf_{k \rightarrow \infty} X_k = \limsup_{k \rightarrow \infty} X_k$  a.s. and

$$E[\limsup_{k \rightarrow \infty} X_k; A] \leq E[X_n; A] \text{ for all } A \in \mathcal{F}_n \text{ and } n \in \mathbb{Z}_{\geq 0}.$$

(ii) Let  $Y_n \in E[\limsup_{k \rightarrow \infty} X_k | \mathcal{F}_n]$  for  $n \in \mathbb{Z}_{\geq 0}$ . Then  $0 \leq Y_n \leq X_n$  a.s. for all  $n \in \mathbb{Z}_{\geq 0}$  and  $X_n - Y_n$  is an  $\mathcal{F}$ -potential.

(iii) If  $E[\limsup_{k \rightarrow \infty} X_k] = E[X_0]$  then  $X_n \in E[\limsup_{k \rightarrow \infty} X_k | \mathcal{F}_n]$  for all  $n \in \mathbb{Z}_{\geq 0}$ .

*Proof.* (i) We see by applying Theorem 5.4 to the non-positive  $\mathcal{F}$ -submartingale  $-X$  that  $\liminf_{k \rightarrow \infty} X_k = \limsup_{k \rightarrow \infty} X_k$  a.s. Let  $n \in \mathbb{Z}_{\geq 0}$  and  $A \in \mathcal{F}_n$ . Since  $E[X_k; A] \leq E[X_n; A]$  for all  $k \geq n$  and  $X_k \geq 0$  a.s., Fatou's lemma shows that

$$E[\limsup_{k \rightarrow \infty} X_k; A] = E[\liminf_{k \rightarrow \infty} X_k; A] \leq \liminf_{k \rightarrow \infty} E[X_k; A] \leq E[X_n; A].$$

(iii) Let  $Y \in E[\limsup_{k \rightarrow \infty} X_k | \mathcal{F}_\infty]$ . It follows that  $Y \leq X_n$  a.s. On the other hand

$$E[X_n] \leq E[X_0] = E[\limsup_{k \rightarrow \infty} X_k] = E[Y].$$

Consequently  $Y = X_n$  a.s. Thus we get  $X_n \in E[\limsup_{k \rightarrow \infty} X_k | \mathcal{F}_n]$ .  $\square$

**5.9 Theorem.** Suppose that  $\mu$  is a probability measure on  $(\Omega, \mathcal{F})$  which is locally  $(\mathcal{F}, P)$ -absolutely continuous and  $X_n \in D(\mu/P | \mathcal{F}_n)$ .

(i)  $\limsup_{k \rightarrow \infty} X_k < +\infty$   $P$ -a.s.,  $\liminf_{k \rightarrow \infty} X_k = \limsup_{k \rightarrow \infty} X_k$   $\mu$ -a.e. and

$$\mu(A \cap \{\limsup_{k \rightarrow \infty} X_k < +\infty\}) = E[\limsup_{k \rightarrow \infty} X_k; A] \text{ for all } A \in \sigma(\mathcal{F}).$$

(ii)  $\bigcup_{n=1}^{\infty} D(\mu/P | \mathcal{F}_n)$  is uniformly integrable  $\Leftrightarrow \mu(\{\limsup_{k \rightarrow \infty} X_k < +\infty\}) = 1$   
 $\Leftrightarrow E[\limsup_{k \rightarrow \infty} X_k] = 1 \Leftrightarrow \mu$  is  $P$ -absolutely continuous on  $\sigma(\mathcal{F})$ .

*Proof.* Since  $E[\limsup_{n \rightarrow \infty} X_n] \leq 1$  by Lemma 5.8, we get  $P(\limsup_{k \rightarrow \infty} X_k = +\infty) = 0$ . We introduce the following probability measure on  $(\Omega, \mathcal{F})$ :

$$\nu(A) := P(A)/2 + \mu(A)/2 \quad A \in \mathcal{F}.$$

Let  $Y \in D(P/\nu | \sigma(\mathcal{F}))$ . Then,  $P$  being  $\nu$ -absolutely continuous, it follows that

$$(\star) \quad P(A) = E^\nu[Y; A] \text{ and } \mu(A) = 2\nu(A) - P(A) = E^\nu[(2 - Y); A] \text{ for all } A \in \sigma(\mathcal{F}).$$

Let  $Y_n \in E^\nu[Y | \mathcal{F}_n]$ . Then we have that

$$P(A) = E^\nu[Y_n; A] \text{ and } \mu(A) = E^\nu[(2 - Y_n); A] \text{ for all } A \in \mathcal{F}_n.$$

On the other hand, since  $X_n \geq 0$   $P$ -a.s., we see by Lemma 1.14 that  $X_n Y_n \geq 0$   $\nu$ -a.s. and

$$E^\nu[X_n Y_n; A] = E[X_n; A] = \mu(A) = E^\nu[(2 - Y_n); A] \text{ for all } A \in \mathcal{F}_n.$$

The second equality holds because  $\mu$  is  $P$ -absolutely continuous on  $\mathcal{F}_n$ . We set

$$\Omega_\infty := \bigcap_{n=1}^{\infty} \{\omega \in \Omega : 0 \leq Y_n(\omega) \leq 2, X_n(\omega)Y_n(\omega) = 2 - Y_n(\omega)\}.$$

It follows that  $\nu(\Omega_\infty) = 1$ . Observe that

$$Y_n > 0 \text{ on } \Omega_\infty \text{ and } X_n = 2/Y_n - 1 \text{ on } \Omega_\infty.$$

Theorem 5.6 claims that  $\liminf_{n \rightarrow \infty} Y_n = Y = \limsup_{n \rightarrow \infty} Y_n$   $\nu$ -a.s. Consequently

$$(\star\star) \quad \limsup_{n \rightarrow \infty} X_n = 2/Y - 1 = \liminf_{n \rightarrow \infty} X_n \text{ } \nu\text{-a.s.}$$

This means that  $\{\limsup_{n \rightarrow \infty} X_n < +\infty\}$  coincides with  $\{Y > 0\}$  up to  $\nu$ -null set. Since  $\mu$  is  $\nu$ -absolutely continuous, it follows that

$$\mu(A \cap \{\limsup_{n \rightarrow \infty} X_n < +\infty\}) = \mu(A \cap \{Y > 0\}) \text{ for all } A \in \mathcal{F}.$$

If  $A \in \sigma(\mathcal{F})$  then the right hand side reads  $E^\nu[(2 - Y); A \cap \{Y > 0\}]$  by  $(\star)$ . We see by  $(\star\star)$  that  $(2 - Y)1_{\{Y > 0\}} = Y \limsup_{n \rightarrow \infty} X_n$   $\nu$ -a.s. Therefore

$$\mu(A \cap \{\limsup_{n \rightarrow \infty} X_n < +\infty\}) = E^\nu[Y \limsup_{n \rightarrow \infty} X_n; A] \text{ for all } A \in \sigma(\mathcal{F}).$$

The right hand side coincides with  $E[\limsup_{n \rightarrow \infty} X_n; A]$  according to Lemma 1.14.  $\square$

*5.10 Remark.* Suppose that  $X$  is an  $\mathcal{F}$ -martingale,  $X_k \geq 0$  a.s. for all  $k \in \mathbb{Z}_{\geq 0}$  and the premeasure  $\bigcup_{n=1}^{\infty} \mathcal{F}_n \rightarrow \mathbb{R}$ ,  $A \mapsto \liminf_{n \rightarrow \infty} E[X_n; A]$  is  $\sigma$ -additive. Then

$$E[\limsup_{k \rightarrow \infty} X_k] + \lim_{a \rightarrow +\infty} \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} E[X_m; \bigcup_{k=n}^m \{X_k > a\}] = 1.$$

$$E[\liminf_{k \rightarrow \infty} X_k] + \lim_{a \rightarrow +\infty} \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} E[X_m; \bigcap_{k=n}^m \{X_k > a\}] = 1.$$

**5.11 Example.** We choose the following product measure space as  $(\Omega, \mathcal{F}, P)$ :

$$\prod(\mathbb{Z}_{\geq 0}, \{0, 1\}), \otimes(\mathbb{Z}_{\geq 0}, \text{Sbset}(\{0, 1\})), \otimes(\mathbb{Z}_{\geq 0}, (\delta_0 + \delta_1)/2).$$

We set  $\mathcal{F}_n := \sigma\{\text{proj}_{\leq n}\}$  for  $n \in \mathbb{Z}_{\geq 0}$  where  $\text{proj}_{\leq n}$  is the canonical mapping

$$\prod(\mathbb{Z}_{\geq 0}, \{0, 1\}) \rightarrow \prod(\mathbb{Z}_{[0, n]}, \{0, 1\}).$$

Then  $\mathcal{F}$  is a filtration on the present probability space. Given  $p \in \mathbb{R}_{(0,1)}$  the measure

$$\mu := \otimes(\mathbb{Z}_{\geq 0}, (1 - p)\delta_0 + p\delta_1)$$

is locally  $(\mathcal{F}, P)$ -absolutely continuous. Indeed the random variable

$$X_n : \omega \mapsto 2^{n+1} p^{\#\{k \leq n : \omega(k) = 1\}} (1 - p)^{\#\{k \leq n : \omega(k) = 0\}}$$

belongs to  $D(\mu/P|\mathcal{F}_n)$  for each  $n \in \mathbb{Z}_{\geq 0}$ . Clearly if  $p = 1/2$  then  $X_n(\omega) = 1$  for all  $\omega$  and  $n$ . In what follows suppose that  $p \neq 1/2$ . Observe that

$$\frac{\log X_n(\omega)}{n + 1} = \log(2 - 2p) + \log \frac{p}{1 - p} \frac{\#\{k \leq n : \omega(k) = 1\}}{n + 1}.$$

By virtue of the strong law of large number the above converges to  $\log \sqrt{4p(1 - p)}$  as  $n$  tends to  $\infty$   $P$ -a.s. Since  $4p(1 - p) < 1$  unless  $p = 1/2$ , it follows that  $\limsup_{n \rightarrow \infty} X_n = 0$   $P$ -a.s.

## 6 Submartingale inequalities

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\mathcal{F}$  be a filtration of sub  $\sigma$ -fields. We directly discuss the continuous time situation. So the index set is  $\mathbb{R}_{\geq 0}$ .

**6.1 Lemma.** *Suppose  $X$  is an  $\mathcal{F}$ -submartingale such that almost every sample path is right continuous and  $\sigma$  is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time such that  $\sigma \leq T$  a.s. for some  $T \in \mathbb{R}_{>0}$ .*

(i)  $P(\sup\{X_s; s \in \mathbb{R}_{[0, \sigma]}\} \geq \lambda) \leq E[\max\{X_\sigma, 0\}]/\lambda$  for all  $\lambda \in \mathbb{R}_{>0}$ .

(ii)  $P(\inf\{X_s; s \in \mathbb{R}_{[0, \sigma]}\} \leq \lambda) \leq (E[X_0] - E[\max\{X_\sigma, 0\}])/\lambda$  for all  $\lambda \in \mathbb{R}_{<0}$ .

*Proof.* There exists a right continuous stochastic process  $Y$  such that  $X_t = Y_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. Then  $Y$  is an  $\mathcal{F} \vee \text{Null}(P)$ -submartingale. Given  $\lambda \in \mathbb{R}_{>0}$ , we write

$$\tau := \inf\{s \in \mathbb{R}_{\geq 0} : Y_s > \lambda\},$$

which is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time. Indeed  $\{\tau < t\} = \bigcup_{s \in \mathbb{Q}: 0 \leq s < t} \{Y_s > \lambda\} \in \mathcal{F}_t \vee \text{Null}(P)$  for all  $t \in \mathbb{R}_{>0}$  due to the right continuity of sample path. We have that

$$\sup\{Y_s; s \in \mathbb{R}_{[0, \sigma]}\} > \lambda \Leftrightarrow \tau < \sigma \text{ or } \tau = \sigma, Y_\sigma > \lambda$$

Clearly  $P(\tau = \sigma, Y_\sigma > \lambda) \leq E[\max\{Y_\sigma, 0\}; \tau = \sigma, Y_\sigma > \lambda]/\lambda$ . On the other hand, since  $Y_\tau \geq \lambda$  on  $\{\tau < +\infty\}$  due to the right continuity of sample path, it follows that

$$P(\tau < \sigma) \leq E[Y_{\tau \wedge \sigma}; \tau < \sigma]/\lambda.$$

According to Lemma 3.15(ii) we have  $\{\tau < \sigma\} \in (\mathcal{F} \vee \text{Null}(P))_{(\tau \wedge \sigma)^+}$ . We see by Theorem 3.9 that

$$E[Y_{\tau \wedge \sigma}; \tau < \sigma] \leq E[Y_\sigma; \tau < \sigma] \leq E[\max\{Y_\sigma, 0\}; \tau < \sigma].$$

Consequently  $P(\tau < \sigma) \leq E[\max\{Y_\sigma, 0\}; \tau < \sigma]/\lambda$ . Thus we get

$$P(\sup\{Y_s; s \in \mathbb{R}_{[0, \sigma]}\} > \lambda) \leq E[\max\{Y_\sigma, 0\}; \sup\{Y_s; s \in \mathbb{R}_{[0, \sigma]}\} > \lambda]/\lambda.$$

We turn to show (ii). This time, given  $\lambda \in \mathbb{R}_{<0}$ , introduce an  $\mathcal{F} \vee \text{Null}(P)$ -optional time by

$$\tau := \inf\{s \in \mathbb{R}_{\geq 0} : Y_s < \lambda\}.$$

We see by Theorem 3.9 that

$$E[Y_0] \leq E[Y_{\tau \wedge \sigma}] = E[Y_\tau; \tau < \sigma] + E[Y_\sigma; \tau = \sigma, Y_\sigma < \lambda] + E[Y_\sigma; \tau \geq \sigma, Y_\sigma \geq \lambda].$$

The term  $E[Y_\sigma; \tau \geq \sigma, Y_\sigma \geq \lambda]$  is dominated by  $E[\max\{Y_\sigma, 0\}]$ . Consequently

$$(E[Y_\tau; \tau < \sigma] + E[Y_\sigma; \tau = \sigma, Y_\sigma < \lambda])/\lambda \leq (E[Y_0] - E[\max\{Y_\sigma, 0\}])/\lambda.$$

The left hand side dominates  $P(\tau < \sigma) + P(\tau = \sigma, Y_\sigma < \lambda)$ . Thus we get (ii).  $\square$

**6.2 Lemma.** *Suppose that  $X$  is an  $\mathcal{F}$ -submartingale whose almost every sample path is right continuous and  $\sigma$  is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time such that  $\sigma \leq T$  a.s. for some  $T \in \mathbb{R}_{\geq 0}$ . Then  $E[\sup\{\max\{X_s, 0\}; s \in \mathbb{R}_{[0, \sigma]}\}^p] \leq \{p/(p-1)\}^p E[\max\{X_\sigma, 0\}^p]$  for all  $p \in \mathbb{R}_{>1}$ .*

*Proof.* Note that  $\max\{X, 0\}$  is an  $\mathcal{F}$ -submartingale. We write

$$Y := \sup\{\max\{X_s, 0\}; s \in \mathbb{R}_{[0, \sigma]}\}.$$

As is shown in the proof of Lemma 6.1, we have that

$$P(Y > \lambda) \leq E[\max\{X_\sigma, 0\}; Y > \lambda]/\lambda \text{ for all } \lambda \in \mathbb{R}_{>0}.$$

Let  $b \in \mathbb{R}_{>0}$ . We observe that

$$E[(Y \wedge b)^p] = \int_0^b p\lambda^{p-1}P(Y > \lambda) d\lambda \leq \int_0^b p\lambda^{p-1}E[\max\{X_\sigma, 0\}; Y > \lambda]/\lambda d\lambda.$$

Invoking Fubini's theorem we see that the right hand side reads

$$pE[\max\{X_\sigma, 0\}(Y \wedge b)^{p-1}/(p-1)].$$

Using Hölder's inequality we infer that

$$E[(Y \wedge b)^p] \leq p/(p-1)E[\max\{X_\sigma, 0\}^p]^{1/p}E[(Y \wedge b)^p]^{1-1/p}.$$

Dividing by  $E[(Y \wedge b)^p]^{1-1/p}$  (provided it is positive) we get

$$E[(Y \wedge b)^p] \leq \{p/(p-1)\}^p E[\max\{X_\sigma, 0\}^p].$$

The inequality trivially holds if  $E[(Y \wedge b)^p] = 0$ . Tending  $b$  to  $+\infty$  we reach the claim.  $\square$

*6.3 Remark.* The function  $\mathbb{R}_{>1} \rightarrow \mathbb{R}, p \mapsto p \log p - p \log(p-1)$  is convex. Indeed the derivative and the second derivative read  $\log p - \log(p-1) - 1/(p-1)$  and  $1/\{p(p-1)^2\}$  respectively. Since the derivative converges to 0 as  $p$  tends to  $+\infty$ , the derivative is negative. Finally the function converges to 1 as  $p$  tends to  $+\infty$ . Thus  $\{p/(p-1)\}^p \geq e$  for all  $p \in \mathbb{R}_{>1}$ .

**6.4 Example.** Let  $M.$  be an  $\mathcal{F}$ -martingale whose almost every sample path is right continuous and  $\sigma$  is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time such that  $\sigma \leq T$  a.s. for some  $T \in \mathbb{R}_{\geq 0}$ .

- (i)  $P(\sup\{|M_s|; s \in \mathbb{R}_{[0,\sigma]}\} \geq \lambda) \leq E[|M_\sigma|^p]/\lambda^p$  for all  $\lambda \in \mathbb{R}_{>0}$  and  $p \in \mathbb{R}_{\geq 1}$ .
- (ii)  $E[\sup\{|M_s|; s \in \mathbb{R}_{[0,\sigma]}\}^p] \leq \{p/(p-1)\}^p E[|M_\sigma|^p]$  for all  $p \in \mathbb{R}_{>1}$ .

*Proof.* (i) Apply Lemma 6.1(i) to the submartingale  $t \mapsto |M_t|^p$ .

(ii) Apply Lemma 6.2 to the submartingale  $t \mapsto |M_t|$ .  $\square$

**6.5 Example.** Let  $W.$  be a complex Wiener process and  $\sigma$  a bounded optional time. Then

- (i)  $E[\max\{|W_t|^p; t \in \mathbb{R}_{[0,\sigma]}\}] \leq p^{p/(p-1)}E[|W_\sigma|^p]$  for all  $p \in \mathbb{R}_{>1}$ .
- (ii)  $E[\max\{|W_t|; t \in \mathbb{R}_{[0,\sigma]}\}] \leq eE[|W_\sigma|]$ .

*Proof.* Observe that  $|W.|^q$  is a submartingale for all  $q \in \mathbb{R}_{>0}$ . It follows by Lemma 6.1 that

$$P(Y > \lambda) \leq E[|W_\sigma|^q; Y > \lambda]/\lambda^q \text{ for all } q \in \mathbb{R}_{>0} \text{ and } \lambda \in \mathbb{R}_{>0}$$

where  $Y := \max\{|W_t|; t \in \mathbb{R}_{[0,\sigma]}\}$ . Let  $b \in \mathbb{R}_{>0}$ . We observe that

$$E[(Y \wedge b)^p] = \int_0^b p\lambda^{p-1}P(Y > \lambda) d\lambda \leq \int_0^b p\lambda^{p-1}E[|W_\sigma|^{p-1}; Y > \lambda]/\lambda^{p-1} d\lambda.$$

The rest of the argument is the same as that in the proof of Lemma 6.2.  $\square$

## 7 Local martingales and uniform integrability

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\mathcal{F}$  be a filtration of sub  $\sigma$ -fields. We discuss local martingales with continuous parameter. So the index space is  $\mathbb{R}_{\geq 0}$ .

$\text{Time}(\mathcal{F})$  denotes the set of all  $\mathcal{F}$ -optional times.

**7.1 Definition.** A stochastic process  $X$  is called an  $\mathcal{F}$ -local submartingale if it is  $\mathcal{F}$ -adapted and there exists a sequence  $S(\cdot)$  of  $\text{Time}(\mathcal{F} \vee \text{Null}(P))$  such that

$$S(n) \leq S(n+1) \text{ a.s.}, \sup_{n \in \mathbb{N}} S(n) = +\infty \text{ a.s.}, E[|X_0|; S(n) > 0] < +\infty \text{ and} \\ X_{\cdot \wedge S(n)} - X_0 \text{ is an } \mathcal{F} \vee \text{Null}(P)\text{-submartingale.}$$

Such  $S(\cdot)$  is called a reducing sequence.

*7.2 Remark.* Let  $t \in \mathbb{R}_{\geq 0}$  and  $a \in \mathbb{R}$ . Observe that the set  $\{X_t < a\}$  coincides with  $\bigcup_{n=1}^{\infty} \{X_t < a, t \leq S(n)\} = \bigcup_{n=1}^{\infty} \{X_{t \wedge S(n)} < a, t \leq S(n)\}$  up to  $\text{Null}(P)$ .

**7.3 Lemma.** (i) All  $\mathcal{F}$ -submartingales are  $\mathcal{F}$ -local submartingales.

(ii) A stochastic process  $X$  is a  $\mathcal{F}$ -local submartingale if and only if it is an  $\mathcal{F}$ -adapted  $\mathcal{F} \vee \text{Null}(P)$ -local submartingale.

*Proof.* (i) We see by Lemma 1.16 that a stochastic process  $X$  is a  $\mathcal{F}$ -submartingale if and only if it is an  $\mathcal{F}$ -adapted  $\mathcal{F} \vee \text{Null}(P)$ -submartingale. Thus as for  $\mathcal{F}$ -submartingales the sequence  $S(n) := +\infty$  serves as a reducing sequence.  $\square$

**7.4 Lemma.** Suppose that  $X$  is a stochastic process such that  $P(|X_0| < +\infty) = 1$  and almost every sample path is right continuous, and  $\sigma \in \text{Time}(\mathcal{F})$ . Then  $X_0 1_{\sigma > 0}$  is  $\mathcal{F}_{0+}$ -measurable,  $E[|X_0|; \sigma > 0] < +\infty$  and  $X_{\cdot \wedge \sigma} - X_0$  is an  $\mathcal{F}$ -submartingale if and only if  $X_{\cdot \wedge \sigma} 1_{\sigma > 0}$  is an  $\mathcal{G}$ -submartingale where  $\mathcal{G}_0 := \mathcal{F}_{0+}$  and  $\mathcal{G}_t := \mathcal{F}_t$  for  $t \in \mathbb{R}_{> 0}$ .

*Proof.* Suppose that  $X_0 1_{\sigma > 0}$  is  $\mathcal{F}_{0+}$ -measurable,  $E[|X_0|; \sigma > 0] < +\infty$  and  $X_{\cdot \wedge \sigma} - X_0$  is an  $\mathcal{F}$ -submartingale. Since  $X_{t \wedge \sigma} - X_0 + X_0 1_{\sigma > 0} = X_{t \wedge \sigma} 1_{\sigma > 0}$ , it follows that

$$X_{\cdot \wedge \sigma} 1_{\sigma > 0} \text{ is an } \mathcal{F}\text{-submartingale with index set } \mathbb{R}_{> 0}$$

and  $E[X_{t \wedge \sigma}; \sigma > 0] \geq E[X_0; \sigma > 0] > -\infty$  for all  $t \in \mathbb{R}_{> 0}$ . We see by Lemma 2.13 that the sequence  $X_{(1/n) \wedge \sigma} 1_{\sigma > 0}$  is uniformly integrable. This sequence converges to  $X_0 1_{\sigma > 0}$  a.s. due to the right continuity of sample path. Let  $A \in \mathcal{F}_{0+}$  and  $t \in \mathbb{R}_{> 0}$ . Then

$$E[X_{(1/n) \wedge \sigma}; \sigma > 0, A] \leq E[X_{t \wedge \sigma}; \sigma > 0, A] \text{ for all } n \in \mathbb{N}_{> 1/t}.$$

Thus, tending  $n$  to  $\infty$ , we conclude that  $E[X_0; \sigma > 0, A] \leq E[X_{t \wedge \sigma}; \sigma > 0, A]$  and hence  $X_{\cdot \wedge \sigma} 1_{\sigma > 0}$  is an  $\mathcal{G}$ -submartingale. The converse implication is easy.  $\square$

**7.5 Lemma.** Suppose that  $X$  and  $Y$  are  $\mathcal{F}$ -local submartingales and  $\tau \in \text{Time}(\mathcal{F} \vee \text{Null}(P))$ .

(i) If  $f$  is nonnegative, bounded and  $\mathcal{F}_0$ -measurable then  $t \mapsto f X_t$  is an  $\mathcal{F}$ -local submartingale and  $t \mapsto -f X_t$  is an  $\mathcal{F}$ -local supermartingale.

(ii) If almost all sample paths of  $X$  and  $Y$  are right continuous then  $X + Y$  is an  $\mathcal{F}$ -local submartingale and  $X_{\cdot \wedge \tau}$  is an  $\mathcal{F} \vee \text{Null}(P)$ -local submartingale.

(ii) If every sample path of  $X$  is right continuous and  $\tau \in \text{Time}(\mathcal{F})$  then  $X_{\cdot \wedge \tau}$  is an  $\mathcal{F}$ -local submartingale.

*Proof.* (ii) We select and fix  $\mathcal{F}$ -reducing sequences  $S(\cdot)$  and  $T(\cdot)$  for  $X$  and  $Y$  respectively. Invoking Corollary 3.17(ii) to the  $\mathcal{F} \vee \text{Null}(P)$ -submartingales  $t \mapsto X_{t \wedge S(n)} - X_0$ , we see that  $S(\cdot)$  also serves as an  $\mathcal{F} \vee \text{Null}(P)$ -reducing sequence for the  $\mathcal{F} \vee \text{Null}(P)$ -adapted process  $X_{\cdot \wedge \tau}$ . Likewise the sequence  $n \mapsto S(n) \wedge T(n)$  serves as an  $\mathcal{F}$ -reducing sequence for the  $\mathcal{F}$ -adapted process  $X + Y$ .  $\square$

**7.6 Lemma.** *Let  $X$  be an  $\mathcal{F}$ -local supermartingale. If almost every sample path is right continuous,  $X_t \geq 0$  a.s. for all  $t \in \mathbb{R}_{\geq 0}$  and  $E[X_0] < +\infty$  then  $X$  is an  $\mathcal{F}$ -supermartingale.*

*Proof.* Let  $S(\cdot)$  be an  $\mathcal{F}$ -reducing sequence for  $X$ . Fix  $s, t \in \mathbb{R}_{\geq 0}$  with  $s < t$ . Then, since  $X_0$  is integrable, it follows that

$$X_{s \wedge S(n)} \in E_{\geq}[X_{t \wedge S(n)} | \mathcal{F}_s \vee \text{Null}(P)] \text{ for all } n \in \mathbb{N}.$$

For each  $n \in \mathbb{N}$  we choose  $Y_n \in E[X_{t \wedge S(n)} | \mathcal{F}_s]$  so that  $X_{s \wedge S(n)} \geq Y_n$  a.s. Let  $Y \in E[X_t | \mathcal{F}_s]$ , that is,  $Y \in E[\liminf_{n \rightarrow \infty} X_{t \wedge S(n)} | \mathcal{F}_s]$ . The right continuity turns  $X_u \geq 0$  a.s.  $\forall u \in \mathbb{R}_{\geq 0}$  into  $\inf_{u \geq 0} X_u \geq 0$  a.s. We see by Corollary 1.11 that

$$X_s = \liminf_{n \rightarrow \infty} X_{s \wedge S(n)} \geq \liminf_{n \rightarrow \infty} Y_n \geq Y \text{ a.s.}$$

Fatou's lemma shows that  $E[X_s] \leq \liminf_{n \rightarrow \infty} E[X_{s \wedge S(n)}] \leq E[X_0]$ , which means the integrability of  $X_s$ . Since it is  $\mathcal{F}_s$ -measurable, we get  $X_s \in E_{\geq}[X_t | \mathcal{F}_s]$ .  $\square$

**7.7 Lemma.** *Suppose that  $X$  is an  $\mathcal{F}$ -supermartingale with almost sure right continuous sample path,  $X_t \geq 0$  a.s. for all  $t \in \mathbb{R}_{\geq 0}$ ,  $\sigma, \tau \in \text{Time}(\mathcal{F} \vee \text{Null}(P))$ .*

(i)  $X_{\tau} \geq 0$  a.s. on  $\{\tau < +\infty\}$ ,  $[X_{\tau}; \tau < +\infty] \leq E[X_0]$  and

$$X_{\sigma \wedge \tau} 1_{\sigma \wedge \tau < +\infty} \in E_{\geq}[X_{\tau} 1_{\tau < +\infty} | (\mathcal{F} \vee \text{Null}(P))_{\sigma+}].$$

(ii) *If  $\tau < +\infty$  a.s. and  $E[X_{\tau}] = E[X_0]$  then  $X_{\sigma \wedge \tau} \in E[X_{\tau} | (\mathcal{F} \vee \text{Null}(P))_{\sigma+}]$ .*

*Proof.* (i) Let  $A \in (\mathcal{F} \vee \text{Null}(P))_{\sigma+}$  and  $n \in \mathbb{N}$ . Then, since  $\sigma \wedge \tau$  is  $(\mathcal{F} \vee \text{Null}(P))_{\sigma+}$ -measurable, we see by Corollary 3.17 that

$$E[X_{\sigma \wedge \tau}; A \cap \{\sigma \wedge \tau < n\}] = E[X_{\sigma \wedge \tau \wedge n}; A \cap \{\sigma \wedge \tau < n\}] \geq E[X_{\tau \wedge n}; A \cap \{\sigma \wedge \tau < n\}].$$

The right continuity turns  $X_t \geq 0$  a.s. for all  $t \in \mathbb{R}_{\geq 0}$  into  $\inf_{t \geq 0} X_t \geq 0$  a.s. Therefore the left hand side is dominated by  $E[X_{\sigma \wedge \tau}; A \cap \{\sigma \wedge \tau < +\infty\}]$ . On the other hand, since  $\{\tau < n\} \subset \{\sigma \wedge \tau < n\}$ , the right hand side dominates  $E[X_{\tau \wedge n}; A \cap \{\tau < n\}]$ , which reads  $E[X_{\tau}; A \cap \{\tau < n\}]$ . It then follows that

$$E[X_{\sigma \wedge \tau}; A \cap \{\sigma \wedge \tau < +\infty\}] \geq E[X_{\tau}; A \cap \{\tau < n\}] \text{ for all } n \in \mathbb{N}.$$

Invoking the monotone convergence theorem, we infer that

$$E[X_{\sigma \wedge \tau}; A \cap \{\sigma \wedge \tau < +\infty\}] \geq E[X_{\tau}; A \cap \{\tau < +\infty\}] \text{ for all } A \in \mathcal{F}_{\sigma+}.$$

Since  $X_{\sigma \wedge \tau} 1_{\sigma \wedge \tau < +\infty}$  is  $(\mathcal{F} \vee \text{Null}(P))_{\sigma+}$ -measurable by Lemma 3.8 and Lemma 3.15, we reach the statement (i).

(ii) Suppose  $\tau < +\infty$  and  $E[X_{\tau}] = E[X_0]$ . We select and fix  $Y \in E[X_{\tau} | (\mathcal{F} \vee \text{Null}(P))_{\sigma+}]$ . Then we have that  $X_{\sigma \wedge \tau} \geq Y$  by (i) and  $E[X_{\sigma \wedge \tau}]$  is dominated by  $E[X_0] = E[X_{\tau}] = E[Y]$ . Thus we get  $Y = X_{\sigma \wedge \tau}$  a.s.  $\square$

**7.8 Remark.** If one takes Lemma 3.13 into account, he can also draw the following statement: Suppose that  $X.$  is a right continuous  $\mathcal{F}$ -supermartingale,  $X_t \geq 0$  a.s. for all  $t \in \mathbb{R}_{\geq 0}$ ,  $\sigma \in \text{Time}(\mathcal{F})$  (respectively  $\sigma$  is an  $\mathcal{F}$ -stopping time),  $\tau \in \text{Time}(\mathcal{F})$  and  $\tau < +\infty$  a.s.

(i)  $1_{\{\sigma \wedge \tau < +\infty\}} X_{\sigma \wedge \tau} \in E_{\geq}[X_{\tau} | \mathcal{F}_{\sigma+}]$  ( $1_{\{\sigma < +\infty\}} X_{\sigma \wedge \tau} \in E_{\geq}[X_{\tau} | \mathcal{F}_{\sigma}]$ ).

(ii) If  $E[X_{\tau}] = E[X_0]$  then  $1_{\{\sigma \wedge \tau < +\infty\}} X_{\sigma \wedge \tau} \in E[X_{\tau} | \mathcal{F}_{\sigma+}]$  ( $1_{\{\sigma < +\infty\}} X_{\sigma \wedge \tau} \in E[X_{\tau} | \mathcal{F}_{\sigma}]$ ).

**7.9 Theorem.** Suppose that  $M.$  is an  $\mathcal{F}$ -local martingale such that almost every sample path is right continuous,  $M_t \geq 0$  a.s. for all  $t \in \mathbb{R}_{\geq 0}$  and  $E[M_0] < +\infty$ .

(i) Let  $\tau \in \text{Time}(\mathcal{F} \vee \text{Null}(P))$ . If  $\tau < +\infty$  a.s. and  $E[M_{\tau}] = E[M_0]$  then  $t \mapsto M_{t \wedge \tau}$  is an  $\mathcal{F} \vee \text{Null}(P)$ -martingale and  $\{M_{\sigma \wedge \tau}; \sigma \in \text{Time}(\mathcal{F})\}$  is uniformly integrable.

(ii) If there exists  $T \in \text{Seq}(\mathbb{R}_{\geq 0})$  such that  $\sup_{n \in \mathbb{N}} T(n) = +\infty$  and  $E[M_{T(n)}] = E[M_0]$  for all  $n \in \mathbb{N}$  then  $M.$  is an  $\mathcal{F}$ -martingale.

*Proof.* Immediate from Lemma 7.6 and Lemma 7.7. □

**7.10 Definition.** Let  $X.$  be an  $\mathcal{F}$ -adapted process with almost sure right continuous path. It is said to be of class DL if  $\{X_{t \wedge \sigma}; \sigma \in \text{Time}(\mathcal{F})\}$  is uniformly integrable for all  $t \in \mathbb{R}_{> 0}$ .

**7.11 Remark.** Suppose  $X.$  and  $Y.$  are  $\mathcal{F}$ -adapted processes with almost sure right continuous path. If  $X_t = Y_t$  a.s. for all  $t \in \mathbb{R}_{\geq 0}$  and  $X.$  is of class DL then so is  $Y.$ . See Corollary 1.25.

**7.12 Lemma.** Let  $X.$  be an  $\mathcal{F}$ -submartingale with almost sure right continuous sample path. If  $X_t \geq 0$  a.s. for all  $t \in \mathbb{R}_{\geq 0}$  or  $X.$  is an  $\mathcal{F}$ -martingale then  $X.$  is of class DL.

*Proof.* Due to the right continuity,  $X_t \geq 0$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. Let  $\sigma$  be an  $\mathcal{F}$ -optional time and  $T \in \mathbb{R}_{> 0}$ . It follows that  $X_{\sigma \wedge T} \geq 0$  a.s. Since  $X_{\sigma \wedge T}$  is  $(\mathcal{F} \vee \text{Null}(P))_{(\sigma \wedge T)+}$ -measurable, we see by Theorem 3.9 that

$$\lambda P(X_{\sigma \wedge T} > \lambda) \leq E[X_{\sigma \wedge T}; X_{\sigma \wedge T} > \lambda] \leq E[X_T; X_{\sigma \wedge T} > \lambda] \leq E[X_T].$$

Thus we get the claim by Corollary 1.19. Indeed if  $\delta \in \mathbb{R}_{> 0}$  and  $E[X_T] \leq \delta \lambda$  then

$$E[X_{\sigma \wedge T}; X_{\sigma \wedge T} > \lambda] \leq \sup_{B \in \mathcal{F}: P(B) \leq \delta} E[X_T; B] \text{ for all } \sigma \in \text{Opt}(\mathcal{F}).$$

The same reasoning also applies to the martingale case. Indeed  $|X.$  is a submartingale. □

**7.13 Corollary.** Let  $M.$  be an  $\mathcal{F}$ -local martingale with almost sure right continuous sample path. Then  $M.$  is an  $\mathcal{F}$ -martingale if and only if it is of class DL.

**7.14 Lemma.** Let  $X.$  be an  $\mathcal{F}$ -local submartingale with almost sure right continuous path. If  $X_t \geq 0$  a.s. for all  $t \in \mathbb{R}_{\geq 0}$  and there exist an  $\mathcal{F}$ -reducing sequence  $S(\cdot)$  and  $p \in \mathbb{R}_{> 1}$  such that  $\sup_{n \in \mathbb{N}} E[X_{t \wedge S(n)}^p] < +\infty$  for all  $t \in \mathbb{R}_{> 0}$  then  $X.^p$  is of class DL.

*Proof.* Apply Lemma 6.2 to the  $\mathcal{F}$ -submartingale  $X_{\cdot \wedge S(n)}$ . □

**7.15 Lemma.** Suppose that  $M.$  is an  $\mathcal{F}$ -local martingale such that almost every sample path is right continuous and  $\tau \in \text{Time}(\mathcal{F} \vee \text{Null}(P))$ .

(i) If  $\tau < +\infty$  a.s. and the family  $\{M_{\sigma \wedge \tau} 1_{\tau > 0}; \sigma \in \text{Time}(\mathcal{F})\}$  is uniformly integrable then

$$M_{\sigma \wedge \tau} 1_{\tau > 0} \in E[M_{\tau} 1_{\tau > 0} | (\mathcal{F} \vee \text{Null}(P))_{\sigma+}] \text{ for all } \sigma \in \text{Time}(\mathcal{F} \vee \text{Null}(P)).$$

(ii) If the family  $\{M_{\sigma \wedge t \wedge \tau} 1_{\tau > 0}; \sigma \in \text{Time}(\mathcal{F})\}$  is uniformly integrable for all  $t \in \mathbb{R}_{\geq 0}$  then  $t \mapsto M_{t \wedge \tau} 1_{\tau > 0}$  is a  $\mathcal{G} \vee \text{Null}(P)$ -martingale where  $\mathcal{G}_0 := \mathcal{F}_{0+}$  and  $\mathcal{G}_t := \mathcal{F}_t$  for  $t \in \mathbb{R}_{> 0}$ .

*Proof.* We select and fix an  $\mathcal{F}$ -reducing sequence  $S(\cdot)$  for  $M$ . Then, since  $(\mathcal{F} \vee \mathcal{N})_{0+} = \mathcal{F}_{0+} \vee \mathcal{N}$  by Lemma 4.8(i) where we write  $\mathcal{N} := \text{Null}(P)$ ,

$$M_{\wedge S(n)} \mathbf{1}_{S(n) > 0} \text{ is a } \mathcal{G} \vee \mathcal{N}\text{-martingale for each } n \in \mathbb{N}$$

according to Lemma 7.4. Observe that  $\text{Time}(\mathcal{F} \vee \mathcal{N}) = \text{Time}(\mathcal{G} \vee \mathcal{N})$ . Let  $\sigma \in \text{Time}(\mathcal{F} \vee \mathcal{N})$ . Then we have that

$$M_{(\sigma \wedge n \wedge \tau) \wedge S(n)} \mathbf{1}_{S(n) > 0} \in E[M_{(\infty \wedge n \wedge \tau) \wedge S(n)} \mathbf{1}_{S(n) > 0} | (\mathcal{G} \vee \mathcal{N})_{\sigma+}].$$

by invoking Corollary 3.17(i). Since  $\{\tau > 0\} \in (\mathcal{F} \vee \mathcal{N})_{0+} = (\mathcal{G} \vee \mathcal{N})_0$ , we see that

$$M_{(\sigma \wedge n \wedge \tau) \wedge S(n)} \mathbf{1}_{S(n) > 0} \mathbf{1}_{\tau > 0} \in E[M_{(\infty \wedge n \wedge \tau) \wedge S(n)} \mathbf{1}_{S(n) > 0} \mathbf{1}_{\tau > 0} | (\mathcal{G} \vee \mathcal{N})_{\sigma+}].$$

Observe that  $n \wedge S(n) \in \text{Time}(\mathcal{F} \vee \mathcal{N})$ . It follows that  $M_{n \wedge S(n) \wedge \tau} \mathbf{1}_{\tau > 0}$  is uniformly integrable and hence so is  $M_{n \wedge S(n) \wedge \tau} \mathbf{1}_{S(n) > 0} \mathbf{1}_{\tau > 0}$ . Moreover  $M_{\sigma \wedge n \wedge S(n) \wedge \tau} \mathbf{1}_{S(n) > 0} \mathbf{1}_{\tau > 0}$  is also uniformly integrable by Theorem 1.23. Therefore, tending  $n$  to  $\infty$ , we reach that

$$M_{\sigma \wedge \tau} \mathbf{1}_{\tau > 0} \in E[M_{\tau} \mathbf{1}_{\tau > 0} | (\mathcal{G} \vee \mathcal{N})_{\sigma+}].$$

Note that  $(\mathcal{G} \vee \mathcal{N})_{\sigma+} = (\mathcal{F} \vee \mathcal{N})_{\sigma+}$ . □

**7.16 Definition.** An  $\mathcal{F}$ -increasing process is an  $\mathcal{F}$ -adapted process whose almost every sample path is finite valued, right continuous and non-decreasing and takes the value 0 at 0.

**7.17 Theorem.** Suppose that  $M$  is an  $\mathcal{F}$ -local martingale with almost sure right continuous sample path,  $A$  is an  $\mathcal{F}$ -increasing process such that  $t \mapsto |M_t|^2 - A_t$  is an  $\mathcal{F}$ -local martingale and  $\tau \in \text{Time}(\mathcal{F} \vee \text{Null}(P))$ . Set  $\mathcal{G}_0 := \mathcal{F}_{0+}$  and  $\mathcal{G}_t := \mathcal{F}_t$  for  $t \in \mathbb{R}_{>0}$ .

- (i)  $E[\sup\{|M_s|^2; s \in \mathbb{R}_{\geq 0}, s \leq \tau\}; B] \leq 4E[|M_0|^2; B] + 4E[A_\tau; B]$  for all  $B \in \mathcal{F}_{0+}$ .
- (ii) If  $E[A_{t \wedge \tau}] < +\infty$  for all  $t \in \mathbb{R}_{\geq 0}$  and  $E[|M_0|^2; \tau > 0] < +\infty$  then  $t \mapsto M_{t \wedge \tau} \mathbf{1}_{\tau > 0}$  and  $t \mapsto |M_{t \wedge \tau}|^2 \mathbf{1}_{\tau > 0} - A_{t \wedge \tau}$  are  $\mathcal{G} \vee \text{Null}(P)$ -martingales.
- (iii) If  $\tau < +\infty$  a.s.,  $E[A_\tau] < +\infty$  and  $E[|M_0|^2; \tau > 0] < +\infty$  then  $E[|M_\tau|; \tau > 0] < +\infty$ ,  $E[M_\tau; \tau > 0] = E[M_0; \tau > 0]$  and  $E[|M_\tau|^2; \tau > 0] = E[|M_0|^2; \tau > 0] + E[A_\tau]$ .

*Proof.* We select and fix  $\mathcal{F}$ -reducing sequences  $S(\cdot)$  and  $T(\cdot)$  for  $M$  and for  $|M|^2 - A$  respectively. We shall write  $\mathcal{N} := \text{Null}(P)$  as usual. Then, according to Lemma 7.4,

$$M_{\wedge S(n)} \mathbf{1}_{S(n) > 0} \text{ as well as } |M_{\wedge T(n)}|^2 \mathbf{1}_{T(n) > 0} - A_{\wedge T(n)} \text{ are } \mathcal{G} \vee \mathcal{N}\text{-martingales.}$$

Note that  $\text{Time}(\mathcal{F} \vee \mathcal{N}) = \text{Time}(\mathcal{G} \vee \mathcal{N})$ . Let  $\sigma$  be a bounded  $\mathcal{F} \vee \mathcal{N}$ -optional time and  $n \in \mathbb{N}$ . Applying Lemma 6.2 to the  $\mathcal{G} \vee \mathcal{N}$ -submartingale  $|M_{\wedge S(n)} \mathbf{1}_{S(n) > 0}|$ , we see that

$$E[\sup_{s \leq \sigma \wedge S(n)} |M_s|^2; S(n) > 0, B] \leq 4E[|M_{\sigma \wedge S(n)}|^2; S(n) > 0, B] \text{ for all } B \in \mathcal{F}_{0+}.$$

On the other hand  $|M_{\sigma \wedge T(n)}|^2 \mathbf{1}_{T(n) > 0} - A_{\sigma \wedge T(n)}$  is integrable and

$$E[|M_{\sigma \wedge T(n)}|^2 - A_{\sigma \wedge T(n)}; T(n) > 0, C] = E[|M_0|^2; T(n) > 0, C] \text{ for all } C \in \mathcal{F}_{0+}$$

by Lemma 3.9. The function  $A_{\sigma \wedge T(n)}$  is  $\mathcal{F}$ -measurable by Lemma 3.8 and, almost every sample path being non-decreasing,  $A_{\sigma \wedge T(n)} \geq 0$  a.s. Consequently we have that

$$\begin{aligned} E[|M_{\sigma \wedge T(n)}|^2; T(n) > 0, C] &= E[|M_{\sigma \wedge T(n)}|^2 - A_{\sigma \wedge T(n)}; T(n) > 0, C] + E[A_{\sigma \wedge T(n)}; C] \\ &= E[|M_0|^2; T(n) > 0, C] + E[A_{\sigma \wedge T(n)}; C] \text{ for all } C \in \mathcal{F}_{0+}. \end{aligned}$$

Combining the two relations above, we infer that

$$\begin{aligned} & E[\sup\{|M_s|^2; s \in \mathbb{R}_{\geq 0}, s \leq \tau \wedge n \wedge T(n) \wedge S(n)\}; S(n) > 0, T(n) > 0, B] \\ & \leq 4E[|M_0|^2; T(n) > 0, S(n) > 0, B] + 4E[A_{\tau \wedge n \wedge S(n) \wedge T(n)}; B]. \end{aligned}$$

Almost every sample path of  $A$  is non-decreasing. Hence, tending  $n$  to  $\infty$ , we get (i) by the monotone convergence theorem. Now suppose that  $E[|M_0|^2; \tau > 0] < +\infty$ . According to (i), if  $E[A_{t \wedge \tau}] < +\infty$  for all  $t \in \mathbb{R}_{\geq 0}$  then  $\{|M_{\sigma \wedge t \wedge \tau}|^2 1_{\tau > 0}; \sigma \in \text{Time}(\mathcal{F})\}$  is uniformly integrable for all  $t \in \mathbb{R}_{\geq 0}$  while if  $E[A_\tau] < +\infty$  then  $\{|M_{\sigma \wedge \tau}|^2 1_{\tau > 0}; \sigma \in \text{Time}(\mathcal{F})\}$  is uniformly integrable. Therefore we get (ii) and (iii) with the help of Lemma 7.15.  $\square$

## 8 Quadratic variation

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\mathcal{F}$  be a filtration of sub  $\sigma$ -fields. The index set is  $\mathbb{R}_{\geq 0}$ . We slightly extend the notion of martingales.

**8.1 Definition.** A stochastic process  $M$  is called a martingale after time  $t_0$  if  $M_t$  is integrable for every  $t \geq t_0$  and  $M_s \in E[M_t | \mathcal{F}_s]$  whenever time parameters satisfy  $t_0 \leq s < t$ . (Note that  $M_t$  is adapted after time  $t_0$ , i.e.,  $\mathcal{F}_t$ -measurable for every  $t \geq t_0$ .)

**8.2 Lemma.** Suppose that  $M$  is an  $\mathcal{F}$ -martingale after  $t(1) \in \mathbb{R}_{\geq 0}$ . If  $\psi$  is an  $\mathcal{F}_{t(1)+}$ -measurable random variable and  $E[|M_t \psi|] < +\infty$  for all  $t \in \mathbb{R}_{\geq t(1)}$  then  $t \mapsto (M_t - M_{t \wedge t(1)})\psi$  and  $t \mapsto (M_{t \wedge t(2)} - M_{t \wedge t(1)})\psi$  are  $\mathcal{F}$ -martingales where  $t(2) \in \mathbb{R}_{> t(1)}$ .

*Proof.* Since both of the processes vanish for  $t \in \mathbb{R}_{[0, t(1)]}$ , they are adapted. Let  $0 \leq s \leq t$ . Then  $s \vee t(1) \leq t \vee t(1)$  and  $\psi$  is  $\mathcal{F}_{s \vee t(1)}$ -measurable. Due to the martingale property of  $M$  we see by Lemma 1.15 that  $M_{s \vee t(1)}\psi \in E[M_{t \vee t(1)}\psi | \mathcal{F}_{s \vee t(1)}]$ . It therefore follows that

$$E[M_{s \vee t(1)}\psi; A] = E[M_{t \vee t(1)}\psi; A] \text{ for all } A \in \mathcal{F}_s.$$

Subtracting  $E[M_{t(1)}\psi; A]$  we get

$$E[(M_{s \vee t(1)} - M_{t(1)})\psi; A] = E[(M_{t \vee t(1)} - M_{t(1)})\psi; A] \text{ for all } A \in \mathcal{F}_s.$$

We see that  $M_{t \vee t(1)} - M_{t(1)}$  coincides with  $M_t - M_{t \wedge t(1)}$  for all  $t \in \mathbb{R}_{\geq 0}$ . Consequently  $t \mapsto (M_t - M_{t \wedge t(1)})\psi$  is a martingale. Finally we observe that  $(M_{t \wedge t(2)} - M_{t \wedge t(1)})\psi$  equals the difference  $(M_t - M_{t \wedge t(1)})\psi - (M_t - M_{t \wedge t(2)})\psi$ .  $\square$

*8.3 Remark.* In particular, by choosing  $\psi = 1$ , we infer that  $t \mapsto M_t - M_{t \wedge t(1)}$  is a martingale. Thus if  $t_0 \leq t(1)$  and  $M$  is an  $\mathcal{F}$ -martingale after  $t_0$  then so is  $t \mapsto M_{t \wedge t(1)}$ .

**8.4 Corollary.** Suppose that  $M$  is an  $\mathcal{F}$ -martingale after  $t(1) \in \mathbb{R}_{\geq 0}$ . If  $\psi$  is a process  $\mathcal{F}$ -adapted after  $t(1)$ ,  $E[|\psi_s M_t|] < +\infty$  for all  $s, t \in \mathbb{R}_{\geq t(1)}$  and  $t(2) \in \mathbb{R}_{> t(1)} \cup \{\infty\}$  then  $t \mapsto M_t(\psi_{t \wedge t(2)} - \psi_{t \wedge t(1)}) - (M_{t \wedge t(2)}\psi_{t \wedge t(2)} - M_{t \wedge t(1)}\psi_{t \wedge t(1)})$  is an  $\mathcal{F}$ -martingale.

*Proof.* Observe that  $(M_t - M_{t \wedge t(1)})\psi_{t \wedge t(1)} = (M_t - M_{t \wedge t(1)})\psi_{t(1)}$  and  $(M_t - M_{t \wedge t(2)})\psi_{t \wedge t(2)} = (M_t - M_{t \wedge t(2)})\psi_{t(2)}$  for all  $t \in \mathbb{R}_{\geq 0}$ . Therefore the process in question coincides with the difference of  $t \mapsto (M_t - M_{t \wedge t(2)})\psi_{t(2)}$  and  $t \mapsto (M_t - M_{t \wedge t(1)})\psi_{t(1)}$ , both of which are martingales according to Lemma 8.2.  $\square$

**8.5 Definition.** An  $\mathcal{F}$ -martingale  $M$  is said to be square integrable if  $E[|M_t|^2] < +\infty$  for all  $t \in \mathbb{R}_{\geq 0}$ .

**8.6 Example.** Suppose that  $M$  and  $N$  are square integrable  $\mathcal{F}$ -martingales after  $s(1) \in \mathbb{R}_{\geq 0}$  respectively  $t(1) \in \mathbb{R}_{\geq 0}$ . Given a bounded  $\mathcal{F}_{s(1) \vee t(1)}$ -measurable function  $\psi$ ,  $s(2) \in \mathbb{R}_{> s(1)}$  and  $t(2) \in \mathbb{R}_{> t(1)}$ . If  $s(1) < t(2)$  and  $t(1) < s(2)$  then the following is an  $\mathcal{F}$ -martingale:

$$\begin{aligned} t \mapsto & (M_{t \wedge s(2)} - M_{t \wedge s(1)})(N_{t \wedge t(2)} - N_{t \wedge t(1)})\psi \\ & - (M_{t \wedge s(2) \wedge t(2)} N_{t \wedge s(2) \wedge t(2)} - M_{t \wedge s(1) \vee t(1)} N_{t \wedge s(1) \vee t(1)})\psi. \end{aligned}$$

Otherwise  $t \mapsto (M_{t \wedge s(2)} - M_{t \wedge s(1)})(N_{t \wedge t(2)} - N_{t \wedge t(1)})\psi$  is an  $\mathcal{F}$ -martingale.

*Proof.* We first discuss the case  $t(2) \leq s(1)$ . We then have that

$$(M_{t \wedge s(2)} - M_{t \wedge s(1)})(N_{t \wedge t(2)} - N_{t \wedge t(1)})\psi = (M_{t \wedge s(2)} - M_{t \wedge s(1)})(N_{t(2)} - N_{t(1)})\psi \text{ for all } t \in \mathbb{R}_{\geq 0}.$$

Since  $(N_{t(2)} - N_{t(1)})\psi$  is  $\mathcal{F}_{s(1)}$ -measurable, it therefore follows by Lemma 8.2 that  $t \mapsto (M_{t \wedge s(2)} - M_{t \wedge s(1)})(N_{t \wedge t(2)} - N_{t \wedge t(1)})\psi$  is an  $\mathcal{F}$ -martingale. The same reasoning works for the case  $s(2) \leq t(1)$ . It remains to discuss the case both  $s(1) < t(2)$  and  $t(1) < s(2)$  hold. So we have that  $t(1) \vee s(1) < t(2) \wedge s(2)$ . We write as follows:

$$\begin{aligned} M_{t \wedge s(2)} - M_{t \wedge s(1)} &= (M_{t \wedge s(2)} - M_{t \wedge u(2)}) + (M_{t \wedge u(2)} - M_{t \wedge u(1)}) + (M_{t \wedge u(1)} - M_{t \wedge s(1)}), \\ N_{t \wedge t(2)} - N_{t \wedge t(1)} &= (N_{t \wedge t(2)} - N_{t \wedge u(2)}) + (N_{t \wedge u(2)} - N_{t \wedge u(1)}) + (N_{t \wedge u(1)} - N_{t \wedge t(1)}). \end{aligned}$$

where  $u(1) := s(1) \vee t(1)$  and  $u(2) := s(2) \wedge t(2)$ . Since either  $s(2) = u(2)$  or  $t(2) = u(2)$ , the cross term  $(M_{t \wedge s(2)} - M_{t \wedge u(2)})(N_{t \wedge t(2)} - N_{t \wedge u(2)})$  never contributes. Neither does the cross term of  $M_{t \wedge u(1)} - M_{t \wedge s(1)}$  and  $N_{t \wedge u(1)} - N_{t \wedge t(1)}$ . On the other hand all of the processes

$$\begin{aligned} t &\mapsto (M_{t \wedge s(2)} - M_{t \wedge u(2)})(N_{t \wedge u(2)} - N_{t \wedge t(1)})\psi, \\ t &\mapsto (M_{t \wedge u(2)} - M_{t \wedge s(1)})(N_{t \wedge t(2)} - N_{t \wedge u(2)})\psi, \\ t &\mapsto (M_{t \wedge u(2)} - M_{t \wedge u(1)})(N_{t \wedge u(1)} - N_{t \wedge t(1)})\psi, \\ t &\mapsto (M_{t \wedge u(1)} - M_{t \wedge s(1)})(N_{t \wedge u(2)} - N_{t \wedge u(1)})\psi \end{aligned}$$

are  $\mathcal{F}$ -martingales by the already established part of the present claim. Consequently all we have to verify is the martingale property of

$$t \mapsto (M_{t \wedge u(2)} - M_{t \wedge u(1)})(N_{t \wedge u(2)} - N_{t \wedge u(1)})\psi - (M_{t \wedge u(2)}N_{t \wedge u(2)} - M_{t \wedge u(1)}N_{t \wedge u(1)})\psi.$$

The process in question is the difference of

$$t \mapsto M_{t \wedge u(2)}(N_{t \wedge u(2)}\psi - N_{t \wedge u(1)}\psi) - (M_{(t \wedge u(2)) \wedge u(2)}N_{t \wedge u(2)}\psi - M_{(t \wedge u(1)) \wedge u(2)}N_{t \wedge u(1)}\psi)$$

and  $t \mapsto (N_{t \wedge u(2)} - N_{t \wedge u(1)})M_{u(1)}\psi$ . The former is an  $\mathcal{F}$ -martingale by Corollary 8.4 while the latter is an  $\mathcal{F}$ -martingale by Lemma 8.2.  $\square$

*8.7 Remark.* Suppose that  $s(1) < s(2)$  and  $t(1) < t(2)$ . Then we have that

$$(s(1), s(2)] \cap (t(1), t(2)] = \begin{cases} (s(1) \vee t(1), s(2) \wedge t(2)] & \text{if } s(1) < t(2) \text{ and } t(1) < s(2) \\ \emptyset & \text{otherwise} \end{cases}$$

Suppose that  $M$  and  $N$  are square integrable  $\mathcal{F}$ -martingales after  $s(1)$  respectively  $t(1)$  and  $\phi$  and  $\psi$  are bounded  $\mathcal{F}_{s(1)}$ - respectively  $\mathcal{F}_{t(1)}$ -measurable function. If  $t(2) \leq s(1)$  or  $s(2) \leq t(1)$  then the random variables  $(M_{s(2)} - M_{s(1)})\phi$  and  $(N_{t(2)} - N_{t(1)})\psi$  are orthogonal.

Suppose that  $M$  is a square integrable  $\mathcal{F}$ -martingale.

Given  $\phi : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}$  such that  $\phi(0) = 0$ ,  $\phi(i) < \phi(i+1)$  for all  $i \in \mathbb{Z}_{\geq 0}$  and  $\sup_{i \in \mathbb{Z}_{\geq 0}} \phi(i) = +\infty$ , we write  $\text{mesh}(\phi) := \sup_{i \in \mathbb{Z}_{\geq 0}} (\phi(i+1) - \phi(i))$  and

$$Q_t^\phi := \sum_{i \in \mathbb{Z}_{\geq 0}} |M_{t \wedge \phi(i+1)} - M_{t \wedge \phi(i)}|^2 (= \sum_{i: \phi(i) < t} |M_{t \wedge \phi(i+1)} - M_{t \wedge \phi(i)}|^2).$$

**8.8 Lemma.** *The process  $t \mapsto |M_t|^2 - Q_t^\phi$  is an  $\mathcal{F}$ -martingale.*

*Proof.* Since  $|M_t|^2 = |M_0|^2 + \sum_{i \in \mathbb{Z}; i \geq 0} (|M_{t \wedge \phi(i+1)}|^2 - |M_{t \wedge \phi(i)}|^2)$ , by Example 8.6.  $\square$

**8.9 Lemma.** *If  $M$  is an  $\mathcal{F}$ -martingale and  $E[|M_t|^4] < +\infty$  for all  $t \in \mathbb{R}_{\geq 0}$  then*

$$E[|Q_t^\phi|^2] \leq \sum_{i: \phi(i) < t} 2E[(|M_t|^2 + |M_{t \wedge \phi(i)}|^2)|M_{t \wedge \phi(i+1)} - M_{t \wedge \phi(i)}|^2].$$

*If there exists  $K \in \mathbb{R}_{>0}$  such that  $|M_t| \leq K$  a.s. for all  $t \in \mathbb{R}_{\geq 0}$  then  $E[|Q_t^\phi|^2] \leq 4K^4$ .*

*Proof.* Since  $\sum_{j: j \geq i+1, \phi(j) < t} |M_{t \wedge \phi(j+1)} - M_{t \wedge \phi(j)}|^2 = Q_t - Q_{t \wedge \phi(i+1)}$ , we have that

$$|Q_t|^2 = \sum_{i: \phi(i) < t} |M_{t \wedge \phi(i+1)} - M_{t \wedge \phi(i)}|^4 + \sum_{i: \phi(i) < t} 2|M_{t \wedge \phi(i+1)} - M_{t \wedge \phi(i)}|^2(Q_t - Q_{t \wedge \phi(i+1)}).$$

We infer by Lemma 8.8 and Lemma 1.15 that

$$E[|M_{t \wedge \phi(i+1)} - M_{t \wedge \phi(i)}|^2(Q_t - Q_{t \wedge \phi(i+1)})] = E[|M_{t \wedge \phi(i+1)} - M_{t \wedge \phi(i)}|^2(|M_t|^2 - |M_{t \wedge \phi(i+1)}|^2)]$$

Observe that  $|M_{t \wedge \phi(i+1)} - M_{t \wedge \phi(i)}|^2 + 2(|M_t|^2 - |M_{t \wedge \phi(i+1)}|^2) \leq 2|M_t|^2 + 2|M_{t \wedge \phi(i)}|^2$ . This implies the desired estimate. If  $|M_t| \leq K$  a.s. for all  $t \in \mathbb{R}_{\geq 0}$  then

$$\sum_{i: \phi(i) < t} E[(|M_t|^2 + |M_{t \wedge \phi(i)}|^2)|M_{t \wedge \phi(i+1)} - M_{t \wedge \phi(i)}|^2] \leq \sum_{i: \phi(i) < t} E[2K^2|M_{t \wedge \phi(i+1)} - M_{t \wedge \phi(i)}|^2].$$

The right hand side equals  $2K^2 E[|M_t|^2 - |M_0|^2]$ , which is dominated by  $2K^4$ .  $\square$

Given  $\delta \in \mathbb{R}_{>0}$  and  $f : I \rightarrow \mathbb{R}$  where  $I$  is an interval of  $\mathbb{R}$ , we write

$$\text{amp}(\delta, f) := \sup_{s, t \in I: |s-t| \leq \delta} |f(s) - f(t)|.$$

**8.10 Lemma.** *If  $\Delta$  is a locally finite refinement of  $\{(\phi(i), \phi(i+1)); i \in \mathbb{Z}_{\geq 0}\}$  then*

$$\sum_{J \in \Delta: \inf J < t} |Q_{t \wedge \sup J}^\phi - Q_{t \wedge \inf J}^\phi|^2 \leq 4(\text{amp}(\text{mesh}(\phi), M|_{[0, t]})^2 \sum_{J \in \Delta: \inf J < t} |M_{t \wedge \sup J} - M_{t \wedge \inf J}|^2).$$

*Proof.* Suppose that  $i \in \mathbb{Z}_{\geq 0}$ ,  $\phi(i) < t$ ,  $J \in \Delta$  and  $J \subset (\phi(i), \phi(i+1)]$ . Then

$$\begin{aligned} Q_{t \wedge \sup J}^\phi - Q_{t \wedge \inf J}^\phi &= |M_{t \wedge \sup J} - M_{t \wedge \phi(i)}|^2 - |M_{t \wedge \inf J} - M_{t \wedge \phi(i)}|^2 \\ &= (M_{t \wedge \sup J} - M_{t \wedge \inf J})(M_{t \wedge \sup J} + M_{t \wedge \inf J} - 2M_{t \wedge \phi(i)}). \end{aligned}$$

It follows that  $\sum_{J \in \Delta: \inf J < t} |Q_{t \wedge \sup J}^\phi - Q_{t \wedge \inf J}^\phi|^2$  equals

$$\sum_{i: \phi(i) < t} \sum_{J \in \Delta: J \subset (\phi(i), \phi(i+1)]} |M_{t \wedge \sup J} - M_{t \wedge \inf J}|^2 |M_{t \wedge \sup J} + M_{t \wedge \inf J} - 2M_{t \wedge \phi(i)}|^2.$$

The second factor in the summand is dominated by  $\{2\text{amp}(\text{mesh}(\phi), M|_{[0, t]})\}^2$ .  $\square$

**8.11 Lemma.** *Suppose that  $M$  is an  $\mathcal{F}$ -martingale, its almost every sample path is right continuous and there exists  $K \in \mathbb{R}_{>0}$  such that  $|M_t| \leq K$  a.s. for all  $t \in \mathbb{R}_{\geq 0}$  and that  $\psi$  is another locally finite partition of  $\mathbb{R}_{\geq 0}$ . If  $\varepsilon, \delta \in \mathbb{R}_{>0}$ ,  $\text{mesh}(\phi) \leq \delta$  and  $\text{mesh}(\psi) \leq \delta$  then*

$$P\left(\sup_{s \in \mathbb{R}: 0 \leq s \leq t} |Q_s^\phi - Q_s^\psi| \geq \varepsilon\right) \leq \frac{32K^2}{\varepsilon^2} \sqrt{E[\{\text{amp}(\delta; M|_{[0,t]})\}^4]}.$$

*Proof.* Being the difference of  $t \mapsto |M_t|^2 - Q_t^\phi$  and  $t \mapsto |M_t|^2 - Q_t^\psi$ , the process  $t \mapsto Q_t^\phi - Q_t^\psi$  is an  $\mathcal{F}$ -martingale by Lemma 8.8. Moreover it is square integrable. It then follows by Example 6.4 that

$$P\left(\sup_{s \in \mathbb{R}: 0 \leq s \leq t} |Q_s^\phi - Q_s^\psi| \geq \varepsilon\right) \leq E[|Q_t^\phi - Q_t^\psi|^2]/\varepsilon^2.$$

Choose a common refinement of  $\{(\phi(i), \phi(i+1)); i \in \mathbb{Z}_{\geq 0}\}$  and  $\{(\psi(i), \psi(i+1)); i \in \mathbb{Z}_{\geq 0}\}$ , for instance the minimal one  $\Delta$ . We see by Example 8.6 that

$$E[|Q_t^\phi - Q_t^\psi|^2] = E\left[\sum_{J \in \Delta: \inf J < t} |Q_{t \wedge \sup J}^\phi - Q_{t \wedge \sup J}^\psi - Q_{t \wedge \inf J}^\phi + Q_{t \wedge \inf J}^\psi|^2\right].$$

The summand is dominated by  $2|Q_{t \wedge \sup J}^\phi - Q_{t \wedge \inf J}^\phi|^2 + 2|Q_{t \wedge \sup J}^\psi - Q_{t \wedge \inf J}^\psi|^2$ . We thus get

$$E[|Q_t^\phi - Q_t^\psi|^2] \leq E[(2+2)4\{\text{amp}(\delta; M|_{[0,t]})\}^2 Q_t^\Delta] \leq 16\sqrt{E[\{\text{amp}(\delta; M|_{[0,t]})\}^4]}E[|Q_t^\Delta|^2]$$

by Lemma 8.10 and Schwarz' inequality. Finally we apply Lemma 8.9.  $\square$

**8.12 Lemma.** *Suppose that  $M$  is an  $\mathcal{F}$ -martingale, its almost every sample path is continuous and there exists  $K \in \mathbb{R}_{>0}$  such that  $|M_t| \leq K$  a.s. for all  $t \in \mathbb{R}_{\geq 0}$ . Then there exists an  $\mathcal{F}$ -adapted process  $A$  such that its almost every sample path is continuous and, as  $\text{mesh}(\phi)$  tends to 0,  $\sup_{s \in \mathbb{R}: 0 \leq s \leq t} |Q_s^\phi - A_s|$  converges to 0 in probability for all  $t \in \mathbb{R}_{\geq 0}$ . A posteriori almost every sample path of  $A$  is nondecreasing and  $t \mapsto |M_t|^2 - A_t$  is an  $\mathcal{F}$ -martingale.*

*Proof.* Let  $\varepsilon \in \mathbb{R}_{>0}$  and  $t \in \mathbb{R}_{\geq 0}$ . We denote by  $\mu$  the probability measure on  $C(\mathbb{R}_{\geq 0}, \mathbb{R})$  induced by  $M$ . Due to the compact inner regularity of probability measures on Polish spaces, there exists a compact subset  $\Lambda$  of  $C(\mathbb{R}_{\geq 0}, \mathbb{R})$  such that  $\mu(\Lambda) > 1 - \varepsilon$ , that is,  $P(M \notin \Lambda) < \varepsilon$ . Invoking the Arzela-Ascoli theorem, we can find  $\delta \in \mathbb{R}_{>0}$  such that  $\text{amp}(\delta; w|_{[0,t]}) \leq \varepsilon$  for all  $w \in \Lambda$ . On the other hand  $\text{amp}(\delta; M|_{[0,t]}) \leq 2K$  a.s. It then follows that

$$\begin{aligned} E[\{\text{amp}(\delta; M|_{[0,t]})\}^4] &= E[\cdots; M \in \Lambda] + E[\cdots; M \notin \Lambda] \\ &\leq \varepsilon^4 P(M \in \Lambda) + (2K)^2 P(M \notin \Lambda) \leq \varepsilon^4 + (2K)^2 \varepsilon. \end{aligned}$$

Consequently we get  $\limsup_{\delta \downarrow 0} E[\{\text{amp}(\delta; M|_{[0,t]})\}^4] = 0$  for all  $t \in \mathbb{R}_{\geq 0}$  and hence

$$(\star) \quad \limsup_{\delta \downarrow 0} \sup_{\phi, \psi: \text{mesh}(\phi), \text{mesh}(\psi) < \delta} P\left(\sup_{s \in \mathbb{R}: 0 \leq s \leq t} |Q_s^\phi - Q_s^\psi| \geq \varepsilon\right) = 0 \text{ for all } \varepsilon \in \mathbb{R}_{>0}, t \in \mathbb{R}_{\geq 0}$$

by Lemma 8.11. We consider a particular sequence of partitions  $\psi(n, i) = i/2^n$ . We can find  $\phi \in \text{Seq}(\mathbb{N}, \uparrow)$  such that

$$P\left(\sup_{s \in \mathbb{R}: 0 \leq s \leq T} |Q_s^{\psi(m, \cdot)} - Q_s^{\psi(n, \cdot)}| \geq 1/2^T\right) \leq 1/2^T \text{ for all } T \in \mathbb{N} \text{ and } m, n \in \mathbb{N}_{\geq \phi(T)}.$$

The Borel-Cantelli lemma shows that there exists  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and

$t \mapsto M_t(\omega)$  is continuous for all  $n \in \mathbb{N}$  and  
 $\sup_{t:t \leq n} |Q_s^{\psi(\phi(n+1), \cdot)}(\omega) - Q_s^{\psi(\phi(n), \cdot)}(\omega)| < 1/2^n$  except for finitely many  $n \in \mathbb{N}$

for all  $\omega \in \Omega_0$ . We set

$$A_t(\omega) := \liminf_{n \rightarrow \infty} Q_t^{\psi(\phi(n), \cdot)}(\omega) \text{ for } t \in \mathbb{R}_{\geq 0} \text{ and } \omega \in \Omega.$$

The process  $t \mapsto a_t$  is  $\mathcal{F}$ -adapted. Let  $\omega \in \Omega_0$  and  $T \in \mathbb{N}$ . Then, since

$$\sum_{k=n}^{\infty} \sup_{t:t \leq T} |Q_s^{\psi(\phi(n+1), \cdot)}(\omega) - Q_s^{\psi(\phi(n), \cdot)}(\omega)| \leq \sum_{k=n}^{\infty} \sup_{t:t \leq k} |\dots| \leq 1/2^{n-1}$$

for all  $n \in \mathbb{N}_{\geq T}$ , it follows that

$$A_t(\omega) - 1/2^{n-1} \leq Q_t^{\psi(\phi(n), \cdot)}(\omega) \leq A_t(\omega) + 1/2^{n-1} \text{ for } t \in \mathbb{R}_{[0, T]} \text{ and } n \in \mathbb{N}_{\geq T}.$$

Therefore the continuity of  $t \mapsto A_t(\omega)$  derives from that of  $t \mapsto Q_t^{\psi(n, \cdot)}(\omega)$ . We verify that  $A(\omega)$  is nondecreasing. Let  $s, t \in \mathbb{Q}^{(2)} := \bigcup_{n \in \mathbb{N}} 2^{-n} \mathbb{Z}_{\geq 0}$  and  $\omega \in \Omega_0$ . Then  $s, t \in 2^{-m} \mathbb{Z}_{\geq 0}$  for some  $m \in \mathbb{N}$ . Since both  $s$  and  $t$  are partition points of  $\psi(n, \cdot)$  for  $n \in \mathbb{N}_{\geq m}$ , we see that  $Q_s^{\psi(\phi(n), \cdot)}(\omega) \leq Q_t^{\psi(\phi(n), \cdot)}(\omega)$  provided  $s < t$  and  $\phi(n) \geq m$ . It therefore follows that

$$A_s(\omega) \leq A_t(\omega) \text{ for all } s, t \in \mathbb{Q}^{(2)} \text{ with } s < t \text{ and } \omega \in \Omega_0.$$

The continuity of  $t \mapsto A_t(\omega)$  for  $\omega \in \Omega_0$  implies that  $t \mapsto A_t(\omega)$  is nondecreasing for  $\omega \in \Omega_0$ . Let  $\phi$  run through locally finite partitions of  $\mathbb{R}_{\geq 0}$ . We see by  $(\star)$  that

$$\limsup_{\delta \downarrow 0} \sup_{\phi: \text{mesh}(\phi) < \delta} P(\sup_{s \in \mathbb{R}: 0 \leq s \leq t} |Q_s^\phi - A_s| \geq \varepsilon) = 0 \text{ for all } \varepsilon \in \mathbb{R}_{> 0}, t \in \mathbb{R}_{\geq 0}.$$

Finally we see by Lemma 8.9 that for each  $t \in \mathbb{R}_{\geq 0}$  the sequence of random variables  $Q_t^{\psi(n, \cdot)}$  is uniformly integrable. Consequently the martingale property of  $t \mapsto |M_t|^2 - A_t$  derives from that of  $t \mapsto |M_t|^2 - Q_t^{\psi(n, \cdot)}$  with the help of Lemma 8.8.  $\square$

**8.13 Lemma.** *Suppose that  $w : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  is continuous and*

$$\sigma : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}, k \mapsto \inf\{s \in \mathbb{R}_{\geq 0} : w(s) > k\}.$$

- (i) *Let  $t \in \mathbb{R}_{> 0} \cup \{+\infty\}$ . Then  $\sup\{w(s) ; s \in \mathbb{R}_{\geq 0}, s \leq t\} \leq k$  if and only if  $t \leq \sigma(k)$ .*
- (ii)  *$\sigma(\cdot)$  is nondecreasing and  $\sup_{k \in \mathbb{R}} \sigma(k) = +\infty$ .*
- (iii) *Suppose that  $w$  is nondecreasing in addition and  $w(+\infty) := \sup\{w(s) ; s \in \mathbb{R}_{\geq 0}\}$ . Then  $w(t \wedge \sigma(k)) = w(t) \wedge k$  for  $t \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$  and  $k \in \mathbb{R}$  with  $k \geq w(0)$ .*

**8.14 Corollary.** (i) *If  $M$  is an  $\mathcal{F}$ -local martingale and its almost every sample path is continuous then  $\tau(K) := \inf\{t \in \mathbb{R}_{\geq 0} : |M_t| > K\}$  forms a reducing sequence.*

(ii) *If  $M$  is an  $\mathcal{F}$ -local martingale with almost sure right continuous sample path,  $A$  is an  $\mathcal{F}$ -increasing process with almost sure continuous sample path and  $t \mapsto |M_t|^2 - A_t$  is an  $\mathcal{F}$ -local martingale then  $\tau(K) := \inf\{t \in \mathbb{R}_{\geq 0} : |A_t| > K\} 1_{|M_0| \leq K}$  forms a common reducing sequence for  $M$  and  $|M|^2 - A$ .*

*Proof.* (i) Let  $K \in \mathbb{R}_{>0}$ . Due to the sample path continuity,  $\tau(K)$  is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time by Example 3.4. Since  $|M_{t \wedge \tau(K)} 1_{\tau(K) > 0}| \leq K$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. by Lemma 8.13(i), it follows by Lemma 7.15 and Lemma 7.4 that  $t \mapsto M_{t \wedge \tau(K)} - M_0$  is an  $\mathcal{F} \vee \text{Null}(P)$ -martingale.

(ii) By the same reasoning as above, we see that  $\tau(K)$  is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time and  $A_{t \wedge \tau(K)} \leq K$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. Since  $E[|M_0|^2; \tau(K) > 0] \leq K^2$ , invoking Theorem 7.17(ii), we infer that both  $t \mapsto M_{t \wedge \tau(K)} - M_0$  and  $t \mapsto |M_{t \wedge \tau(K)}|^2 - A_{t \wedge \tau(K)} - |M_0|^2$  are  $\mathcal{F} \vee \text{Null}(P)$ -martingales.  $\square$

**8.15 Theorem.** *Suppose that  $M$  is an  $\mathcal{F}$ -local martingale with almost sure continuous sample path. Then there exists an  $\mathcal{F}$ -increasing process  $A$  such that its almost every sample path is continuous and  $\sup_{s \in \mathbb{R}: 0 \leq s \leq t} |Q_s^\phi - A_s|$  converges to 0 in probability as  $\text{mesh}(\phi)$  tends to 0 for all  $t \in \mathbb{R}_{\geq 0}$ . A posteriori the process  $t \mapsto |M_t|^2 - A_t$  is an  $\mathcal{F}$ -local martingale. If  $M$  is a square integrable  $\mathcal{F}$ -martingale then  $t \mapsto |M_t|^2 - A_t$  is an  $\mathcal{F}$ -martingale.*

*Proof.* The following form a reducing sequence for the  $\mathcal{F}$ -local martingale  $M$ :

$$\tau(K) := \inf\{t \in \mathbb{R}_{\geq 0} : |M_t| > K\}, \quad K \in \mathbb{R}_{>0},$$

that is,  $\tau(K) \leq \tau(K+1)$ ,  $\sup_{K \in \mathbb{N}} \tau(K) = +\infty$  a.s. and

$$M_{\cdot \wedge \tau(K)} 1_{\tau(K) > 0} \text{ is an } \mathcal{G} \vee \text{Null}(P)\text{-martingale.}$$

where  $\mathcal{G}_0 := \mathcal{F}_{0+}$  and  $\mathcal{G}_t := \mathcal{F}_t$  for  $t \in \mathbb{R}_{>0}$  by Corollary 8.14(i). Its proof shows that

$$|M_{t \wedge \tau(K)} 1_{\tau(K) > 0}| \leq K \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

Let  $\phi$  and  $\psi$  run through locally finite partitions of  $\mathbb{R}_{\geq 0}$ . Observe that

$$Q_{t \wedge \tau(K)}^\phi = \sum_i |M_{t \wedge \phi(i+1) \wedge \tau(K)} 1_{\tau(K) > 0} - M_{t \wedge \phi(i) \wedge \tau(K)} 1_{\tau(K) > 0}|^2 \text{ and}$$

$$P(\sup_{s \leq t} |Q_s^\phi - Q_s^\psi| \geq \varepsilon) \leq P(\sup_{s \leq t} |Q_{s \wedge \tau(K)}^\phi - Q_{s \wedge \tau(K)}^\psi| \geq \varepsilon) + P(\tau(K) < t).$$

The proof of Lemma 8.12 shows that the first term of the left hand side in the second line converges to 0 as  $\text{mesh}(\phi)$  and  $\text{mesh}(\psi)$  tend to 0. Therefore it follows that

$$\limsup_{\delta \downarrow 0} \sup_{\phi, \psi: \text{mesh}(\phi), \text{mesh}(\psi) < \delta} P(\sup_{s \leq t} |Q_s^\phi - Q_s^\psi| \geq \varepsilon) \leq P(\tau(K) < t) \text{ for all } K \in \mathbb{R}_{>0}.$$

Since  $P(\tau(K) < t)$  converges to 0 as  $K$  tends to  $+\infty$ , we thus get

$$\limsup_{\delta \downarrow 0} \sup_{\phi, \psi: \text{mesh}(\phi), \text{mesh}(\psi) < \delta} P(\sup_{s \leq t} |Q_s^\phi - Q_s^\psi| \geq \varepsilon) = 0 \text{ for all } \varepsilon \in \mathbb{R}_{>0}, t \in \mathbb{R}_{\geq 0}.$$

The rest of the argument for drawing the existence of  $A$  process is the same as in the proof of Lemma 8.12. It remains to show the local martingale property of  $|M_\cdot|^2 - A_\cdot$ . We see that

$$\limsup_{\delta \downarrow 0} \sup_{\phi: \text{mesh}(\phi) < \delta} P(\sup_{s \leq t} |Q_{s \wedge \tau(K)}^\phi - A_{s \wedge \tau(K)}| \geq \varepsilon) = 0 \text{ for all } \varepsilon \in \mathbb{R}_{>0}, t \in \mathbb{R}_{\geq 0}.$$

It therefore follows by Lemma 8.12 that

$$t \mapsto |M_{t \wedge \tau(K)} 1_{\tau(K) > 0}|^2 - A_{t \wedge \tau(K)} \text{ is an } \mathcal{G} \vee \text{Null}(P)\text{-martingale for each } K \in \mathbb{R}_{>0}.$$

Thus  $K \mapsto \tau(K)$  forms a reducing sequence for  $t \mapsto |M_t|^2 - A_t$  in view of Lemma 7.4. Finally suppose that  $M$  is a square integrable  $\mathcal{F}$ -martingale. By choosing a sequence  $\psi(n)$  of locally finite partitions, we can realize an almost sure convergence  $A_t = \liminf_{n \rightarrow \infty} Q_t^{\psi(n)}$ . Since  $t \mapsto |M_t|^2 - Q_t^{\psi(n)}$  is an  $\mathcal{F}$ -martingale by Lemma 8.8, Fatou's lemma shows that

$$E[A_t] \leq \liminf_{n \rightarrow \infty} E[Q_t^{\psi(n)}] = E[|M_t|^2] - E[|M_0|^2] < +\infty.$$

Thus  $t \mapsto |M_t|^2 - A_t$  is an  $\mathcal{F}$ -martingale by Theorem 7.17(ii). Indeed it is  $\mathcal{F}$ -adapted.  $\square$

Given  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  and a left open and right closed interval  $I$  of  $\mathbb{R}_{\geq 0}$ , we write

$$\text{var}[df; I] := \sup \left\{ \sum_{J \in \Delta} |f(\sup J) - f(\inf J)|; \Delta \text{ finite partition of } I \right\}.$$

**8.16 Lemma.** *If  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  is of finite variation and  $w : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  is continuous then  $t \mapsto \sum_{i: \phi(i) < t} \{f(t \wedge \phi(i+1)) - f(t \wedge \phi(i))\} \{w(t \wedge \phi(i+1)) - w(t \wedge \phi(i))\}$  converges to 0 locally uniformly as  $\text{mesh}(\phi)$  tends to 0.*

*Proof.* The sum  $\sum_{i: \phi(i) < s} \dots$  is dominated by  $\text{var}(df, (0, s]) \text{amp}[w]_{|[0, s]}$ .  $\square$

**8.17 Corollary.** *Let  $M$  be an  $\mathcal{F}$ -local martingale. If its almost every sample path is continuous and of finite variation then  $M_t = M_0$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s.*

*Proof.* We may regard that  $M_0 = 0$  a.s. Theorem 8.15 and Lemma 8.16 show that  $t \mapsto |M_t|^2$  is an  $\mathcal{F}$ -local martingale. Fix  $t \in \mathbb{R}_{\geq 0}$ . Let  $\tau(K)$  be as in the proof of Theorem 8.15. It follows that  $E[|M_{t \wedge \tau(K)}|^2] = 0$ . Invoking Fatou's lemma, we see that

$$E[|M_t|^2] = E[\liminf_{K \rightarrow \infty} |M_{t \wedge \tau(K)}|^2] \leq \liminf_{K \rightarrow \infty} E[|M_{t \wedge \tau(K)}|^2] = 0,$$

which implies  $M_t = 0$  a.s. We reach the conclusion due to the sample path continuity.  $\square$

**8.18 Corollary.** *Suppose that  $M$  is an  $\mathcal{F}$ -local martingale whose almost every sample path is continuous,  $B$  is a stochastic process whose almost every sample path is continuous and of finite variation and  $X_t := M_t + B_t$ . Then there exists an  $\mathcal{F}$ -adapted process  $A$  such that its almost every sample path is continuous and as  $\text{mesh}(\phi)$  tends to 0*

$$\sup_{s \in \mathbb{R}: 0 \leq s \leq t} \left| \sum_{i: \phi(i) < s} |X_{s \wedge \phi(i+1)} - X_{s \wedge \phi(i)}|^2 - A_s \right| \text{ converges to 0 in probability for all } t \in \mathbb{R}_{\geq 0}.$$

*A posteriori almost every sample path of  $A$  is nondecreasing and the process  $t \mapsto |M_t|^2 - A_t$  is an  $\mathcal{F}$ -local martingale.*

*Proof.* The sum  $\sum_{i: \phi(i) < s} |X_{s \wedge \phi(i+1)} - X_{s \wedge \phi(i)}|^2$  equals

$$Q_s^\phi + 2 \sum_{i: \phi(i) < s} (M_{s \wedge \phi(i+1)} - M_{s \wedge \phi(i)})(B_{s \wedge \phi(i+1)} - B_{s \wedge \phi(i)}) + \sum_{i: \phi(i) < s} |B_{s \wedge \phi(i+1)} - B_{s \wedge \phi(i)}|^2.$$

The second term and the third term converge to 0 by Lemma 8.16. As for the first term we apply Theorem 8.15.  $\square$

- 8.19 Definition.** (i) An  $\mathcal{F}$ -finite variation process  $A$  is an  $\mathcal{F}$ -adapted process such that almost every sample path is right continuous and of finite variation.  
(ii) An  $\mathcal{F}$ -semimartingale is an  $\mathcal{F}$ -adapted stochastic process which is indistinguishable from a sum of an  $\mathcal{F}$ -local martingale whose almost every sample path is continuous and an  $\mathcal{F}$ -finite variation process whose almost every sample path is continuous.  
(iii) Given an  $\mathcal{F}$ -semimartingale  $X$ , a process which shares the same property as the  $A$  process in Corollary 8.18 is called a quadratic variation of  $X$ .

*8.20 Remark.* The usage of the term semimartingale deviates from the standard one: an  $\mathcal{F}$ -local martingale with almost sure right continuous path plus an  $\mathcal{F}$ -finite variation process.

Given an  $\mathcal{F}$ -semimartingale  $X$  we write

$$\begin{aligned} \text{Mart}[X.; \mathcal{F}.] &:= \{M. : \mathcal{F}\text{-local martingale with a.s.-continuous sample path,} \\ &\quad M_0 = 0 \text{ a.s., } t \mapsto X_t - M_t \text{ is of finite variation a.s.}\}, \\ \text{Qvar}[X.; \mathcal{F}.] &:= \{A. : \mathcal{F}\text{-quadratic variation of } X.\}. \end{aligned}$$

**8.21 Theorem.** *Suppose that  $X, Y$  are  $\mathcal{F}$ -semimartingales.*

- (i) *If  $M, N \in \text{Mart}[X., \mathcal{F}.]$  then  $M_t = N_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. If  $M. \in \text{Mart}[X.; \mathcal{F}.]$ ,  $N.$  is an  $\mathcal{F}$ -adapted process and  $M_t = N_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. then  $N. \in \text{Mart}[X.; \mathcal{F}.]$*   
(ii) *If  $\text{Mart}[X.; \mathcal{F} \vee \text{Null}(P)] \cap \text{Mart}[Y.; \mathcal{F} \vee \text{Null}(P)] \neq \emptyset$  then  $t \mapsto X_t - Y_t$  is of finite variation a.s. If  $t \mapsto X_t - Y_t$  is of finite variation a.s. then  $\text{Mart}[X.; \mathcal{F}.] = \text{Mart}[Y.; \mathcal{F}.]$*   
(iii)  *$\text{Qvar}[X.; \mathcal{F}.] \neq \emptyset$ . If  $A, B. \in \text{Qvar}[X.; \mathcal{F}.]$  then  $A_t = B_t \forall t \in \mathbb{R}_{\geq 0}$  a.s. If  $A.$  is an  $\mathcal{F}$ -adapted process,  $B. \in \text{Qvar}[X.; \mathcal{F}.]$  and  $A_t = B_t \forall t \in \mathbb{R}_{\geq 0}$  a.s. then  $A. \in \text{Qvar}[X.; \mathcal{F}.]$*

*Proof.* (i) The first part follows from Corollary 8.17. (iii) The first part is by Corollary 8.18. The second part is due to the uniqueness of limits in probability.  $\square$

## 9 Semimartingale defined by Riemann-Stieltjes integration

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\mathcal{F}$  be a filtration of sub  $\sigma$ -fields. The index set is  $\mathbb{R}_{\geq 0}$ . We recall some facts from Riemann-Stieltjes integration.

**9.1 Lemma.** *All continuous functions are Riemann-Stieltjes integrable with respect to a right (left) continuous integrator of finite variation. All functions of finite variation are Riemann-Stieltjes integrable with respect to a continuous integrator.*

*Proof.* Suppose that  $a < b$ ,  $f : \mathbb{R}_{[a,b]} \rightarrow \mathbb{R}$  is of finite variation and  $w : \mathbb{R}_{[a,b]} \rightarrow \mathbb{R}$  is continuous. Let  $\Delta$  be a tagged partition of  $\mathbb{R}_{[a,b]}$ , by which we mean a set such that  $x \in \mathbb{R}$ ,  $J$  is an left open and right closed interval and  $\inf J \leq x \leq \sup J$  for all  $(x, J) \in \Delta$  and  $(x, J) \mapsto J$  is a finite partition of  $\mathbb{R}_{(a,b)}$ . Then we have that

$$\left| w(t) - \sum_{(x,J) \in \Delta} w(x) 1_J(t) \right| \leq \sup_{(x,J) \in \Delta} \sup_{s \in J} |w(s) - w(x)| \leq \text{amp}(\text{mesh}(\Delta), w)$$

for all  $t \in \mathbb{R}_{[a,b]}$ . Suppose that  $f$  is right continuous. It follows that

$$\left| \int_{(a,b)} w df - \sum_{(x,J) \in \Delta} w(x) \{f(\sup J) - f(\inf J)\} \right| \leq \text{amp}(\text{mesh}(\Delta), w) \text{var}(df, \mathbb{R}_{(a,b)}).$$

Therefore the function  $w$  is Riemann-Stieltjes integrable with respect to the integrator  $f$  and the integral coincides with the counter part  $\int_{(a,b]} w df$  in the measure theory. If  $f$  is left continuous then  $\int_a^b w df = \int_{[a,b)} w df$ . On the other hand

$$\begin{aligned} & \left| \sum_{(x,J) \in \Delta} f(\inf J + o) \{w(\sup J) - w(\inf J)\} - \sum_{(x,J) \in \Delta} f(x) \{w(\sup J) - w(\inf J)\} \right| \\ & \leq \sum_{(x,J) \in \Delta} \text{var}(df, J) |w(\sup J) - w(\inf J)| \leq \text{var}(df, \mathbb{R}_{(a,b)}) \text{amp}(\text{mesh}(\Delta), w). \end{aligned}$$

Observe that  $\sum_{(x,J) \in \Delta} f(\inf J + o) \{w(\sup J) - w(\inf J)\}$  coincides with

$$w(b) \bar{f}^r(b) - w(a) \bar{f}^r(a) - \sum_{(x,J) \in \Delta} w(\sup J) \{\bar{f}^r(\sup J) - \bar{f}^r(\inf J)\},$$

where we write  $\bar{f}^r(x) = f(x + o)$  for  $x \in \mathbb{R}_{[a,b]}$ . Thus we infer that

$$\begin{aligned} & \left| w(b) \bar{f}^r(b) - w(a) \bar{f}^r(a) - \int_{(a,b]} w d\bar{f}^r - \sum_{(x,J) \in \Delta} f(x) \{w(\sup J) - w(\inf J)\} \right| \\ & \leq 2 \text{var}(df, \mathbb{R}_{(a,b)}) \text{amp}(\text{mesh}(\Delta), w). \end{aligned}$$

Consequently  $f$  is Riemann-Stieltjes integrable with respect to the integrator  $w$  and

$$(9.2) \quad \int_a^b f dw = w(b) \bar{f}^r(b) - w(a) \bar{f}^r(a) - \int_{(a,b]} w d\bar{f}^r = w(b) \bar{f}^l(b) - w(a) \bar{f}^l(a) - \int_{[a,b)} w d\bar{f}^l.$$

Here  $\bar{f}^l(x) = f(x - o)$  for  $x \in \mathbb{R}_{[a,b]}$  and the second relation is obtained by changing the roll of the left and the right. Note that

$$\int_a^b f dw = \int_a^b \bar{f}^r dw = \int_a^b \bar{f}^l dw \text{ and } w(b) \bar{f}^r(b) - \int_{(a,b]} w d\bar{f}^r = w(b) \bar{f}^l(b) - \int_{[a,b)} w d\bar{f}^l$$

The latter implies that  $\{(s, t) : s < t\} \rightarrow \mathbb{R}, (s, t) \mapsto \int_s^t f dw$  is continuous.  $\square$

Given  $t \in \mathbb{R}_{\geq 0}$  and finite valued processes  $f$  and  $X$ , we set

$$R[f dX]_t := \liminf_{n \rightarrow \infty} \sum_{i \in \mathbb{Z}: 0 \leq i < 2^n t} (X_{t \wedge (i+1)/2^n} - X_{t \wedge i/2^n}) f_{i/2^n}.$$

**9.3 Corollary.** *Suppose that  $X$  is an  $\mathcal{F}$ -adapted process whose almost every sample path is finite valued and continuous and  $f$  is an  $\mathcal{F}$ -adapted process whose almost every sample path is of finite variation. Then  $t \mapsto R[f dX]_t$  is an  $\mathcal{F}$ -adapted process whose almost every sample path is finite valued and continuous.*

*Proof.* The adaptedness is clear by definition. We select and fix  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$ ,  $X(\omega)$  is finite valued and continuous and  $f(\omega)$  is right continuous and of finite variation for every  $\omega \in \Omega_0$ . Then  $R[f dX]_t(\omega) = \int_0^t f(\omega) dX(\omega)$  for all  $\omega \in \Omega_0$  and  $t \in \mathbb{R}_{\geq 0}$ . The function  $t \mapsto \int_0^t f(\omega) dX(\omega)$  is continuous for every  $\omega \in \Omega_0$  according to the proof of Lemma 9.1.  $\square$

$\mathfrak{Mart}^c(\mathcal{F})$  stands for the space of all  $\mathcal{F}$ -local martingales with almost sure continuous sample path.

**9.4 Lemma.** *Suppose  $M \in \mathfrak{Mart}^c(\mathcal{F})$ ,  $A \in \text{Qvar}[M; \mathcal{F}]$  and  $f$  is an  $\mathcal{F}$ -adapted process such that  $\sup_{\omega \in \Omega} |f_t(\omega)| < +\infty$  for all  $t \in \mathbb{R}_{\geq 0}$ . If  $\phi$  is a locally finite partition of  $\mathbb{R}_{\geq 0}$  then*

$$I : t \mapsto \sum_{i:\phi(i) < t} (M_{t \wedge \phi(i+1)} - M_{t \wedge \phi(i)}) f_{\phi(i)} \text{ is an } \mathcal{F}\text{-local martingale}$$

and  $t \mapsto \sum_{i:\phi(i) < t} (A_{t \wedge \phi(i+1)} - A_{t \wedge \phi(i)}) |f_{\phi(i)}|^2 \in \text{Qvar}[I; \mathcal{F}]$ .

*Proof.* Let  $S(\cdot)$  be a reducing sequence for the  $\mathcal{F}$ -local martingale  $M$ . It follows that

$$t \mapsto \sum_{i:\phi(i) < T} (M_{t \wedge \phi(i+1) \wedge S(k)} - M_{t \wedge \phi(i) \wedge S(k)}) f_{\phi(i)} \text{ is an } \mathcal{F} \vee \text{Null}(P)\text{-martingale}$$

for each  $T \in \mathbb{R}_{> 0}$  and  $k \in \mathbb{N}$  by Lemma 8.2. The above coincides with

$$t \mapsto \sum_{i:\phi(i) < t} (M_{t \wedge S(k) \wedge \phi(i+1)} - M_{t \wedge S(k) \wedge \phi(i)}) f_{\phi(i)}$$

on the time interval  $[0, T]$ .  $T$  being arbitrary, this means that the latter process is an  $\mathcal{F} \vee \text{Null}(P)$ -martingale for each  $k \in \mathbb{N}$ . On the other hand the process  $t \mapsto I_t$  is  $\mathcal{F}$ -adapted. Thus we verified that  $S(\cdot)$  also serves as a reducing sequence for the  $\mathcal{F}$ -local martingale  $I$ . Let  $\Delta$  be a refinement of the partition  $\{\phi(i); i \in \mathbb{Z}_{\geq 0}\}$ . Then

$$\sum_{J \in \Delta: \inf J < t} |I_{t \wedge \sup J} - I_{t \wedge \inf J}|^2 = \sum_{i:\phi(i) < t} \sum_{J \in \Delta: J \subset (\phi(i), \phi(i+1))} |M_{t \wedge \sup J} - M_{t \wedge \inf J}|^2 |f_{\phi(i)}|^2.$$

According to Theorem 8.15 the term  $\sum_{J \in \Delta: J \subset (\phi(i), \phi(i+1))} |M_{t \wedge \sup J} - M_{t \wedge \inf J}|^2$  converges to  $A_{t \wedge \phi(i+1)} - A_{t \wedge \phi(i)}$  locally uniformly in probability as  $\text{mesh}(\Delta)$  tends to 0. Therefore

$$\sup_{s \in \mathbb{R}: 0 \leq s \leq t} \left| \sum_{J \in \Delta: \inf J < t} |I_{t \wedge \sup J} - I_{t \wedge \inf J}|^2 - \sum_{i:\phi(i) < t} (A_{t \wedge \phi(i+1)} - A_{t \wedge \phi(i)}) |f_{\phi(i)}|^2 \right|$$

converges to 0 in probability as  $\text{mesh}(\Delta)$  tends to 0 for all  $t \in \mathbb{R}_{\geq 0}$ .  $\square$

**9.5 Lemma.** *Suppose that  $M^n$  is a sequence of  $\mathfrak{Mart}^c(\mathcal{F})$ . If there exists a process  $M$  such that its almost every sample path is right continuous and  $\sup_{s \leq t} |M_s^n - M_s|$  converges to 0 in probability for all  $t \in \mathbb{R}_{\geq 0}$  then  $t \mapsto M_t \in \mathfrak{Mart}^c(\mathcal{F} \vee \text{Null}(P))$ .*

*Proof.* By choosing a subsequence we may assume that

$$\limsup_{n \rightarrow \infty} \sup_{s \leq t} |M_s^n - M_s| = 0 \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

It follows that  $M_t = \liminf_{n \rightarrow \infty} M_t^n$  a.s. for all  $t \in \mathbb{R}_{\geq 0}$  and hence  $M$  is  $\mathcal{F} \vee \text{Null}(P)$ -adapted. Let  $K \in \mathbb{R}_{> 0}$ . We introduce the following  $\mathcal{F} \vee \text{Null}(P)$ -optional times:

$$\sigma(n, K) := \inf\{t \in \mathbb{R}_{\geq 0} : |M_t^n| > K\}, n \in \mathbb{N}; \quad \tau(K) := \inf\{t \in \mathbb{R}_{\geq 0} : |M_t| > K\}.$$

The former constitutes a reducing sequence for  $M^n$  as an  $\mathcal{F} \vee \text{Null}(P)$ -local martingale for each  $n \in \mathbb{N}$  by Corollary 8.14(i). In particular we have by Corollary 3.17(ii) that

$t \mapsto M_{t \wedge \tau(K) \wedge \sigma(n, K+1)}^n - M_0^n$  is an  $\mathcal{F} \vee \text{Null}(P)$ -martingale.

Since  $\sup_{s \leq t \wedge \tau(K)} |M_s| \leq K$  on  $\{\tau(K) > 0\}$  by Lemma 8.13(i),

$$\sup_{s \leq t \wedge \tau(K)} |M_s^n| \leq \sup_{s \leq t \wedge \tau(K)} |M_s| + \sup_{s \leq t \wedge \tau(K)} |M_s^n - M_s| \leq K + \sup_{s \leq t} |M_s^n - M_s| \text{ on } \{\tau(K) > 0\}.$$

Suppose that  $\varepsilon \in \mathbb{R}_{(0,1]}$ . Then, invoking Lemma 8.13(i) again, we infer that

$$\tau(K) > 0 \text{ and } \sup_{s \leq t} |M_s^n - M_s| \leq \varepsilon \Rightarrow t \wedge \tau(K) \leq \sigma(n, K+1).$$

If  $\tau(K) = 0$  then  $t \wedge \tau(K) \leq \sigma(n, K+1)$  holds. Hence

$$\sup_{s \leq t} |M_s^n - M_s| \leq \varepsilon \Rightarrow \sup_{s \leq t} |M_{s \wedge \tau(K) \wedge \sigma(n, K+1)}^n - M_{s \wedge \tau(K)}| \leq \varepsilon.$$

Therefore we obtain that

$$P(\sup_{s \leq t} |M_{s \wedge \tau(K) \wedge \sigma(n, K+1)}^n - M_{s \wedge \tau(K)}| > \varepsilon) \leq P(\sup_{s \leq t} |M_s^n - M_s| > \varepsilon).$$

This implies that

$$M_{t \wedge \tau(K) \wedge \sigma(n, K+1)}^n - M_0^n \text{ converges to } M_{t \wedge \tau(K)} - M_0 \text{ in probability}$$

as  $n$  tends to  $\infty$ . Taking into account that  $|M_{t \wedge \tau(K) \wedge \sigma(n, K+1)}^n - M_0^n| \leq 2(K+1)$  a.s., we conclude that  $t \mapsto M_{t \wedge \tau(K)} - M_0$  is an  $\mathcal{F} \vee \text{Null}(P)$ -martingale.  $\square$

**9.6 Lemma.** *Suppose that  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  is locally bounded and admits left hand limits everywhere, and  $v : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  is of finite variation and right continuous. Then*

$$\sum_{i: 0 \leq i < 2^n t} \{v(t \wedge (i+1)/2^n) - v(t \wedge i/2^n)\} f(i/2^n) \text{ converges to } \int_{(0,t]} f(\cdot - o) dv \quad \forall t \in \mathbb{R}_{\geq 0}$$

where  $f(t - o) := \liminf_{s \uparrow t: s \in \mathbb{Q}} f(s)$ .

*Proof.* The function  $s \mapsto f(s - o)$  is left continuous by Lemma 14.3(ii).  $\square$

**9.7 Corollary.** *Suppose that  $f$  is an  $\mathcal{F}_+$ -adapted process whose almost every sample path is locally bounded and admits left hand limits everywhere, and  $A$  is an  $\mathcal{F}$ -finite variation process. Then  $R[f dA]$  is an  $\mathcal{F}$ -finite variation process. If  $A$  admits almost sure continuous sample path then so does  $R[f dA]$ .*

**9.8 Lemma.** *Suppose  $f : \mathbb{R}_{[a,b]} \rightarrow \mathbb{R}$  is of finite variation and  $w : \mathbb{R}_{[a,b]} \rightarrow \mathbb{R}$  is continuous.*

(i) *If  $g : \mathbb{R}_{[a,b]} \rightarrow \mathbb{R}$  is of finite variation then so are  $f + g$  and  $fg : \mathbb{R}_{[a,b]} \rightarrow \mathbb{R}$ , and*

$$\int_a^b (f + g) dw = \int_a^b f dw + \int_a^b g dw \text{ and } \int_a^b fg dw = \int_a^b f dv$$

where  $v$  is the continuous function  $t \mapsto \int_a^t g dw$ .

(ii) *If  $v : \mathbb{R}_{[a,b]} \rightarrow \mathbb{R}$  is continuous then*

$$\int_a^b f d(v + w) = \int_a^b f dv + \int_a^b g dw \text{ and } \int_a^b f d(vw) = \int_a^b g dw + w(b) \int_a^b f dv$$

where  $g$  is the function of finite variation  $t \mapsto \int_{[a,t)} v df(\cdot - o) + f(a - o)v(a)$ .

(iii) *If  $\ell : \mathbb{R} \rightarrow \mathbb{R}$  is Lipschitz continuous then the composition  $\ell \circ f$  is of finite variation.*

*Proof.* (i) Since  $\text{var}(d(fg), \mathbb{R}_{(a,b)}) \leq \sup_{[a,b]} |f| \text{var}(dg, \mathbb{R}_{(a,b)}) + \sup_{[a,b]} |g| \text{var}(df, \mathbb{R}_{(a,b)})$ , the product  $fg$  is of finite variation. Suppose that  $a \leq s < t \leq b$ . We see by (9.2) that

$$g(s+o)(w(t) - w(s)) - (v(t) - v(s)) = \int_{(s,t]} (w(t) - w) dg(\cdot + o).$$

It therefore follows that  $|g(s+o)(w(t) - w(s)) - (v(t) - v(s))| \leq \text{amp}[t-s, w] \text{var}(dg, \mathbb{R}_{(s,t]})$ .

(ii) We repeatedly apply the integration by parts formula (9.2). Taking into account that  $g(b) + \int_a^b f dv = f(b-o)v(b)$  and  $\int_{[a,b)} w dg = \int_{[a,b)} wv df(\cdot - o)$ , we get

$$\begin{aligned} \int_a^b g dw + w(b) \int_a^b f dv &= g(b)w(b) - g(a)w(a) - \int_{[a,b)} w dg + w(b) \int_a^b f dv \\ &= f(b-o)v(b)w(b) - f(a-o)v(a)w(a) - \int_{[a,b)} wv df(\cdot - o). \end{aligned}$$

The right hand side equals  $\int_a^b f d(vw)$ . □

**9.9 Lemma.** *Suppose that  $M. \in \mathfrak{Mart}^c(\mathcal{F})$  and  $A.$  is an  $\mathcal{F}$ -adapted process. Then  $A. \in \text{Qvar}[M.; \mathcal{F}]$  if and only if almost every sample path of  $A.$  starts from 0 and is continuous and of finite variation, and  $t \mapsto |M_t|^2 - A_t$  is an  $\mathcal{F}$ -local martingale.*

*Proof.* Theorem 8.15 shows that  $\text{Qvar}[M.; \mathcal{F}]$  is not void. Choose an element  $B.$  from it. Then it is  $\mathcal{F}$ -adapted, its almost every sample path starts from 0 and is continuous and of finite variation. Moreover  $|M.|^2 - B.$  is an  $\mathcal{F}$ -local martingale by Theorem 8.15. The implication  $\Rightarrow$  is thus established. To prove the converse suppose that  $A.$  shares the same property of  $B.$  described above. It follows that  $A. - B.$  is an  $\mathcal{F}$ -local martingale whose almost every sample path starts from 0, and is continuous and of finite variation. With the help of Corollary 8.17 and Theorem 8.21(iv) we infer that  $A. \in \text{Qvar}[M.; \mathcal{F}]$ . □

**9.10 Theorem.** *Suppose that  $M. \in \mathfrak{Mart}^c(\mathcal{F})$ ,  $A. \in \text{Qvar}[M.; \mathcal{F}]$  and  $f.$  is an  $\mathcal{F}$ -adapted process whose almost every sample path is of finite variation. Then the process  $t \mapsto R[f dM]_t$  belongs to  $\mathfrak{Mart}^c(\mathcal{F})$  and  $R[|f|^2 dA.] \in \text{Qvar}[R[f dM].; \mathcal{F}]$ .*

*Proof.* The sample path continuity of the processes in question is mentioned in Corollary 9.3. We next show the local martingale property of  $t \mapsto R[f dM]_t$  with additionally assuming that  $\sup_{\omega \in \Omega} |f_t(\omega)| < +\infty$  for all  $t \in \mathbb{R}_{\geq 0}$ . Let  $n \in \mathbb{N}$ . It follows by Lemma 9.4 that

$$I^n : t \mapsto \sum_{i \in \mathbb{Z}: 0 \leq i < 2^n t} (M_{t \wedge (i+1)/2^n} - M_{t \wedge i/2^n}) f_{i/2^n}.$$

is an  $\mathcal{F}$ -local martingale and

$$B^n : t \mapsto \sum_{i \in \mathbb{Z}: 0 \leq i < 2^n t} (A_{t \wedge (i+1)/2^n} - A_{t \wedge i/2^n}) |f_{i/2^n}|^2 \in \text{Qvar}[I^n.; \mathcal{F}].$$

According to Theorem 8.15,

$$t \mapsto |I_t^n|^2 - B_t^n \text{ is an } \mathcal{F}\text{-local martingale.}$$

We select and fix  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$ ,

$f.(\omega)$  is of finite variation,  $M.(\omega)$  is finite valued and continuous, and  $A.(\omega)$  is finite valued, nondecreasing and continuous for every  $\omega \in \Omega_0$ .

It then follows by Lemma 9.1 that

$$\begin{aligned} \sum_{i \in \mathbb{Z}: 0 \leq i < 2^{nt}} (M_{t \wedge (i+1)/2^n}(\omega) - M_{t \wedge i/2^n}(\omega)) f_{i/2^n}(\omega) &\text{ converges to } \int_0^t f.(\omega) dM.(\omega), \\ \sum_{i \in \mathbb{Z}: 0 \leq i < 2^{nt}} (A_{t \wedge (i+1)/2^n}(\omega) - A_{t \wedge i/2^n}(\omega)) |f_{i/2^n}(\omega)|^2 &\text{ converges to } \int_0^t |f.(\omega)|^2 dA.(\omega) \end{aligned}$$

locally uniformly on  $\mathbb{R}_{\geq 0}$  for each  $\omega \in \Omega_0$ . Invoking Lemma 9.5, we infer that

$$R[f dM]. \text{ and } |R[f dM].|^2 - R[|f|^2 dA]. \text{ are } \mathcal{F} \vee \text{Null}(P)\text{-local martingales.}$$

Actually, being  $\mathcal{F}$ -adapted by definition, they are  $\mathcal{F}$ -local martingales. The local martingale property of the second process implies that  $R[|f|^2 dA]. \in \text{Qvar}[R[f dM].; \mathcal{F}.]$  in view of Corollary 9.7 and Lemma 9.9.

We relax the boundedness of the integrand. Let  $n \in \mathbb{N}$ . Since  $x \mapsto \max\{\min\{x, n\}, -n\}$  is Lipschitz continuous,  $f^n : t \mapsto \max\{\min\{f_t, n\}, -n\}$  is an  $\mathcal{F}$ -finite variation process by Lemma 9.8(iii) and  $|f_t^n| \leq n$  for all  $t \in \mathbb{R}_{\geq 0}$ . We see by the preceding paragraph that

$$t \mapsto R[f^n dM]_t \text{ and } t \mapsto |R[f^n dM]_t|^2 - R[|f^n|^2 dA]_t \text{ are } \mathcal{F}\text{-local martingales.}$$

Let  $\Omega_0$  be as above. If  $\omega \in \Omega_0$  then  $\sup_{s \in \mathbb{R}: 0 \leq s \leq t} |f_s(\omega)| \leq n$  and hence  $f^n(\omega) = f.(\omega)$  on  $\mathbb{R}_{[0,t]}$  for sufficiently large  $n \in \mathbb{N}$ . Thus, with the help of Lemma 9.5, we get the full claim.  $\square$

**9.11 Corollary.** *Suppose that  $X.$  is an  $\mathcal{F}$ -semimartingale and  $f.$  is an  $\mathcal{F}$ -adapted process whose almost every sample path is of finite variation. Then the process  $t \mapsto R[f dX]_t$  is an  $\mathcal{F}$ -semimartingale. Moreover if  $M. \in \text{Mart}[X.; \mathcal{F}.]$  and  $A. \in \text{Qvar}[X.; \mathcal{F}.]$  then*

$$R[f dM]. \in \text{Mart}[R[f dX].; \mathcal{F}.] \text{ and } R[|f|^2 dA]. \in \text{Qvar}[R[f dX].; \mathcal{F}.].$$

*Proof.*  $R[f dX]_t = R[f dM]_t + R[f d(X - M)]_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. by Lemma 9.8(ii).  $\square$

## 10 Product of semimartingales is a semimartingale

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\mathcal{F}$  be a filtration of sub  $\sigma$ -fields.  $\text{Mart}^c(\mathcal{F}.)$  denotes the space of all  $\mathcal{F}$ -local martingales with almost sure continuous sample path.

**10.1 Lemma.** *Suppose that  $X., Y.$  are  $\mathcal{F}$ -semimartingales. If  $f$  is a bounded  $\mathcal{F}_0$ -measurable function then  $fX.$  is an  $\mathcal{F}$ -semimartingale.  $X. + Y.$  is an  $\mathcal{F}$ -semimartingale.*

*Proof.* Lemma 7.5(i) and Lemma 7.5(iii).  $\square$

**10.2 Lemma.** *Let  $X.$  be an  $\mathcal{F}_+ \vee \text{Null}(P)$ -adapted process. If its almost every sample path is left continuous then there exists an  $\mathcal{F}$ -adapted process  $\tilde{X}.$  such that  $X_t = \tilde{X}_t \forall t \in \mathbb{R}_{\geq 0}$  a.s.*

*Proof.* We select and fix an  $\mathcal{F}_+$ -adapted modification  $Y$  of  $X$ . There exists  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and  $X_t(\omega) = Y_t(\omega)$  for all  $t \in \mathbb{Q}_{\geq 0}$  and  $\omega \in \Omega_0$ . Since  $X$  is left continuous a.s., we may assume that  $t \mapsto X_t(\omega)$  is left continuous for all  $\omega \in \Omega_0$ . It follows that

$$\liminf_{s \uparrow t: s \in \mathbb{Q}} Y_s(\omega) = \liminf_{s \uparrow t: s \in \mathbb{Q}} X_s(\omega) = X_t(\omega) \text{ for all } t \in \mathbb{R}_{>0} \text{ and } \omega \in \Omega_0.$$

Thus the process  $\tilde{X} : t \mapsto \liminf_{s \uparrow t: s \in \mathbb{Q}} Y_s$  is a desired one.  $\square$

**10.3 Lemma.** *Suppose that  $X$  and  $Y$  are  $\mathcal{F}$ -semimartingales. Then there exists an  $\mathcal{F}$ -adapted process  $A$  such that its almost every sample path is continuous and*

$$\sup_{s \in \mathbb{R}: 0 \leq s \leq t} \left| \sum_{i: \phi(i) < s} (X_{s \wedge \phi(i+1)} - X_{s \wedge \phi(i)})(Y_{s \wedge \phi(i+1)} - Y_{s \wedge \phi(i)}) - A_s \right|,$$

where  $\phi$  runs through locally finite partitions of  $\mathbb{R}_{\geq 0}$ , converges to 0 in probability as  $\text{mesh}(\phi)$  tends to 0 for all  $t \in \mathbb{R}_{\geq 0}$ . A posteriori almost every sample path of  $A$  is of finite variation and  $t \mapsto (M_t + m)(N_t + n) - A_t \in \mathfrak{Mart}^c(\mathcal{F})$  for  $M \in \text{Mart}[X; \mathcal{F}]$  and  $N \in \text{Mart}[Y; \mathcal{F}]$  where  $m, n$  are finite valued  $\mathcal{F}_0$ -measurable random variables.

*Proof.* We see that  $X + Y$  and  $X - Y$  are  $\mathcal{F}$ -semimartingales, and

$$M + N \in \text{Mart}[X + Y; \mathcal{F}] \text{ and } M - N \in \text{Mart}[X - Y; \mathcal{F}].$$

Corollary 8.18 claims  $\text{Qvar}[X + Y; \mathcal{F}]$  and  $\text{Qvar}[X - Y; \mathcal{F}]$  are not void. We select elements of them, say  $B$  and  $C$  respectively. Due to the polarization identities

$$XY = (X + Y)^2 - (X - Y)^2 \text{ and } MN = (M + N)^2 - (M - N)^2,$$

the process  $\tilde{A} := (B - C)/4$  shares the same property as the desired  $A$  process except for the  $\mathcal{F}$ -adaptedness. Lemma 10.2 takes care of the  $\mathcal{F}$ -adaptedness.  $\square$

**10.4 Definition.** Given two  $\mathcal{F}$ -semimartingales  $X$  and  $Y$ , a process which shares the same property as the  $A$  process in Lemma 10.3 is called a cross variation of  $X$  and  $Y$ .

Given  $\mathcal{F}$ -semimartingales  $X, Y$  we write

$$\text{Crv}[X, Y; \mathcal{F}] := \{A : \mathcal{F}\text{-cross variation of } X \text{ and } Y\}.$$

Clearly we have that  $\text{Crv}[X, X; \mathcal{F}] = \text{Qvar}[X; \mathcal{F}]$  and  $\text{Crv}[X, Y; \mathcal{F}] = \text{Crv}[Y, X; \mathcal{F}]$ .

**10.5 Corollary.** *Suppose that  $X$  and  $Y$  are  $\mathcal{F}$ -semimartingales. Then  $t \mapsto X_t Y_t$  is an  $\mathcal{F}$ -semimartingale. If  $M \in \text{Mart}[X; \mathcal{F}]$ ,  $N \in \text{Mart}[Y; \mathcal{F}]$  and  $A \in \text{Crv}[X, Y; \mathcal{F}]$  then*

$$M.N - A + R[(X - M)dN] + R[(Y - N)dM] \in \text{Mart}[X.Y; \mathcal{F}].$$

*Proof.* There is nothing new to mention about the sample path continuity. The process  $t \mapsto M_t N_t - A_t$  is an  $\mathcal{F}$ -local martingale by Lemma 10.3. Both  $B : t \mapsto X_t - M_t$  and  $C : t \mapsto Y_t - N_t$  are  $\mathcal{F}$ -finite variation processes. Their product  $t \mapsto B_t C_t$  is also an  $\mathcal{F}$ -finite variation process by Lemma 9.8(i). We see in (9.2) that

$$B_t N_t = R[BdN]_t + R[NdB]_t, C_t M_t = R[CdM]_t + R[MdC]_t \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

According to Theorem 9.10, the terms  $R[BdN]_t$  and  $R[CdM]_t$  define  $\mathcal{F}$ -local martingales while  $R[NdB]_t$  and  $R[MdC]_t$  constitute  $\mathcal{F}$ -finite variation processes. Thus we verified that  $t \mapsto M_t N_t + B_t N_t + C_t M_t + B_t C_t$  is an  $\mathcal{F}$ -semimartingale and its local martingale part is given by  $t \mapsto M_t N_t - A_t + R[BdN]_t + R[CdM]_t$ .  $\square$

We postpone the discussion on  $\text{Qvar}[X.Y; \mathcal{F}]$  after the introduction of stochastic integration. See Example 12.3.

**10.6 Lemma.** *Suppose that  $X, Y$  and  $Z$  are  $\mathcal{F}$ -semimartingales.*

- (i)  $\text{Crv}[X, Y, \mathcal{F}] \neq \emptyset$ . If  $A, B \in \text{Crv}[X, Y; \mathcal{F}]$  then  $A_t = B_t \forall t \in \mathbb{R}_{\geq 0}$  a.s. If  $A$  is  $\mathcal{F}$ -adapted,  $B \in \text{Crv}[X, Y; \mathcal{F}]$  and  $A_t = B_t \forall t \in \mathbb{R}_{\geq 0}$  a.s. then  $A \in \text{Crv}[X, Y; \mathcal{F}]$ .
- (ii) Suppose that  $A$  is an  $\mathcal{F}$ -adapted process,  $M \in \text{Mart}[X, ; \mathcal{F}]$ ,  $N \in \text{Mart}[Y, ; \mathcal{F}]$  and  $m, n$  are finite valued  $\mathcal{F}_0$ -measurable random variables. Then  $A \in \text{Crv}[X, Y; \mathcal{F}]$  if and only if almost every sample path of  $A$  starts from 0 and is continuous and of finite variation, and  $t \mapsto (M_t + m)(N_t + n) - A_t$  is an  $\mathcal{F}$ -local martingale.
- (iii) If  $f$  is a bounded  $\mathcal{F}_0$ -measurable function then  $f\text{Crv}[X, Z; \mathcal{F}] = \text{Crv}[fX, Z; \mathcal{F}]$ .  $\text{Crv}[X, Z; \mathcal{F}] + \text{Crv}[Y, Z; \mathcal{F}] = \text{Crv}[X + Y, Z; \mathcal{F}]$ .
- (iv) If  $\text{Crv}[X, Z; \mathcal{F}] \cap \text{Crv}[Y, Z; \mathcal{F}] \neq \emptyset$  then  $0 \in \text{Crv}[X - Y, Z; \mathcal{F}]$ . Conversely if  $0 \in \text{Crv}[X - Y, Z; \mathcal{F}]$  then  $\text{Crv}[X, Z; \mathcal{F}] = \text{Crv}[Y, Z; \mathcal{F}]$ .

*Proof.* (i) Lemma 10.3 shows that  $\text{Crv}[X, Y; \mathcal{F}]$  is not void. The second part is due to the uniqueness of limits in probability.

(ii) Lemma 10.3 shows the implication  $\Rightarrow$ . To prove the converse suppose that  $A$  is  $\mathcal{F}$ -adapted, its almost every sample path starts from 0 and is continuous and of finite variation, and  $M.N - A$  is an  $\mathcal{F}$ -local martingale. Choose an element  $B$  from  $\text{Crv}[X, Y; \mathcal{F}]$ . It follows that the difference  $A - B$  is an  $\mathcal{F}$ -local martingale whose almost every sample path starts from 0, and is continuous and of finite variation. With the help of Corollary 8.17 and (i) we infer that  $A \in \text{Crv}[X, Y; \mathcal{F}]$ .

(iii) If  $A \in \text{Crv}[X, Z; \mathcal{F}]$ ,  $B \in \text{Crv}[Y, Z; \mathcal{F}]$  and  $c \in \mathbb{R}$  then

$$A + B \in \text{Crv}[X + Y, Z; \mathcal{F}] \text{ and } cA \in \text{Crv}[cX, Z; \mathcal{F}].$$

(iv) Combine (i) and (iii). □

**10.7 Lemma.** *Suppose that  $X$  and  $Y$  are  $\mathcal{F}$ -semimartingales and  $\tau$  is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time. Then  $X_{\cdot \wedge \tau}$  is an  $\mathcal{F} \vee \text{Null}(P)$ -semimartingale,  $M_{\cdot \wedge \tau} \in \text{Mart}[X_{\cdot \wedge \tau}; \mathcal{F} \vee \text{Null}(P)]$  for all  $M \in \text{Mart}[X; \mathcal{F}]$  and  $A_{\cdot \wedge \tau} \in \text{Crv}[X_{\cdot \wedge \tau}, Y; \mathcal{F} \vee \text{Null}(P)]$  for all  $A \in \text{Crv}[X, Y; \mathcal{F}]$ .*

*Proof.* Lemma 7.5(iii). The result about the cross variation derives by definition. □

**10.8 Theorem.** (i) *If  $X, Y$  and  $Z$  are  $\mathcal{F}$ -semimartingales and  $t \mapsto X_t - Y_t$  is of finite variation a.s. then  $\text{Crv}[X, Z; \mathcal{F}] = \text{Crv}[Y, Z; \mathcal{F}]$ .*

(ii) *If  $X$  is an  $\mathcal{F}$ -semimartingale and  $0 \in \text{Qvar}[X; \mathcal{F}]$  then  $X$  is of finite variation a.s.*

(iii) *If  $X$  is an  $\mathcal{F} \vee \text{Null}(P)$ -semimartingale and  $0 \in \text{Crv}[X, M; \mathcal{F} \vee \text{Null}(P)]$  for all bounded  $M \in \mathfrak{Mart}^c(\mathcal{F})$  then  $X$  is of finite variation a.s.*

(iv) *If  $X$  and  $Y$  are  $\mathcal{F} \vee \text{Null}(P)$ -semimartingales and  $\text{Crv}[X, M; \tilde{\mathcal{F}}] \cap \text{Crv}[Y, M; \tilde{\mathcal{F}}] \neq \emptyset$  for all bounded  $M \in \mathfrak{Mart}^c(\mathcal{F})$  then  $t \mapsto X_t - Y_t$  is of finite variation a.s.*

*Proof.* (i) Lemma 8.16 shows that  $0 \in \text{Crv}[X - Y, Z; \mathcal{F}]$ . Hence we see by Lemma 10.6(iv) that  $\text{Crv}[X, Z; \mathcal{F}] = \text{Crv}[Y, Z; \mathcal{F}]$ .

(ii) Let  $M \in \text{Mart}[X; \mathcal{F}]$ . Then  $t \mapsto |M_t|^2$  is an  $\mathcal{F}$ -local martingale by Theorem 10.6(ii). Such process must satisfy  $M_t = 0$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. according to the proof of Corollary 8.17. Thus we reach  $0 \in \text{Mart}[X; \mathcal{F}]$ , i.e.,  $X$  is of finite variation a.s.

(iii) Let  $N. \in \text{Mart}[X.; \mathcal{F} \vee \text{Null}(P)]$ . According to Corollary 8.14(i)

$$\tau(k) := \inf\{t \in \mathbb{R}_{\geq 0} : |N_t| > k\}, k \in \mathbb{N}$$

forms a reducing sequence for the  $\mathcal{F} \vee \text{Null}(P)$ -local martingale  $N.$ . It follows by Lemma 10.2 that there exists an  $\mathcal{F}$ -adapted process  $M.$  such that

$$M_t = N_{t \wedge \tau(k)} \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

Then  $M.$  is an  $\mathcal{F}$ -martingale with almost sure continuous sample path. Since  $|N_{t \wedge \tau(k)}| \leq k$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. by Lemma 8.13(i), we may assume that  $|M_t(\omega)| \leq k$  for all  $t \in \mathbb{R}_{\geq 0}$  and  $\omega \in \Omega$ . Invoking (i) we infer that

$$\text{Crv}[X., M.; \mathcal{F} \vee \text{Null}(P)] = \text{Crv}[X., N_{\cdot \wedge \tau(k)}; \dots] = \text{Crv}[N., N_{\cdot \wedge \tau(k)}; \mathcal{F} \vee \text{Null}(P)].$$

Observe that the left hand side contains 0. Let  $A. \in \text{Qvar}[N.; \mathcal{F} \vee \text{Null}(P)]$ . Lemma 10.7 claims that the right hand side contains  $A_{\cdot \wedge \tau(k)}$ . It follows by Lemma 10.6(i) that

$$A_{t \wedge \tau(k)} = 0 \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s. for all } k \in \mathbb{N}.$$

Since  $\tau(k) \leq \tau(k+1)$  and  $\sup_{k \in \mathbb{N}} \tau(k) = +\infty$ , we infer that  $A_t = 0$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. Applying (i) again we reach that  $0 \in \text{Qvar}[N.; \mathcal{F} \vee \text{Null}(P)] = \text{Qvar}[X.; \mathcal{F} \vee \text{Null}(P)]$ , which implies that  $W.$  is of finite variation a.s. by (ii).

(iv) Combine (iii) in the present statement and Lemma 10.6(iv). □

**10.9 Corollary.** (i) If  $X. \in \text{Mart}^c(\mathcal{F}.)$  and  $0 \in \text{Qvar}[X.; \mathcal{F}]$  then  $X_t = X_0 \forall t \in \mathbb{R}_{\geq 0}$  a.s.

(ii) If  $X. \in \text{Mart}^c(\mathcal{F} \vee \text{Null}(P))$  and  $0 \in \text{Crv}[X., M.; \mathcal{F} \vee \text{Null}(P)]$  for all bounded  $M. \in \text{Mart}^c(\mathcal{F}.)$  then  $X_t = X_0$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s.

(iii) If  $X., Y. \in \text{Mart}^c(\mathcal{F} \vee \text{Null}(P))$  and  $\text{Crv}[X., M.; \mathcal{F} \vee \text{Null}(P)] \cap \text{Crv}[Y., M.; \mathcal{F} \vee \text{Null}(P)] \neq \emptyset$  for all bounded  $M. \in \text{Mart}^c(\mathcal{F}.)$  then  $X_t - X_0 = Y_t - Y_0$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s.

*Proof.* Theorem 10.8 and Corollary 8.17. □

**10.10 Theorem.** Suppose that  $X.$  and  $Y.$  are  $\mathcal{F}$ -semimartingales,  $f.$  and  $g.$  are  $\mathcal{F}$ -adapted process whose almost every sample path is of finite variation. If  $A. \in \text{Crv}[X., Y.; \mathcal{F}]$  then

$$R[f dA]. \in \text{Crv}[R[f dX]., Y.; \mathcal{F}] \text{ and } R[g dA]. \in \text{Crv}[R[f dX]., R[g dY].; \mathcal{F}].$$

*Proof.* Let  $M. \in \text{Mart}[X.; \mathcal{F}]$  and  $N. \in \text{Mart}[Y.; \mathcal{F}]$ . Then Corollary 9.11 shows that

$$R[f dM]. \in \text{Mart}[R[f dX].; \mathcal{F}] \text{ and } R[g dN]. \in \text{Mart}[R[g dY].; \mathcal{F}].$$

Moreover it follows by Corollary 10.8(i) that

$$\text{Crv}[X., Y.; \mathcal{F}] = \text{Crv}[M., N.; \mathcal{F}] \text{ and } \text{Crv}[R[f dX]., Y.; \mathcal{F}] = \text{Crv}[R[f dM]., N.; \mathcal{F}].$$

Thus we may assume that  $X.$  and  $Y.$  are  $\mathcal{F}$ -local martingales. We select and fix  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$ , and  $f.(\omega)$  is of finite variation,  $X_0(\omega) = 0$ , and  $X.(\omega)$  and  $Y.(\omega)$  are finite valued and continuous for every  $\omega \in \Omega_0$ . Then we have that

$$Y_t(\omega) \int_0^t f(\omega) dX(\omega) = \int_0^t f(\omega) d(X(\omega)Y(\omega)) - \int_0^t h(\omega) dY(\omega) \text{ for all } \omega \in \Omega_0$$

by Lemma 9.8(ii) where we write

$$h_t(\omega) := \begin{cases} \int_{[0,t)} X(\omega) df_{\cdot-o}(\omega) & \omega \in \Omega_0 \\ 0 & \omega \notin \Omega_0 \end{cases}$$

The process  $t \mapsto h_t$  is  $\mathcal{F} \vee \text{Null}(P)$ -adapted and its almost every sample path is left continuous and of finite variation. It therefore follows by Lemma 9.8(ii) that

$$R[fdX]_t Y_t - R[fdA]_t = R[fd(XY - A)]_t - R[h dY]_t \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

Lemma 10.6(ii) shows  $t \mapsto X_t Y_t - A_t \in \mathfrak{Mart}^c(\mathcal{F})$ . Thus, taking  $\mathcal{F}$ -adaptedness into account, we see by Theorem 9.10 that  $t \mapsto R[fdX]_t Y_t - R[fdA]_t$  is an  $\mathcal{F}$ -local martingale. Invoking Corollary 9.7 and Lemma 10.6(ii), we reach that  $R[fdA]_{\cdot} \in \text{Crv}[R[fdX]_{\cdot}, Y; \mathcal{F}]$ . Finally, since  $R[fdR[gdA]]_t = R[fgdA]_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. by Lemma 9.8(i), we obtain that  $R[fgdA]_{\cdot} \in \text{Crv}[R[fdX]_{\cdot}, R[gdY]_{\cdot}; \mathcal{F}]$ .  $\square$

## 11 Stochastic integration – beyond Riemann-Stieltjes

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\mathcal{F}$  be a filtration of sub  $\sigma$ -fields.  $\mathfrak{Mart}^c(\mathcal{F})$  denotes the space of all  $\mathcal{F}$ -local martingales with almost sure continuous sample path.

**11.1 Theorem.** *Suppose that  $M$  is an  $\mathcal{F}$ -local martingale such that its almost every sample path is right continuous, and that  $A$  is an  $\mathcal{F}$ -increasing process with almost sure continuous sample path such that  $t \mapsto |M_t|^2 - A_t$  is an  $\mathcal{F}$ -local martingale. Then*

$$P(\sup\{|M_t|; t \in \mathbb{R}_{\geq 0}, t \leq \tau\} > \lambda) \leq \frac{E[A_\tau \wedge \varepsilon]}{\lambda^2} + P(A_\tau > \varepsilon) + \frac{E[|M_0|^2 \wedge \lambda^2]}{\lambda^2}$$

for all  $\mathcal{F} \vee \text{Null}(P)$ -optional times  $\tau$  and  $\lambda, \varepsilon \in \mathbb{R}_{> 0}$ .

*Proof.* Given  $\lambda, \varepsilon \in \mathbb{R}_{> 0}$ , we introduce the following  $\mathcal{F} \vee \text{Null}(P)$ -optional times:

$$S := \inf\{t \in \mathbb{R}_{\geq 0} : A_t > \varepsilon\}, \quad T := \begin{cases} S & \text{if } |M_0| \leq \lambda \\ 0 & \text{otherwise} \end{cases}.$$

Since  $A_0 = 0 < \varepsilon$  and  $t \mapsto A_t$  is continuous a.s., we have that  $S > 0$  a.s. and hence

$$\{|M_0| \leq \lambda\} = \{T > 0\} \quad \text{mod Null}(P).$$

Observe that if  $T > 0$  then  $T = S$ . Therefore we infer that

$$\begin{aligned} (\star) \quad & P(\sup\{|M_t|; t \leq \tau\} > \lambda, \tau \leq S, |M_0| \leq \lambda) \leq P(\sup\{|M_t|; t \leq \tau \wedge S\} > \lambda, |M_0| \leq \lambda) \\ & = P(\sup\{|M_t 1_{T>0}|; t \leq \tau \wedge T\} > \lambda, T > 0) \leq P(\sup\{|M_t 1_{T>0}|; t \leq \tau \wedge T\} > \lambda). \end{aligned}$$

We write  $\mathcal{G}_0 := \mathcal{F}_{0+}$  and  $\mathcal{G}_t := \mathcal{F}_t$  for  $t \in \mathbb{R}_{> 0}$ . According to Lemma 8.13(iii),

$$A_{t \wedge T} \leq A_{t \wedge S} = A_t \wedge \varepsilon \leq \varepsilon \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

Since  $E[|M_0|^2; T > 0] = E[|M_0|^2; |M_0| \leq \lambda] < +\infty$ , it follows by Theorem 7.17(ii) that

$t \mapsto M_{t \wedge T} 1_{T > 0}$  is a square integrable  $\mathcal{G} \vee \text{Null}(P)$ -martingale.

Therefore Example 6.4 shows that

$$P(\sup\{|M_t 1_{T > 0}|; t \leq \tau \wedge n \wedge T\} > \lambda) \leq E[|M_{\tau \wedge n \wedge T} 1_{T > 0}|^2] / \lambda^2 \text{ for all } n \in \mathbb{N}.$$

On the other hand we see by Theorem 7.17(iii) that

$$E[|M_{\tau \wedge n \wedge T}|^2; T > 0] = E[|M_0|^2; T > 0] + E[A_{\tau \wedge n \wedge T}] \leq E[|M_0|^2; |M_0| \leq \lambda] + E[A_\tau \wedge \varepsilon]$$

where we used that  $\{|M_0| \leq \lambda\} = \{T > 0\}$  and  $A_{\tau \wedge n \wedge T} \leq A_\tau \wedge \varepsilon$  a.s. Hence

$$P(\sup\{|M_t 1_{T > 0}|; t \leq \tau \wedge n \wedge T\} > \lambda) \leq (E[|M_0|^2; |M_0| \leq \lambda] + E[A_\tau \wedge \varepsilon]) / \lambda^2.$$

Tending  $n$  to  $\infty$ , we reach

$$P(\sup\{|M_t 1_{T > 0}|; t \leq \tau \wedge T\} > \lambda) \leq (E[|M_0|^2; |M_0| \leq \lambda] + E[A_\tau \wedge \varepsilon]) / \lambda^2.$$

The left hand side dominates  $P(\sup\{|M_t|; t \leq \tau\} > \lambda, \tau \leq S, |M_0| \leq \lambda)$  by  $(\star)$ . Thus

$$P(\sup\{|M_t|; t \leq \tau\} > \lambda) \leq \frac{E[|M_0|^2; |M_0| \leq \lambda]}{\lambda^2} + \frac{E[A_\tau \wedge \varepsilon]}{\lambda^2} + P(\tau > S) + P(|M_0| > \lambda).$$

Finally, taking Lemma 8.13(i) into account, we get the claim.  $\square$

**11.2 Corollary.** *Suppose that  $M^n$  is a sequence of  $\mathfrak{Mart}^c(\mathcal{F})$ ,  $A^n \in \text{Qvar}[M^n; \mathcal{F}]$  and  $\tau$  is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time. If both  $M_0^n$  and  $A_\tau^n$  converge to 0 in probability then so does the sequence  $\sup\{|M_t^n|; t \in \mathbb{R}_{\geq 0}, t \leq \tau\}$ .*

**11.3 Lemma.** *Suppose that  $f$  is an  $\mathcal{F}_+$ -adapted process whose almost every sample path is locally bounded and admits left hand limits everywhere,  $A$  is an  $\mathcal{F}$ -finite variation process, and  $B$  is an  $\mathcal{F}$ -increasing process. Then for each  $t \in \mathbb{R}_{\geq 0}$*

$$\begin{aligned} \limsup_{n \rightarrow \infty} \sup_{s \in \mathbb{R}: 0 \leq s \leq t} |R[f([n \cdot]/n, \star) dA]_s - R[f dA]_s| &= 0 \text{ a.s.}, \\ \limsup_{n \rightarrow \infty} R[|f([n \cdot]/n, \star) - f|^2 dB]_t &= 0 \text{ a.s. and} \\ \limsup_{m, n \rightarrow \infty} R[|f([m \cdot]/m, \star) - f([n \cdot]/n, \star)|^2 dB]_t &= 0 \text{ a.s.} \end{aligned}$$

where  $[x] = \max(\{i \in \mathbb{Z} : i < x\} \cup \{0\})$  for  $x \in \mathbb{R}$ .

*Proof.* Select and fix  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$ , and  $f(\cdot, \omega)$  is locally bounded and admits left hand limits everywhere,  $A(\omega)$  is of finite variation and right continuous, and  $B(\omega)$  is finite valued, nondecreasing and right continuous for every  $\omega \in \Omega_0$ . Let  $n \in \mathbb{N}$ . Then

$$R[f([n \cdot]/n, \star) dA]_t - R[f dA]_t(\omega) = \int_{(0, t]} \{f([n \cdot]/n, \omega) - f(\cdot - o, \omega)\} dA(\omega)$$

for all  $t \in \mathbb{R}_{\geq 0}$  and  $\omega \in \Omega_0$  by Lemma 9.6. The right hand side is dominated by

$$\int_{(0, t]} |f([n \cdot]/n, \omega) - f(\cdot - o, \omega)| \text{var}(dA(\omega), \cdot).$$

Observe that  $f([nt]/n, \omega)$  converges to  $f(t - o, \omega)$  as  $n$  tends to  $\infty$  for each  $t \in \mathbb{R}_{\geq 0}$  and  $\omega \in \Omega_0$ . Due to the local boundedness of  $f(\cdot, \omega)$ , we see that

$$\limsup_{n \rightarrow \infty} \int_{(0,t]} |f([n\cdot]/n, \omega) - f(\cdot - o, \omega)| \text{var}(dA(\omega), \cdot) = 0 \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ and } \omega \in \Omega_0$$

by the dominated convergence theorem. The argument for other limits is similar.  $\square$

**11.4 Theorem.** *Suppose that  $M. \in \mathfrak{Mart}^c(\mathcal{F})$  and  $f$  is an  $\mathcal{F}$ -adapted process whose almost every sample path is locally bounded and admits left hand limits everywhere. Then there exists  $I. \in \mathfrak{Mart}^c(\mathcal{F})$  such that  $I_0 = 0$  a.s. and*

$$R[f dA]. \in \text{Crv}[I., N.; \mathcal{F}.] \text{ for all } N. \in \mathfrak{Mart}^c(\mathcal{F}.) \text{ and } A. \in \text{Crv}[M., N.; \mathcal{F}.]$$

*Proof.* We introduce the following sequence of  $\mathcal{F}$ -finite variation processes:

$$f^n : t \mapsto f([nt]/n, \star), \quad n \in \mathbb{N} \text{ where } [x] = \max(\{i \in \mathbb{Z} : i < x\} \cup \{0\}).$$

Let  $n \in \mathbb{N}$ . The process  $t \mapsto R[f^n dM]_t$  belongs to  $\mathfrak{Mart}^c(\mathcal{F})$  by Theorem 9.10 and

$$R[f^n dA]. \in \text{Crv}[R[f^n dM]., N.; \mathcal{F}.]$$

by Theorem 10.10. Let  $B. \in \text{Qvar}[M.; \mathcal{F}.]$ . We have by Lemma 9.8(i) that

$$R[f^m dM]_t - R[f^n dM]_t = R[(f^m - f^n) dM]_t \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

According to Theorem 9.10,

$$R[|f^m - f^n|^2 dB] \in \text{Qvar}[R[(f^m - f^n) dM]; \mathcal{F}.]$$

Since the almost sure convergence implies the convergence in probability, it follows that

$$\limsup_{m, n \rightarrow \infty} P(R[|f^m - f^n|^2 dB]_t > \varepsilon) = 0 \text{ for all } \varepsilon \in \mathbb{R}_{> 0} \text{ and } t \in \mathbb{R}_{\geq 0}$$

by Lemma 11.3. Thus, invoking Theorem 11.1, we infer that

$$\limsup_{m, n \rightarrow \infty} P\left(\sup_{s \in \mathbb{R}: 0 \leq s \leq t} |R[f^m dM]_s - R[f^n dM]_s| > \lambda\right) = 0 \text{ for all } \lambda \in \mathbb{R}_{> 0} \text{ and } t \in \mathbb{R}_{\geq 0}.$$

We can find  $\phi \in \text{Seq}(\mathbb{N}, \uparrow)$  such that

$$P\left(\sup_{s \in \mathbb{R}: 0 \leq s \leq T} |R[f^m dM]_s - R[f^n dM]_s| \geq 1/2^T\right) \leq 1/2^T \text{ for all } T \in \mathbb{N} \text{ and } m, n \in \mathbb{N}_{\geq \phi(T)}.$$

We write  $I_t^n := R[f^n dM]_t$  for simplicity. The Borel-Cantelli lemma shows that there exists  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and

$$t \mapsto R[f^n dM]_t(\omega) \text{ is continuous for all } n \in \mathbb{N} \text{ and} \\ \sup_{t: t \leq n} |I_t^{\phi(n+1)}(\omega) - I_t^{\phi(n)}(\omega)| < 1/2^n \text{ except for finitely many } n \in \mathbb{N}$$

for all  $\omega \in \Omega_0$ . We set

$$I_t(\omega) := \liminf_{n \rightarrow \infty} R[f^{\phi(n)} dM]_t(\omega) \text{ for } t \in \mathbb{R}_{\geq 0} \text{ and } \omega \in \Omega.$$

The process  $t \mapsto I_t$  is  $\mathcal{F}$ -adapted. Let  $\omega \in \Omega_0$  and  $T \in \mathbb{N}$ . Then, since

$$\sum_{k=n}^{\infty} \sup_{t:t \leq T} |I_t^{\phi(k+1)}(\omega) - I_t^{\phi(k)}(\omega)| \leq \sum_{k=n}^{\infty} \sup_{t:t \leq k} |I_t^{\phi(k+1)}(\omega) - I_t^{\phi(k)}(\omega)| \leq 1/2^{n-1}$$

for all  $n \in \mathbb{N}_{\geq T}$ , it follows that

$$I_t(\omega) - 1/2^{n-1} \leq I_t^{\phi(n)}(\omega) \leq I_t(\omega) + 1/2^{n-1} \text{ for } t \in \mathbb{R}_{[0,T]} \text{ and } n \in \mathbb{N}_{\geq T}.$$

Therefore the continuity of  $t \mapsto I_t(\omega)$  derives from that of  $t \mapsto I_t^n(\omega)$ . Moreover Lemma 9.5 shows that  $I$  is an  $\mathcal{F}$ -local martingale. We see by Lemma 11.3 that

$$\limsup_{n \rightarrow \infty} P\left(\sup_{s \in \mathbb{R}: 0 \leq s \leq t} |R[f^n dA]_s - R[f dA]_s| > \varepsilon\right) = 0 \text{ for all } \varepsilon \in \mathbb{R}_{>0} \text{ and } t \in \mathbb{R}_{\geq 0}.$$

We get by Lemma 9.5, Corollary 9.7 and Lemma 10.6(ii) that  $R[f dA] \in \text{Crv}[I, N, ; \mathcal{F}]$ .  $\square$

Given  $M \in \mathfrak{Mart}^c(\mathcal{F})$  and an  $\mathcal{F}$ -adapted process  $f$  whose almost every sample path is locally bounded and admits left hand limits everywhere, we set

$$\text{It}\hat{\circ}[fdM; \mathcal{F}] := \{I \in \mathfrak{Mart}^c(\mathcal{F}) : I_0 = 0 \text{ a.s.}, \text{ and } R[f dA] \in \text{Crv}[I, N, ; \mathcal{F}] \\ \text{for any bounded } N \in \mathfrak{Mart}^c(\mathcal{F}) \text{ and } A \in \text{Crv}[M, N, ; \mathcal{F}]\}.$$

*Stochastic integral of  $f$  with integrator  $M$ .*

**11.5 Corollary.** *Suppose  $M \in \mathfrak{Mart}^c(\mathcal{F})$ , and  $f$  is an  $\mathcal{F}$ -adapted process whose almost every sample path is locally bounded and admits left hand limits everywhere.*

- (i)  $\text{It}\hat{\circ}[fdM; \mathcal{F}] \neq \emptyset$ . If  $I, L \in \text{It}\hat{\circ}[fdM; \mathcal{F}]$  then  $I_t = L_t \forall t \in \mathbb{R}_{\geq 0}$  a.s. If  $I$  is  $\mathcal{F}$ -adapted,  $L \in \text{It}\hat{\circ}[fdM; \mathcal{F}]$  and  $I_t = L_t \forall t \in \mathbb{R}_{\geq 0}$  a.s. then  $I \in \text{It}\hat{\circ}[fdM; \mathcal{F}]$ .
- (ii) Let  $I \in \text{It}\hat{\circ}[fdM; \mathcal{F}]$ . If  $A \in \text{Qvar}[M, ; \mathcal{F}]$ ,  $N \in \mathfrak{Mart}^c(\mathcal{F})$  and  $B \in \text{Crv}[M, N, ; \mathcal{F}]$  then  $R[|f|^2 dA] \in \text{Qvar}[I, ; \mathcal{F}]$  and  $R[f dB] \in \text{Crv}[I, N, ; \mathcal{F}]$ .
- (iii)  $\text{It}\hat{\circ}[fdM; \mathcal{F}] \subset \text{It}\hat{\circ}[fdM; \mathcal{F} \vee \text{Null}(P)]$ .

*Proof.* (i) Let  $I, L \in \text{It}\hat{\circ}[fdM]$ . Then  $\text{Crv}[I, N, ; \mathcal{F}] \cap \text{Crv}[L, N, ; \mathcal{F}] \neq \emptyset$  for any bounded  $N \in \mathfrak{Mart}^c(\mathcal{F})$ . It follows that  $I_t = L_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. by Corollary 10.9(iii). Conversely suppose that  $I$  is  $\mathcal{F}$ -adapted,  $L \in \text{It}\hat{\circ}[fdM]$  and  $I_t = L_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. Clearly  $I \in \mathfrak{Mart}^c(\mathcal{F})$ . Theorem 10.8(i) shows that  $\text{Crv}[I, N, ; \mathcal{F}]$  coincides with  $\text{Crv}[L, N, ; \mathcal{F}]$  for any bounded  $N \in \mathfrak{Mart}^c(\mathcal{F})$ . Hence  $I \in \text{It}\hat{\circ}[fdM]$ .

(ii) Combining (i) and Theorem 11.4, we get  $R[f dB] \in \text{Crv}[I, N, ; \mathcal{F}]$ . In particular we have that  $R[f dA] \in \text{Crv}[I, M, ; \mathcal{F}] = \text{Crv}[M, I, ; \mathcal{F}]$ . According to Lemma 9.6,

$$R[|f|^2 dA]_t = R[f dR[f dA]]_t \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

It therefore follows by Lemma 10.6(i) that  $R[|f|^2 dA] \in \text{Crv}[I, I, ; \mathcal{F}] = \text{Qvar}[I, ; \mathcal{F}]$ .

(iii) We select  $J \in \text{It}\hat{\circ}[fdM; \tilde{\mathcal{F}}] \neq \emptyset$  where  $\tilde{\mathcal{F}} := \mathcal{F} \vee \text{Null}(P)$ . Let  $I \in \text{It}\hat{\circ}[fdM; \mathcal{F}]$ . Then we infer that

$\text{Crv}[I, N.; \tilde{\mathcal{F}}] \cap \text{Crv}[J, N.; \tilde{\mathcal{F}}] \neq \emptyset$  for any bounded  $N. \in \mathfrak{Mart}^c(\mathcal{F})$ .

Indeed, given  $A. \in \text{Crv}[M., N.; \mathcal{F}] \subset \text{Crv}[M., N.; \tilde{\mathcal{F}}]$ , we have that

$$R[f dA] \in \text{Crv}[I, N.; \mathcal{F}] \subset \text{Crv}[I, N.; \tilde{\mathcal{F}}] \text{ and } R[f dA] \in \text{Crv}[J, N.; \tilde{\mathcal{F}}]$$

It therefore follows that  $I_t = J_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. by Corollary 10.9(iii). Hence we conclude that  $I. \in \text{It}\hat{o}[fdM; \tilde{\mathcal{F}}]$  by using (i) in the present corollary.  $\square$

**11.6 Lemma.** *Suppose that  $M., N. \in \mathfrak{Mart}^c(\mathcal{F})$ , and  $f, g$  are  $\mathcal{F}$ -adapted processes whose almost sure sample paths are locally bounded and admit left hand limits everywhere.*

(i) *Let  $A \in \text{Qvar}[M.; \mathcal{F}]$ . Then  $0 \in \text{It}\hat{o}[fdM; \mathcal{F}]$  if and only if  $R[|f|^2 dA]_t = 0$  a.s.  $\forall t \in \mathbb{R}_{\geq 0}$ .*

(ii) *For  $h$  bounded and  $\mathcal{F}_0$ -measurable  $h \text{It}\hat{o}[fdM; \mathcal{F}] = \text{It}\hat{o}[hfdM; \mathcal{F}] = \text{It}\hat{o}[fd(hM); \mathcal{F}]$ .*

(iii)  $\text{It}\hat{o}[fdM; \mathcal{F}] + \text{It}\hat{o}[gdM; \mathcal{F}] = \text{It}\hat{o}[(f + g)dM; \mathcal{F}]$ .

$$\text{It}\hat{o}[fdM; \mathcal{F}] + \text{It}\hat{o}[fdN; \mathcal{F}] = \text{It}\hat{o}[fd(M + N); \mathcal{F}].$$

(iv) *If  $J. \in \text{It}\hat{o}[gdM; \mathcal{F}]$  then  $\text{It}\hat{o}[fgdM; \mathcal{F}] = \text{It}\hat{o}[fdJ; \mathcal{F}]$ .*

(v) *If  $\sigma \in \text{Time}(\mathcal{F})$  and  $I. \in \text{It}\hat{o}[fdM; \mathcal{F}]$  then  $I_{\wedge \sigma} \in \text{It}\hat{o}[1_{(0, \sigma]}fdM; \mathcal{F} \vee \text{Null}(P)]$  more precisely  $I_{t \wedge \sigma} = J_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. for  $J. \in \text{It}\hat{o}[1_{(0, \sigma]}fdM; \mathcal{F}]$ .*

(vi) *If  $\sigma \in \text{Time}(\mathcal{F} \vee \text{Null}(P))$  then  $\text{It}\hat{o}[1_{(0, \sigma]}fdM; \mathcal{F} \vee \text{Null}(P)] = \text{It}\hat{o}[fdM_{\wedge \sigma}; \mathcal{F} \vee \text{Null}(P)]$ .*

*Proof.* Let  $I. \in \text{It}\hat{o}[fdM]$  and  $J. \in \text{It}\hat{o}[gdM]$ . (i) Since  $R[|f|^2 dA]. \in \text{Qvar}[I.; \mathcal{F}]$  by Corollary 11.5(ii), we get the following series of equivalence relations:

$$0 \in \text{It}\hat{o}[fdM] \Leftrightarrow I_t = 0 \forall t \in \mathbb{R}_{\geq 0} \text{ a.s.} \Leftrightarrow 0 \in \text{Qvar}[I.; \mathcal{F}] \Leftrightarrow R[|f|^2 dA]_t = 0 \forall t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

by Corollary 11.5(i), Corollary 10.9(i) and Theorem 8.21(iii) respectively.

(iii) and (iv) Given  $L. \in \mathfrak{Mart}^c(\mathcal{F})$  and  $B. \in \text{Crv}[M., L.; \mathcal{F}]$ . Then we have that

$$R[(f + g)dB]_t = R[f dB]_t + R[g dB]_t \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.,}$$

$$R[f dB]. \in \text{Crv}[I., L.; \mathcal{F}] \text{ and } R[g dB]. \in \text{Crv}[J., L.; \mathcal{F}].$$

It follows by Lemma 10.6(i), (iii) that

$$R[(f + g)dB]. \in \text{Crv}[I., L.; \mathcal{F}] + \text{Crv}[J., L.; \mathcal{F}] = \text{Crv}[I. + J., L.; \mathcal{F}].$$

This means  $I. + J. \in \text{It}\hat{o}[(f + g)dM]$  and hence

$$\text{It}\hat{o}[fdM] + \text{It}\hat{o}[gdM] \subset \text{It}\hat{o}[(f + g)dM].$$

Actually the equality holds by Corollary 11.5(i). Similar discussion also works for the second relation in (iii). To show (iv) we select and fix  $K. \in \text{It}\hat{o}[fdJ]$ . Since

$$R[fgdB]_t = R[fdR[gdB]]_t \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

by Lemma 9.6 and  $R[fdR[gdB]]. \in \text{Crv}[K., L.; \mathcal{F}]$ , it follows by Lemma 10.6(i) that

$$R[fgdB]. \in \text{Crv}[K., L.; \mathcal{F}].$$

This means  $K. \in \text{It}\hat{o}[fgdM]$ . Clearly we can revert the argument above.

(v) Let  $N. \in \mathfrak{Mart}^c(\tilde{\mathcal{F}})$  and  $A. \in \text{Crv}[M., N.; \tilde{\mathcal{F}}]$  where  $\tilde{\mathcal{F}} := \mathcal{F} \vee \text{Null}(P)$ . Then, since  $I. \in \text{It}\hat{\circ}[fdM; \tilde{\mathcal{F}}]$  according to Corollary 11.5(iii), we have that  $R[fdA]. \in \text{Crv}[I., N.; \tilde{\mathcal{F}}]$ . It follows by Lemma 10.7 that

$$R[fdA]_{\cdot \wedge \sigma} \in \text{Crv}[I_{\cdot \wedge \sigma}, N.; \tilde{\mathcal{F}}].$$

On the other hand  $R[fdA]_{t \wedge \sigma} = R[1_{(0, \sigma]} fdA]_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. Lemma 10.6(i) shows that

$$R[1_{(0, \sigma]} fdA] \in \text{Crv}[I_{\cdot \wedge \sigma}, N.; \tilde{\mathcal{F}}].$$

Thus  $I_{\cdot \wedge \sigma} \in \text{It}\hat{\circ}[1_{(0, \sigma]} fdM; \tilde{\mathcal{F}}]$ . Finally, since  $J. \in \text{It}\hat{\circ}[1_{(0, \sigma]} fdM; \tilde{\mathcal{F}}]$  by Corollary 11.5(iii), it follows that  $I_{t \wedge \sigma} = J_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. by Corollary 11.5(i).  $\square$

**11.7 Corollary.** *Suppose  $X.$  is an  $\mathcal{F}$ -semimartingale, and  $f$  is an  $\mathcal{F}$ -adapted process whose almost every sample path is locally bounded and admits left hand limits everywhere. Then*

$$R[fd(X - M)]. + \text{It}\hat{\circ}[fdM; \mathcal{F}.] = R[fd(X - N)]. + \text{It}\hat{\circ}[fdN; \mathcal{F}.]$$

for all  $M., N. \in \text{Mart}[X.; \mathcal{F}.]$ .

*Proof.*  $R[fd(X - M)]_t = R[fd(X - N)]_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. and  $\text{It}\hat{\circ}[fdM] = \text{It}\hat{\circ}[fdN]$ .  $\square$

Given an  $\mathcal{F}$ -semimartingale  $X.$ , and an  $\mathcal{F}$ -adapted process  $f$  whose almost every sample path is locally bounded and admits left hand limits everywhere, we set

$$\text{It}\hat{\circ}[fdX; \mathcal{F}.] := R[fd(X - M)]. + \text{It}\hat{\circ}[fdM; \mathcal{F}.]$$

where  $M. \in \text{Mart}[X.; \mathcal{F}.]$ . *Stochastic integral of  $f$  with integrator  $X.$*

## 12 Semimartingale and Itô's formula

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\mathcal{F}$  be a filtration of sub  $\sigma$ -fields.  $\mathfrak{Mart}^c(\mathcal{F})$  denotes the space of all  $\mathcal{F}$ -local martingales with almost sure continuous sample path.

**12.1 Lemma.** *Suppose that  $X., Y.$  are  $\mathcal{F}$ -semimartingales, and  $f, g$  are  $\mathcal{F}$ -adapted processes whose almost sure sample paths are locally bounded and admit left hand limits everywhere.*

- (i) For  $h$  bounded  $\mathcal{F}_0$ -measurable  $h \text{It}\hat{\circ}[fdX; \mathcal{F}.] = \text{It}\hat{\circ}[hfdX; \mathcal{F}.] = \text{It}\hat{\circ}[fd(hX); \mathcal{F}.]$ .
- (ii)  $\text{It}\hat{\circ}[fdX; \mathcal{F}.] + \text{It}\hat{\circ}[gdX; \mathcal{F}.] = \text{It}\hat{\circ}[(f + g)dX; \mathcal{F}.]$   
 $\text{It}\hat{\circ}[fdX; \mathcal{F}.] + \text{It}\hat{\circ}[fdY; \mathcal{F}.] = \text{It}\hat{\circ}[fd(X + Y); \mathcal{F}.]$
- (iii) If  $Z. \in \text{It}\hat{\circ}[gdX; \mathcal{F}.]$  then  $\text{It}\hat{\circ}[fgdX; \mathcal{F}.] = \text{It}\hat{\circ}[fdZ; \mathcal{F}.]$ .
- (iv)  $I. \in \text{It}\hat{\circ}[fdX; \mathcal{F}.], J. \in \text{It}\hat{\circ}[gdY; \mathcal{F}.], A. \in \text{Crv}[X, Y; \mathcal{F}.] \Rightarrow R[fgdA]. \in \text{Crv}[I, J; \mathcal{F}.]$ .
- (v) If  $\sigma$  is an  $\mathcal{F}$ -optional time and  $I. \in \text{It}\hat{\circ}[fdX; \mathcal{F}.]$  then  $I_{\cdot \wedge \sigma} \in \text{It}\hat{\circ}[1_{(0, \sigma]} fdX; \mathcal{F} \vee \text{Null}(P)]$  more precisely  $I_{t \wedge \sigma} = J_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. for  $J. \in \text{It}\hat{\circ}[1_{(0, \sigma]} fdX; \mathcal{F}.]$ .

**12.2 Lemma.** *Suppose that  $X.$  is an  $\mathcal{F}$ -semimartingale, and  $f$  is an  $\mathcal{F}$ -adapted process whose almost every sample paths is locally bounded and admits left hand limits everywhere.*

- (i) If almost every sample path of  $X$  or  $f$  is of finite variation then  $R[fdX]. \in \text{It}\hat{\circ}[fdX]$ .
- (ii) Let  $I. \in \text{It}\hat{\circ}[fdX]$ . Then  $\sup_{s \leq t} |R[f([n \cdot]/n, \star)dX]_s - I_s|$  converges to 0 in probability where  $[x] := \max(\{i \in \mathbb{Z} : i < x\} \cup \{0\})$  for  $x \in \mathbb{R}$ .

*Proof.* Let  $M. \in \text{Mart}[X.; \mathcal{F}].$  (i) We have that  $R[f dM]. \in \text{It}\hat{\circ}[f dM]$  by Theorem 10.10.

(ii) Let  $A. \in \text{Qvar}[M.; \mathcal{F}].$  Set  $J_t := I_t - R[f d(X - M)]_t.$  Then  $J. \in \text{It}\hat{\circ}[f dM].$  We see by Theorem 10.10 and Corollary 11.5(ii) that

$$R[f([n\cdot]/n, \star) f dA]. \in \text{Crv}[R[f([n\cdot]/n, \star) dM]., J.; \mathcal{F} \vee \text{Null}(P)].$$

This implies that  $R[[f([n\cdot]/n, \star) - f]^2 dA].$  is a quadratic variation of the local martingale  $R[f([n\cdot]/n, \star) dM]. - J..$  Invoking Lemma 11.3 and Corollary 11.2, we infer that

$$\limsup_{n \rightarrow \infty} P(\sup_{s \leq t} |R[f([n\cdot]/n, \star) dM]_s - J_s| > \varepsilon) = 0 \text{ for all } \varepsilon \in \mathbb{R}_{\geq 0} \text{ and } t \in \mathbb{R}_{\geq 0}.$$

On the other hand Lemma 9.8(ii) shows that

$$R[f([n\cdot]/n, \star) dX]_s = R[f([n\cdot]/n, \star) dM]_s + R[f([n\cdot]/n, \star) d(X - M)]_s \text{ for all } s \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

The second term in the right hand side converges to  $R[f d(X - M)].$  locally uniformly a.s. by Lemma 11.3. Thus we get the claim.  $\square$

**12.3 Example.** Suppose that  $X.$  and  $Y.$  are  $\mathcal{F}.$ -semimartingales. Then

$$\begin{aligned} X.Y. &\in X_0 Y_0 + \text{It}\hat{\circ}[X dY] + \text{It}\hat{\circ}[Y dX] + \text{Crv}[X., Y.; \mathcal{F}] \\ \text{Mart}[X.Y.; \mathcal{F}] &= \text{It}\hat{\circ}[X dN] + \text{It}\hat{\circ}[Y dM] \end{aligned}$$

for  $M. \in \text{Mart}[X.; \mathcal{F}]$  and  $N. \in \text{Mart}[Y.; \mathcal{F}].$

*Proof.* Let  $A. \in \text{Crv}[X., Y.; \mathcal{F}].$  According to Lemma 10.3,

$$\limsup_{n \rightarrow \infty} P(\sup_{s \in \mathbb{R}: 0 \leq s \leq t} |\sum_{i:i < ns} (X_{s \wedge (i+1)/n} - X_{s \wedge i/n})(Y_{s \wedge (i+1)/n} - Y_{s \wedge i/n}) - A_s| > \varepsilon) = 0$$

for all  $\varepsilon \in \mathbb{R}_{> 0}$  and  $t \in \mathbb{R}_{\geq 0}.$  Arranging the summation we have that

$$\begin{aligned} X_s Y_s - X_0 Y_0 - \sum_{i:i < ns} (X_{s \wedge (i+1)/n} - X_{s \wedge i/n})(Y_{s \wedge (i+1)/n} - Y_{s \wedge i/n}) \\ = \sum_{i:i < ns} (X_{s \wedge (i+1)/n} - X_{s \wedge i/n}) Y_{i/n} + \sum_{i:i < ns} (Y_{s \wedge (i+1)/n} - Y_{s \wedge i/n}) X_{i/n}. \end{aligned}$$

The left hand side converges to  $X_s Y_s - X_0 Y_0 - A_s$  locally uniformly in probability while the right hand equals  $R[Y_{[n\cdot]/n} dX]_s + R[X_{[n\cdot]/n} dY]_s$  for all  $s \in \mathbb{R}_{\geq 0}$  a.s. where we write  $[x] := \max(\{i \in \mathbb{Z} : i < x\} \cup \{0\})$  for  $x \in \mathbb{R}.$  Thus we get the claim by invoking Lemma 12.2(ii).  $\square$

**12.4 Lemma.** Suppose that  $f^n, f$  are  $\mathcal{F}.$ -adapted processes with almost sure left continuous and locally bounded sample path,  $A.$  is an  $\mathcal{F}.$ -finite variation process, and  $X.$  is an  $\mathcal{F}.$ -semimartingale. If  $f^n(t, \cdot)$  converges to  $f(t, \cdot)$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. and there exist an  $\mathcal{F}.$ -increasing process  $g$  such that  $|f^n(t, \cdot)| \leq g(t, \cdot)$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. then for each  $t \in \mathbb{R}_{\geq 0}$

$$\begin{aligned} \limsup_{n \rightarrow \infty} \sup_{s \in \mathbb{R}: 0 \leq s \leq t} |R[f^n dA]_s - R[f dA]_s| &= 0 \text{ a.s.}, \\ \sup_{s \in \mathbb{R}: 0 \leq s \leq t} |I_s^n - I_s| &\text{ converges to 0 in probability.} \end{aligned}$$

where  $I^n \in \text{It}\hat{\circ}[f^n dX]$  and  $I. \in \text{It}\hat{\circ}[f dX].$

*Proof.* Let  $B. \in \text{Qvar}[X.; \mathcal{F}]$ . The argument in the proof of Corollary 11.3 shows that

$$\limsup_{n \rightarrow \infty} \sup_{s \in \mathbb{R}: 0 \leq s \leq t} |R[f^n dA]_s - R[f dA]_s| = 0 \text{ a.s. and } \limsup_{n \rightarrow \infty} R[|f^n - f|^2 dB]_t = 0 \text{ a.s.}$$

Let  $M. \in \text{Mart}[X.; \mathcal{F}]$ . The above applies to  $A_t = X_t - M_t$ . Set

$$J_t^n := I_t^n - R[f^n d(X - M)]_t - I_t + R[f d(X - M)]_t.$$

Then we see by Lemma 12.1(ii) and Corollary 11.5(ii) that

$$J^n \in \text{It}\hat{\circ}[(f^n - f)dM] \text{ and } R[|f^n - f|^2 dB]. \in \text{Qvar}[J^n; \mathcal{F}].$$

Invoking Corollary 11.2, we reach the convergence of  $\sup_{s \leq t} |J_s^n|$  in probability.  $\square$

**12.5 Definition.** Let  $d \in \mathbb{N}$ . A  $d$ -dimensional  $\mathcal{F}$ -semimartingale is an  $\mathbb{R}^d$ -valued process each component of which is an  $\mathcal{F}$ -semimartingale. A complex  $\mathcal{F}$ -semimartingale is a  $\mathbb{C}$ -valued process whose real part and well as imaginary part are  $\mathcal{F}$ -semimartingales.

**12.6 Theorem.** Suppose that  $X.$  is a  $d$ -dimensional  $\mathcal{F}$ -semimartingale with components  $X^i$ ,  $i \in \mathbb{N}_{\leq d}$ . Then, given  $f \in C^2(\mathbb{R}^d)$ , the composition  $f \circ X.$  is an  $\mathcal{F}$ -semimartingale and

$$f(X.) \in f(X_0) + \sum_{i=1}^d \text{It}\hat{\circ}[\nabla_i f(X.) dX^i] + \frac{1}{2} \sum_{i,j=1}^d R[\nabla_{ij}^2 f(X.) dA^{ij}].$$

where  $A^{ij} \in \text{Crv}[X^i, X^j; \mathcal{F}]$ . It $\hat{\circ}$ 's formula

*Proof.* The statement holds for affine functions. Denote by  $\mathcal{D}$  the set of all  $C^2$ -functions for which the statement holds. We first show that  $\mathcal{D}$  is an algebra over  $\mathbb{R}$ . Since the differential operators  $\nabla$  and  $\nabla^2$  are linear, the linearity of  $\mathcal{D}$  follows by Lemma 12.1(i), (ii). Now suppose that  $f, g \in \mathcal{D}$ , that is,

$$\begin{aligned} f(X.) &\in f(X_0) + \sum_{i=1}^d \text{It}\hat{\circ}[\nabla_i f(X.) dX^i] + \frac{1}{2} \sum_{i,j=1}^d R[\nabla_{ij}^2 f(X.) dA^{ij}]. \text{ and} \\ g(X.) &\in g(X_0) + \sum_{i=1}^d \text{It}\hat{\circ}[\nabla_i g(X.) dX^i] + \frac{1}{2} \sum_{i,j=1}^d R[\nabla_{ij}^2 g(X.) dA^{ij}]. \end{aligned}$$

It then follows by Lemma 12.1(ii), (iii) and Lemma 12.2(i) that

$$\sum_{i=1}^d \text{It}\hat{\circ}[f(X.) \nabla_i g(X.) dX^i] + \frac{1}{2} \sum_{i,j=1}^d R[f(X.) \nabla_{ij}^2 g(X.) dA^{ij}]. = \text{It}\hat{\circ}[f(X.) dg(X.)].$$

On the other hand Lemma 12.1(iv), Lemma 10.6(iii) and Lemma 8.16 tell us that

$$\sum_{i,j=1}^d R[\nabla_i f(X.) \nabla_j g(X.) dA^{ij}]. \in \text{Crv}[f(X.), g(X.); \tilde{\mathcal{F}}]$$

Observe that  $\nabla_i(f+g) = \nabla_i f + \nabla_i g$  and  $\nabla_{ij}^2(fg) = f\nabla_{ij}^2 g + g\nabla_{ij}^2 f + \nabla_i f \nabla_j g + \nabla_i g \nabla_j f$ . Taking into account that  $A_t^{ij} = A_t^{ji}$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s., we infer by Example 12.3 that

$$f(X.)g(X.) \in f(X_0)g(X_0) + \sum_{i=1}^d \text{It}\hat{\circ}[\nabla_i(fg)(X.)dX^i] + \frac{1}{2} \sum_{i,j=1}^d R[\nabla_{ij}^2(fg)(X.)dA^{ij}].$$

Thus we verified that  $fg \in \mathcal{D}$ . In particular it follows that  $\mathcal{D}$  contains all polynomial functions. With the help of Lemma 12.4 one can verify that the space  $\mathcal{D}$  is closed with respect to the  $C^2$ -topology. Consequently  $\mathcal{D} = C^2(\mathbb{R}^d)$  by Weierstrass approximation theorem.  $\square$

**12.7 Example.** Suppose that  $X$  is an  $\mathcal{F}$ -semimartingale and  $A \in \text{Qvar}[X.; \mathcal{F}]$ . Then

$$\exp\{X. - \frac{A.}{2}\} \in \exp\{X_0\} + \text{It}\hat{\circ}[\exp\{X. - \frac{A.}{2}\}dX].$$

In particular if  $X \in \mathfrak{Mart}^c(\mathcal{F})$  then  $\exp\{X. - A./2\} \in \mathfrak{Mart}^c(\mathcal{F})$ .

*Proof.* We apply Theorem 12.6 to the semimartingale  $t \mapsto X_t - A_t/2$  and the function  $f : x \mapsto e^x$ . We see that  $\text{Qvar}[X. - A./2; \mathcal{F}] = \text{Qvar}[X.; \mathcal{F}]$  by Theorem 10.8(i). Since  $f' = f$  and  $f'' = f$ , it then follows that

$$\exp\{X. - \frac{A.}{2}\} \in \exp\{X_0\} + \text{It}\hat{\circ}[\exp\{X. - \frac{A.}{2}\}d(X - \frac{A.}{2})] + \frac{1}{2}R[\exp\{X. - \frac{A.}{2}\}dA].$$

The right hand side equals  $\exp\{X_0\} + \text{It}\hat{\circ}[\exp\{X. - A./2\}dX]$  by Lemma 12.2(i).  $\square$

**12.8 Definition.** A  $d$ -dimensional standard  $\mathcal{F}$ -Brownian motion  $X$  is a  $d$ -dimensional process such that it is  $\mathcal{F}$ -adapted, its almost every sample path is continuous,  $X_0 = 0$  a.s., and for each pair  $s, t \in \mathbb{R}_{\geq 0}$  with  $s < t$  the increment  $X_t - X_s$  is independent of  $\mathcal{F}_s$  and normally distributed with mean 0 and covariance  $(t-s)\delta_{ij}$ .

We denote the non-random increasing process  $t \mapsto t$  by  $\lambda$ .

**12.9 Theorem.** Suppose that  $X$  is a  $d$ -dimensional process whose almost every sample path is continuous. Then  $X$  is a standard  $\mathcal{F}$ -Brownian motion if and only if it is an  $\mathcal{F}$ -local martingale,  $X_0 = 0$  a.s. and  $\delta_{ij}\lambda \in \text{Crv}[X^i, X^j; \mathcal{F}]$ . Lévy's characterization

*Proof.* Suppose that  $X$  is a  $d$ -dimensional  $\mathcal{F}$ -local martingale with almost sure continuous sample path,  $X_0 = 0$  a.s. and  $t \mapsto \delta_{ij}t \in \text{Crv}[X^i, X^j; \mathcal{F}]$ . Given  $\xi \in \mathbb{C}^d$ , we consider the following  $\mathbb{C}$ -valued  $C^\infty$ -function

$$f : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{C}, (x, y) \mapsto \exp\{\sqrt{-1}\xi \cdot x + q(\xi)y/2\}.$$

Here  $q : \mathbb{C}^d \rightarrow \mathbb{C}$  is the standard quadratic form  $\xi \mapsto \sum_{i=1}^d (\xi_i)^2$ . We apply Theorem 12.6 to the semimartingale  $t \mapsto (X_t, \lambda_t)$  and the function  $f$ . Observe that

$$\nabla_i f = \sqrt{-1}\xi_i f \text{ for } i \in \mathbb{N}_{\leq d}, \nabla_{d+1} f = q(\xi)f/2 \text{ and } \nabla_i \nabla_j f = -\xi_i \xi_j f \text{ for } i, j \in \mathbb{N}_{\leq d}.$$

Since  $0 \in \text{Crv}[X^i, \lambda; \mathcal{F}]$  and  $0 \in \text{Qvar}[\lambda; \mathcal{F}]$ , we have that

$$\begin{aligned} f(X., \lambda.) &\in f(0, 0) + \sum_{i=1}^d \sqrt{-1} \xi_i \text{It}\hat{o}[f(X., \lambda.) dX^i] \\ &\quad + \frac{q(\xi)}{2} \text{It}\hat{o}[f(X., \lambda.) d\lambda] + \frac{1}{2} \sum_{i,j=1}^d -\xi_i \xi_j R[f(X., \lambda.) d(\delta_{ij} \lambda)]. \end{aligned}$$

The last two terms cancel each other by Lemma 12.2(i). It follows that  $f(X., \lambda.)$  is a complex local martingale. If  $\xi \in \mathbb{R}^d$  then  $f(X., \lambda.)$  is bounded and hence it is a martingale. Suppose that  $s, t \in \mathbb{R}_{\geq 0}$  and  $s \leq t$ . Then we have that

$$\begin{aligned} \exp\{\sqrt{-1} \xi \cdot X_s + q(\xi) s / 2\} &= f(X_s, \lambda_s) \\ &\in E[f(X_t, \lambda_t) | \mathcal{F}_s] = E[\exp\{\sqrt{-1} \xi \cdot X_t + q(\xi) t / 2\} | \mathcal{F}_s] \text{ for all } \xi \in \mathbb{R}^d. \end{aligned}$$

Consequently we reach that

$$\exp\{-q(\xi)(t-s)/2\} \in E[\exp\{\sqrt{-1} \xi \cdot (X_t - X_s)\} | \mathcal{F}_s] \text{ for all } \xi \in \mathbb{R}^d.$$

This means that  $X_t - X_s$  is independent of  $\mathcal{F}_s$  and normally distributed with mean 0 and covariance  $(t-s)\delta_{ij}$ .  $\square$

**12.10 Corollary.** *Suppose that  $X.$  is a  $d$ -dimensional stochastic process whose almost every sample path is continuous. Then  $X. - X_0$  is a standard  $\mathcal{F}$ -Brownian motion if and only if  $X.$  is  $\mathcal{F}$ -adapted and  $f(X.) - R[\Delta f(X) d\lambda] / 2$  is an  $\mathcal{F}$ -martingale for each  $f \in C_0^\infty(\mathbb{R}^d)$ .*

*Proof.* Let  $g \in C^\infty(\mathbb{R}^d)$ . We show that  $t \mapsto g(X_t) - R[\Delta g(X) d\lambda]_t / 2$  is an  $\mathcal{F}$ -local martingale. Once this is established, choosing  $g$  as coordinate functions  $x \mapsto x^i$  and their quadratic monomials  $x \mapsto x^i x^j$ , we infer with the help of Lemma 10.6(ii) that

$$X. \text{ is an } \mathcal{F}\text{-local martingale and } \delta_{ij} \lambda. \in \text{Crv}[X^i, X^j; \mathcal{F}].$$

Then the claim derives by Theorem 12.9. The followings are  $\mathcal{F} \vee \text{Null}(P)$ -optional times:

$$\tau(k) := \inf\{t \in \mathbb{R}_{\geq 0} : |X_t| > k\}, \quad k \in \mathbb{N}.$$

Given  $k \in \mathbb{N}$ , there exists  $f \in C_0^\infty(\mathbb{R}^d)$  such that  $g(x) = f(x)$  for all  $x \in \mathbb{R}^d$  with  $|x| < k+1$ . We select and fix  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and  $t \mapsto X_t(\omega)$  is continuous for all  $\omega \in \Omega_0$ . If  $\omega \in \Omega_0$  and  $\tau(k, \omega) > 0$  then  $|X_t(\omega)| \leq k$  for all  $t \in \mathbb{R}_{\geq 0}$  with  $t \leq \tau(k, \omega)$ . It follows that

$$g(X_{t \wedge \tau}) - g(X_0) = f(X_{t \wedge \tau}) - f(X_0) \text{ and } \int_0^{t \wedge \tau(k)} \Delta g(X_s) ds = \int_0^{t \wedge \tau(k)} \Delta f(X_s) ds$$

for all  $t \in \mathbb{R}_{\geq 0}$  on  $\Omega_0$ . This means that

$$\begin{aligned} &g(X_{t \wedge \tau(k)}) - g(X_0) - \frac{1}{2} R[\Delta g(X) d\lambda]_{t \wedge \tau(k)} \\ &= f(X_{t \wedge \tau(k)}) - f(X_0) - \frac{1}{2} R[\Delta f(X) d\lambda]_{t \wedge \tau(k)} \end{aligned} \quad \text{for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

The right hand side is a stopped process of the  $\mathcal{F}$ -martingale  $f(X.) - R[\Delta f(X) d\lambda] / 2 - f(X_0)$ . Therefore  $g(X_{\cdot \wedge \tau(k)}) - \frac{1}{2} R[\Delta g(X) d\lambda]_{\cdot \wedge \tau(k)} - g(X_0)$  is an  $\mathcal{F} \vee \text{Null}(P)$ -martingale. On the other hand  $E[|g(X_0)|; \tau(k) > 0] \leq \sup_{x: |x| \leq k} |g(x)| < +\infty$ . It follows that  $k \mapsto \tau(k)$  is a reducing sequence for the  $\mathcal{F}$ -local martingale  $g(X.) - R[\Delta g(X) d\lambda] / 2$ .  $\square$

### 13 Burkholder-Davis-Gundy inequality

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\mathcal{F}$  be a filtration of sub  $\sigma$ -fields. The index set is  $\mathbb{R}_{\geq 0}$ . The following is known with the keyword *domination relation* and the inequality (i) is called the Lenglart inequality.

**13.1 Lemma.** *Suppose that  $X$  is an  $\mathcal{F}$ -adapted process such that almost every sample path is right continuous and  $X_t \geq 0$  a.s. for all  $t \in \mathbb{R}_{\geq 0}$ ,  $A$  is an  $\mathcal{F}$ -increasing process such that almost every sample path is continuous and  $\tau$  is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time. If*

$$E[X_\sigma; X_0 \leq \lambda] \leq E[X_0; X_0 \leq \lambda] + E[A_\sigma]$$

for all bounded  $\mathcal{F}$ -optional times  $\sigma$  and  $\lambda \in \mathbb{R}_{>0}$  then the following holds:

$$(i) \quad P(\sup_{t \leq \tau} X_t > \lambda) \leq \frac{E[X_0 \wedge \lambda]}{\lambda} + \frac{E[A_\tau \wedge \varepsilon]}{\lambda} + P(A_\tau > \varepsilon) \quad \forall \varepsilon \in \mathbb{R}_{>0}.$$

$$(ii) \quad E[f(\sup_{t \leq \tau} X_t)] \leq E[X_0 \int_{[X_0, +\infty)} \frac{df}{x} + f(X_0); X_0 > 0] \\ + E[A_\tau \int_{[A_\tau, +\infty)} \frac{df}{x} + 2f(A_\tau); A_\tau > 0]$$

where  $f$  is any left continuous, non-decreasing function  $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  with  $f(0) = 0 = f(0+)$ .

$$(iii) \quad E[\sup_{t \leq \tau} X_t^p] \leq \frac{1}{1-p} E[X_0^p] + \frac{2-p}{1-p} E[A_\tau^p] \text{ for all } p \in \mathbb{R}_{(0,1)}.$$

*Proof.* There exists  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$ , the following holds for every  $\omega \in \Omega_0$ :

$X(\omega)$  is right continuous and  $X(\omega) \geq 0$  on  $\mathbb{R}_{\geq 0}$ ,

$A(\omega)$  is finite valued, continuous and non-decreasing and  $A_0(\omega) = 0$ .

The process with sample path  $t \mapsto 1_{\Omega_0} X_t$  (respectively  $t \mapsto 1_{\Omega_0} A_t$ ) is  $\mathcal{F} \vee \text{Null}(P)$ -adapted. Given  $\lambda \in \mathbb{R}_{>0}$  and  $\varepsilon \in \mathbb{R}_{>0}$ , we introduce the following  $\mathcal{F} \vee \text{Null}(P)$ -optional times:

$$T := \inf\{t \in \mathbb{R}_{\geq 0} : 1_{\Omega_0} X_t > \lambda\}, \quad S := \inf\{t \in \mathbb{R}_{\geq 0} : 1_{\Omega_0} A_t > \varepsilon\}.$$

According to Lemma 4.8, there exist  $\mathcal{F}$ -optional times  $\tilde{T}$ ,  $\tilde{S}$  and  $\tilde{\tau}$  such that

$$T = \tilde{T} \text{ a.s.}, \quad S = \tilde{S} \text{ a.s.} \text{ and } \tau = \tilde{\tau} \text{ a.s.}$$

Observe that  $\{T < +\infty\} \subset \Omega_0$ . Due to the right continuity of  $t \mapsto 1_{\Omega_0} X_t$ ,

$$X_T \geq \lambda \text{ on } \{T < +\infty\}.$$

To save the space we write  $B := \{X_0 \leq \lambda\}$ . Let  $n \in \mathbb{N}$ . We have that  $X_{T \wedge \tau \wedge n} \geq 0$  on  $\Omega_0$ . It therefore follows that

$$P(T \leq \tau, T < n, B) \leq E[X_{T \wedge \tau \wedge n}; T \leq \tau, T < n, B] / \lambda \leq E[X_{T \wedge \tau \wedge n}; B \cap \Omega_0] / \lambda.$$

On the other hand,  $\tilde{T} \wedge \tilde{\tau} \wedge n$  being a bounded  $\mathcal{F}$ -optional time,

$$E[X_{\tilde{T} \wedge \tilde{\tau} \wedge n}; B] \leq E[X_0; B] + E[A_{\tilde{T} \wedge \tilde{\tau} \wedge n}] \\ = E[X_0; B] + E[A_{\tilde{T} \wedge \tilde{\tau} \wedge n}; \Omega_0] \leq E[X_0; B] + E[A_{\tilde{\tau}}; \Omega_0].$$

Here the last inequality is due to the monotonicity of  $A$ . on  $\Omega_0$ . Consequently

$$P(T \leq \tau, T < n, X_0 \leq \lambda) \leq E[X_0; X_0 \leq \lambda]/\lambda + E[A_\tau; \Omega_0]/\lambda \text{ for all } n \in \mathbb{N}.$$

The monotone convergence theorem shows

$$P(T \leq \tau, T < +\infty, X_0 \leq \lambda) \leq E[X_0; X_0 \leq \lambda]/\lambda + E[A_\tau; \Omega_0]/\lambda.$$

Thus we reach

$$(\star) \quad P(\sup_{t \leq \tau} X_t > \lambda, X_0 \leq \lambda, \Omega_0) \leq E[X_0; X_0 \leq \lambda]/\lambda + E[A_\tau; \Omega_0]/\lambda$$

since  $\sup\{1_{\Omega_0} X_t; t \leq \tau\} > \lambda$  implies  $T < \tau$  or ' $T = \tau, T < +\infty$  and  $X_T > \lambda$ '. Applying  $(\star)$  with  $\tau$  replaced by  $\tau \wedge S$ , we get

$$P(\sup_{t \leq \tau \wedge S} X_t > \lambda, X_0 \leq \lambda, \Omega_0) \leq E[X_0; X_0 \leq \lambda]/\lambda + E[A_{\tau \wedge S}; \Omega_0]/\lambda.$$

The left hand side dominates

$$P(\sup_{t \leq \tau} X_t > \lambda, \tau \leq S, X_0 \leq \lambda, \Omega_0).$$

Consequently it follows that

$$P(\sup_{t \leq \tau} X_t > \lambda, \Omega_0) \leq \frac{E[X_0; X_0 \leq \lambda]}{\lambda} + \frac{E[A_{\tau \wedge S}; \Omega_0]}{\lambda} + P(\tau > S) + P(X_0 > \lambda).$$

According to Lemma 8.13(iii),  $A_{\tau \wedge S} = A_\tau \wedge \varepsilon$  on  $\Omega_0$  and  $\{\tau > S\} = \{A_\tau > \varepsilon\} \cap \Omega_0$ .

(ii) Put  $\varepsilon = \lambda$  and apply Lemma 13.3.

(iii) If  $f : x \mapsto x^p$  then  $f(a) + a \int_{[a, +\infty)} df/x = a^p/(1-p)$  for  $a \in \mathbb{R}_{>0}$ .  $\square$

*13.2 Remark.* Let  $X$ . be an  $\mathcal{F}$ -adapted process whose every sample path is continuous and non-negative valued,  $A$ . an  $\mathcal{F}$ -adapted process whose every sample path is finite valued, continuous and non-decreasing and takes the value 0 at time 0 and  $\tau$  an  $\mathcal{F}$ -stopping time. If  $E[X_\sigma; X_0 \leq \lambda] \leq E[X_0; X_0 \leq \lambda] + E[A_\sigma]$  for all bounded  $\mathcal{F}$ -stopping times  $\sigma$  then

$$P(\sup_{t < \tau} X_t > \lambda) \leq E[X_0 \wedge \lambda]/\lambda + E[A_\tau \wedge \varepsilon]/\lambda + P(A_\tau \geq \varepsilon).$$

*Proof.* We introduce the following stopping times:

$$T := \inf\{t \in \mathbb{R}_{\geq 0} : X_t \geq \lambda\}, \quad S := \inf\{t \in \mathbb{R}_{\geq 0} : A_t \geq \varepsilon\}.$$

Note that  $\sup\{X_t; t \in \mathbb{R}_{[0, \tau)}\} > \lambda$  implies  $T < \tau$  and that  $\{\tau \geq S\} = \{A_\tau \geq \varepsilon\}$ .  $\square$

**13.3 Lemma.** *Let  $X$  be a non-negative random variable and  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  be a left continuous and non-decreasing function such that  $f(0+) = 0$  ( $f(+\infty) := \sup f$ ). Then*

$$E[f(X); X > 0] = \int_{(0, +\infty)} P(X > \cdot) df, \quad E[X \int_{[X, +\infty)} \frac{df}{x}; X > 0] = \int_{(0, +\infty)} \frac{E[X; X \leq x]}{x} df.$$

*Proof.* Fubini's theorem.  $\square$

**13.4 Lemma.** *Suppose that  $M$  is an  $\mathcal{F}$ -adapted process with almost sure right continuous sample path and  $A$  is an  $\mathcal{F}$ -increasing process such that  $|M|^2 - A$  is an  $\mathcal{F}$ -local martingale. If  $\sigma$  is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time and  $\sigma < +\infty$  a.s. then*

$$E[|M_\sigma|^2; B] \leq E[|M_0|^2; B] + E[A_\sigma; B] \leq E[\sup\{|M_t|^2; t \leq \sigma\}; B] \text{ for all } B \in \mathcal{F}_{0+}.$$

*Proof.* See the proof of Theorem 7.17(iii). Invoke Fatou's lemma instead.  $\square$

**13.5 Theorem.** *Suppose that  $M$  is an  $\mathcal{F}$ -adapted process with almost sure right continuous sample path,  $A$  is an  $\mathcal{F}$ -increasing process such that  $|M|^2 - A$  is an  $\mathcal{F}$ -local martingale, and  $\tau$  is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time.*

(i) *If almost every sample path of  $A$  is continuous then*

$$P(\sup_{t \leq \tau} |M_t| > \lambda) \leq \frac{E[|M_0|^2 \wedge \lambda^2]}{\lambda^2} + \frac{E[A_\tau \wedge \varepsilon]}{\lambda^2} + P(A_\tau > \varepsilon) \text{ for all } \lambda, \varepsilon \in \mathbb{R}_{>0}.$$

$$E[\sup_{t \leq \tau} |M_t|^p] \leq \frac{2}{2-p} E[|M_0|^p] + \frac{4-p}{2-p} E[A_\tau^{p/2}] \text{ for all } p \in \mathbb{R}_{(0,2)}.$$

(ii) *If almost every sample path of  $M$  is continuous and  $M_0 = 0$  a.s. then*

$$P(A_\tau > \lambda) \leq \frac{E[\sup_{t \leq \tau} |M_t|^2 \wedge \varepsilon]}{\lambda} + P(\sup_{t \leq \tau} |M_t|^2 > \varepsilon) \text{ for all } \lambda, \varepsilon \in \mathbb{R}_{>0}.$$

$$E[A_\tau^{p/2}] \leq \frac{4-p}{2-p} E[\sup_{t \leq \tau} |M_t|^p] \text{ for all } p \in \mathbb{R}_{(0,2)}.$$

*Proof.* (i) We get the claim by Lemma 13.4 and Lemma 13.1.

(ii) Since  $t \mapsto \sup\{|M_s|^2; s \leq t\}$  is an  $\mathcal{F}$ -increasing process with almost sure continuous sample path, we get the claim by Lemma 13.4 and Lemma 13.1.  $\square$

Recall that  $\mathfrak{Mart}^c(\mathcal{F})$  denotes the space of all  $\mathcal{F}$ -local martingales with almost sure continuous sample path.

**13.6 Example.** Suppose that  $M^n$  is a sequence of  $\mathfrak{Mart}^c(\mathcal{F})$ ,  $A^n \in \text{Qvar}[M^n; \mathcal{F}]$  and  $\tau \in \text{Time}(\mathcal{F} \vee \text{Null}(P))$ . Then  $\sup\{|M_t^n|; t \in \mathbb{R}_{\geq 0}, t \leq \tau\}$  converges to 0 in probability if and only if both  $M_0^n$  and  $A_\tau^n$  converge to 0 in probability.

So far the martingale property of  $M$  itself is redundant. The lack of martingale property prevents us from extending the moment estimate to the case  $p = 2$  in Theorem 13.5.

**13.7 Example.** Suppose that  $M$  is an  $\mathcal{F}$ -local martingale such that almost every sample path is right continuous,  $A$  is an  $\mathcal{F}$ -increasing process such that almost every sample path is continuous and  $|M|^2 - A$  is an  $\mathcal{F}$ -local martingale and  $\tau \in \text{Time}(\mathcal{F} \vee \text{Null}(P))$ .

(i) If  $E[|M_0|; \tau > 0] < +\infty$  and  $E[\sqrt{A_{t \wedge \tau}}] < +\infty$  for all  $t \in \mathbb{R}_{>0}$  then  $t \mapsto M_{t \wedge \tau} - M_0$  is an  $\mathcal{F} \vee \text{Null}(P)$ -martingale.

(ii) If  $\tau < +\infty$  a.s.,  $E[|M_0|; \tau > 0] < +\infty$  and  $E[\sqrt{A_\tau}] < +\infty$  then  $E[|M_\tau|; \tau > 0] < +\infty$ ,  $E[M_\tau; \tau > 0] = E[M_0; \tau > 0]$  and  $E[|M_\tau|^2; \tau > 0] = E[|M_0|^2; \tau > 0] + E[A_\tau]$ .

*Proof.* We show (ii). According to Theorem 13.5(i), the family  $\{M_{\sigma \wedge \tau} 1_{\tau > 0}; \sigma \in \text{Time}(\mathcal{F})\}$  is uniformly integrable. Let  $\sigma \in \text{Time}(\mathcal{F} \vee \text{Null}(P))$ . we infer by Lemma 7.15 that

$$M_{\sigma \wedge \tau} 1_{\tau > 0} \in E[M_{\tau} 1_{\tau > 0} | (\mathcal{F} \vee \text{Null}(P))_{\sigma+}].$$

In particular  $E[M_{\tau} 1_{\tau > 0}] = E[M_0 1_{\tau > 0}]$ . Theorem 1.13 tells us that

$$E[|M_{\sigma \wedge \tau} 1_{\sigma \wedge \tau > 0}|^2] \leq E[|M_{\sigma \wedge \tau} 1_{\tau > 0}|^2] \leq E[|M_{\tau} 1_{\tau > 0}|^2].$$

The left hand side equals  $E[|M_0 1_{\sigma \wedge \tau > 0}|^2] + E[A_{\sigma \wedge \tau}]$  provided the latter is finite according to Theorem 7.17(iii). For each  $n \in \mathbb{N}$  the following is a bounded  $\mathcal{F} \vee \text{Null}(P)$ -optional time:

$$\sigma(n) := \begin{cases} \inf\{t \in \mathbb{R}_{\geq 0} : A_t > n\} & \text{if } |M_0| \leq n \\ 0 & \text{if } |M_0| > n \end{cases}.$$

Lemma 8.13(ii) shows that  $\sigma(\cdot)$  is non-decreasing and  $\sup \sigma(\cdot) = +\infty$ . Hence  $E[A_{\sigma(n) \wedge \tau}]$  tends to  $E[A_{\tau}]$  by the monotone convergence theorem. Thus

$$E[|M_0 1_{\tau > 0}|^2] + E[A_{\tau}] \leq E[|M_{\tau} 1_{\tau > 0}|^2].$$

The converse inequality  $E[|M_{\tau} 1_{\tau > 0}|^2] \leq E[|M_0 1_{\tau > 0}|^2] + E[A_{\tau}]$  holds by Lemma 13.4.  $\square$

*13.8 Remark.* The difference between Theorem 7.17 and the present situation is that  $A$  is merely right continuous in the former while  $A$  is continuous in the latter.

**13.9 Lemma.** *Suppose that  $M \in \mathfrak{Mart}^c(\mathcal{F})$ ,  $M_0 = 0$  a.s.,  $A \in \text{Qvar}[M; \mathcal{F}]$  and  $\sigma$  is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time with  $\sigma < +\infty$  a.s. Then the followings hold:*

- (i) *If  $E[\sqrt{A_{\sigma}}] < +\infty$  then  $E[A_{\sigma}^p] \leq p^p E[|M_{\sigma}|^{2p}]$  for all  $p \in \mathbb{R}_{\geq 1}$ .*
- (ii)  *$E[|M_{\sigma}|^{2p}] \leq \{p(2p-1)\}^p \{2p/(2p-1)\}^{2p(p-1)} E[A_{\sigma}^p]$  for all  $p \in \mathbb{R}_{\geq 1}$ .*
- (iii)  *$E[|M_{\sigma}|^{2p}] \leq \{p(2p-1)\}^p E[A_{\sigma}^p]$  for all  $p \in \mathbb{R}_{\geq 3/2}$ .*

*Proof.* (i) Let  $p \in \mathbb{R}_{> 1}$ . According to Example 12.3,  $|M_{\cdot}|^2 \in \text{It}\hat{\circ}[2MdM] + A_{\cdot}$ . Since  $A$  is continuous and nondecreasing a.s., we have that  $pR[A^{p-1}dA]_t = A_t^p$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. We therefore see by Example 12.3 that

$$|M_{\cdot}|^2 A^{p-1} \in \text{It}\hat{\circ}[A^{p-1}2MdM] + \frac{1}{p} A^p + R[|M|^2 dA^{p-1}].$$

with Lemma 12.1(ii), (iii), Lemma 12.2(i) and Lemma 8.16 taking into account. The second and the third terms are increasing processes. Let  $\tau$  be a bounded  $\mathcal{F} \vee \text{Null}(P)$ -optional time such that

$$\sup\{|M_t(\omega)|; t \in \mathbb{R}_{[0, \tau]}, \omega \in \Omega\} < +\infty \text{ and } \sup\{|A_t(\omega)|; t \in \mathbb{R}_{[0, \tau]}, \omega \in \Omega\} < +\infty.$$

It then follows that  $E[R[|M|^2 dA^{p-1}]_{\tau}] < +\infty$ . This implies that

$$\frac{1}{p} E[A_{\tau}^p] \leq E[R[|M|^2 dA^{p-1}]_{\tau}] + \frac{1}{p} E[A_{\tau}^p] = E[|M_{\tau}|^2 A_{\tau}^{p-1}]$$

by Corollary 11.5(ii) and Example 13.7(ii). Invoking Hölder's inequality we get

$$E[A_\tau^p] \leq pE[|M_\tau|^{2p}]^{1/p} E[A_\tau^p]^{(p-1)/p}.$$

If  $E[A_\tau^p] = 0$  then  $E[|M_\tau|^2] \leq E[A_\tau] = 0$  by Lemma 13.4 and hence  $M_\tau = 0$  a.s. Otherwise dividing by  $E[A_\tau^p]^{(p-1)/p}$  we get the claim (i) for optional times  $\tau$  with additional requirements. As for general optional times with  $E[\sqrt{A_\sigma}] < +\infty$ , since  $M_0 = 0$  a.s., it follows by the proof of Example 13.7 that  $M_\sigma$  is integrable and

$$M_{\tau \wedge \sigma} \in E[M_\sigma | (\mathcal{F} \vee \text{Null}(P))_{(\tau \wedge \sigma)_+}].$$

Invoking the claim (i) for the optional time  $\tau \wedge \sigma$  and Theorem 1.13, we see that

$$E[A_{\tau \wedge \sigma}^p] \leq p^p E[|M_{\tau \wedge \sigma}|^{2p}] \leq p^p E[|M_\sigma|^{2p}].$$

The inequality above is valid for  $\tau(k)$  in place of  $\tau$  where

$$\tau(k) := \inf\{t \geq 0 : |M_t| > k\} \wedge \inf\{t \geq 0 : A_t > k\} \wedge k \text{ for } k \in \mathbb{N}.$$

Tending  $k$  to  $\infty$  we get the full statement by the monotone convergence theorem.

(ii) & (iii) Fix  $\delta \in \mathbb{R}_{>0}$  and set  $f : x \mapsto (\delta + x^2)^p$  where  $p > 1$ . It follows that

$$f'(x) = 2p(\delta + x^2)^{p-1}x \text{ and } f''(x) = 2p(\delta + x^2)^{p-2}(\delta + (2p-1)x^2).$$

Invoking Itô's formula we get

$$(\delta + |M_\cdot|^2)^p \in \delta^p + \text{It}\hat{o}[2p(\delta + |M|^2)^{p-1}M dM] + R[p(\delta + |M|^2)^{p-2}(\delta + (2p-1)|M|^2)dA]..$$

Let  $\tau$  be as above. It follows that

$$E[(\delta + |M_\tau|^2)^p] = \delta^p + pE[R[(\delta + |M|^2)^{p-2}(\delta + (2p-1)|M|^2)dA]_\tau].$$

Since  $(2p-1) > 1$ , the integrand in the last term is dominated by  $(2p-1)(\delta + M_s^2)^{p-1}$ . Therefore tending  $\delta$  to 0 we get by the dominated convergence theorem that

$$E[|M_\tau|^{2p}] = p(2p-1)E[R[|M|^{2p-2}dA]_\tau].$$

Up to the factor  $p(2p-1)$ , the right hand side is dominated by

$$E[\sup\{|M_s|; s \in \mathbb{R}_{[0,\tau]}\}^{2p-2} A_\tau] \leq E[\sup\{|M_s|; s \in \mathbb{R}_{[0,\tau]}\}^{2p(p-1)/p} E[A_\tau^p]^{1/p}].$$

where Hölder's inequality is applied. We reach that

$$E[|M_\tau|^{2p}] \leq p(2p-1)\{2p/(2p-1)\}^{2p-2} E[|M_\tau|^{2p}]^{(p-1)/p} E[A_\tau^p]^{1/p}$$

by invoking Lemma 6.2 to the  $\mathcal{F}$ -submartingale  $|M_{\cdot \wedge \tau}|$ . When  $p \in \mathbb{R}_{\geq 3/2}$ , since  $|M_{\cdot \wedge \tau}|^{2p-2}$  is an  $\mathcal{F}$ -submartingale, another discussion works. First observe that as  $n$  tends to  $\infty$

$$\sum_{i:i < n\tau} |M_{\tau \wedge (i+1)/n}|^{2p-2} (A_{\tau \wedge (i+1)/n} - A_{\tau \wedge i/n}) \text{ converges to } R[|M|^{2p-2}dA]_\tau \text{ a.s.}$$

Since the integrand is non-negative, it follows by Fatou's lemma that

$$ER[|M|^{2p-2}dA]_{\tau} \leq \liminf_{n \rightarrow \infty} \sum_{i=0}^{\infty} E[|M_{\tau \wedge (i+1)/n}|^{2p-2}(A_{\tau \wedge (i+1)/n} - A_{\tau \wedge i/n})].$$

The submartingale property of  $|M_{\cdot \wedge \tau}|^{2p-2}$  and monotonicity of  $A$  yield that

$$E[|M_{\tau \wedge (i+1)/n}|^{2p-2}(A_{\tau \wedge (i+1)/n} - A_{\tau \wedge i/n})] \leq E[|M_{\tau}|^{2p-2}(A_{\tau \wedge (i+1)/n} - A_{\tau \wedge i/n})].$$

Consequently we obtain that

$$E[R[|M|^{2p-2}dA]_{\tau}] \leq E[|M_{\tau}|^{2p-2}A_{\tau}] \leq E[|M_{\tau}|^{2p}]^{(p-1)/p} E[A_{\tau}]^{1/p}.$$

If  $M_{\tau} = 0$  a.s. then  $E[A_{\tau}] = E[|M_{\tau}|^2] = 0$  and hence  $A_{\tau} = 0$  a.s. Otherwise dividing by  $E[|M_{\tau}|^{2p}]^{(p-1)/p}$  we get the claims (ii) and (iii). As for general optional times  $\sigma$  we start from

$$E[|M_{\sigma \wedge \tau(k)}|^{2p}] \leq \{p(2p-1)\}^p E[A_{\sigma \wedge \tau(k)}]^p \leq \{p(2p-1)\}^p E[A_{\sigma}]^p.$$

We reach the full statement by Fatou's lemma. □

**13.10 Example.** *The Burkholder-Davis-Gundy inequality:* Suppose that  $M \in \mathfrak{Mart}^c(\mathcal{F})$ ,  $M_0 = 0$  a.s.,  $A \in \text{Qvar}[M; \mathcal{F}]$ ,  $\tau \in \text{Time}(\mathcal{F} \vee \text{Null}(P))$ ,  $q \in \mathbb{R}_{\geq 3/2}$  and  $p \in \mathbb{R}_{(0,2q)}$ . Then

$$E[(A_{\tau})^{p/2}] \leq \frac{4q-p}{2q-p} q^{p/2} E[\sup_{t < \tau} |M_t|^p] \text{ and } E[\sup_{t < \tau} |M_t|^p] \leq \frac{4q-p}{2q-p} (q(2q-1))^{p/2} E[(A_{\tau})^{p/2}].$$

*Proof.* The claim (i) and (iii) in Lemma 13.9 read

$$E[A_{\sigma}^q] \leq q^q E[|M_{\sigma}|^{2q}] \leq q^q E[\sup_{t < \sigma} |M_t|^{2q}] \text{ and } E[|M_{\sigma}|^{2q}] \leq (q(2q-1))^q E[A_{\sigma}^q]$$

for all  $\mathcal{F}$ -optional times  $\sigma$  with  $\sigma < +\infty$  a.s. and  $E[A_{\sigma}] < +\infty$ . Thus we get the present claim by Lemma 13.1(ii). □

## 14 Sample path regularity of submartingale

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\mathcal{F}$  be a filtration of sub  $\sigma$ -fields. We discuss submartingales with continuous parameter. So the index space is  $\mathbb{R}_{\geq 0}$ .

**14.1 Lemma.** *Suppose that  $Q$  is a subset of  $\mathbb{R}$ ,  $f \in \text{Map}(Q, \overline{\mathbb{R}})$  and  $a$  and  $b$  is a pair of real numbers with  $a < b$ . If  $\liminf_{s \uparrow t} f(s) < a$  and  $\limsup_{s \uparrow t} f(s) > b$  for some  $t \in \overline{Q}^{\text{left}}$  or  $\liminf_{s \downarrow t} f(s) < a$  and  $\limsup_{s \downarrow t} f(s) > b$  for some  $t \in \overline{Q}^{\text{right}}$  then*

$$\sup\{\#\Delta; \Delta \text{ family of disjoint closed intervals s.t.} \\ \min J \in Q, \max J \in Q, f(\min J) < a, f(\max J) > b \forall J \in \Delta\} = +\infty.$$

**14.2 Lemma.** *Suppose that  $Q$  is a subset of  $\mathbb{R}$ ,  $f \in \text{Map}(Q, \overline{\mathbb{R}})$  and  $a$  and  $b$  is a pair of real numbers with  $a < b$ . If either  $\#\{t \in \overline{Q}^{\text{left}} \cap \overline{Q}^{\text{right}} : \liminf_{s \uparrow t} f(s) < a, \limsup_{s \downarrow t} f(s) > b\} = \infty$  or  $\#\{t \in \overline{Q}^{\text{left}} \cap \overline{Q}^{\text{right}} : \liminf_{s \downarrow t} f(s) < a, \limsup_{s \uparrow t} f(s) > b\} = \infty$  then*

$$\sup\{\#\Delta; \Delta \text{ family of disjoint closed intervals s.t.} \\ \min J \in Q, \max J \in Q, f(\min J) < a, f(\max J) > b \forall J \in \Delta\} = +\infty.$$

**14.3 Lemma.** *Let  $f \in \text{Map}(\mathbb{Q}_{\geq 0}, \overline{\mathbb{R}})$ . Define  $g : \mathbb{R}_{\geq 0} \rightarrow \overline{\mathbb{R}}$  and  $h : \mathbb{R}_{> 0} \rightarrow \overline{\mathbb{R}}$  as follows:*

$$g(t) := \liminf_{s \downarrow t} f(s) \text{ for } t \in \mathbb{R}_{\geq 0} \text{ and } h(t) := \liminf_{s \uparrow t} f(s) \text{ for } t \in \mathbb{R}_{> 0}.$$

- (i)  $\liminf_{s \downarrow t} g(s) \geq g(t)$  for all  $t \in \mathbb{R}_{\geq 0}$  and  $g$  is Borel measurable. If  $g(t) = \limsup_{s \downarrow t} f(s)$  for all  $t \in \mathbb{R}_{\geq 0}$  then  $g$  is right continuous.
- (ii)  $\liminf_{s \uparrow t} h(s) \geq h(t)$  for all  $t \in \mathbb{R}_{> 0}$  and  $h$  is Borel measurable. If  $h(t) = \limsup_{s \uparrow t} f(s)$  for all  $t \in \mathbb{R}_{> 0}$  then  $h$  is left continuous.

*Proof.* The discussion being the same, it suffices to show (i). Let  $t \in \mathbb{R}_{\geq 0}$  and  $n \in \mathbb{N}$ . Then

$$\inf_{u \in \mathbb{Q}: t < u < t+1/n} f(u) \leq \inf_{u \in \mathbb{Q}: s < u < t+1/n} f(u) \leq \liminf_{u \downarrow s} f(u) = g(s) \text{ for all } s \in \mathbb{R}_{(t, t+1/n)}.$$

This implies that  $\inf_{u \in \mathbb{Q}: t < u < t+1/n} f(u) \leq \inf_{s: t < s < t+1/n} g(s)$ . Therefore, tending  $n$  to  $\infty$ , we get  $g(s) \leq \liminf_{s \downarrow t} g(s)$ . To see the measurability we observe that

$$\{t \in \mathbb{R}_{\geq 0} : \inf_{s \in \mathbb{Q}: t < s \leq t+1/n} f(s) < a\} = \bigcup_{s \in \mathbb{Q}: s > 0, f(s) < a} [\max\{s - 1/n, 0\}, s) \in \text{Borel}(\mathbb{R}_{\geq 0})$$

for all  $a \in \mathbb{R}$ . It follows that  $g$  is Borel measurable. Changing the role of inf and sup we get

$$\limsup_{s \downarrow t} g'(s) \leq g'(t) \text{ for all } t \in \mathbb{R}_{\geq 0}$$

where  $g' : \mathbb{R}_{\geq 0} \rightarrow \overline{\mathbb{R}}$ ,  $t \mapsto \limsup_{s \downarrow t} f(s)$ . Thus if  $g = g'$  then  $g$  is right continuous.  $\square$

**14.4 Lemma.** *Suppose that  $\mathcal{F}$  is a filtration with index space  $\mathbb{Q}_{> 0}$  and  $X$  is an  $\mathcal{F}$ -adapted stochastic process with index space  $\mathbb{Q}_{> 0}$ . Then  $(t, \omega) \mapsto \liminf_{s \downarrow t} X_s(\omega)$  is  $\mathcal{F}_{\cdot+}$ -progressively measurable where  $\mathcal{F}_{t+} := \bigcap_{s \in \mathbb{Q}: s > t} \mathcal{F}_s$  for  $t \in \mathbb{R}_{\geq 0}$ .*

*Proof.* Let  $t \in \mathbb{R}_{>0}$ . If  $s \in \mathbb{R}_{(0,t)}$ ,  $n \in \mathbb{N}$  and  $n \geq 1/(t-s)$  then

$$\begin{aligned} & \{(u, \omega) \in \mathbb{R}_{[0,t]} \times \Omega : u \leq s, \inf_{q \in \mathbb{Q}: u < q \leq u+1/n} X_q(\omega) < a\} \\ &= \bigcup_{q \in \mathbb{Q}: 0 < q \leq t} \{u \in \mathbb{R}_{[0,t]} : u \leq s, q - 1/n \leq u < q\} \times \{\omega \in \Omega : X_q(\omega) < a\} \end{aligned}$$

for all  $a \in \mathbb{R}$ . The right hand side is a  $\text{Borel}(\mathbb{R}_{[0,t]}) \otimes \mathcal{F}_t$ -measurable set where we write  $\mathcal{F}_t := \sigma(\bigcup_{q \in \mathbb{Q}: 0 < q \leq t} \mathcal{F}_q)$ . (The right hand coincides with the old  $\mathcal{F}_t$  if  $t \in \mathbb{Q}$ ). It follows that

$$\{(u, \omega) \in \mathbb{R}_{[0,t]} \times \Omega : u \leq s, \liminf_{q \downarrow u} X_q(\omega) > b\} \in \text{Borel}(\mathbb{R}_{[0,t]}) \otimes \mathcal{F}_t$$

for all  $s \in \mathbb{R}_{(0,t)}$  and  $b \in \mathbb{R}$ . Since  $\{\omega \in \Omega : \liminf_{q \downarrow t} X_q(\omega) > b\} \in \mathcal{F}_{t+}$ , we get

$$\{(u, \omega) \in \mathbb{R}_{[0,t]} \times \Omega : \liminf_{q \downarrow u} X_q(\omega) > b\} \in \text{Borel}(\mathbb{R}_{[0,t]}) \otimes \mathcal{F}_{t+}$$

for all  $b \in \mathbb{R}$ . Consequently  $(t, \omega) \mapsto \liminf_{s \downarrow t} X_s(\omega)$  is  $\mathcal{F}_{+}$ -progressively measurable.  $\square$

**14.5 Corollary.** *Suppose that  $X$  is an  $\mathcal{F}_{+} \vee \text{Null}(P)$ -adapted process whose almost every sample path is right continuous (left continuous). Then there exists an  $\mathcal{F}_{+}$ -progressively ( $\mathcal{F}_{-}$ -progressively) measurable process  $Y$  such that  $X_t = Y_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s.*

*Proof.* There exists  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and

$$t \mapsto X_t(\omega) \text{ is right continuous for each } \omega \in \Omega_0.$$

Given  $t \in \mathbb{Q}_{\geq 0}$ , we select an  $\mathcal{F}_{t+}$ -measurable random variable  $\tilde{X}_t$  so that  $\tilde{X}_t = X_t$  a.s. By choosing a proper subset of  $\Omega_0$  if necessary, we have that  $\tilde{X}_t(\omega) = X_t(\omega)$  for all  $t \in \mathbb{Q}_{\geq 0}$  and  $\omega \in \Omega_0$ . The process  $t \mapsto \liminf_{s \downarrow t: s \in \mathbb{Q}} \tilde{X}_s$  is a desired one by Lemma 14.4.  $\square$

**14.6 Lemma.** *Suppose that  $X$  is an  $\mathcal{F}$ -submartingale,  $a$  and  $b$  is a pair of real numbers with  $a < b$  and  $Q$  is a countable subset of  $\mathbb{R}_{\geq 0}$ .*

$$\begin{aligned} N(Q) &:= \sup\{\#\Delta; \Delta \text{ family of disjoint closed intervals s.t.} \\ &\quad \min J \in Q, \max J \in Q, X_{\min J} < a, X_{\max J} > b \forall J \in \Delta\} \end{aligned}$$

Then  $N(Q)$  is  $\mathcal{F}$ -measurable and

$$E[N(Q)] \leq (\sup_{t \in Q} E[\max\{X_t, a\}] - E[\max\{X_0, a\}]) / (b - a).$$

*Proof.* If  $Q$  is a finite set then the claim holds according to Lemma 5.2. Given a countably infinite set  $Q$  there exists a sequence  $F_n$  of finite sets such that

$$F_n \subset F_{n+1} \text{ for all } n \in \mathbb{N} \text{ and } \bigcup_{n=1}^{\infty} F_n = Q.$$

Then, since  $N(F_n) \leq N(F_{n+1})$  and  $N(Q) = \sup_{n \in \mathbb{N}} N(F_n)$ , we get the claim by the monotone convergence theorem.  $\square$

**14.7 Corollary.** *Suppose that  $X$  is an  $\mathcal{F}$ -submartingale with a.s.-right continuous sample path,  $a$  and  $b$  is a pair of real numbers with  $a < b$  and  $T \in \mathbb{R}_{>0}$ . Then*

$$N := \sup\{\#\Delta; \Delta \text{ family of disjoint closed intervals s.t.} \\ \max J \leq T, X_{\min J} < a, X_{\max J} > b \forall J \in \Delta\}$$

is  $\mathcal{F}$ -measurable and  $E[N] \leq (E[\max\{X_T, a\}] - E[\max\{X_0, a\}])/(b - a)$ .

*Proof.* There exists  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and  $t \mapsto X_t(\Omega)$  is right continuous for all  $\omega \in \Omega_0$ . Due to the right continuity we have that  $N = N(\mathbb{Q}_{[0,T]} \cup \{T\})$  on  $\Omega_0$ . Thus, invoking Lemma 14.6, we get the claim. Note that  $\text{Null}(P) \subset \mathcal{F}$ .  $\square$

**14.8 Theorem.** *Let  $X$  be an  $\mathcal{F}$ -submartingale with a.s.-right continuous sample path.*

(i) *Almost every sample path is locally bounded on  $\mathbb{R}_{\geq 0}$ , admits left-hand limits everywhere and at most countably many discontinuous points on  $\mathbb{R}_{>0}$ .*

(ii) *If  $\sup_{t \geq 0} E[\max\{X_t, 0\}] < +\infty$  then  $X_\infty := \liminf_{t \rightarrow +\infty} X_t = \limsup_{t \rightarrow +\infty} X_t$  a.s. and  $E[|X_\sigma|] < +\infty$  for all  $\mathcal{F} \vee \text{Null}(P)$ -optional times  $\sigma$ .*

(iii) *If  $X$  is uniformly integrable then  $X$  converges to  $X_\infty$  in  $L^1$ -sense and*

$$X_\sigma \in E_{\leq}[X_\tau | (\mathcal{F} \vee \text{Null}(P))_{\sigma+}]$$

for any pair  $\sigma$  and  $\tau$  of  $\mathcal{F} \vee \text{Null}(P)$ -optional times with  $\sigma \leq \tau$  a.s.

*Proof.* (i) derives from Lemma 6.1, Lemma 14.1, Lemma 14.2 and Corollary 14.7. The discussion for (ii) and (iii) is the same as in the proof of Theorem 5.4. Few technical points we must take into account: Since  $|X_\sigma| = \liminf_{n \rightarrow \infty} |X_{\sigma \wedge n}|$  a.s.,

$$E[|X_\sigma|] = E[\liminf_{n \rightarrow \infty} |X_{\sigma \wedge n}|] \leq \liminf_{n \rightarrow \infty} E[|X_{\sigma \wedge n}|] \leq 2 \sup_{t \geq 0} E[\max\{X_t, 0\}] - E[X_0].$$

We also have that  $E[\liminf_{t \rightarrow +\infty} X_t; A] = E[\liminf_{n \rightarrow \infty} X_n; A]$  for all  $A \in \mathcal{F}$ .

$$E[X_\sigma; A \cap \{\sigma < n\}] = E[X_{\sigma \wedge n}; A \cap \{\sigma < n\}] \leq E[X_{\tau \wedge n}; A \cap \{\sigma < n\}].$$

for  $A \in (\mathcal{F} \vee \text{Null}(P))_{\sigma+}$  and  $n \in \mathbb{N}$  where we use Corollary 3.17(i) this time.  $\square$

**14.9 Remark.** Let  $X$  be an  $\mathcal{F}$ -submartingale with almost sure right continuous sample path. Unlike the discrete parameter case, even if the family  $\{X_t; t \in \mathbb{R}_{\geq 0}\}$  is uniformly integrable the family  $\{X_\sigma; \sigma \mathcal{F}$ -stopping times $\}$  need *not* be uniformly integrable. This is the subtle point related to the Doob-Meyer decomposition theorem.

**14.10 Lemma.** *Suppose that  $X$  is an  $\mathcal{F}$ -submartingale.*

(i)  $\sup\{|X_s|; s \in \mathbb{Q}_{[0,T]}\} < +\infty$  for all  $T \in \mathbb{R}_{>0}$  a.s.

(ii)  $\liminf_{s \uparrow t: s \in \mathbb{Q}} X_s = \limsup_{s \uparrow t: s \in \mathbb{Q}} X_s$  for all  $t \in \mathbb{R}_{>0}$  a.s.

(iii)  $E[|\liminf_{s \uparrow t: s \in \mathbb{Q}} X_s|] < +\infty$  and  $\liminf_{s \uparrow t: s \in \mathbb{Q}} X_s \in E_{\leq}[X_t | \mathcal{F}_{t-}]$  for all  $t \in \mathbb{R}_{>0}$ .

(iv)  $\liminf_{s \downarrow t: s \in \mathbb{Q}} X_s = \limsup_{s \downarrow t: s \in \mathbb{Q}} X_s$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s.

(v)  $E[|\liminf_{s \downarrow t: s \in \mathbb{Q}} X_s|] < +\infty$  and  $X_t \in E_{\leq}[\liminf_{s \downarrow t: s \in \mathbb{Q}} X_s | \mathcal{F}_t]$  for all  $t \in \mathbb{R}_{\geq 0}$ .

(vi)  $\liminf_{s \downarrow t: s \in \mathbb{Q}} X_s \in E_{\leq}[X_T | \mathcal{F}_{t+}]$  for all  $t \in \mathbb{R}_{\geq 0}$  and  $T \in \mathbb{R}_{>t}$ .

*Proof.* We get (i) by Lemma 6.1. (ii) and (iv) derive from Lemma 14.1 and Lemma 14.6.

To show (iii) given  $t \in \mathbb{R}_{>0}$ , fix a sequence  $q(\cdot)$  of positive rational numbers such that  $q(n) < q(n+1)$  and  $\sup_{n \in \mathbb{N}} q(n) = t$ . We have that

$\liminf_{s \uparrow t: s \in \mathbb{Q}} X_s = \liminf_{n \rightarrow \infty} X_{q(n)}$  a.s. and  $|\liminf_{s \uparrow t: s \in \mathbb{Q}} X_s| = \liminf_{n \rightarrow \infty} |X_{q(n)}|$  a.s.

Using the latter relation we deduce that  $E[|\liminf_{s \uparrow t: s \in \mathbb{Q}} X_s|] < +\infty$  as in the proof of Theorem 5.6. Let  $Y_n \in E[X_t | \mathcal{F}_{q(n)}]$  for  $n \in \mathbb{N}$ . It then follows by Theorem 5.6 that  $\liminf_{n \rightarrow \infty} Y_n \in E[X_t | \mathcal{F}_{t-}]$ . Since  $X_{q(n)} \leq Y_n$  a.s. for all  $n \in \mathbb{N}$  and  $\liminf_{s \uparrow t: s \in \mathbb{Q}} X_s$  is  $\mathcal{F}_{t-}$ -measurable, we infer that  $\liminf_{s \uparrow t: s \in \mathbb{Q}} X_s \in E_{\leq}[X_t | \mathcal{F}_{t-}]$ .

To show (v) and (vi) given  $t \in \mathbb{R}_{\geq 0}$  and  $T \in \mathbb{R}_{> t}$ , fix a sequence  $q(\cdot)$  of rational numbers such that  $T \geq q(n) > q(n+1)$  and  $\inf_{n \in \mathbb{N}} q(n) = t$ . According to (iv),

the sequence  $X_{q(\cdot)}$  converges to  $\liminf_{s \downarrow t: s \in \mathbb{Q}} X_s$  a.s.

On the other hand, its uniform integrability derives as in the proof of Lemma 3.8. Hence  $\liminf_{s \downarrow t: s \in \mathbb{Q}} X_s$  is integrable. Let  $A \in \mathcal{F}_t$  and  $B \in \mathcal{F}_{t+}$ . Observe that

$$E[X_t; A] \leq E[X_{q(n)}; A] \text{ and } E[X_{q(n)}; B] \leq E[X_T; B] \text{ for all } n \in \mathbb{N}.$$

We thus get  $E[X_t; A] \leq E[\liminf_{s \downarrow t: s \in \mathbb{Q}} X_s; A]$  and  $E[\liminf_{s \downarrow t: s \in \mathbb{Q}} X_s; B] \leq E[X_T; B]$ .  $\square$

**14.11 Theorem.** *Suppose that  $X$  is an  $\mathcal{F}$ -submartingale.*

(i) *The process  $t \mapsto \liminf_{s \downarrow t: s \in \mathbb{Q}} X_s$  is an  $\mathcal{F}_+$ -progressively measurable  $\mathcal{F}_+$ -submartingale such that almost every sample path is locally bounded and right continuous on  $\mathbb{R}_{\geq 0}$ , admits left-hand limits on  $\mathbb{R}_{> 0}$  and at most countably many discontinuous points.*

(ii) *If  $X$  is right continuous in probability or  $t \mapsto E[X_t]$  is right continuous and  $\mathcal{F}_{t+} \subset \mathcal{F}_t \vee \text{Null}(P)$  for all  $t \in \mathbb{R}_{\geq 0}$  then the process  $t \mapsto \liminf_{s \downarrow t: s \in \mathbb{Q}} X_s$  is a modification of  $X$ . (Being an  $\mathcal{F}_+$ -adapted modification of an  $\mathcal{F}_+$ -submartingale,  $X$  is an  $\mathcal{F}_+$ -submartingale.)*

*Proof.* (i) We get the progressive measurability by Lemma 14.4, the almost sure right continuity by Lemma 14.3 and Lemma 14.10(iv), and the submartingale property by Lemma 14.10(v) and (vi). Theorem 14.8 shows the other properties.

(ii) The stochastic right continuity implies  $X_t = \liminf_{s \downarrow t: s \in \mathbb{Q}} X_s$  a.s. for all  $t \in \mathbb{R}_{\geq 0}$  by Lemma 14.10(iv). We discuss the other sufficient condition. Let  $t \in \mathbb{R}_{\geq 0}$ . Then, since  $\mathcal{F}_{t+} \subset \mathcal{F}_t \vee \text{Null}(P)$ , we see by Lemma 14.10(v) that

$$X_t \leq \liminf_{s \downarrow t: s \in \mathbb{Q}} X_s \text{ a.s.}$$

On the other hand Lemma 14.10(vi) shows

$$E[\liminf_{s \downarrow t: s \in \mathbb{Q}} X_s] \leq E[X_T] \text{ for all } T \in \mathbb{R}_{> t}.$$

Since  $t \mapsto E[X_t]$  is right continuous, it follows that  $E[\liminf_{s \downarrow t: s \in \mathbb{Q}} X_s] \leq E[X_t]$ . Consequently we obtain  $X_t = \liminf_{s \downarrow t: s \in \mathbb{Q}} X_s$  a.s.  $\square$

Recall that our definition of  $\mathcal{F}_{\sigma+}$  is not  $\{A \in \sigma(\mathcal{F}) \vee \text{Null}(P) : A \cap \{\sigma < t\} \in \mathcal{F}_t \forall t \in \mathbb{R}_{> 0}\}$  but  $\{A \in \mathcal{F} : A \cap \{\sigma < t\} \in \mathcal{F}_t \forall t \in \mathbb{R}_{> 0}\}$ , where there is no prescript for  $A \cap \{\sigma = +\infty\}$ .

**14.12 Corollary.** *Let  $X$  be an integrable random variable. Then there exists an  $\mathcal{F}_+$ -progressively measurable stochastic process  $X$  such that almost every sample path is right continuous on  $\mathbb{R}_{\geq 0}$ , admits left-hand limits everywhere on  $\mathbb{R}_{> 0}$ , and  $X_t \in E[X | \mathcal{F}_{t+}]$  for all  $t \in \mathbb{R}_{\geq 0}$ . If  $X$  is  $\sigma(\mathcal{F}) \vee \text{Null}(P)$ -measurable and  $\sigma$  is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time then*

$$X_\sigma \in E[X | (\mathcal{F} \vee \text{Null}(P))_{\sigma+}] \text{ where } X_\infty = X.$$

*Proof.* Set  $X_\infty := \liminf_{t \rightarrow +\infty} X_t$ . Then we see by Theorem 14.8(iii) that

$$X_\sigma \in E[X_\infty | (\mathcal{F} \vee \text{Null}(P))_{\sigma+}].$$

Let  $A \in \bigcup_{t \geq 0} \mathcal{F}_t$ . There exists  $n \in \mathbb{N}$  such that  $A \in \mathcal{F}_{n+}$ . It follows that

$$E[X_\infty; A] = E[X_n; A] = E[X; A].$$

The monotone class theorem shows that  $E[X_\infty; A] = E[X; A]$  holds for all  $A \in \sigma(\mathcal{F})$ . Since  $X$  is  $\sigma(\mathcal{F}) \vee \text{Null}(P)$ -measurable, this implies that  $X_\infty = X$  a.s.  $\square$

## 15 Natural increasing process and martingale

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\mathcal{F}$  be a filtration of sub  $\sigma$ -fields. We discuss the relation between martingales and finite variation processes.

Given  $v : \mathbb{R}_{\geq 0} \rightarrow \overline{\mathbb{R}}$  and  $t \in \mathbb{R}_{\geq 0}$  we set

$$\text{var}_+(v)_t := \begin{cases} \lim_{s \downarrow t: s \in \mathbb{Q}} \text{var}_+(dv|_{\mathbb{Q}}; \mathbb{R}_{(0,s]}) & \lim_{s \downarrow 0: s \in \mathbb{Q}} \text{var}_+(dv|_{\mathbb{Q}}; \mathbb{R}_{(0,s]}) = 0 \\ 0 & \lim_{s \downarrow 0: s \in \mathbb{Q}} \text{var}_+(dv|_{\mathbb{Q}}; \mathbb{R}_{(0,s]}) \neq 0 \end{cases}.$$

$$\text{var}_-(v)_t := \begin{cases} \lim_{s \downarrow t: s \in \mathbb{Q}} \text{var}_-(dv|_{\mathbb{Q}}; \mathbb{R}_{(0,s]}) & \lim_{s \downarrow 0: s \in \mathbb{Q}} \text{var}_-(dv|_{\mathbb{Q}}; \mathbb{R}_{(0,s]}) = 0 \\ 0 & \lim_{s \downarrow 0: s \in \mathbb{Q}} \text{var}_-(dv|_{\mathbb{Q}}; \mathbb{R}_{(0,s]}) \neq 0 \end{cases}.$$

Here  $\text{var}_+(dv|_{\mathbb{Q}}; \mathbb{R}_{(0,s]}) = +\infty$  if  $v(a) \in \{+\infty, -\infty\}$  for some  $a \in \mathbb{Q}_{(0,s]}$ .

**15.1 Lemma.** (i) *The functions  $t \mapsto \text{var}_+(v)_t$  and  $t \mapsto \text{var}_-(v)_t$  are right continuous and non-decreasing and takes the value 0 at time 0.*

(ii) *Suppose that  $v : \mathbb{R}_{\geq 0} \rightarrow \overline{\mathbb{R}}$  is right continuous and of finite variation. Then*

$$\text{var}_+(dv; \mathbb{R}_{(0,t]}) = \text{var}_+(v)_t \text{ and } \text{var}_-(dv; \mathbb{R}_{(0,t]}) = \text{var}_-(v)_t \text{ for all } t \in \mathbb{R}_{\geq 0}.$$

*If  $v$  is continuous in addition then  $t \mapsto \text{var}_+(v)_t$  and  $t \mapsto \text{var}_-(v)_t$  are continuous.*

**15.2 Corollary.** *Let  $A$  be an  $\mathcal{F}$ -finite variation process.*

(i) *The processes  $\text{var}_+(A)_\cdot$  and  $\text{var}_-(A)_\cdot$  are  $\mathcal{F}_\cdot$ -progressively measurable as well as  $\mathcal{F} \vee \text{Null}(P)$ -progressively measurable, and right continuous almost surely.*

(ii)  *$\text{var}_+(dA; \mathbb{R}_{(0,t]}) = \text{var}_+(A)_t$  and  $\text{var}_-(dA; \mathbb{R}_{(0,t]}) = \text{var}_-(A)_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s.*

(iii) *If  $A$  is continuous a.s. then so are  $\text{var}_+(A)_\cdot$  and  $\text{var}_-(A)_\cdot$ .*

(iv) *If every sample path of  $A$  is right continuous and of finite variation then  $\text{var}_+(A)_\cdot$  and  $\text{var}_-(A)_\cdot$  are  $\mathcal{F}$ -adapted.*

*Proof.* There exists  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and  $t \mapsto A_t(\omega)$  is right continuous and of finite variation for all  $\omega \in \Omega_0$ . We then have that

$$\text{var}_+(A(\omega))_t = \text{var}_+(dA(\omega); \mathbb{R}_{(0,t]}) = \text{var}_+(dA(\omega)|_{\mathbb{Q} \cup \{t\}}; \mathbb{R}_{(0,t]})$$

for all  $t \in \mathbb{R}_{\geq 0}$  and  $\omega \in \Omega_0$ .  $\square$

Given a metrizable topological space  $E$  we denote by  $D(\mathbb{R}_{\geq 0}, E)$  the set of all right continuous mappings  $\mathbb{R}_{\geq 0} \rightarrow E$  which admit left-hand limits everywhere.

**15.3 Lemma.** *If  $w \in D(\mathbb{R}_{\geq 0}, \mathbb{R})$  then  $w$  is locally bounded and  $\{t \in \mathbb{R}_{> 0} : |w(t) - w(t-)| > \varepsilon\}$  is locally finite for all  $\varepsilon \in \mathbb{R}_{> 0}$ .*

*Proof.* Suppose that  $T \in \mathbb{N}$ ,  $g : \mathbb{R}_{[0, T]} \rightarrow \mathbb{R}$  is right continuous and  $\sup_{t \in [0, T]} |g(t)| = +\infty$ . Then  $\{t \in \mathbb{Q}_{[0, T]} : |g(t)| > k\}$  is a non-empty set for all  $k \in \mathbb{N}$ . Exploiting an enumeration of  $\mathbb{Q}$  (this is simply to avoid the axiom of choice) we can choose a sequence  $a.$  such that

$$a_k \in \mathbb{Q}_{[0, T]} \text{ and } |g(a_k)| > k \text{ for all } k \in \mathbb{N}.$$

Then  $\text{clust}(a.) \neq \emptyset$  due to the sequential compactness of  $\mathbb{R}_{[0, T]}$ . Let  $t \in \text{clust}(a.)$ . Since  $t \in \mathbb{R}_{[0, T]}$  and  $g$  is right continuous,  $\{k \in \mathbb{N} : t \leq a_k < t + \delta\}$  is a finite set for some  $\delta \in \mathbb{R}_{> 0}$ . Hence  $\{k \in \mathbb{N} : t - \varepsilon \leq a_k < t\}$  is an infinite set for all  $\varepsilon \in \mathbb{R}_{> 0}$ . Thus we infer that  $t > 0$  and  $\limsup_{s \uparrow t} |g(s)| = +\infty$ , which denies the existence of left-hand limit at  $t \in \mathbb{R}_{(0, T]}$ .

Suppose that  $\varepsilon \in \mathbb{R}_{> 0}$  and  $a \in \mathbb{R}_{> 0}$ . Due to the right continuity and the existence of left-hand limit there exists  $\delta \in \mathbb{R}_{(0, a)}$  such that

$$|w(t) - w(a)| \leq \varepsilon/2 \text{ for all } t \in \mathbb{R}_{[a, a+\delta]} \text{ and } |w(t) - w(a-)| \leq \varepsilon/2 \text{ for all } t \in \mathbb{R}_{[a-\delta, a]}$$

It follows that  $|w(t-) - w(a)| \leq \varepsilon/2$  for all  $t \in \mathbb{R}_{(a, a+\delta]}$  and  $|w(t-) - w(a-)| \leq \varepsilon/2$  for all  $t \in \mathbb{R}_{(a-\delta, a)}$ . We thus infer that  $|w(t) - w(t-)| \leq \varepsilon$  for all  $t \in \mathbb{R}_{(a, a+\delta]}$  and  $t \in \mathbb{R}_{(a-\delta, a)}$ .  $\square$

**15.4 Corollary.** *Suppose that  $E$  is metrizable,  $\rho$  is a compatible metric on  $E$ ,  $w \in D(\mathbb{R}_{\geq 0}, E)$  and  $F$  is a closed set. Then*

$$\inf\{t \in \mathbb{R}_{\geq 0} : w(s) \in F\} \leq t \text{ or } \inf\{t \in \mathbb{R}_{> 0} : w(s-o) \in F\} \leq t \Leftrightarrow \inf_{s \leq t} \rho(w(s), F) = 0$$

*Proof.* If  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  is right continuous and admits left-hand limits everywhere then

$$\exists s \in \mathbb{R}_{[0, t]} \text{ s.t. } m := \inf_{u \in [0, t]} f(u) = f(s) \text{ or } \exists s \in \mathbb{R}_{(0, t]} \text{ s.t. } m = f(s-o).$$

Indeed we may assume that  $f(0) > m$  and  $f(t) > m$  else the above is obvious. Since  $m$  is finite by Lemma 15.3, there exists a sequence  $b.$  such that  $b_k \in \mathbb{Q}_{(0, t)}$  and  $f(b_k) < m - 1/k$  for all  $k \in \mathbb{N}$ . Let  $s \in \text{clust}(b.) \neq \emptyset$ . It follows that either  $\{k \in \mathbb{N} : s \leq a_k < s + \varepsilon\}$  is an infinite set for all  $\varepsilon \in \mathbb{R}_{> 0}$  or  $\{k \in \mathbb{N} : s - \varepsilon \leq a_k < s\}$  is an infinite set for all  $\varepsilon \in \mathbb{R}_{> 0}$ . If the former holds then  $f(s) = m$  and  $0 < s < t$ . Otherwise  $f(s-o) = m$  and  $0 < s \leq t$ . We apply this result to  $t \mapsto \rho(w(t), F)$ . Then we get

$$\inf_{s \leq t} \rho(w(s), F) = 0 \Leftrightarrow \exists s \in \mathbb{R}_{[0, t]} \text{ s.t. } w(s) \in F \text{ or } \exists s \in \mathbb{R}_{(0, t]} \text{ s.t. } w(s-o) \in F.$$

Taking into account that  $\lim_{s \downarrow t} w(s-o) = w(t)$  for all  $t \in \mathbb{R}_{\geq 0}$ , we reach the statement.  $\square$

**15.5 Remark.** If  $w : \mathbb{R}_{\geq 0} \rightarrow E$  is right continuous then

$$\inf_{s \leq t} \rho(w(s), F) = \min\left\{ \inf_{s \in \mathbb{Q} : s \leq t} \rho(w(s), F), \rho(w(t), F) \right\}.$$

**15.6 Lemma.** *Suppose  $w \in D(\mathbb{R}_{\geq 0}, \mathbb{R})$  and  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  is of finite variation*

(i) *If  $f$  is left continuous then  $\sum_{i: i < 2^n t} \{w(t \wedge (i+1)/2^n) - w(t \wedge i/2^n)\} f(i/2^n)$  converges to  $-\int_{[0, t]} w df + f(t+o)w(t) - f(0)w(0)$  as  $n$  tends to  $\infty$  for all  $t \in \mathbb{R}_{\geq 0}$ .*

(ii) *If  $f$  is right continuous then  $\sum_{i: i < 2^n t} \{w(t \wedge (i+1)/2^n) - w(t \wedge i/2^n)\} f(i/2^n)$  converges to  $-\int_{(0, t]} w df + f(t)w(t) - f(0)w(0)$  as  $n$  tends to  $\infty$  for all  $t \in \mathbb{R}_{\geq 0}$ .*

*Proof.* (i) Fix  $n \in \mathbb{N}$  and write  $\phi(i) = i/2^n$  for  $i \in \mathbb{Z}_{\geq 0}$ . We see that

$$\sum_{i:\phi(i)<t} \{w(t \wedge \phi(i+1)) - w(\phi(i))\}f(\phi(i)) + \sum_{i:\phi(i)<t} \{f(t \wedge \phi(i+1)) - f(\phi(i))\}w(t \wedge \phi(i+1))$$

equals  $f(t)w(t) - f(0)w(0)$ . Note that  $-\int_{[0,t]} w df + f(t)w(t) = -\int_{[0,t]} w df + f(t+o)w(t)$ .  $\square$

**15.7 Corollary.** *Suppose that  $f$  is an  $\mathcal{F}_+$ -adapted process whose almost every sample path is of finite variation and left continuous (right continuous). If  $X_\cdot$  is an  $\mathcal{F}$ -adapted process with almost sure  $D(\mathbb{R}_{\geq 0}, \mathbb{R})$ -sample path then so is  $t \mapsto R[f dX]_t$ .*

We see that  $R[1_{(a,b]} dX]_t = X_{t \wedge b} - X_{t \wedge a} = R[1_{[a,b]} dX]_t$  and  $R[1_{[0,b]} dX]_t = X_{t \wedge b} - X_0$ .

Given  $v : \mathbb{R}_{\geq 0} \rightarrow \overline{\mathbb{R}}$  and  $t \in \mathbb{R}_{\geq 0}$  we set  $\text{var}(v)_t := \text{var}_+(v)_t + \text{var}_-(v)_t$ .

**15.8 Lemma.** *Suppose that  $M_\cdot$  is an  $\mathcal{F}$ -submartingale ( $\mathcal{F}$ -local martingale) with almost sure right continuous sample path,  $f_\cdot$  is a non-negative  $\mathcal{F}$ -finite variation process ( $\mathcal{F}$ -finite variation process) and  $\sigma$  is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time. If*

$$E\left[\sup_{s \in \mathbb{Q}: s \leq t \wedge \sigma} |M_s|(\text{var}(f_\cdot)_{t \wedge \sigma} + |f_0|); \sigma > 0\right] < +\infty \text{ for all } t \in \mathbb{R}_{\geq 0}$$

then  $t \mapsto R[f dM]_{t \wedge \sigma}$  is an  $\mathcal{F} \vee \text{Null}(P)$ -submartingale ( $\mathcal{F} \vee \text{Null}(P)$ -martingale).

*Proof.* We see by Corollary 15.7 and Corollary 3.14 that  $t \mapsto R[f dM]_{t \wedge \sigma}$  is  $\mathcal{F} \vee \text{Null}(P)$ -adapted. Set  $\tilde{M}_\cdot := M_{\cdot \wedge \sigma}$  and  $\tilde{f}_\cdot := f_{\cdot \wedge \sigma}$ . Then we see by Lemma 15.6(ii) that

$$(\star) \quad \sum_{i \in \mathbb{Z}: 0 \leq i < 2^n t} (\tilde{M}_{t \wedge (i+1)/2^n} - \tilde{M}_{t \wedge i/2^n}) \tilde{f}_{i/2^n} \text{ converges to } R[f dM]_{t \wedge \sigma} \text{ a.s.}$$

Since  $\tilde{M}_\cdot$  is an  $\mathcal{F} \vee \text{Null}(P)$ -submartingale by Corollary 3.17,  $\tilde{f}_{i/2^n}$  is  $\mathcal{F}_{i/2^n} \vee \text{Null}(P)$ -measurable and non-negative a.s. and both  $\tilde{M}_{t \wedge (i+1)/2^n} \tilde{f}_{i/2^n}$  and  $\tilde{M}_{t \wedge i/2^n} \tilde{f}_{i/2^n}$  are integrable, it follows by the proof of Lemma 8.2 that

$$t \mapsto (\tilde{M}_{t \wedge (i+1)/2^n} - \tilde{M}_{t \wedge i/2^n}) \tilde{f}_{i/2^n} \text{ is an } \mathcal{F} \vee \text{Null}(P)\text{-submartingale.}$$

Let  $n \in \mathbb{N}$ . Invoking Abel's trick, we see that

$$\begin{aligned} & \sum_{i \in \mathbb{Z}: 0 \leq i < 2^n t} (\tilde{M}_{t \wedge (i+1)/2^n} - \tilde{M}_{t \wedge i/2^n}) \tilde{f}_{i/2^n} \\ &= \tilde{M}_t \tilde{f}_t - \sum_{i \in \mathbb{Z}: 0 \leq i < 2^n t} (\tilde{f}_{t \wedge (i+1)/2^n} - \tilde{f}_{t \wedge i/2^n}) \tilde{M}_{t \wedge (i+1)/2^n} - \tilde{M}_0 \tilde{f}_0. \end{aligned}$$

Since  $2 \sup_{s \leq t \wedge \sigma} |M_s|(\text{var}(f_\cdot)_{t \wedge \sigma} + |f_0|)1_{\sigma > 0}$  is a dominating function, the above convergence  $(\star)$  occurs in  $L^1$ . Being an  $L^1$ -limit of a sequence of submartingales, the process  $t \mapsto R[f dM]_{t \wedge \sigma}$  is an  $\mathcal{F} \vee \text{Null}(P)$ -submartingale.

Let  $S(\cdot)$  be a reducing sequence for the local martingale  $M_\cdot$ . It follows that

$$t \mapsto R[f dM]_{t \wedge \sigma \wedge S(k)} \text{ is an } \mathcal{F} \vee \text{Null}(P)\text{-martingale.}$$

The function  $2 \sup_{s \leq t \wedge \sigma} |M_s|(\text{var}(f_\cdot)_{t \wedge \sigma} + |f_0|)1_{\sigma > 0}$  serves as a dominating function. Tending  $k$  to  $\infty$ , we infer that  $t \mapsto R[f dM]_{t \wedge \sigma}$  is an  $\mathcal{F} \vee \text{Null}(P)$ -martingale.  $\square$

**15.9 Lemma.** Suppose  $f$  is an  $\mathcal{F}$ -adapted process whose almost every sample path is locally bounded and admits left hand limits everywhere,  $M$  is not only an  $\mathcal{F}$ -local martingale but also an  $\mathcal{F}$ -finite variation process and  $\sigma$  is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time. If

$$E\left[\sup_{s \in \mathbb{Q}: s < t \wedge \sigma} |f_s| \text{var}(M.)_{t \wedge \sigma}\right] < +\infty \text{ for all } t \in \mathbb{R}_{>0}$$

then  $R[f dM]_{\cdot \wedge \sigma}$  is an  $\mathcal{F} \vee \text{Null}(P)$ -martingale.

*Proof.* Let  $S(\cdot)$  be a reducing sequence for  $M$ . Then  $t \mapsto M_{t \wedge S(n)} - M_0$  is an  $\mathcal{F} \vee \text{Null}(P)$ -martingale. In what follows set  $\tilde{M} := M_{\cdot \wedge \sigma \wedge S(n)}$  and  $\tilde{f} := f_{\cdot \wedge \sigma \wedge S(n)}$ . Then

$$\sum_{i \in \mathbb{Z}: 0 \leq i < 2^n t} (\tilde{M}_{t \wedge (i+1)/2^n} - \tilde{M}_{t \wedge i/2^n}) \tilde{f}_{i/2^n} \text{ converges to } R[f dM]_{t \wedge \sigma \wedge S(n)} \text{ a.s.}$$

as  $n$  tends to  $\infty$  by Lemma 9.6. Since  $\sup_{s < t \wedge \sigma} |f_s| \text{var}(M.)_{t \wedge \sigma}$  is a dominating function, the above convergence occurs in  $L^1$ . Repeating the discussion for Lemma 15.8, we infer that  $R[f dM]_{\cdot \wedge \sigma}$  is an  $\mathcal{F} \vee \text{Null}(P)$ -martingale.  $\square$

**15.10 Lemma.** (i) If  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  is locally bounded and right continuous, and  $v : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  is of finite variation and right continuous then

$$\sum_{i: 0 \leq i < 2^n t} \{v(t \wedge (i+1)/2^n) - v(t \wedge i/2^n)\} f(t \wedge (i+1)/2^n) \text{ converges to } \int_{(0,t]} f dv \quad \forall t \in \mathbb{R}_{\geq 0}.$$

(ii) If  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  is of finite variation and right continuous, and  $w \in D(\mathbb{R}_{\geq 0}, \mathbb{R})$  then

$$\sum_{i: 0 \leq i < 2^n t} \{w(t \wedge (i+1)/2^n) - w(t \wedge i/2^n)\} f(t \wedge (i+1)/2^n)$$

converges to  $f(t)w(t) - f(0)w(0) - \int_{(0,t]} w(\cdot - o) df$  for all  $t \in \mathbb{R}_{\geq 0}$ .

Given  $t \in \mathbb{R}_{\geq 0}$  and finite valued processes  $f$  and  $X$ , we set

$$R\{f dX\}_t := \liminf_{n \rightarrow \infty} \sum_{i \in \mathbb{Z}: 0 \leq i < 2^n t} (X_{t \wedge (i+1)/2^n} - X_{t \wedge i/2^n}) f_{t \wedge (i+1)/2^n}.$$

**15.11 Corollary.** Suppose that  $Y$  is an  $\mathcal{F}$ -adapted processes whose almost every sample path is right continuous and locally bounded, and  $G$  is an  $\mathcal{F}$ -finite variation process.

(i)  $t \mapsto R\{Y dG\}_t$  is an  $\mathcal{F}$ -finite variation process and

$$Y_t G_t - Y_0 G_0 - R\{Y dG\}_t = R[G dY]_t \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

(ii) If almost every sample path of  $Y$  admits left hand-limits everywhere then

$$Y_t G_t - Y_0 G_0 - R[Y dG]_t = R\{G dY\}_t \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

*Proof.* Lemma 15.6(ii) and Lemma 15.10.  $\square$

Given  $t \in \mathbb{R}_{\geq 0}$  and locally bounded processes  $Y$  and  $Z$ ., we set

$$\text{jmpv}(Y, G)_t := \sum_{s:0 < s \leq t} \max\{(Y_s - Y_{s-})(Z_s - Z_{s-}), 0\} + \sum_{s:0 < s \leq t} \min\{(Y_s - Y_{s-})(Z_s - Z_{s-}), 0\}$$

with convention  $Y_{t-} := \liminf_{s \uparrow t; s \in \mathbb{Q}} Y_s$  for  $t \in \mathbb{R}_{> 0}$  and  $\infty - \infty = -\infty$ .

**15.12 Lemma.** *Let  $Y$  be an  $\mathcal{F}$ -adapted processes with almost sure  $D(\mathbb{R}_{\geq 0}, \mathbb{R})$ -sample path and  $G$  an  $\mathcal{F}$ -finite variation process.*

(i) *If  $Z$  is an  $\mathcal{F}$ -adapted process such that  $t \mapsto Z_t - G_t$  is continuous almost surely then*

$$R\{YdG\}_t - \text{jmpv}(Y, Z)_t = R[YdG]_t \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

*In particular  $t \mapsto \text{jmpv}(Y, Z)_t$  is indistinguishable from an  $\mathcal{F}$ -finite variation process.*

(ii) *If  $f$  is an  $\mathcal{F}$ -adapted process whose almost every sample path is locally bounded and admits left-hand limits everywhere then*

$$\text{jmpv}(Y, R[fdG])_t = R[f\text{djmpv}(Y, G)]_t \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

*Proof.* (i) There exists  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$ ,  $Y(\omega)$  is right continuous admits left-hand limits everywhere for every  $\omega \in \Omega_0$ ,  $G(\omega)$  is right continuous and of finite variation, and  $Z(\omega) - G(\omega)$  is continuous for every  $\omega \in \Omega_0$ . Let  $\omega \in \Omega_0$  and  $t \in \mathbb{R}_{\geq 0}$ . Then Lemma 9.6 and Lemma 15.10 show that

$$R\{YdG\}_t(\omega) - R[YdG]_t(\omega) = \int_{(0,t]} Y_s(\omega) dG_s(\omega) - \int_{(0,t]} Y_{s-}(\omega) dG_s(\omega).$$

According to Lemma 15.3 the set  $\{s \in \mathbb{R}_{(0,t]} : Y_s(\omega) - Y_{s-}(\omega) \neq 0\}$  is countable. Therefore the right hand side reads  $\sum_{s:0 < s \leq t} (Y_s(\omega) - Y_{s-}(\omega))(G_s(\omega) - G_{s-}(\omega)) = \text{jmpv}(Y, Z)_t(\omega)$ .  $\square$

$\mathfrak{Mart}_b(\mathcal{F})$  stands for the space of all bounded  $\mathcal{F}$ -martingales with almost sure right continuous sample path.

**15.13 Definition.** An  $\mathcal{F}$ -finite variation process  $A$  is said to be natural if

$$E[\text{var}(A)_t] < +\infty \text{ and } E[R\{MdA\}_t] = E[R[MdA]_t] \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ and } M \in \mathfrak{Mart}_b(\mathcal{F}_+).$$

**15.14 Lemma.** *Let  $A$  be an  $\mathcal{F}$ -finite variation process. If its almost every sample path is continuous and  $E[\text{var}(A)_t] < +\infty$  for all  $t \in \mathbb{R}_{\geq 0}$  then it is natural.*

*Proof.* Let  $M \in \mathfrak{Mart}_b(\mathcal{F}_+)$ . According to Theorem 14.8, almost every sample path of  $M$  admits left-hand limits everywhere. Since  $t \mapsto A_t$  is continuous a.s.,  $\text{jmpv}(M, A)_t = 0$  a.s. On the other hand  $R\{MdA\}_t - \text{jmpv}(M, A)_t = R[MdA]_t$  a.s. by Lemma 15.12(i).  $\square$

We prove the uniqueness part of the Doob-Meyer decomposition theorem.

**15.15 Lemma.** *If  $A$  and  $B$  are natural  $\mathcal{F}$ -finite variation processes then so is  $A - B$ .*

*Proof.* Let  $M \in \mathfrak{Mart}_b(\mathcal{F}_+)$ . There exists  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and for each  $\omega \in \Omega_0$ ,

$t \mapsto M_t(\omega)$  is right continuous and admits left-hand limits everywhere,  
 $t \mapsto A_t(\omega)$  and  $t \mapsto B_t(\omega)$  are right continuous and of finite variation.

Let  $t \in \mathbb{R}_{\geq 0}$ . If  $\omega \in \Omega_0$  then  $R\{Md(A - B)\}_t(\omega) = R\{MdA\}_t(\omega) - R\{MdB\}_t(\omega)$ . Indeed

$$\int_{(0,t]} M_s(\omega) d(A_s(\omega) - B_s(\omega)) = \int_{(0,t]} M_s(\omega) dA_s(\omega) - \int_{(0,t]} M_s(\omega) dB_s(\omega).$$

Similarly we get  $R[Md(A - B)]_t(\omega) = [MdA]_t(\omega) - R[MdB]_t(\omega)$  for all  $\omega \in \Omega_0$ . On the other hand we have that  $\text{var}(A - B)_t(\omega) \leq \text{var}(A)_t(\omega) + \text{var}(B)_t(\omega)$  for all  $\omega \in \Omega_0$ . Consequently the difference  $A - B$  is natural.  $\square$

*15.16 Remark.* We also infer that linear combinations of natural  $\mathcal{F}$ -finite variation processes are natural. We shall later verify that a natural  $\mathcal{F}$ -finite variation process admits  $\mathcal{F}_+$ -natural positive variation part in Corollary 18.9(iv).

Given finite valued processes  $f$ . and  $X$ ., set

$$R[f dX]_\infty := \liminf_{t \rightarrow +\infty} R[f dX]_t \text{ and } R\{f dX\}_\infty := \liminf_{t \rightarrow +\infty} R\{f dX\}_t.$$

**15.17 Lemma.** *Suppose that  $M$ . is an  $\mathcal{F}_+ \vee \text{Null}(P)$ -martingale whose almost every sample path is right continuous and  $\sigma$  is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time. Then there exists an  $\mathcal{F}_+$ -martingale  $N$ . such that  $M_{t \wedge \sigma} = N_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s.*

*Proof.* Lemma 4.8(i) shows that  $(\mathcal{F} \vee \text{Null}(P))_+ = \mathcal{F}_+ \vee \text{Null}(P)$ . In particular the set of  $\mathcal{F} \vee \text{Null}(P)$ -optional times coincides with the set of  $\mathcal{F}_+ \vee \text{Null}(P)$ -stopping times. We see by Corollary 3.17(ii) that  $t \mapsto M_{t \wedge \sigma}$  is an  $\mathcal{F}_+ \vee \text{Null}(P)$ -martingale, whose almost every sample path is right continuous. Corollary 14.5 shows that there exists an  $\mathcal{F}_+$ -progressively measurable process  $N$ . such that  $M_{t \wedge \sigma} = N_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s.  $\square$

**15.18 Lemma.** *Suppose that  $A$ . is an  $\mathcal{F}$ -finite variation process,  $M$ . is an  $\mathcal{F}_+$ -local martingale with almost sure right continuous sample path,  $\sigma$  is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time,  $E[\text{var}(A)_\sigma] < +\infty$  and  $\sup_{s:s \leq \sigma} |M_s|$  is bounded. Set  $M_\infty := \liminf_{t \rightarrow +\infty} M_t$ .*

- (i) *If  $[|A_0|; \sigma > 0] < +\infty$  then  $E[M_\sigma A_\sigma; \sigma > 0] = E[M_0 A_0; \sigma > 0] + E[R\{MdA\}_\sigma]$ .*
- (ii) *If  $A$ . is natural then  $E[R\{MdA\}_\sigma] = E[R[MdA]_\sigma]$ .*

*Proof.* (i) We see that  $t \mapsto M_{t \wedge \sigma} A_{t \wedge \sigma} - M_0 A_0 - R\{MdA\}_{t \wedge \sigma}$  is an  $\mathcal{F}_+ \vee \text{Null}(P)$ -martingale by Corollary 15.11(i) and Lemma 15.8. Clearly this is dominated by the integrable random variable  $2 \sup_{s:s \leq \sigma} |M_s| (|A_0| + \text{var}(A)_\sigma) \mathbf{1}_{\sigma > 0}$ . Thus, taking Theorem 14.8(ii) into account, we reach the equality  $E[M_\sigma A_\sigma - M_0 A_0 - R\{MdA\}_\sigma] = 0$ .

(ii) According to Lemma 15.17,  $t \mapsto M_{t \wedge \sigma}$  is indistinguishable from a bounded  $\mathcal{F}_+$ -martingale, say,  $t \mapsto N_t$ . Since  $A$ . is a natural  $\mathcal{F}$ -finite variation process, we infer that

$$E[R\{M_{\cdot \wedge \sigma} dA\}_t] = E[R\{NdA\}_t] = E[R[NdA]_t] = E[R[M_{\cdot \wedge \sigma} dA]_t] \text{ for all } t \in \mathbb{R}_{\geq 0}.$$

There exists  $K \in \mathbb{R}_{> 0}$  and  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and for each  $\omega \in \Omega_0$ ,

$t \mapsto A_t(\omega)$  is right continuous and of finite variation,  $\text{var}(dA(\omega); (0, \sigma(\omega))) < +\infty$   
 $t \mapsto M_t(\omega)$  is right continuous and admits left-hand limits,  $|M_t(\omega)| \leq K$ .

Here read  $(0, +\infty] = \mathbb{R}_{>0}$ . Let  $\omega \in \Omega_0$ . Then we see that

$$\lim_{s \uparrow t} M_{s \wedge \sigma}(\omega) = \begin{cases} M_{t-}(\omega) & 0 < t \leq \sigma(\omega) \\ M_{\sigma}(\omega) & t > \sigma(\omega) \end{cases}.$$

Therefore for each  $t \in \mathbb{R}_{\geq 0}$  the difference  $R\{M_{\cdot \wedge \sigma} dA\}_t(\omega) - R[M_{\cdot \wedge \sigma} dA]_t(\omega)$  equals

$$\int_{(0,t]} \{M_{s \wedge \sigma}(\omega) - (M_{\cdot \wedge \sigma})_{s-}(\omega)\} dA_s(\omega) = \int_{(0,t \wedge \sigma]} \{M_s(\omega) - M_{s-}(\omega)\} dA_s(\omega).$$

The right hand side coincides with  $R\{MdA\}_{t \wedge \sigma}(\omega) - R[MdA]_{t \wedge \sigma}(\omega)$ . Consequently

$$E[R\{MdA\}_{t \wedge \sigma}] = E[R[MdA]_{t \wedge \sigma}] \text{ for all } t \in \mathbb{R}_{\geq 0}.$$

If  $\omega \in \Omega_0$  then  $R\{MdA\}_{t \wedge \sigma}(\omega)$  converges to  $\int_{(0,\sigma(\omega))} M_s(\omega) dA_s(\omega)$  while  $R[MdA]_{t \wedge \sigma}(\omega)$  converges to  $\int_{(0,\sigma(\omega))} M_{s-}(\omega) dA_s(\omega)$  as  $t$  tends to  $+\infty$ . Both of them are dominated by the integrable random variable  $\sup_{s:s \leq \sigma} |M_s| \text{var}(A)_{\sigma}$ . Tending  $t$  to  $+\infty$  we get the claim.  $\square$

**15.19 Theorem.** *Suppose that  $A$  and  $B$  are natural  $\mathcal{F}$ -finite variation processes. If  $A - B$  is an  $\mathcal{F}$ -martingale and  $A_0 = B_0$  a.s. then  $A_t = B_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s.*

*Proof.* Let  $\Lambda \in \mathcal{F}$ . According to Corollary 14.12, there exist a stochastic process  $X$  such that almost every sample path is right continuous and

$$X_t \in E[1_{\Lambda} | \mathcal{F}_{t+}] \text{ for all } t \in \mathbb{R}_{\geq 0}.$$

We may regard that  $0 \leq X_t(\omega) \leq 1$  for all  $\omega \in \Omega$ . Indeed  $(X \vee 0) \wedge 1$  is a desired modification. Hence  $X$  is a bounded  $\mathcal{F}_+$ -martingale. Let  $t \in \mathbb{R}_{\geq 0}$ . Since  $A - B$  is a natural  $\mathcal{F}$ -finite variation process by Lemma 15.15 and  $A_0 = B_0$  a.s., it follows by Lemma 15.18 that

$$E[A_t - B_t; \Lambda] = E[X_t(A_t - B_t)] - E[X_0(A_0 - B_0)] = E[R[Xd(A - B)]_t].$$

The right hand side vanishes by Lemma 15.9. Indeed  $A - B$  is an  $\mathcal{F}_+$ -martingale due to the  $\mathcal{F}$ -martingale property and the right continuity. Thus we conclude that

$$E[A_t - B_t; \Lambda] = 0 \text{ for all } \Lambda \in \mathcal{F}, t \in \mathbb{R}_{\geq 0}.$$

This together with the right continuity implies that  $A_t - B_t = 0$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s.  $\square$

## 16 Local martingale with finite variation path

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\mathcal{F}$  be a filtration of sub  $\sigma$ -fields. We discuss the relation between martingales with continuous sample path and martingales with finite variation sample path. So the index space is  $\mathbb{R}_{\geq 0}$ .

$\mathfrak{Mart}(\mathcal{F})$  denotes the set of all  $\mathcal{F}$ -local martingales with almost sure right continuous sample paths.  $\mathfrak{Mart}^c(\mathcal{F}) := \{M \in \mathfrak{Mart}(\mathcal{F}) : t \mapsto M_t \text{ continuous a.s.}\}$

**16.1 Lemma.** Suppose that  $M, N \in \mathfrak{Mart}^c(\mathcal{F})$ ,  $F$  and  $G$  are  $\mathcal{F}$ -finite variation processes, and  $C \in \text{Crv}[M, N; \mathcal{F}]$ . Set  $Y := t \mapsto M_t + F_t$  and  $Z := t \mapsto N_t + G_t$ .

(i) There exists an  $\mathcal{F}$ -finite variation process  $Q$  such that

$$t \mapsto Y_t Z_t - R[YdG]_t - R[ZdF]_t - Q_t \in Y_0 Z_0 + \text{It}\hat{o}[YdN] + \text{It}\hat{o}[ZdM].$$

(ii) Let  $Q$  be an  $\mathcal{F}$ -finite variation process with  $Q_0 = 0$  a.s. Then

$$\begin{aligned} & t \mapsto Y_t Z_t - R[YdG]_t - R[ZdF]_t - Q_t \in \mathfrak{Mart}^c(\mathcal{F}) \\ \Leftrightarrow & Q_t = \text{jmpv}(F, G)_t + C_t \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.} \\ \Leftrightarrow & t \mapsto Y_t Z_t - R[YdG]_t - R[ZdF]_t - Q_t \in \mathfrak{Mart}(\mathcal{F}) \\ & Q_t - Q_{t-} = (F_t - F_{t-})(G_t - G_{t-}) \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.} \end{aligned}$$

*Proof.* (i) Since  $t \mapsto M_t$  and  $t \mapsto N_t$  are continuous a.s., it follows that  $\text{jmpv}(M, G)_t = 0$  and  $\text{jmpv}(N, F)_t = 0$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. We get

$$M_t G_t - M_0 G_0 = R[MdG]_t + R[GdM]_t, N_t F_t - N_0 F_0 = R[NdF]_t + R[FdN]_t$$

for all  $t \in \mathbb{R}_{\geq 0}$  a.s. by Corollary 15.11(i) and Lemma 15.12(i). On the other hand

$$F_t G_t - F_0 G_0 - \text{jmpv}(F, G)_t = R[FdG]_t + R[GdF]_t \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

Since  $Y_t Z_t - M_t N_t = F_t N_t + M_t G_t + F_t G_t$ , the process  $t \mapsto Y_t Z_t - R[YdG]_t - R[ZdF]_t$  is indistinguishable from

$$t \mapsto Y_0 Z_0 + M_t N_t - M_0 N_0 + \text{jmpv}(F, G)_t + R[GdM]_t + R[FdN]_t.$$

Finally  $M.N \in M_0 N_0 + \text{It}\hat{o}[MdN] + \text{It}\hat{o}[NdM] + \text{Crv}[M, N; \mathcal{F}]$  by Example 12.3.

(ii) Write  $X := t \mapsto Y_t Z_t - R[YdG]_t - R[ZdF]_t$ . Then we see that

$$\begin{aligned} & t \mapsto X_t - Q_t \in \mathfrak{Mart}(\mathcal{F}) \Leftrightarrow t \mapsto C_t + \text{jmpv}(F, G)_t - Q_t \in \mathfrak{Mart}(\mathcal{F} \vee \text{Null}(P)) \\ & t \mapsto X_t - Q_t \text{ continuous a.s.} \Leftrightarrow t \mapsto \text{jmpv}(F, G)_t - Q_t \text{ continuous a.s.} \end{aligned}$$

On the other hand any local martingale whose almost every sample paths is continuous and of finite variation, and starts from 0 is evanescent according to Corollary 8.17.  $\square$

**16.2 Lemma.** Suppose that  $f, g : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  are right continuous and of finite variation and  $\Delta$  runs through locally finite partitions of  $\mathbb{R}_{\geq 0}$ . Then as  $\text{mesh}(\Delta)$  tends to 0

$$\sum_{J \in \Delta: \inf J < t} \{f(t \wedge \sup J) - f(t \wedge \inf J)\} \{g(t \wedge \sup J) - g(t \wedge \inf J)\}$$

converges to  $\sum_{s: s \leq t} \{f(s) - f(s-o)\} \{g(s) - g(s-o)\}$  locally uniformly on  $\mathbb{R}_{\geq 0}$ .

*Proof.* We denote the union of discontinuous points of  $f$  and  $g$  by  $D$ , which is countable set. Suppose that  $T \in \mathbb{R}_{> 0}$  and  $0 \leq a < b \leq T$ . Then

$$\begin{aligned} & f(b)g(b) - f(a)g(a) - \sum_{s \in D: a < s \leq b} \{f(s) - f(s-o)\} \{g(s) - g(s-o)\} \\ & = \int_{(a,b]} f(\cdot - o) dg + \int_{(a,b]} g(\cdot - o) df. \end{aligned}$$

Since  $f(b)g(a) + f(a)g(b) - 2f(a)g(a) = \int_{(a,b]} f(a) dg + \int_{(a,b]} g(a) df$ , it follows that

$$\begin{aligned} & \{f(b) - f(a)\}\{g(b) - g(a)\} - \sum_{s \in D: a < s \leq b} \{f(s) - f(s-o)\}\{g(s) - g(s-o)\} \\ &= \int_{(a,b]} \{f(\cdot - o) - f(a)\} dg + \int_{(a,b]} \{g(\cdot - o) - g(a)\} df. \end{aligned}$$

Let  $\Delta$  be a locally finite partition of  $\mathbb{R}_{\geq 0}$  and  $t \in \mathbb{R}_{[0,T]}$ . The absolute value of the difference

$$\begin{aligned} & \sum_{J \in \Delta: \inf J < t} \{f(t \wedge \sup J) - f(t \wedge \inf J)\}\{g(t \wedge \sup J) - g(t \wedge \inf J)\} \\ & - \sum_{s \in D: s \leq t} \{f(s) - f(s-o)\}\{g(s) - g(s-o)\} \end{aligned}$$

is dominated by

$$\int_{(0,T]} |f(\cdot - o) - \sum_{J \in \Delta} f(\inf J) 1_J| \text{var}^*(dg; \cdot) + \int_{(0,T]} |g(\cdot - o) - \sum_{J \in \Delta} g(\inf J) 1_J| \text{var}^*(df; \cdot).$$

This converges to 0 as  $\text{mesh}(\Delta)$  tends to 0.  $\square$

**16.3 Example.** Let  $N$  be an  $\mathcal{F}$ -Poisson process. If  $\Delta$  runs through locally finite partitions of  $\mathbb{R}_{\geq 0}$  and  $\text{mesh}(\Delta)$  tends to 0 then  $\sum_{J \in \Delta: \inf J < t} (N_{t \wedge \sup J} - N_{t \wedge \inf J})^2$  converges to  $N_t$  locally uniformly on  $\mathbb{R}_{\geq 0}$  a.s. Indeed  $\sum_{s: s \leq t} (N_s - N_{s-})^2 = N_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s.

**16.4 Corollary.** Suppose that  $M, N \in \mathfrak{Mart}^c(\mathcal{F})$ ,  $F$  and  $G$  are  $\mathcal{F}$ -finite variation processes, and  $C \in \text{Crv}[M, N; \mathcal{F}]$ . Set  $Y : t \mapsto M_t + F_t$  and  $Z : t \mapsto N_t + G_t$ . If  $\Delta$  runs through locally finite partitions of  $\mathbb{R}_{\geq 0}$  and  $\text{mesh}(\Delta)$  tends to 0 then

$$\sup_{s: s \leq t} \left| \sum_{J \in \Delta: \inf J < s} (Y_{s \wedge \sup J} - Y_{s \wedge \inf J})(Z_{s \wedge \sup J} - Z_{s \wedge \inf J}) - C_s - \text{jmpv}(F, G)_t \right|$$

converges to 0 in probability for all  $t \in \mathbb{R}_{\geq 0}$ .

*Proof.* Lemma 10.3, Lemma 8.16 and Lemma 16.2.  $\square$

**16.5 Definition.** An  $\mathcal{F}$ -locally bounded process is an  $\mathcal{F}$ -adapted process  $f$  such that there exist a sequence  $S(\cdot)$  of  $\mathcal{F} \vee \text{Null}(P)$ -optional times and  $K \in \text{Seq}(\mathbb{R})$  such that

$$S(n) \leq S(n+1) \text{ a.s.}, \sup_{n \in \mathbb{N}} S(n) = +\infty \text{ a.s. and } |f_{t \wedge S(n)} 1_{S(n) > 0}| \leq K_n \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

Such pair  $S(\cdot), K$  is called a reducing sequence.

**16.6 Example.** (i) Any left continuous function  $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  which admits right hand limits everywhere is locally bounded on  $\mathbb{R}_{\geq 0}$ .

(ii) If  $f$  is an  $\mathcal{F}$ -adapted process and its almost every sample path is left continuous and locally bounded then it is an  $\mathcal{F}$ -locally bounded process.

*Proof.* (i) Invoke Lemma 15.3 with the role of the left and the right interchanged.

(ii) Set  $\tau(k) := \inf\{t \in \mathbb{R}_{\geq 0} : |f_t| > k\}$ . We select and fix  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and  $f(\omega)$  is left continuous and locally bounded for all  $\omega \in \Omega_0$ . Let  $\omega \in \Omega_0$ . If  $k$  is sufficiently large then  $|f_t(\omega)| \leq k$  for all  $t \in \mathbb{R}_{[0,1]}$ , which means that  $\tau(k, \omega) > 0$  and, due to the left continuity,  $|f_{t \wedge \tau(k)}(\omega)| \leq k$  for all  $t \in \mathbb{R}_{\geq 0}$ . (Different from continuous case it may happen that  $\tau(k) < +\infty$  and  $|f_{\tau(k)}| < k$ .) The local boundedness also implies that  $\sup_{k \in \mathbb{N}} \tau(k, \omega) = +\infty$ . Thus the pair  $k \mapsto \tau(k)$ ,  $k \mapsto k$  is a reducing sequence.  $\square$

$$\mathfrak{Mart}^{\text{fv}}(\mathcal{F}) := \{F \in \mathfrak{Mart}(\mathcal{F}) : t \mapsto F_t \text{ of finite variation a.s.}\}$$

**16.7 Lemma.** *Suppose  $F \in \mathfrak{Mart}^{\text{fv}}(\mathcal{F})$  and  $\tau(k) := \inf\{t \in \mathbb{R}_{\geq 0} : \text{var}(F)_t > k\}$  for  $k \in \mathbb{N}$ .*

(i) *If  $\sigma$  is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time and  $t \mapsto F_{t \wedge \sigma} - F_0$  is an  $\mathcal{F} \vee \mathcal{N}$ -martingale then*

$$E[\{\text{var}(F)_{t \wedge \sigma \wedge \tau(k)}\}^p]^{1/p} \leq 2k + E[|F_{t \wedge \sigma} - F_0|^p]^{1/p} \text{ for all } t \in \mathbb{R}_{\geq 0}, k \in \mathbb{N} \text{ and } p \in \mathbb{R}_{\geq 1}.$$

(ii) *There exists a reducing sequence  $S(\cdot)$  such that  $E[\text{var}(F)_{t \wedge S(n)}] < +\infty \forall t \in \mathbb{R}_{\geq 0} \forall n \in \mathbb{N}$ .*

*Proof.* We select and fix  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and  $F(\omega)$  is right continuous and of finite variation for all  $\omega \in \Omega_0$ . Let  $\omega \in \Omega_0$  and  $t \in \mathbb{R}_{\geq 0}$ .

$$\begin{aligned} \text{var}(dF(\omega); \mathbb{R}_{(0,t]}) &= \text{var}(dF(\omega); \mathbb{R}_{(0,t)}) + |F_t(\omega) - F_{t-}(\omega)| \\ &\leq \text{var}(dF(\omega); \mathbb{R}_{(0,t)}) + |F_{t-}(\omega) - F_0(\omega)| + |F_t(\omega) - F_0(\omega)| \\ &\leq 2\text{var}(dF(\omega); \mathbb{R}_{(0,t)}) + |F_t(\omega) - F_0(\omega)|. \end{aligned}$$

We see that  $\text{var}(F)_{(t \wedge \tau(k))^-}(\omega) \leq k$ . It therefore follows that

$$\text{var}(F)_{t \wedge \tau(k)} \leq 2k + |F_{t \wedge \tau(k)} - F_0| \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

Since  $t \mapsto F_{t \wedge \sigma} - F_0$  is a martingale, we have by Corollary 3.17(i) that

$$F_{t \wedge \sigma \wedge \tau(k)} - F_0 \in E[F_{t \wedge \sigma} - F_0 | (\mathcal{F} \vee \text{Null}(P))_{\tau(k)+}].$$

Lemma 1.13(i) shows that  $E[|F_{t \wedge \sigma \wedge \tau(k)} - F_0|^p] \leq E[|F_{t \wedge \sigma} - F_0|^p]$  for  $p \in \mathbb{R}_{\geq 1}$ .  $\square$

**16.8 Theorem.** *Suppose that  $F, G \in \mathfrak{Mart}^{\text{fv}}(\mathcal{F})$ .*

(i) *If  $\ell$  is an  $\mathcal{F}$ -adapted process whose almost every sample path is locally bounded and admits left-hand limits then  $R[\ell dF]_t \in \mathfrak{Mart}^{\text{fv}}(\mathcal{F})$ .*

(ii)  *$t \mapsto F_t G_t - \text{jmpv}(F, G)_t \in \mathfrak{Mart}^{\text{fv}}(\mathcal{F} \vee \text{Null}(P))$ .*

(iii) *If  $\ell$  is as in (i) then  $t \mapsto R[\ell dF]_t G_t - R[\ell d \text{jmpv}(F, G)]_t \in \mathfrak{Mart}^{\text{fv}}(\mathcal{F} \vee \text{Null}(P))$ .*

*Proof.* (i) With the help of Example 16.6 and Lemma 16.7, we apply Lemma 15.9.

(ii)  $t \mapsto F_t G_t - \text{jmpv}(F, G)_t$  is indistinguishable from  $t \mapsto F_0 G_0 + R[F dG]_t + R[G dF]_t$ .

(iii)  $\text{jmpv}(R[\ell dF], G)_t = R[\ell d \text{jmpv}(F, G)]_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. by Corollary 15.11(iii).  $\square$

$\mathfrak{Mart}_2(\mathcal{F})$  stands for the space of all square integrable  $\mathcal{F}$ -martingales with almost sure right continuous sample path.

**16.9 Theorem.** Suppose  $M, N \in \mathfrak{Mart}^c(\mathcal{F})$ ,  $C \in \text{Crv}[M, N; \mathcal{F}]$  and  $F, G \in \mathfrak{Mart}^{\text{fv}}(\mathcal{F})$ . Set  $Y : t \mapsto M_t + F_t$  and  $Z : t \mapsto N_t + G_t$ .

- (i) There exists an  $\mathcal{F}$ -finite variation process indistinguishable from  $t \mapsto C_t + \text{jmpv}(F, G)_t$ .  
(ii) An  $\mathcal{F}$ -finite variation process  $Q$  is indistinguishable from  $t \mapsto C_t + \text{jmpv}(F, G)_t$

$$\Leftrightarrow t \mapsto Y_t Z_t - Q_t \in \mathfrak{Mart}(\mathcal{F}), Q_0 = 0, Q_t - Q_{t-} = (F_t - F_{t-})(G_t - G_{t-}) \forall t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

- (iii) If  $Y \in \mathfrak{Mart}_2(\mathcal{F})$  then  $M, F \in \mathfrak{Mart}_2(\mathcal{F})$  and  $E[\text{jmpv}(F, F)_t] < +\infty$  for all  $t \in \mathbb{R}_{\geq 0}$ .  
(iv) If  $Y, Z \in \mathfrak{Mart}_2(\mathcal{F})$  then  $t \mapsto Y_t Z_t - C_t - \text{jmpv}(F, G)_t$  is an  $\mathcal{F} \vee \text{Null}(P)$ -martingale.

*Proof.* (i) By Lemma 16.1(i). (ii) By Lemma 16.1(ii) and Theorem 16.8(ii).

(iii) We select and fix a reducing sequence  $S(\cdot)$  for  $t \mapsto |Y_t|^2 - A_t - \text{jmpv}(F, F)_t$  where  $A \in \text{Qvar}[M; \mathcal{F}]$ . Since  $t \mapsto |Y_t|^2$  is a submartingale, it follows that

$$E[|Y_0|^2] + E[A_{t \wedge S(n)} + \text{jmpv}(F, F)_{t \wedge S(n)}] = E[|Y_{t \wedge S(n)}|^2] \leq E[|Y_t|^2] < +\infty$$

for all  $t \in \mathbb{R}_{\geq 0}$  and  $n \in \mathbb{N}$ . Note that  $t \mapsto A_t + \text{jmpv}(F, F)_t$  is an increasing process. The monotone convergence theorem shows that

$$E[A_t + \text{jmpv}(F, F)_t] \leq E[|Y_t|^2] - E[|Y_0|^2] < +\infty \text{ for all } t \in \mathbb{R}_{\geq 0}.$$

This implies the martingale property of  $t \mapsto |Y_t|^2 - A_t - \text{jmpv}(F, F)_t$  by Theorem 7.17(ii). On the other hand we also have that  $E[A_t] < +\infty$  for all  $t \in \mathbb{R}_{\geq 0}$ . Invoking Theorem 7.17(ii) again we infer that  $M \in \mathfrak{Mart}_2(\mathcal{F})$ .

(iv) The polarization identity  $4\text{jmpv}(F, G)_t = \text{jmpv}(F+G, F+G)_t - \text{jmpv}(F-G, F-G)_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. establishes the statement with the help of Lemma 10.6(iii).  $\square$

## 17 Predictable process and martingale with finite variation path

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\mathcal{F}$  be a filtration of sub  $\sigma$ -fields.

Assume that  $F$  is an  $\mathcal{F}$ -local martingale whose every (not almost every) sample path is of finite variation and right continuous.

If  $\ell$  is an  $\mathcal{F}$ -adapted process whose every sample path is locally bounded and left continuous then

$$R[\ell dF]_t(\omega) = \int_{(0,t]} \ell(\cdot, \omega) dF(\omega) \text{ for all } (t, \omega) \in \mathbb{R}_{\geq 0} \times \Omega$$

and hence the process with finite variation sample path  $t \mapsto \int_{(0,t]} \ell(\cdot, \omega) dF(\omega)$  is an  $\mathcal{F}$ -local martingale by Theorem 16.8(i). We shall extend this result with keeping our eye on the measurability.

**17.1 Definition.** We denote by  $\text{Pred}(\mathcal{F})$  the  $\sigma$ -field on  $\mathbb{R}_{\geq 0} \times \Omega$  generated by the set of all left continuous  $\mathcal{F}$ -adapted processes. A stochastic process  $X$  is said to be  $\mathcal{F}$ -predictable if the mapping  $(t, \omega) \mapsto X_t(\omega)$  is  $\text{Pred}(\mathcal{F})$ -measurable.

- 17.2 Lemma.** (i) If  $\tau$  is an  $\mathcal{F}$ -optional time then  $\{(t, \omega) : \tau(\omega) < t\} \in \text{Pred}(\mathcal{F})$ .  
(ii)  $\text{Pred}(\mathcal{F})$  is generated by  $\{\mathbb{R}_{(u,v]} \times A; u, v \in \mathbb{R}, 0 \leq u < v, A \in \mathcal{F}_u\} \cup \{\{0\} \times A; A \in \mathcal{F}_0\}$ .  
(iii)  $\text{Pred}(\mathcal{F})$  is generated by  $\{\{(t, \omega) : \sigma(\omega) < t\}, \{0\} \times \{\sigma = 0\}; \sigma \text{ } \mathcal{F}\text{-stopping time}\}$ .  
(iv)  $\text{Pred}(\mathcal{F})$  is generated by the set of all continuous  $\mathcal{F}$ -adapted processes.

*Proof.* (i) We see that  $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ ,  $t \mapsto t \wedge \tau(\omega)$  is continuous for each  $\omega \in \Omega$  and  $\Omega \rightarrow \mathbb{R}$ ,  $\omega \mapsto t \wedge \tau(\omega)$  is  $\mathcal{F}_t$ -measurable for each  $t \in \mathbb{R}_{\geq 0}$  (see the prof of Lemma 3.13). It then follows that  $(t, \omega) \mapsto t \wedge \tau(\omega)$  is  $\text{Pred}(\mathcal{F})$ -measurable. Therefore we have that

$$\{(t, \omega) : \tau(\omega) < t\} = \{(t, \omega) : t \wedge \tau(\omega) < t\} \in \text{Pred}(\mathcal{F}).$$

(ii) We write  $[x] := \max\{i \in \mathbb{Z} : i < x\}$  for  $x \in \mathbb{R}$ . If  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  is left continuous then  $f([2^n t]/2^n)$  converges to  $f(t)$  for all  $t \in \mathbb{R}_{> 0}$ . Note that  $\max\{t - 1/2^n, 0\} \leq [2^n t]/2^n < t$ .

(iii) Suppose that  $u, v \in \mathbb{R}$ ,  $0 \leq u < v$  and  $A \in \mathcal{F}_{u+}$ . The function  $\sigma$  such that  $\sigma = u$  on  $A$  and  $\sigma = v$  on  $\Omega \setminus A$  is an  $\mathcal{F}$ -optional time ( $\mathcal{F}$ -stopping time if  $A \in \mathcal{F}_u$ ). Indeed

$$\{(t, \omega) : \sigma(\omega) < t\} = (\mathbb{R}_{(u,v]} \times A) \cup (\mathbb{R}_{>v} \times \Omega)$$

We see that  $\mathbb{R}_{(u,v]} \times A = \{(t, \omega) : \sigma(\omega) < t\} \cap (\mathbb{R}_{\leq v} \times \Omega)$ .

(iv) Suppose that  $u, v \in \mathbb{R}$ ,  $0 \leq u < v$  and  $A \in \mathcal{F}_u$ . For each  $n \in \mathbb{N}$  the function  $f_n : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ ,  $t \mapsto \max\{\min\{n(t - u), 1, n(v - t) + 1\}, 0\}$  is continuous and the sequence  $f_n$  converges to  $1_{(u,v]}$  point wise. Since  $f_n(t) = 0$  for  $t \leq u$ , the process with sample path  $t \mapsto f_n(t)1_A(\omega)$  is  $\mathcal{F}$ -adapted. Finally observe that for each  $n \in \mathbb{N}$  the function  $g_n : t \mapsto \max\{1 - nt, 0\}$  is continuous and the sequence  $g_n$  converges to  $1_{\{0\}}$  point wise.  $\square$

**17.3 Lemma.** *Suppose that  $\mathcal{F}$  is a filtration with index space  $\mathbb{Q}_{\geq 0}$  and  $X$  is an  $\mathcal{F}$ -adapted stochastic process with index space  $\mathbb{Q}_{\geq 0}$ . Then  $(t, \omega) \mapsto \liminf_{s \uparrow t} X_s(\omega)$  is  $\mathcal{G}$ -predictable where  $\liminf_{s \uparrow 0} X_s(\omega) := X_0(\omega)$ ,  $\mathcal{G}_t := \bigcap_{s \in \mathbb{Q}: s < t} \mathcal{F}_s$  for  $t \in \mathbb{R}_{> 0}$  and  $\mathcal{G}_0 := \mathcal{F}_0$ .*

*Proof.* Let  $s \in \mathbb{R}_{> 0}$ . If  $n \in \mathbb{N}$  and  $n > 1/s$  then

$$\begin{aligned} & \{(t, \omega) \in \mathbb{R}_{\geq 0} \times \Omega : t > s, \inf_{q \in \mathbb{Q}: t-1/n \leq q < t} X_q(\omega) < a\} \\ &= \bigcup_{q \in \mathbb{Q}: q \geq 0} \{t \in \mathbb{R}_{\geq 0} : t > s, q < t \leq q + 1/n\} \times \{\omega \in \Omega : X_q(\omega) < a\} \text{ for all } a \in \mathbb{R}. \end{aligned}$$

The right hand side belongs to  $\text{Pred}(\mathcal{G})$  according to Lemma 17.2(ii). It follows that

$$\{(t, \omega) \in \mathbb{R}_{\geq 0} \times \Omega : t > s, \liminf_{q \uparrow t} X_q(\omega) > b\} \in \text{Pred}(\mathcal{G})$$

for all  $s \in \mathbb{R}_{> 0}$  and  $b \in \mathbb{R}$ . Since  $\{X_0 > b\} \in \mathcal{F}_0$ , we get by Lemma 17.2(ii) that

$$\begin{aligned} & \{(t, \omega) \in \mathbb{R}_{\geq 0} \times \Omega : \liminf_{q \uparrow t} X_q(\omega) > b\} \\ &= \{(t, \omega) \in \mathbb{R}_{\geq 0} \times \Omega : t > 0, \liminf_{q \uparrow t} X_q(\omega) > b\} \cup (\{0\} \times \{X_0 > b\}) \in \text{Pred}(\mathcal{G}) \end{aligned}$$

for all  $b \in \mathbb{R}$ . Consequently  $(t, \omega) \mapsto \liminf_{s \uparrow t} X_s(\omega)$  is  $\mathcal{G}$ -predictable.  $\square$

**17.4 Corollary.** *Suppose that  $X$  is an  $\mathcal{F}_+ \vee \text{Null}(P)$ -adapted process whose almost every sample path is left continuous. If  $X_0$  is  $\mathcal{F}_0 \vee \text{Null}(P)$ -measurable then there exists an  $\mathcal{F}$ -predictable process  $Y$  such that  $X_t = Y_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s.*

*Proof.* There exists  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and

$$t \mapsto X_t(\omega) \text{ is left continuous for each } \omega \in \Omega_0.$$

Given  $t \in \mathbb{Q}_{>0}$ , we select an  $\mathcal{F}_{t+}$ -measurable random variable  $\tilde{X}_t$  so that  $\tilde{X}_t = X_t$  a.s. Set  $\tilde{X}_0 := X_0$ . By choosing a proper subset of  $\Omega_0$  if necessary, we have that  $\tilde{X}_t(\omega) = X_t(\omega)$  for all  $t \in \mathbb{Q}_{\geq 0}$  and  $\omega \in \Omega_0$ . The process  $t \mapsto \liminf_{s \uparrow t: s \in \mathbb{Q}} \tilde{X}_s$  is a desired one by Lemma 17.3.  $\square$

We first establish the  $\mathcal{F}$ -adaptedness of the process with sample path

$$t \mapsto \int_{(0,t]} \ell(\cdot, \omega) dF(\omega)$$

for each  $\mathcal{F}$ -predictable process  $\ell$  with locally bounded sample path. See Corollary 17.8.

**17.5 Definition.**  $\text{Prog}(\mathcal{F}) := \{A \subset \mathbb{R}_{\geq 0} \times \Omega : A \cap (\mathbb{R}_{[0,t]} \times \Omega) \in \text{Borel}(\mathbb{R}_{\geq 0}) \otimes \mathcal{F}_t \forall t \in \mathbb{R}_{\geq 0}\}$ .

**17.6 Lemma.** (i)  $\text{Prog}(\mathcal{F})$  is a  $\sigma$ -field on  $\mathbb{R}_{\geq 0} \times \Omega$ .  $\text{Pred}(\mathcal{F}) \subset \text{Prog}(\mathcal{F})$ .

(ii) A stochastic process  $X$  with state space  $E$  is  $\mathcal{F}$ -progressively measurable if and only if  $\mathbb{R}_{\geq 0} \times \Omega \rightarrow E$ ,  $(t, \omega) \mapsto X_t(\omega)$  is  $\text{Prog}(\mathcal{F})$ -measurable.

(iii)  $\tau : \Omega \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$  is an  $\mathcal{F}$ -stopping time if and only if  $\{(t, \omega) : \tau(\omega) \leq t\} \in \text{Prog}(\mathcal{F})$ . If  $\tau$  is an  $\mathcal{F}$ -optional time then  $\{(t, \omega) : \tau(\omega) \leq t\} \in \text{Prog}(\mathcal{F}_+)$ .

*Proof.* (i)  $\mathbb{R}_{[0,t]} \times \Omega \in \text{Borel}(\mathbb{R}_{\geq 0}) \otimes \mathcal{F}_t$ . Lemma 3.12 and Lemma 17.6.

(ii) Let  $\mathcal{B}_1$  respectively  $\mathcal{B}_2$  be  $\sigma$ -fields on  $T_1$  and  $T_2$ . Given  $\varphi_1 : S_1 \rightarrow T_1$  and  $\varphi_2 : S_2 \rightarrow T_2$ ,

$$(\varphi_1^* \mathcal{B}_1) \otimes (\varphi_2^* \mathcal{B}_2) = \sigma(\varphi_1^* \mathcal{B}_1 * \varphi_2^* \mathcal{B}_2) = \sigma((\varphi_1 \times \varphi_2)^*(\mathcal{B}_1 * \mathcal{B}_2)) = (\varphi_1 \times \varphi_2)^*(\mathcal{B}_1 \otimes \mathcal{B}_2).$$

Let  $t \in \mathbb{R}_{\geq 0}$  and  $\iota$  be the canonical mapping  $\mathbb{R}_{[0,t]} \rightarrow \text{Borel}(\mathbb{R}_{\geq 0})$ . Since  $\iota^* \text{Borel}(\mathbb{R}_{\geq 0}) = \text{Borel}(\mathbb{R}_{[0,t]})$  and  $\mathbb{R}_{[0,t]} \times \Omega \in \text{Borel}(\mathbb{R}_{\geq 0}) \otimes \mathcal{F}_t$ , it follows that

$$\text{Borel}(\mathbb{R}_{[0,t]}) \otimes \mathcal{F}_t = \{A \in \text{Borel}(\mathbb{R}_{\geq 0}) \otimes \mathcal{F}_t : A \subset \mathbb{R}_{[0,t]} \times \Omega\} \text{ for all } t \in \mathbb{R}_{\geq 0}.$$

Consequently, given  $B \subset E$ , we have that  $\{(s, \omega) : s \leq t, X_s(\omega) \in B\} \in \text{Borel}(\mathbb{R}_{[0,t]}) \otimes \mathcal{F}_t$  if and only if  $\{(s, \omega) : X_s(\omega) \in B\} \cap (\mathbb{R}_{[0,t]} \times \Omega) \in \text{Borel}(\mathbb{R}_{\geq 0}) \otimes \mathcal{F}_t$ .

(iii) Let  $\tau$  be an  $\mathcal{F}$ -stopping time. We see that  $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ ,  $t \mapsto 1_{[\tau(\omega), +\infty)}(t)$  is right continuous for each  $\omega \in \Omega$  and that  $\Omega \rightarrow \mathbb{R}$ ,  $\omega \mapsto 1_{[0,t]}(\tau(\omega))$  is  $\mathcal{F}_t$ -measurable for each  $t \in \mathbb{R}_{\geq 0}$ . It follows by Lemma 3.12 that  $(t, \omega) \mapsto 1_{[\tau(\omega), +\infty)}(t)$  is  $\mathcal{F}$ -progressively measurable.  $\square$

**17.7 Lemma.** Let  $t \in \mathbb{R}_{\geq 0}$  and  $v : \mathbb{R}_{[0,t]} \times \Omega \rightarrow \overline{\mathbb{R}}$ . Suppose  $v(\cdot, \omega)$  is finite valued, right continuous and non-decreasing for each  $\omega \in \Omega$  and  $v(s, \cdot)$  is  $\mathcal{F}_t$ -measurable for each  $s \in \mathbb{R}_{[0,t]}$ .

(i)  $(s, \omega) \mapsto dv(A \cap \mathbb{R}_{[0,s]}, \omega)$  is  $\text{Borel}(\mathbb{R}_{[0,t]}) \otimes \mathcal{F}_t$ -measurable for every  $A \in \text{Borel}(\mathbb{R}_{[0,t]})$ .

(ii) The function  $\mathbb{R}_{[0,t]} \times \Omega \rightarrow \overline{\mathbb{R}}$ ,  $(s, \omega) \mapsto \int_{(0,s]} 1_B(\cdot, \omega) dv(\cdot, \omega)$  is  $\text{Borel}(\mathbb{R}_{[0,t]}) \otimes \mathcal{F}_t$ -measurable for every  $B \in \text{Borel}(\mathbb{R}_{[0,t]}) \otimes \mathcal{F}_t$ .

*Proof.* (i) We set  $\mathcal{J} := \{\mathbb{R}_{(a,b]} ; a, b \in \mathbb{R}_{[0,t]}, a < b\} \cup \{\emptyset\}$  and consider the following family:

$$\mathcal{D} := \{A \in \text{Borel}(\mathbb{R}_{[0,t]}) : (s, \omega) \mapsto dv(A \cap \mathbb{R}_{[0,s]}, \omega) \text{ is } \text{Borel}(\mathbb{R}_{[0,t]}) \otimes \mathcal{F}_t\text{-measurable}\}.$$

Clearly  $\mathcal{J}$  is closed under intersection and  $\mathcal{J} \cup \{\mathbb{R}_{[0,t]}\} \subset \mathcal{D}$ . The monotone convergence theorem shows that  $\mathcal{D}$  is a Dynkin system. Consequently  $\text{Borel}(\mathbb{R}_{[0,t]}) \subset \mathcal{D}$ .

(ii) The body of the discussion is the same as that for (i). The Dynkin system

$$\{B \in \text{Borel}(\mathbb{R}_{[0,t]}) \otimes \mathcal{F}_t : (s, \omega) \mapsto \int_{(0,s]} 1_B(\cdot, \omega) dv(\cdot, \omega) \text{ is } \text{Borel}(\mathbb{R}_{[0,t]}) \otimes \mathcal{F}_t\text{-measurable}\}$$

contains  $\text{Borel}(\mathbb{R}_{[0,t]}) * \mathcal{F}_t$  by (i).  $\square$

**17.8 Corollary.** Let  $A$  be an  $\mathcal{F}$ -adapted process whose every sample path is finite valued, right continuous and non-decreasing and  $X$  a non-negative  $\mathcal{F}$ -progressively measurable process. Then  $t \mapsto \int_{(0,t]} X_s dA_s$  is  $\mathcal{F}$ -progressively measurable and hence  $\mathcal{F}$ -adapted.

*17.9 Remark.* The sample path  $t \mapsto \int_{(0,t]} X_s dA_s$  is not necessarily right continuous. It may happen that  $\int_{(0,t]} X_s dA_s < +\infty$  but  $\int_{(0,u]} X_s dA_s = +\infty$  for all  $u > t$ .

**17.10 Definition.** An  $\mathcal{F}$ -increasing process  $A$  is said to be  $\mathcal{F}$ -locally integrable if there exist a sequence  $S(\cdot)$  of  $\mathcal{F} \vee \text{Null}(P)$ -optional times such that

$$S(n) \leq S(n+1) \text{ a.s.}, \sup_{n \in \mathbb{N}} S(n) = +\infty \text{ a.s. and } E[A_{t \wedge S(n)}] < +\infty \text{ for all } t \in \mathbb{R}_{\geq 0}.$$

Such  $S(\cdot)$  is called a reducing sequence.

$\mathfrak{Mart}(\mathcal{F})$  denotes the set of all  $\mathcal{F}$ -local martingales with almost sure right continuous sample paths.  $\mathfrak{Mart}^{\text{fv}}(\mathcal{F}) := \{M \in \mathfrak{Mart}(\mathcal{F}) : t \mapsto M_t \text{ of finite variation a.s.}\}$

**17.11 Lemma.** Suppose that  $A$  is an  $\mathcal{F}$ -locally integrable increasing process and  $S(\cdot)$  is a reducing sequence for  $A$ . If  $M \in \mathfrak{Mart}(\mathcal{F})$ ,  $E[|M_0|] < +\infty$  and  $|M_t - M_0| \leq A_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. then  $S(\cdot)$  is a reducing sequence for  $M$  as well.

*Proof.* Let  $\tau(\cdot)$  be a reducing sequence for  $M$ . Since  $M_0$  is integrable, it follows that  $t \mapsto M_{t \wedge \tau(k)}$  is an  $\mathcal{F} \vee \text{Null}(P)$ -martingale for each  $k \in \mathbb{N}$ . Fix  $n \in \mathbb{N}$ . We see that

$$t \mapsto M_{t \wedge S(n) \wedge \tau(k)} \text{ is an } \mathcal{F} \vee \text{Null}(P)\text{-martingale for each } k \in \mathbb{N}$$

by Corollary 3.17(ii). On the other hand

$$|M_{t \wedge S(n) \wedge \tau(k)}| \leq |M_0| + A_{t \wedge S(n) \wedge \tau(k)} \leq |M_0| + A_{t \wedge S(n)} \text{ a.s. for all } t \in \mathbb{R}_{\geq 0} \text{ and } k \in \mathbb{N}.$$

Thus  $t \mapsto M_{t \wedge S(n)}$  is an  $\mathcal{F} \vee \text{Null}(P)$ -martingale by the dominated convergence theorem.  $\square$

**17.12 Corollary.** Suppose  $M^n$  is a sequence of  $\mathfrak{Mart}(\mathcal{F})$ ,  $M$  is a process with almost sure right continuous path, and  $\sup_{s \leq t} |M_s^n - M_s|$  converges to 0 in probability for all  $t \in \mathbb{R}_{\geq 0}$ . Then  $t \mapsto M_t \in \mathfrak{Mart}(\mathcal{F} \vee \text{Null}(P))$  provided there exists a sequence  $A^k$  of  $\mathcal{F}$ -locally integrable increasing processes and a sequence  $S(\cdot)$  of  $\mathcal{F} \vee \text{Null}(P)$ -optional times such that

$$S(k) \leq S(k+1) \text{ a.s.}, \sup_{n \in \mathbb{N}} S(k) = +\infty \text{ a.s. and } |M_{t \wedge S(k)}^n| \leq A_t^k \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

*Proof.* By choosing a subsequence we may assume that

$$\limsup_{n \rightarrow \infty} \sup_{s \leq t} |M_s^n - M_s| = 0 \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

It follows that  $M_t = \liminf_{n \rightarrow \infty} M_t^n$  a.s. for all  $t \in \mathbb{R}_{\geq 0}$  and hence  $M$  is  $\mathcal{F} \vee \text{Null}(P)$ -adapted. Fix  $k \in \mathbb{N}$ . It suffices to show that

$$N : t \mapsto M_{t \wedge S(k)} \in \mathfrak{Mart}(\mathcal{F} \vee \text{Null}(P)).$$

We write  $N^n : t \mapsto M_{t \wedge S(k)}^n$  for each  $n \in \mathbb{N}$ , which is an  $\mathcal{F} \vee \text{Null}(P)$ -local martingale by Lemma 7.5(ii). Let  $\tau(\cdot)$  be a reducing sequence for  $A^k$ . We have that  $|N_t^n| \leq A_t^k$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. In particular  $N_0^n = 0$  a.s. Therefore, according to Lemma 17.11,

$\tau(\cdot)$  also serves as a reducing sequence of  $t \mapsto N_t^n$  for each  $n \in \mathbb{N}$ .

On the other hand we see that

$$|N_{t \wedge \tau(i)}^n - N_{t \wedge \tau(i)}| \leq \sup_{s \leq t} |N_s^n - N_s| \leq \sup_{s \leq t} |M_s^n - M_s| \text{ and } |N_{t \wedge \tau(i)}^n| \leq A_{t \wedge \tau(i)}^k.$$

Tending  $n$  to  $\infty$ , we infer that  $t \mapsto N_{t \wedge \tau(i)}$  is an  $\mathcal{F} \vee \text{Null}(P)$ -martingale for each  $i \in \mathbb{N}$  by the dominated convergence theorem. Thus  $\tau(\cdot)$  also serves as a reducing sequence of the  $\mathcal{F} \vee \text{Null}(P)$ -local martingale  $t \mapsto N_t$ .  $\square$

**17.13 Definition.** A family of  $\mathcal{F}$ -locally bounded processes is said to be equi-locally bounded if there exist a common reducing sequence.

**17.14 Lemma.** *Suppose that  $\ell_n$  is an  $\mathcal{F}$ -equi locally bounded sequence of  $\mathcal{F}$ -predictable process with locally bounded sample path and the sequence converges to a function  $\ell$  point wise. If the process with sample path  $t \mapsto \int_{(0,t]} \ell_n(\cdot, \omega) dF(\omega)$  is an  $\mathcal{F}$ -local martingale for each  $n \in \mathbb{N}$  then so is the process with sample path  $t \mapsto \int_{(0,t]} \ell(\cdot, \omega) dF(\omega)$ .*

*Proof.* Corollary 17.8 shows that the process in question is  $\mathcal{F}$ -adapted. There exist  $\Omega_0 \in \mathcal{F}$ , a sequence  $S(\cdot)$  of  $\text{Time}(\mathcal{F} \vee \text{Null}(P))$  and a sequence  $K_i$  of  $\mathbb{R}$  such that  $P(\omega_0) = 1$ ,

$$S(i, \omega) \leq S(i+1, \omega), \sup_{i \in \mathbb{N}} S(i, \omega) = +\infty \text{ for all } \omega \in \Omega_0 \text{ and } |\ell_n| \leq K_i \text{ on } \{(t, \omega) : \omega \in \Omega_0, 0 < t \leq S(i, \omega)\}$$

By Lemma 16.7 the  $\mathcal{F}$ -increasing process  $t \mapsto \text{var}(F)_t$  is locally integrable. Moreover

$$\left| \int_{(0, t \wedge S(i, \omega)]} \ell_n(\cdot, \omega) dF(\omega) \right| \leq K_i \text{var}(F)_t(\omega) \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ and } \omega \in \Omega_0.$$

If  $t \in \mathbb{R}_{> 0}$  and  $\omega \in \Omega_0$  then  $t \leq S(i, \omega)$  for sufficiently large  $i$  and hence

$$\int_{(0,t]} \ell_n(\cdot, \omega) dF(\omega) \text{ converges to } \int_{(0,t]} \ell(\cdot, \omega) dF(\omega) \text{ as } n \text{ tends to } \infty$$

by the dominated convergence theorem. According to Corollary 17.12, the limit process inherits the  $\mathcal{F} \vee \text{Null}(P)$ -local martingale property.  $\square$

**17.15 Theorem.** *If  $\ell$  is an  $\mathcal{F}$ -locally bounded  $\mathcal{F}$ -predictable process with locally bounded sample path then the process with finite variation sample path  $I : t \mapsto \int_{(0,t]} \ell(\cdot, \omega) dF(\omega)$  is an  $\mathcal{F}$ -local martingale.  $\text{jmpv}(I, I)_t(\omega) = \int_{(0,t]} |\ell(\cdot, \omega)|^2 d\text{jmpv}(F, F)(\omega)$*

*Proof.* We write  $\mathcal{C} := \{\mathbb{R}_{(u,v]} \times A; u, v \in \mathbb{R}, 0 \leq u < v, A \in \mathcal{F}_u\} \cup \{\{0\} \times A; A \in \mathcal{F}_0\}$ . Since

$$(u, v] \cap (u', v'] = \begin{cases} (u \vee u', v \wedge v'] & \text{if } u < v' \text{ and } u' < v \\ \emptyset & \text{otherwise} \end{cases}$$

the family  $\mathcal{C}$  is closed under intersections. If  $A \in \mathcal{C}$  then the process with sample path  $t \mapsto 1_A(t, \omega)$  is a bounded  $\mathcal{F}$ -adapted process with left continuous sample path. Hence  $\mathcal{C}$  is included in the following family  $\mathcal{D}$  by Theorem 16.8(i):

$$\mathcal{D} := \{A \in \text{Pred}(\mathcal{F}) : t \mapsto \int_{(0,t]} 1_A(\cdot, \omega) dF(\omega) \text{ is an } \mathcal{F}\text{-local martingale}\}.$$

We see by Lemma 17.14 that  $\mathcal{D}$  is a Dynkin system. On the other hand the  $\sigma$ -field  $\text{Pred}(\mathcal{F})$  is generated by  $\mathcal{C}$  as Lemma 17.2(ii) shows. Thus  $\mathcal{D} = \text{Pred}(\mathcal{F})$ . Invoking the dominated convergence theorem and Lemma 17.14, we get the statement.  $\square$

**17.16 Lemma.** *Suppose that a process  $X$  is indistinguishable from an  $\mathcal{F}$ -predictable process. Then there exists  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and  $t \mapsto X_t$  is an  $\mathcal{F} \cap \Omega_0$ -predictable process on the probability space  $(\Omega_0, \mathcal{F} \cap \Omega_0, P|_{\Omega_0})$ .*

*Proof.* Let  $Y$  be an  $\mathcal{F}$ -predictable process indistinguishable from  $X$ .

There exists  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and  $X_t(\omega) = Y_t(\omega)$  for all  $t \in \mathbb{R}_{\geq 0}$  and  $\omega \in \Omega_0$ .

We work on the probability space  $(\Omega_0, \mathcal{F} \cap \Omega_0, P|_{\Omega_0})$  equipped with the filtration  $\mathcal{F} \cap \Omega_0$ . Accordingly introduce the system

$$\mathcal{C} := \{\mathbb{R}_{(u,v)} \times B; u, v \in \mathbb{R}, 0 \leq u < v, B \in \mathcal{F}_u\} \cup \{\{0\} \times B; B \in \mathcal{F}_0\}$$

so that we have  $\text{Pred}(\mathcal{F}) = \sigma(\mathcal{C})$  by Lemma 17.2(ii). On the other hand let  $\iota$  be the canonical mapping  $\mathbb{R}_{\geq 0} \times \Omega_0 \rightarrow \mathbb{R}_{\geq 0} \times \Omega$ . Since

$$\iota^* \mathcal{C} := \{\mathbb{R}_{(u,v)} \times B; u, v \in \mathbb{R}, 0 \leq u < v, B \in \mathcal{F}_u \cap \Omega_0\} \cup \{\{0\} \times B; B \in \mathcal{F}_0 \cap \Omega_0\},$$

it follows that  $\text{Pred}(\mathcal{F} \cap \Omega_0) = \sigma(\iota^* \mathcal{C}) = \iota^* \sigma(\mathcal{C}) = \text{Pred}(\mathcal{F}) \cap (\mathbb{R}_{\geq 0} \times \Omega_0)$ . Therefore

$$t \mapsto Y_t(\omega) \text{ is an } \mathcal{F} \cap \Omega_0\text{-predictable process on } (\Omega_0, \mathcal{F} \cap \Omega_0, P|_{\Omega_0}).$$

The statement is now trivial.  $\square$

**17.17 Corollary.** *Suppose  $M \in \mathfrak{Mart}^{\text{fv}}(\mathcal{F})$ ,  $f$  is an  $\mathcal{F}$ -locally bounded  $\mathcal{F}$ -adapted process with almost sure right continuous sample path. If  $f$  is indistinguishable from an  $\mathcal{F} \vee \text{Null}(P)$ -predictable process then  $t \mapsto R\{fdM\}_t \in \mathfrak{Mart}^{\text{fv}}(\mathcal{F})$  and  $\text{jmpv}(R\{fdM\}, R\{fdM\})_t = R\{|f|^2 d\text{jmpv}(M, M)\}_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s.*

*Proof.* We see by Lemma 17.16 that there exists  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and

$$\begin{aligned} t \mapsto M_t(\omega) &\text{ is of finite variation and right continuous for each } \omega \in \Omega_0, \\ t \mapsto f_t(\omega) &\text{ is locally bounded and right continuous for each } \omega \in \Omega_0, \text{ and} \\ t \mapsto f_t(\omega) &\text{ is an } (\mathcal{F} \vee \text{Null}(P)) \cap \Omega_0\text{-predictable process on } (\Omega_0, \mathcal{F} \cap \Omega_0, P|_{\Omega_0}). \end{aligned}$$

Moreover  $f$  is  $(\mathcal{F} \vee \text{Null}(P)) \cap \Omega_0$ -locally bounded. Since

$$\int_{(0,t]} f(\omega) dM(\omega) = R\{fdM\}_t(\omega) \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ and } \omega \in \Omega_0,$$

we see by Theorem 17.15 that

$$t \mapsto R\{fdM\}_t \text{ is an } (\mathcal{F} \vee \text{Null}(P)) \cap \Omega_0\text{-local martingale on } (\Omega_0, \mathcal{F} \cap \Omega_0, P|_{\Omega_0}).$$

As a process on  $(\Omega, \mathcal{F}, P)$ , being  $\mathcal{F}$ -adapted, it is an  $\mathcal{F}$ -local martingale.  $\square$

**17.18 Theorem.** Suppose that  $M. \in \mathfrak{Mart}^{\text{fv}}(\mathcal{F})$ ,  $f$  is an  $\mathcal{F}$ -adapted process with almost sure locally bounded sample path, and  $\sigma \in \text{Time}(\mathcal{F} \vee \text{Null}(P))$ .

(i) If almost every sample path of  $f$  admits left-hands limits everywhere, and

$$E[R[|f|d\text{var}(M)]_{t \wedge \sigma}] < +\infty \text{ for all } t \in \mathbb{R}_{\geq 0}$$

then  $t \mapsto R[f dM]_{t \wedge \sigma}$  is an  $\mathcal{F} \vee \text{Null}(P)$ -martingale.

(ii) If almost every sample path of  $f$  is right continuous it is indistinguishable from an  $\mathcal{F} \vee \text{Null}(P)$ -predictable process and

$$E[R\{|f|d\text{var}(M)\}_{t \wedge \sigma}] < +\infty \text{ for all } t \in \mathbb{R}_{\geq 0}$$

then  $t \mapsto R\{f dM\}_{t \wedge \sigma}$  is an  $\mathcal{F} \vee \text{Null}(P)$ -martingale.

*Proof.* The argument being parallel, we concentrate on the claim (ii). We introduce a sequence of functions  $\phi_n : \mathbb{R} \rightarrow \mathbb{R}$ ,  $x \mapsto \max\{\min\{x, n\}, -n\}$ . The composition process  $t \mapsto \max\{\min\{f_t, n\}, -n\}$  is right continuous almost surely and  $|\phi_n(f_t)| \leq n$  for all  $t \in \mathbb{R}_{\geq 0}$ . Moreover it is also indistinguishable from an  $\mathcal{F} \vee \text{Null}(P)$ -predictable process. We see by Corollary 17.17 (as for (i) use Theorem 16.8 instead) that

$$t \mapsto R\{\phi_n(f)dM\}_t \in \mathfrak{Mart}^{\text{fv}}(\mathcal{F}).$$

The stopped process  $t \mapsto R\{\phi_n(f)dM\}_{t \wedge \sigma}$  is an  $\mathcal{F} \vee \text{Null}(P)$ -martingale. Indeed

$$|R\{\phi_n(f)dM\}_{t \wedge \sigma}| \leq R\{|f|d\text{var}(M)\}_{t \wedge \sigma} \text{ a.s.}$$

and Lemma 17.11 shows the martingale property. On the other hand

$$|R\{\phi_n(f)dM\}_{t \wedge \sigma} - R\{f dM\}_{t \wedge \sigma}| \leq R\{|\phi_n(f) - f|d\text{var}(M)\}_{t \wedge \sigma} \leq R\{|f|d\text{var}(M)\}_{t \wedge \sigma} \text{ a.s.}$$

Therefore we infer that  $R\{\phi_n(f)dM\}_{t \wedge \sigma}$  converges to  $R\{f dM\}_{t \wedge \sigma}$  in  $L_1$ . Thus we get the martingale property of the stopped process  $t \mapsto R\{f dM\}_{t \wedge \sigma}$ .  $\square$

## 18 Doob-Meyer decomposition theorem

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\mathcal{F}$  be a filtration of sub  $\sigma$ -fields.

Let  $X$  be an  $\mathcal{F}$ -submartingale with stochastic right continuity.

According to Theorem 14.11(ii),  $X$  is an  $\mathcal{F}_+$ -submartingale as well. It follows that if  $t, s \in \mathbb{R}_{\geq 0}$ ,  $s < t$  and  $Y \in E[X_t - X_s | \mathcal{F}_{s+}]$  then  $Y \geq 0$  a.s.

**18.1 Lemma.** Let  $n \in \mathbb{Z}_{\leq 0}$ . If  $Y_q \in E[X_q - X_{q-2^n} | \mathcal{F}_{(q-2^n)_+}]$  for  $q \in 2^n \mathbb{N}$  then

$$E\left[\sum_{q \leq T} Y_q; \sum_{q \leq T} Y_q > a\right] \leq 6 \sup_{\sigma \in \text{Time}(\mathcal{F}, \mathbb{Q})} \sup_{B \in \mathcal{F}: P(B) \leq \delta} E[|X_{T \wedge \sigma}|; B].$$

for all  $T \in \mathbb{N}$ ,  $a \in \mathbb{R}_{> 0}$  and  $\delta \in \mathbb{R}_{> 0}$  with  $2E[X_T - X_0] \leq \delta a$  where

$$\text{Time}(\mathcal{F}, \mathbb{Q}) := \{\sigma \in \text{Time}(\mathcal{F}) : \sigma(\omega) \in \mathbb{Q} \cup \{\infty\} \forall \omega \in \Omega\}.$$

*Proof.* Since  $\max\{Y_q, 0\}$  is a modification of  $Y_q$ , we may regard that  $Y_q(\omega) \geq 0$  for all  $\omega \in \Omega$ . We have that  $E[Y_q] < +\infty$  for all  $q \in 2^n\mathbb{N}$ .

The process  $t \mapsto X_t - \sum_{q \leq t} Y_q$  with index set  $2^n\mathbb{Z}_{\geq 0}$  is an  $\mathcal{G}$ -martingale.

Here  $\mathcal{G}$  denotes the filtration  $q \mapsto \mathcal{F}_{q+}$  with index set  $2^n\mathbb{Z}_{\geq 0}$ . For each  $a \in \mathbb{R}_{>0}$  we set

$$\sigma(a) := \inf\{t \in 2^n\mathbb{Z}_{\geq 0} : \sum_{q \leq t+2^n} Y_q > a\}.$$

Then  $\sigma(a)$  is a  $\mathcal{G}$ -stopping time since  $\sigma(a) \in 2^n\mathbb{Z}_{\geq 0} \cup \{+\infty\}$  and

$$\{\sigma(a) \leq t\} = \{\sum_{q \leq t+2^n} Y_q > a\} \in \mathcal{F}_{t+} \text{ for all } t \in 2^n\mathbb{Z}_{\geq 0}.$$

We fix  $T \in \mathbb{N}$  and  $a \in \mathbb{R}_{>0}$ . The  $\mathcal{G}$ -martingale property of  $t \mapsto X_t - \sum_{q \leq t} Y_q$  shows

$$\begin{aligned} E[\sum_{q \leq T} Y_q; \sum_{q \leq T} Y_q > a] &= E[\sum_{q \leq T} Y_q; \sigma(a) < T] \\ &= E[\sum_{q \leq T \wedge \sigma(a)} Y_q; \sigma(a) < T] + E[X_T; \sigma(a) < T] - E[X_{T \wedge \sigma(a)}; \sigma(a) < T]. \end{aligned}$$

Clearly  $\sum_{q \leq T \wedge \sigma(a)} Y_q \leq \sum_{q \leq T} Y_q$ . On the other hand

$$\sum_{q \leq \sigma(a/2)} Y_q \leq a/2 \text{ on } \{\sigma(a/2) < +\infty\} \text{ and } \sum_{q \leq T} Y_q > a \text{ on } \{\sigma(a) < T\}.$$

Since  $\sigma(a/2) \leq \sigma(a)$ , it follows that

$$\sum_{q \leq T \wedge \sigma(a)} Y_q \leq 2 \sum_{q \leq T} Y_q - 2 \sum_{q \leq T \wedge \sigma(a/2)} Y_q \text{ on } \{\sigma(a) < T\}.$$

Taking into account that  $\sum_{q \leq T \wedge \sigma(a/2)} Y_q \leq \sum_{q \leq T} Y_q$ , we get

$$E[\sum_{q \leq T \wedge \sigma(a)} Y_q; \sigma(a) < T] \leq 2E[\sum_{\sigma(a/2) < q \leq T} Y_q] = 2E[X_T - X_{T \wedge \sigma(a/2)}].$$

The right hand side reads  $2E[X_T; \sigma(a/2) < T] - 2E[X_{T \wedge \sigma(a/2)}; \sigma(a/2) < T]$ . Consequently

$$\begin{aligned} E[\sum_{q \leq T} Y_q; \sum_{q \leq T} Y_q > a] &\leq E[|X_T|; \sigma(a) < T] + E[|X_{T \wedge \sigma(a)}|; \sigma(a) < T] \\ &\quad + 2E[|X_T|; \sigma(a/2) < T] + 2E[|X_{T \wedge \sigma(a/2)}|; \sigma(a/2) < T]. \end{aligned}$$

Invoking Markov's inequality we see that

$$P(\sum_{q \leq T} Y_q > a/2) \leq \frac{2}{a} E[\sum_{q \leq T} Y_q] = \frac{2}{a} E[X_T - X_0].$$

Suppose that  $\delta \in \mathbb{R}_{>0}$  and  $2E[X_T - X_0] \leq \delta a$ . Then we have that

$$P(\sigma(a) < T) \leq P(\sigma(a/2) < T) = P(\sum_{q \leq T} Y_q > a/2) \leq \delta.$$

Finally  $\sigma(a)$  is an  $\mathcal{F}$ -optional time. Indeed  $\{\sigma(a) < t\} = \bigcup_{q < t} \{\sigma(a) \leq q\}$  for all  $t \in \mathbb{R}_{\geq 0}$ .  $\square$

**18.2 Definition.** Let  $X.$  be an  $\mathcal{F}$ -adapted process. It is said to be of class DL over  $\mathbb{Q}$  if the system  $\{X_{T \wedge \sigma}; \sigma \in \text{Time}(\mathcal{F}, \mathbb{Q})\}$  is uniformly integrable for each  $T \in \mathbb{N}$ .

For each  $n \in \mathbb{Z}_{\leq 0}$  choose a stochastic process  $Y.^n$  with index space  $2^n \mathbb{N}$  such that

$$Y_q^n \geq 0, E[Y_q^n] < +\infty \text{ and } Y_q^n \in E[X_q - X_{q-2^n} | \mathcal{F}_{(q-2^n)_+}] \text{ for all } q \in 2^n \mathbb{N}.$$

**18.3 Corollary.** If  $X.$  is of class DL over  $\mathbb{Q}$  then there exist  $\phi \in \text{Map}(\mathbb{N}, \mathbb{Z}_{\leq 0})$  and a sequence of integrable random variables  $A^T$  such that  $\phi(k) > \phi(k+1)$  and  $\sum_{q \in 2^{\phi(k)} \mathbb{N}; q \leq T} Y_q^{\phi(k)}$  converges to  $A^T$   $L^\infty$ -weakly as  $k \rightarrow \infty$  for all  $T \in \mathbb{N}$ .

*Proof.* According to Lemma 18.1, the sequence of random variables  $\sum_{q \in 2^n \mathbb{N}; q \leq T} Y_q^n$  is uniformly integrable for each  $T \in \mathbb{N}$ . Invoking the Dunford-Pettis compactness criterion and the diagonal trick we get the claim.  $\square$

Let  $\phi$  and  $A^T$  be as in Corollary 18.3. For each  $T \in \mathbb{N}$  choose a stochastic process  $Z.^T$  with index space  $\mathbb{R}_{[0, T)}$  such that almost every sample path is right continuous and

$$Z_t^T \in E[X_T - A^T | \mathcal{F}_{t+}] \text{ for all } t \in \mathbb{R}_{[0, T)},$$

which exists by Corollary 14.12.  $A_t^T := \liminf_{s \downarrow t; s \in \mathbb{Q}} X_s - Z_t^T$ .  $\mathbb{Q}^{(2)} := \bigcup_{n \in \mathbb{Z}; n \leq 0} 2^n \mathbb{Z}$ .

**18.4 Lemma.** If  $X.$  is of class DL over  $\mathbb{Q}$  then  $\sum_{q \in 2^{\phi(k)} \mathbb{N}; q \leq t} Y_q^{\phi(k)}$  converges to  $A_t^T$   $L^\infty$ -weakly as  $n \rightarrow \infty$  for all  $T \in \mathbb{N}$  and  $t \in \mathbb{Q}_{[0, T)}^{(2)}$ .

*Proof.* If  $n \in \mathbb{Z}_{\leq 0}$  is sufficiently negative so that  $t \in 2^n \mathbb{Z}_{\geq 0}$  then

$$X_t - \sum_{q \in 2^n \mathbb{N}; q \leq t} Y_q^n \in E[X_T - \sum_{q \in 2^n \mathbb{N}; q \leq T} Y_q^n | \mathcal{F}_{t+}].$$

Let  $f$  be a bounded  $\mathcal{F}$ -measurable function and  $g \in E[f | \mathcal{F}_{t+}]$ . We may suppose that  $g$  is bounded. If  $k$  is sufficiently large so that  $t \in 2^{\phi(k)} \mathbb{Z}_{\geq 0}$  then

$$E[(X_t - \sum_{q \in 2^{\phi(k)} \mathbb{N}; q \leq t} Y_q^{\phi(k)})f] = E[(X_T - \sum_{q \in 2^{\phi(k)} \mathbb{N}; q \leq T} Y_q^{\phi(k)})g].$$

The right hand side converges to  $E[(X_T - A^T)g]$  as  $k$  tends to  $\infty$ . Since  $g \in E[f | \mathcal{F}_{t+}]$ ,  $Z_t^T \in E[X_T - A^T | \mathcal{F}_{t+}]$  and  $A_t^T = X_t - Z_t^T$  a.s. by Theorem 14.11(ii), we have that

$$E[(X_T - A^T)g] = E[Z_t^T f] = E[(X_t - A_t^T)f].$$

The bounded  $\mathcal{F}$ -measurable function  $f$  being arbitrary, we infer that  $\sum_{q \in 2^{\phi(k)} \mathbb{N}; q \leq t} Y_q^{\phi(k)}$  converges to  $A_t^T$   $L^\infty$ -weakly as  $k \rightarrow \infty$ .  $\square$

$\mathfrak{Mart}_b(\mathcal{F})$  stands for the space of all bounded  $\mathcal{F}$ -martingales with almost sure right continuous sample path.

We prove the existence part of the Doob-Meyer decomposition theorem. The uniqueness part is discussed at Theorem 15.19. Before we start we note the following:

18.5 Remark. We have that  $(\mathcal{F}_+)_t = \mathcal{F}_t$  for all  $t \in \mathbb{R}_{\geq 0}$ . Thus an  $\mathcal{F}_+$ -increasing process  $A$  is natural if  $E[R\{MdA\}_t] = E[R[MdA]_t]$  for all  $t \in \mathbb{R}_{\geq 0}$  whenever  $M \in \mathfrak{Mart}_b(\mathcal{F}_+)$ .

**18.6 Theorem.** *Suppose that  $X$  is an  $\mathcal{F}$ -submartingale whose almost every sample path is right continuous. If  $X$  is of class DL then there exists a natural  $\mathcal{F}_+$ -increasing process  $A$  such that  $t \mapsto X_t - A_t$  is an  $\mathcal{F}_+$ -martingale.*

*Proof.* For each  $T \in \mathbb{N}$ , according to Lemma 18.4,  $A_0^T = 0$  a.s. and if  $s, t \in \mathbb{Q}_{[0, T]}^{(2)}$  and  $s \leq t$  then  $A_s^T \leq A_t^T$  a.s. Moreover almost every sample path of  $A^T$  is right continuous by Theorem 14.11(i). Hence there exists  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and

$$t \mapsto A_t^T(\omega) \text{ is finite valued, right continuous and non-decreasing and } A_0^T(\omega) = 0$$

for each  $\omega \in \Omega_0$  and  $T \in \mathbb{N}$ . Suppose that  $S, T \in \mathbb{N}$  and  $S < T$ . If  $t \in \mathbb{Q}_{[0, S]}^{(2)}$  then, since both  $A_t^S$  and  $A_t^T$  are  $L^\infty$ -weak limits of  $\sum_{q \in 2^{\phi(k)}\mathbb{N}: q \leq t} Y_q^{\phi(k)}$ , they must coincide a.s. We set

$$\tilde{\Omega} := \bigcap_{t \in \mathbb{Q}^{(2)}: t \geq 0} \bigcap_{S, T \in \mathbb{N}: S > t, T > t} \{\omega \in \Omega_0 : A_t^S(\omega) = A_t^T(\omega)\} \text{ and } A_t := \inf\{A_t^T; T \in \mathbb{N}, T > t\}.$$

Then  $\tilde{\Omega} \in \mathcal{F}$ ,  $P(\tilde{\Omega}) = 1$  and, each  $A_t^T$  being  $\mathcal{F}_{t+}$ -measurable,  $A$  is  $\mathcal{F}_+$ -adapted. Moreover

$$A_t(\omega) = A_t^T(\omega) \text{ for all } \omega \in \tilde{\Omega}, T \in \mathbb{N} \text{ and } t \in \mathbb{R}_{[0, T]}.$$

Consequently  $t \mapsto A_t(\omega)$  is finite valued, right continuous and non-decreasing and  $A_0(\omega) = 0$  for all  $\omega \in \tilde{\Omega}$ . Thus  $A$  is an  $\mathcal{F}_+$ -increasing process. Since  $X_t - A_t^T = Z_t^T$  a.s. for all  $t \in \mathbb{R}_{[0, T]}$  by Theorem 14.11(ii) and  $X_t$  is  $\mathcal{F}_{t+}$ -measurable, we see that

$$X_t - A_t \in E[X_T - A^T | \mathcal{F}_{t+}] \text{ for all } T \in \mathbb{N} \text{ and } t \in \mathbb{R}_{[0, T]},$$

which means that  $t \mapsto X_t - A_t$  is an  $\mathcal{F}_+$ -martingale. Given  $M \in \mathfrak{Mart}_b(\mathcal{F}_+)$  and  $t \in \mathbb{Q}_{> 0}^{(2)}$ . Suppose that  $n \in \mathbb{Z}_{\leq 0}$  and  $t \in 2^n \mathbb{Z}_{\geq 0}$ . Then

$$E[M_t Y_q^n] = E[M_{q-2^n} (X_q - X_{q-2^n})] = E[M_{q-2^n} (A_q - A_{q-2^n})] \text{ for all } q \in 2^n \mathbb{N} \text{ with } q \leq t.$$

Indeed if  $q \in 2^n \mathbb{N}$  and  $q \leq t$  then  $M_{q-2^n} \in E[M_t | \mathcal{F}_{(q-2^n)_+}]$ ,  $Y_q^n \in E[X_q - X_{q-2^n} | \mathcal{F}_{(q-2^n)_+}]$  and  $X_{q-2^n} - A_{q-2^n} \in E[X_q - A_q | \mathcal{F}_{(q-2^n)_+}]$ . On the other hand by Lemma 18.4

$$\sum_{q \in 2^{\phi(k)}\mathbb{N}: q \leq t} Y_q^{\phi(k)} \text{ converges to } A_t \text{ } L^\infty\text{-weakly as } k \rightarrow \infty \text{ for all } t \in \mathbb{Q}_{\geq 0}^{(2)}.$$

We thus get  $E[M_t A_t] = E[R[MdA]_t]$ , which hold for all  $t \in \mathbb{R}_{\geq 0}$  due to the right continuity. Since  $E[M_t A_t] = E[R\{MdA\}_t]$  by Lemma 15.18(i), it follows that  $A$  is natural.  $\square$

The conclusion in Theorem 18.6 remains valid for  $\mathcal{F}$ -submartingales of class DL over  $\mathbb{Q}$  with stochastic right continuity.

Given a stochastic process  $X$ , we set

$$\text{DM}[X, \mathcal{F}] := \{A : \text{natural } \mathcal{F}\text{-increasing process, } t \mapsto X_t - A_t \text{ } \mathcal{F}\text{-martingale with stochastic right continuity}\}.$$

- 18.7 Theorem.** (i) If  $A, B \in \text{DM}[X, \mathcal{F}]$  then  $A_t = B_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s.  
(ii) If  $A \in \text{DM}[X, \mathcal{F}]$ ,  $B$  is an  $\mathcal{F}$ -adapted process with almost sure right continuous path and  $A_t = B_t$  a.s. for all  $t \in \mathbb{R}_{\geq 0}$  then  $B \in \text{DM}[X, \mathcal{F}]$ .  
(iii) If  $\text{DM}[X, \mathcal{F}] \cap \text{DM}[Y, \mathcal{F}] \neq \emptyset$  then  $\text{DM}[X, \mathcal{F}] = \text{DM}[Y, \mathcal{F}]$ .  
(iv)  $\text{DM}[X, \mathcal{F}] \subset \text{DM}[X, \mathcal{F}_+]$ .  $X$  is  $\mathcal{F}$ -adapted and  $\text{DM}[X, \mathcal{F}_+] \neq \emptyset$  if and only if  $X$  is an  $\mathcal{F}$ -submartingale of class DL over  $\mathbb{Q}$  with stochastic right continuity.  
(v) If  $X$  and  $Y$  are  $\mathcal{F}$ -adapted and  $\text{DM}[X, \mathcal{F}_+] \cap \text{DM}[Y, \mathcal{F}_+] \neq \emptyset$  then  $t \mapsto X_t - Y_t$  is an  $\mathcal{F}$ -martingale with stochastic right continuity. If  $t \mapsto X_t - Y_t$  is an  $\mathcal{F}$ -martingale with stochastic right continuity then  $\text{DM}[X, \mathcal{F}] = \text{DM}[Y, \mathcal{F}]$ .  
(vi) If  $X_0 = 0$  a.s.,  $X_t \geq 0$  a.s.  $\forall t \in \mathbb{R}_{\geq 0}$  and  $0 \in \text{DM}[X, \mathcal{F}]$  then  $X_t = 0$  a.s.  $\forall t \in \mathbb{R}_{\geq 0}$ .

*Proof.* (iv) An  $\mathcal{F}$ -martingale with stochastic right continuity is an  $\mathcal{F}_+$ -martingale by Theorem 14.11(ii). This shows the first implication.  $\square$

Let  $X$  be an  $\mathcal{F}$ -submartingale (with almost sure right continuous path). According to Lemma 7.12, if  $X_t \geq 0$  a.s. for all  $t \in \mathbb{R}_{\geq 0}$  then  $X$  is of class DL over  $\mathbb{Q}$  (of class DL).

- 18.8 Definition.** (i) An  $\mathcal{F}$ -increasing process  $A$  is said to be integrable if  $E[A_t] < +\infty$  for all  $t \in \mathbb{R}_{\geq 0}$ . (Recall that  $A_0 = 0$  a.s. by definition.)  
(ii) Let  $X$  be an integrable  $\mathcal{F}$ -increasing process. An  $\mathcal{F}$ -compensator of  $X$  is a natural  $\mathcal{F}$ -increasing process  $A$  such that  $t \mapsto X_t - A_t$  is an  $\mathcal{F}$ -martingale.  
(iii) Let  $X$  be an  $\mathcal{F}$ -finite variation process such that  $E[\text{var}(X)_t] < +\infty$  for all  $t \in \mathbb{R}_{\geq 0}$ . An  $\mathcal{F}$ -natural projection of  $X$  is a natural  $\mathcal{F}$ -finite variation process  $Y$  such that  $Y_0 = X_0$  a.s. and  $t \mapsto X_t - Y_t$  is an  $\mathcal{F}$ -martingale.

- 18.9 Corollary.** Let  $X$  be an  $\mathcal{F}$ -finite variation process with  $E[\text{var}(X)_t] < +\infty \forall t \in \mathbb{R}_{\geq 0}$ .  
(i) An  $\mathcal{F}_+$ -natural projection of  $X$  exists.  $\mathcal{F}_+$ -natural projections are indistinguishable.  
(ii) If  $t \mapsto X_t$  is non-decreasing a.s. then so are  $\mathcal{F}_+$ -natural projections.  
(iii) If  $Y$  is an  $\mathcal{F}_+$ -natural projection of  $X$  then  $t \mapsto \text{var}_+(X)_t - \text{var}_+(Y)_t$  as well as  $t \mapsto \text{var}_-(X)_t - \text{var}_-(Y)_t$  are  $\mathcal{F}_+$ -submartingales.  
(iv) If  $X$  is natural then both  $\text{var}_+(X)$  and  $\text{var}_-(X)$  are natural  $\mathcal{F}_+$ -increasing processes.

*Proof.* (i) An integrable  $\mathcal{F}_+$ -increasing process is an  $\mathcal{F}_+$ -submartingale and, by Lemma 7.12, it is of class DL. We apply Theorem 18.6 to  $\mathcal{F}_+$ -increasing processes  $t \mapsto \text{var}_+(X)_t$  and  $t \mapsto \text{var}_-(X)_t$ . There exist natural  $\mathcal{F}_+$ -increasing processes  $A$  and  $B$  such that  $t \mapsto \text{var}_+(X)_t - A_t$  and  $t \mapsto \text{var}_-(X)_t - B_t$  are  $\mathcal{F}_+$ -martingales. According to Lemma 15.15,  $Y : t \mapsto X_0 + A_t - B_t$  is a natural  $\mathcal{F}_+$ -finite variation process. The uniqueness is due to Theorem 15.19. (iii) We have that  $\text{var}_+(Y)_t - \text{var}_+(Y)_s \leq A_t - A_s$  for  $s \leq t$  a.s. Hence

$$E[\text{var}_+(Y)_t - \text{var}_+(Y)_s; C] \leq E[A_t - A_s; C] = E[\text{var}_+(X)_t - \text{var}_+(X)_s; C]$$

for  $s \leq t$ ,  $C \in \mathcal{F}_{s+}$ . The equality is due to that  $t \mapsto \text{var}_+(X)_t - A_t$  is an  $\mathcal{F}_+$ -martingale. Consequently  $t \mapsto \text{var}_+(X)_t - \text{var}_+(Y)_t$  is an  $\mathcal{F}_+$ -submartingale.

(iv) Suppose that  $X$  is natural. It is a natural  $\mathcal{F}_+$ -finite variation process as well. Due to the uniqueness, we have that  $X_t = Y_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. It follows that  $\text{var}_+(X)_t \leq A_t$  a.s. and  $E[\text{var}_+(X)_t] = E[A_t]$  for all  $t \in \mathbb{R}_{\geq 0}$ . We thus infer that  $\text{var}_+(X)_t = A_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. Consequently  $t \mapsto \text{var}_+(X)_t$  is a natural  $\mathcal{F}_+$ -increasing process.  $\square$

**18.10 Definition.** An  $\mathcal{F}$ -Poisson process  $N$  is an  $\mathcal{F}$ -adapted process such that its almost every sample path is right continuous,  $N_0 = 0$  a.s., and for each pair  $s, t \in \mathbb{R}_{\geq 0}$  with  $s < t$  the increment  $N_t - N_s$  is independent of  $\mathcal{F}_s$  and Poisson distributed with mean  $t - s$ .

**18.11 Example.** (i) Let  $N$  be an  $\mathcal{F}$ -Poisson process. Then  $t \mapsto N_t(N_t - 1) \cdots (N_t - n)$  is an  $\mathcal{F}$ -increasing process with  $\mathcal{F}$ -compensator  $t \mapsto (n + 1)R[N.(N. - 1) \cdots (N. - n + 1)d\lambda]_t$  for each  $n \in \mathbb{Z}_{\geq 0}$  where  $\lambda. : (t, \omega) \mapsto t$ . Moreover  $\lambda. \in \text{DM}[(N. - \lambda.)^2, \mathcal{F}]$ .  
(ii) Let  $N$  and  $M$  be two mutually independent Poisson processes. Then  $X. : t \mapsto N_t - M_t$  is a martingale, and  $\text{var}_+(X.)_t = N_t$  and  $\text{var}_-(X.)_t = M_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s.

*Proof.* Since  $k(k - 1) \cdots (k - n) = (n + 1) \sum_{i=1}^k (i - 1) \cdots (i - n)$  for all  $k \in \mathbb{N}$ , we have that

$$N_t(N_t - 1) \cdots (N_t - n) = (n + 1) \int_{(0,t]} N._(N._ - 1) \cdots (N._ - n + 1) dN.$$

Taking into account that  $\lambda.$  is an  $\mathcal{F}$ -compensator of  $N$ ., we get the first statement by Lemma 15.9. Indeed the right hand side below is integrable:

$$\sup_{s \leq t} |N_{s-}(N_{s-} - 1) \cdots (N_{s-} - n + 1)| \text{var}(N. - \lambda.)_t \leq N_t(N_t - 1) \cdots (N_t - n + 1)(N_t + t).$$

The  $\mathcal{F}$ -martingale property of  $t \mapsto (N_t - t)^2 - t$  derives from

$$(N_t - t)^2 = N_t(N_t - 1) - 2 \int_0^t N. d\lambda. - 2 \int_{(0,t]} \lambda. d(N. - \lambda.) + N_t$$

The second term of the right hand side is a compensator of the first term.

$$(ii) \{t : N_t(\omega) \neq N_{t-}(\omega)\} \cap \{t : M_t(\omega) \neq M_{t-}(\omega)\} = \emptyset \text{ } P\text{-a.s.} \quad \square$$

## 19 Compensator of bounded increasing process

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\mathcal{F}$  be a filtration of sub  $\sigma$ -fields. Given a submartingale  $M$  we write  $M_\infty := \liminf_{t \rightarrow +\infty} M_t$  provided  $\sup_{t \geq 0} E[\max\{M_t, 0\}] < +\infty$ .

**19.1 Lemma.** *Suppose that  $X$  is an  $\mathcal{F}$ -finite variation process such that  $E[\text{var}(X)_t] < +\infty$  for all  $t \in \mathbb{R}_{\geq 0}$ ,  $A$  is a  $\mathcal{F}_+$ -natural projection of  $X$ .,  $M$  is an  $\mathcal{F}_+$ -martingale whose almost every sample path is right continuous, and  $\sigma$  is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time.*

(i) *If  $M$  is bounded and  $[|X_0|; \sigma > 0] + E[\text{var}(X)_\sigma] < +\infty$  then  $E[\text{var}(A)_\sigma] < +\infty$  and*

$$E[M_\sigma A_\sigma; \sigma > 0] = E[M_0 X_0; \sigma > 0] + E[R[MdX]_\sigma].$$

(ii) *If  $t \mapsto X_t$  is non-decreasing a.s.,  $X_0 \geq 0$  a.s.,  $M_t \geq 0$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. and  $\sigma \leq T$  a.s. for some  $T \in \mathbb{R}_{> 0}$  then*

$$E[M_\sigma A_\sigma; \sigma > 0] = E[M_0 X_0; \sigma > 0] + E[R[MdX]_\sigma].$$

*Proof.* (i) Lemma 4.8(i) shows that  $(\mathcal{F} \vee \text{Null}(P))_+ = \mathcal{F}_+ \vee \text{Null}(P)$ . In particular the set of  $\mathcal{F} \vee \text{Null}(P)$ -optional times coincides with that of  $\mathcal{F}_+ \vee \text{Null}(P)$ -stopping times. According to Corollary 18.9(iii),  $t \mapsto \text{var}(X)_t - \text{var}(A)_t$  is an  $\mathcal{F}_+$ -submartingale. It follows that

$$E[\text{var}(A)_{t \wedge \sigma}] \leq E[\text{var}(A)_{t \wedge \sigma}] + E[\text{var}(X)_{t \wedge \sigma} - \text{var}(A)_{t \wedge \sigma}] = E[\text{var}(X)_{t \wedge \sigma}]$$

by Corollary 3.17(ii). Since  $E[\text{var}(X)_\sigma]$  dominates the right hand side, we get

$$E[\text{var}(A)_\sigma] \leq E[\text{var}(X)_\sigma] < +\infty$$

by invoking the dominated convergence theorem. Therefore Lemma 15.18 shows that

$$E[M_\sigma A_\sigma; \sigma > 0] = E[M_0 X_0; \sigma > 0] + E[R[MdA]_\sigma].$$

On the other hand  $t \mapsto R[Md(X - A)]_{t \wedge \sigma}$  is an  $\mathcal{F}_+ \vee \text{Null}(P)$ -martingale by Lemma 15.9 and this is dominated by the integrable random variable  $\sup_{s:s \geq 0} |M_s|(\text{var}(A)_\sigma + \text{var}(X)_\sigma)$ . This implies that  $E[R[MdX]_\sigma] - E[R[MdA]_\sigma] = E[R[Md(X - A)]_\sigma] = 0$ .

(ii) Observe that  $E[\text{var}(X)_\sigma] \leq E[\text{var}(X)_T] < +\infty$ . Let  $n \in \mathbb{N}$ . There exist a stochastic process  $M^n$  such that almost every sample path is right continuous and

$$M_t^n \in E[\min\{M_T, n\} | \mathcal{F}_{t+}] \text{ for all } t \in \mathbb{R}_{\geq 0}$$

according to Corollary 14.12. We may regard that  $0 \leq M_t^n(\omega) \leq n$  for all  $\omega \in \Omega$ . Indeed  $(M^n \vee 0) \wedge n$  is a desired modification. Hence  $M^n$  is a bounded  $\mathcal{F}_+$ -martingale. We get

$$E[M_\sigma^n (A_\sigma - X_0); \sigma > 0] = E[R[M^n dX]_\sigma] \text{ for all } n \in \mathbb{N}$$

by applying (i) to the pair  $X - X_0$  and  $A - A_0$ . Since  $E[M_\sigma^n; \Lambda] = E[M_0^n; \Lambda]$  for all  $\Lambda \in \mathcal{F}_{0+}$  and the integrands are non-negative, we see by the monotone convergence theorem that

$$E[M_\sigma^n X_0; \sigma > 0] = E[M_0^n X_0; \sigma > 0].$$

Adding side by side we infer that

$$E[M_\sigma^n A_\sigma; \sigma > 0] = E[M_0^n X_0; \sigma > 0] + E[R[M^n dX]_\sigma] \text{ for all } n \in \mathbb{N}.$$

There exists  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and

$t \mapsto M_t^n(\omega)$  is right continuous and admits left-hand limits everywhere for each  $\omega \in \Omega_0$  and  $n \in \mathbb{N}$ .

We infer by Lemma 1.8 that

$$M_t^n(\omega) \leq M_t^{n+1}(\omega) \leq M_t(\omega) \text{ for all } t \in \mathbb{R}_{[0,T]}, n \in \mathbb{N} \text{ and } \omega \in \Omega_0$$

where we replace  $\Omega_0$  by a smaller set with full measure from  $\mathcal{F}$  if necessary. It follows that

$$M_{t-}^n(\omega) \leq M_{t-}^{n+1}(\omega) \text{ and } 0 \leq M_{t-}(\omega) - \sup_{k \in \mathbb{N}} M_{t-}^k(\omega) \leq M_{t-}(\omega) - M_{t-}^n(\omega)$$

for all  $t \in \mathbb{R}_{(0,T]}$ ,  $n \in \mathbb{N}$  and  $\omega \in \Omega_0$ . Let  $\varepsilon \in \mathbb{R}_{>0}$ . Example 6.4 shows that

$$P\left(\sup_{t \in [0,T]} |M_t^n - M_t| \geq \varepsilon\right) \leq E[|M_T^n - M_T|] / \varepsilon \leq E[M_T; M_T > n] / \varepsilon \text{ for all } n \in \mathbb{N}.$$

The right hand side vanishes at the limit  $n$  tends to  $\infty$ . Consequently

$$P\left(\sup_{t \in [0,T]} \left| \sup_{k \in \mathbb{N}} M_{t-}^k - M_{t-} \right| \geq \varepsilon, \Omega_0\right) = 0 \text{ for all } \varepsilon \in \mathbb{R}_{>0},$$

which means that

$$\sup_{k \in \mathbb{N}} M_{t-}^k(\omega) = M_{t-}(\omega) \text{ for all } t \in \mathbb{R}_{(0,T]} \text{ and } \omega \in \Omega_0.$$

Here we choose a smaller set with full measure from  $\mathcal{F}$  if necessary. Similarly we have that

$$\sigma(\omega) \leq T, M_\sigma^n(\omega) \leq M_\sigma^{n+1}(\omega) \text{ and } \sup_{k \in \mathbb{N}} M_\sigma^k(\omega) = M_\sigma(\omega) \text{ for all } \omega \in \Omega_0$$

with a necessary modification of  $\Omega_0$ . Thus the monotone convergence theorem shows that

$$\sup_{n \in \mathbb{N}} E[R[M^n dX]_\sigma] = \sup_{n \in \mathbb{N}} E\left[\int_{(0,\sigma]} M_{s-}^n dX_s; \Omega_0\right] = E\left[\int_{(0,\sigma]} M_{s-} dX_s; \Omega_0\right] = E[R[M dX]_\sigma].$$

Similarly  $\sup_{n \in \mathbb{N}} E[M_\sigma^n A_\sigma; \sigma > 0]$  and  $\sup_{n \in \mathbb{N}} E[M_0^n X_0; \sigma > 0]$  equal  $E[M_\sigma A_\sigma; \sigma > 0]$  and  $E[M_0 X_0; \sigma > 0]$  respectively.  $\square$

**19.2 Lemma.** *Suppose that  $X$  and  $Y$  are integrable  $\mathcal{F}$ -increasing processes,  $A \in \text{DM}[X, \mathcal{F}_+]$ ,  $B \in \text{DM}[Y, \mathcal{F}_+]$ ,  $\sigma$  is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time, and  $Y_\infty \leq K$  a.s. for some  $K \in \mathbb{R}_{\geq 0}$ . Then  $E[A_\sigma B_\sigma] + E[(Y_\infty - Y_\sigma)A_\sigma] \leq E[X_\sigma B_{\sigma-}] + KE[X_\sigma]$  and  $E[(B_\sigma)^2] \leq 2KE[Y_\sigma] < +\infty$  where  $B_{0-} := B_0$  and  $B_{t-} := \liminf_{s \uparrow t: s \in \mathbb{Q}} B_s$  for  $t \in \mathbb{R}_{>0}$ .*

*Proof.* Let  $n \in \mathbb{N}$ . According to Corollary 14.12, there exist a stochastic process  $M$  such that almost every sample path is right continuous and

$$M_t \in E[B_n | \mathcal{F}_{t+}] \text{ for all } t \in \mathbb{R}_{\geq 0}.$$

We may regard that  $0 \leq M_t(\omega)$  for all  $\omega \in \Omega$ . Indeed  $\max\{M, 0\}$  is a desired modification. Hence  $M$  is a non-negative  $\mathcal{F}_+$ -martingale. We get by Lemma 19.1(ii) that

$$E[B_n A_{\sigma \wedge n}] = E[M_{\sigma \wedge n} A_{\sigma \wedge n}] = E[R[M dX]_{\sigma \wedge n}].$$

Let  $t \in \mathbb{R}_{\geq 0}$ . Then  $E[B_n | \mathcal{F}_{t+}] = E[Y_n | \mathcal{F}_{t+}] + B_{t \wedge n} - Y_{t \wedge n}$  implies that

$$E[M_t; \Lambda] = E[B_n; \Lambda] = E[Y_n; \Lambda] + E[B_{t \wedge n} - Y_{t \wedge n}; \Lambda] \text{ for all } \Lambda \in \mathcal{F}_{t+}.$$

The right hand side is dominated by  $E[K + B_{t \wedge n} - Y_{t \wedge n}; \Lambda]$ . It follows that

$$M_t \leq K + B_t - Y_t \text{ for all } t \in \mathbb{R}_{[0,n]} \text{ a.s.}$$

due to the right continuity of sample paths and hence

$$M_{t-} \leq K + B_{t-} - Y_{t-} \leq K + B_{(\sigma \wedge n)-} \text{ for all } t \in \mathbb{R}_{(0, \sigma \wedge n]} \text{ a.s.}$$

due to the left hand regularity of sample paths. We thus get

$$R[M dX]_{\sigma \wedge n} \leq (K + B_{(\sigma \wedge n)-})X_{\sigma \wedge n} \leq (K + B_{\sigma-})X_\sigma \text{ a.s.}$$

Consequently we infer that

$$E[B_{\sigma \wedge n} A_{\sigma \wedge n}] + E[(Y_n - Y_{\sigma \wedge n})A_{\sigma \wedge n}] = E[B_n A_{\sigma \wedge n}] \leq KE[X_\sigma] + E[X_\sigma B_{\sigma-}] \text{ for all } n \in \mathbb{N}.$$

Tending  $n$  to  $\infty$  we reach the statement by the monotone convergence theorem. Finally, since  $E[B_{\sigma \wedge n}] = E[Y_{\sigma \wedge n}]$ , we have that  $E[B_\sigma] = E[Y_\sigma]$ . On the other hand  $E[B_{\sigma-}] \leq E[B_\sigma]$ .  $\square$

$\mathfrak{Mart}_2(\mathcal{F}_.)$  stands for the space of all square integrable  $\mathcal{F}_.$ -martingales with almost sure right continuous sample path.

**19.3 Lemma.** *Let  $X_.$  be an  $\mathcal{F}_.$ -increasing process such that  $X_\infty \leq K$  a.s. for some  $K \in \mathbb{R}_{\geq 0}$ ,  $A. \in \text{DM}[X_., \mathcal{F}_.]$ , and  $\sigma$  a bounded  $\mathcal{F}_. \vee \text{Null}(P)$ -optional time. Then  $E[(A_\sigma)^2] \leq 2K^2$  and*

$$E[M_\sigma X_\sigma] = E[R\{M dX\}_\sigma] \text{ and } E[M_\sigma A_\sigma] = E[R[M dX]_\sigma] \text{ for all } M. \in \mathfrak{Mart}_2(\mathcal{F}_.).$$

*Proof.* Let  $M. \in \mathfrak{Mart}_2(\mathcal{F}_.)$  and  $n \in \mathbb{N}$ . According to Corollary 14.12 there exist a stochastic process  $M_t^n$  such that almost every sample path is right continuous and

$$M_t^n \in E[\max\{\min\{M_T, n\}, -n\} | \mathcal{F}_{t+}] \text{ for all } t \in \mathbb{R}_{\geq 0}.$$

We may regard that  $|M_t^n(\omega)| \leq n$  for all  $\omega \in \Omega$ . Indeed  $\max\{\min\{M_T, n\}, -n\}$  is a desired modification. Hence  $M_t^n$  is a bounded  $\mathcal{F}_.$ -martingale. We get that

$$E[M_\sigma^n X_\sigma] = E[R\{M_t^n dX\}_\sigma] \text{ and } E[M_\sigma^n A_\sigma] = E[R[M_t^n dX]_\sigma] \text{ for all } n \in \mathbb{N}$$

by Lemma 15.18(i) and Lemma 19.1. Choose  $T \in \mathbb{R}_{> 0}$  such that  $\sigma \leq T$  a.s. Then

$$|M_\sigma^n - M_\sigma| \leq \sup_{t \in [0, T]} |M_t^n - M_t| \text{ a.s. and } |R[M_t^n dX]_\sigma - R[M dX]_\sigma| \leq \sup_{t \in [0, T]} |M_t^n - M_t| X_T.$$

According to Lemma 19.2,  $E[(A_\sigma)^2] \leq 2KE[X_\sigma] < +\infty$  while Example 6.4 shows that

$$E[\sup_{t \in [0, T]} |M_t^n - M_t|^2] \leq 4E[|M_T^n - M_T|^2] \leq 4E[|M_T|^2; M_T > n] \text{ for all } n \in \mathbb{N}.$$

The right hand side vanishes at the limit  $n$  tends to  $\infty$ . Consequently we reach the claim.  $\square$

**19.4 Lemma.** *If  $A \in \text{Prog}(\mathcal{F}_.)$  then  $\{\omega \in \Omega : \exists s \in \mathbb{R}_{[0, t]} \text{ s.t. } (s, \omega) \in A\} \in \mathcal{F}_t \vee \text{Null}(P)$ .*

**19.5 Corollary.** *Suppose that  $E$  is a metrizable topological space and  $X_.$  is an  $E$ -valued process indistinguishable from an  $\mathcal{F}_. \vee \text{Null}(P)$ -progressively measurable process. Then for each  $F \in \text{Borel}(E)$  both the entry time  $\inf\{t \in \mathbb{R}_{\geq 0} : X_t \in F\}$  to  $F$  and the hitting time  $\inf\{t \in \mathbb{R}_{> 0} : X_t \in F\}$  to  $F$  are  $\mathcal{F}_. \vee \text{Null}(P)$ -optional times.*

*Proof.* We may assume that  $X_.$  itself is  $\mathcal{F}_.$ -progressively measurable. We have that

$$A := \{(s, \omega) \in \mathbb{R}_{\geq 0} \times \Omega : X_s \in F\} \in \text{Prog}(\mathcal{F}_.).$$

On the other hand if  $\tau := \inf\{s \geq 0 : X_s \in F\}$  and  $t \in \mathbb{R}_{> 0}$  then

$$\{\tau < t\} = \{\omega \in \Omega : \exists s \in \mathbb{R}_{[0, t]} \text{ s.t. } (s, \omega) \in A\}.$$

Thus we get  $\{\tau < t\} \in \mathcal{F}_t \vee \text{Null}(P)$  for all  $t \in \mathbb{R}_{> 0}$ .  $\square$

$\mathfrak{Mart}_2^c(\mathcal{F}_.) := \{M. \in \mathfrak{Mart}_2(\mathcal{F}_.) : t \mapsto M_t \text{ continuous a.s.}\}.$   
 $\mathfrak{Mart}_2^{\text{fv}}(\mathcal{F}_.) := \{M. \in \mathfrak{Mart}_2(\mathcal{F}_.) : t \mapsto M_t \text{ of finite variation a.s.}\}.$

**19.6 Theorem.** *Suppose that  $M. \in \mathfrak{Mart}_2(\mathcal{F}_.)$ . Then  $M.$  is indistinguishable from a process in  $\mathfrak{Mart}_2^c(\mathcal{F}_.)$  if and only if  $t \mapsto M_t F_t$  is an  $\mathcal{F}_.$ -martingale whenever  $F. \in \mathfrak{Mart}_2^{\text{fv}}(\mathcal{F}_.)$ .*

*Proof.* We have that  $\mathfrak{Mart}_2(\mathcal{F}) \subset \mathfrak{Mart}_2(\mathcal{F}_+)$  by Theorem 14.11(ii). Thus the implication  $\Rightarrow$  is stated in Theorem 16.9(iv) with the filtration  $\mathcal{F}$  replaced by  $\mathcal{F}_+$ .

Conversely suppose that  $t \mapsto M_t F_t$  is an  $\mathcal{F}_+$ -martingale for all  $F \in \mathfrak{Mart}_2^{\text{fv}}(\mathcal{F}_+)$ . Given an  $\mathcal{F} \vee \text{Null}(P)$ -optional time  $\sigma$  there exists an  $\mathcal{F}$ -optional time  $\tau$  such that  $\sigma = \tau$  a.s. by Lemma 4.8(iii). We see that

$$X : t \mapsto 1_{[\tau, \infty)}(t) - 1_{\tau=0} \text{ is a bounded } \mathcal{F}_+\text{-increasing process.}$$

Theorem 18.6 shows that  $\text{DM}[X, \mathcal{F}_+] \neq \emptyset$ . We select and fix  $A \in \text{DM}[X, \mathcal{F}_+]$ . Then

$$F : t \mapsto X_t - A_t \in \mathfrak{Mart}_2(\mathcal{F}_+).$$

The square integrability derives by Lemma 19.2. Clearly it is of finite variation almost surely. It then follows that  $t \mapsto M_t F_t$  is an  $\mathcal{F}_+$ -martingale and hence

$$E[M_T X_T] = E[M_T A_T] + E[M_T F_T] = E[M_T A_T] + E[M_0 F_0] = E[M_T A_T] \text{ for all } T \in \mathbb{R}_{>0}.$$

The left hand side equals  $E[R\{MdX\}_T]$  while the right hand side equals  $E[R[MdX]_T]$  according to Lemma 19.3. Consequently we get that

$$(\star) \quad E[R\{Md(1_{[\tau, \infty)} - 1_{\tau=0})\}_T] = E[R[Md(1_{[\tau, \infty)} - 1_{\tau=0})]_T] \text{ for all } T \in \mathbb{R}_{>0}.$$

We select and fix  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and  $t \mapsto M_t(\omega)$  is right continuous and admits left-hand limits for each  $\omega \in \Omega_0$ . Let  $\omega \in \Omega_0$ . We see that

$$\int_{(0, t]} M_{s-}(\omega) d1_{[\tau, \infty)}(s) = \begin{cases} 0 & \tau(\omega) = 0 \text{ or } \tau(\omega) > t \\ M_{\tau-}(\omega) & 0 < \tau(\omega) \leq t \end{cases}.$$

Let  $T \in \mathbb{R}_{>0}$ . Since  $R[Md(1_{[\tau, \infty)} - 1_{\tau=0})]_T(\omega) = \int_{(0, T]} M_{s-}(\omega) d1_{[\tau, \infty)}(s)$ , we get by  $(\star)$

$$E[M_\sigma; 0 < \sigma \leq T] = E[M_\tau; 0 < \tau \leq T] = E[M_{\tau-}; 0 < \tau \leq T] = E[M_{\sigma-}; 0 < \sigma \leq T].$$

Now for each  $\varepsilon \in \mathbb{R}_{>0}$  we introduce the following:

$$\sigma(\varepsilon) := \inf\{t \in \mathbb{R}_{>0} : M_t - M_{t-} > \varepsilon\} \text{ and } \tau(\varepsilon) := \inf\{t \in \mathbb{R}_{>0} : M_t - M_{t-} < -\varepsilon\}$$

which are  $\mathcal{F} \vee \text{Null}(P)$ -optional times by Corollary 19.5. We see that  $\sigma(\varepsilon, \omega) > 0$  and  $\tau(\varepsilon, \omega) > 0$  for all  $\omega \in \Omega_0$  by Lemma 15.3. It also shows that if  $\omega \in \{\sigma(\varepsilon) < +\infty\} \cap \Omega_0$  then  $M_{\sigma(\varepsilon)}(\omega) - M_{\sigma(\varepsilon)-}(\omega) > \varepsilon$ . Therefore

$$0 \leq \varepsilon P(\sigma(\varepsilon) \leq T) = \varepsilon P(0 < \sigma(\varepsilon) \leq T) \leq E[M_{\sigma(\varepsilon)} - M_{\sigma(\varepsilon)-}; 0 < \sigma(\varepsilon) \leq T] = 0,$$

which implies that  $P(\sigma(\varepsilon) \leq T) = 0$  for all  $T \in \mathbb{R}_{>0}$ , that is,  $P(\sigma(\varepsilon) < +\infty) = 0$ . Similarly we obtain that  $P(\tau(\varepsilon) < +\infty) = 0$ . The number  $\varepsilon \in \mathbb{R}_{>0}$  being arbitrary, we conclude that  $M_t = M_{t-}$  for all  $t \in \mathbb{R}_{>0}$  a.s. Consequently  $M$  is indistinguishable from  $t \mapsto \liminf_{s \uparrow t; s \in \mathbb{Q}} M_s$ , which is  $\mathcal{F}$ -adapted and continuous almost surely.  $\square$

**19.7 Lemma.** *Suppose that  $\tau$  is an  $\mathcal{F}$ -optional time,  $Y$  is an integrable  $\mathcal{F}$ -increasing process,  $A \in \text{DM}[1_{[\tau, \infty)} - 1_{\tau=0}, \mathcal{F}_+]$ ,  $B \in \text{DM}[Y, \mathcal{F}_+]$ ,  $\sigma$  is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time. If there exists  $K \in \mathbb{R}_{\geq 0}$  such that  $Y_\infty := \sup_{t: t \geq 0} Y_t \leq K$  a.s. then*

$$E[A_\sigma B_\sigma] + E[A_\sigma(Y_\infty - Y_\sigma)] \leq E[B_{\tau-} + K - Y_{\tau-}; 0 < \tau \leq \sigma, \tau < +\infty].$$

*Proof.* Since  $E[B_\infty] = \sup_{t \geq 0} E[B_t] = \sup_{t \geq 0} E[Y_t] \leq K$ , the  $\mathcal{F}_+$ -martingale  $t \mapsto Y_t - B_t$  is uniformly integrable. It follows by Theorem 14.8 that

$$Y_t - B_t \in E[Y_\infty - B_\infty | \mathcal{F}_{t+}] \text{ for all } t \in \mathbb{R}_{\geq 0}.$$

According to Corollary 14.12, there exist a stochastic process  $M$  and  $\Omega_0 \in \mathcal{F}$  such that

$$M_s \in E[B_\infty | \mathcal{F}_{s+}] \text{ for all } s \in \mathbb{R}_{\geq 0}, P(\Omega_0) = 1 \text{ and } t \mapsto M_t(\omega) \text{ is right continuous and admits left-hand limits for each } \omega \in \Omega_0.$$

Observe that  $M$  is a non-negative  $\mathcal{F}_+$ -martingale. Let  $\omega \in \Omega_0$ . We see that

$$\int_{(0,t]} M_{s-}(\omega) d1_{[\tau,\infty)}(s) = \begin{cases} 0 & \tau(\omega) = 0 \text{ or } \tau(\omega) > t \\ M_{\tau-}(\omega) & 0 < \tau(\omega) \leq t \end{cases}.$$

Since  $R[Md(1_{[\tau,\infty)} - 1_{\tau=0})]_{\sigma \wedge T}(\omega) = \int_{(0,\sigma \wedge T]} M_{s-}(\omega) d1_{[\tau,\infty)}(s)$ , Lemma 19.1(ii) shows that

$$E[B_\infty A_{\sigma \wedge T}] = E[M_\sigma A_{\sigma \wedge T}] = E[M_{\tau-}; 0 < \tau \leq \sigma \wedge T].$$

The first equality is due to that  $E[B_\infty; \Lambda] = E[M_{\sigma \wedge T}; \Lambda]$  for all  $\Lambda \in \mathcal{F}_{(\sigma \wedge T)+}$  and  $A_{\sigma \wedge T}$  is non-negative. Let  $s \in \mathbb{R}_{\geq 0}$ . Then  $E[B_\infty | \mathcal{F}_{s+}] = E[Y_\infty | \mathcal{F}_{s+}] + B_s - Y_s$  implies that

$$M_{s-} \leq K + B_{s-} - Y_{s-} \text{ for all } s \in \mathbb{R}_{>0} \text{ a.s.}$$

according to the proof of Lemma 19.2. Thus we get

$$E[B_\infty A_{\sigma \wedge T}] \leq E[K + B_{\tau-} - Y_{\tau-}; 0 < \tau \leq \sigma \wedge T].$$

The left hand side equals  $E[(B_{\sigma \wedge T} + Y_\infty - Y_{\sigma \wedge T})A_{\sigma \wedge T}]$ . □

**19.8 Corollary.** Let  $\sigma, \tau \in \text{Time}(\mathcal{F})$ ,  $A \in \text{DM}[1_{[\sigma,\infty)}, \mathcal{F}_+]$  and  $B \in \text{DM}[1_{[\tau,\infty)}, \mathcal{F}_+]$ .

- (i)  $E[A_t] = P(0 < \sigma \leq t) = E[A_{t \wedge \sigma}]$  for all  $t \in \mathbb{R}_{\geq 0}$ .  $A_{t \wedge \sigma} = A_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s.
- (ii)  $E[A_t B_t] + E[A_t; t < \tau < +\infty] \leq E[B_{\sigma-}; 0 < \sigma \leq t] + P(0 < \sigma \leq t \wedge \tau)$  for all  $t \in \mathbb{R}_{\geq 0}$ .
- (iii) If  $\sigma \leq \tau$  a.s. then  $E[A_t B_t] \leq E[A_t; \tau = \infty] + E[A_t; \tau \leq t] + E[B_{\sigma-}; 0 < \sigma \leq t]$  and  $E[(1_{\sigma \leq t} - A_t)(1_{\tau \leq t} - B_t)] \leq P(\tau = 0) + E[A_t; \tau = \infty] + E[B_{\sigma-}; 0 < \sigma \leq t] + E[B_t; \sigma > t]$ .
- (iv) If  $\tau < \sigma$  a.s. then  $E[A_t B_t] + E[A_t; t < \tau < +\infty] \leq E[B_\tau; 0 < \sigma \leq t]$ .

*Proof.* (i) We see that  $E[A_t] = P(\sigma \leq t) - P(\sigma = 0) = P(0 < \sigma \leq t) \leq 1$ . The compensator of  $t \mapsto A_{t \wedge \sigma}$  is given by  $t \mapsto 1_{[\sigma,\infty)}(t \wedge \sigma) = 1_{[\sigma,\infty)}(t)$ .

(ii) Lemma 19.7.

(iii)  $P(\tau \leq t) = P(\tau = 0) + E[B_t]$ .  $B_{\sigma-} = (B_{\cdot \wedge \tau})_{\sigma-} = B_\tau$  on  $\{\sigma > \tau\}$ . □

## 20 Stable spaces of square integrable martingales

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\mathcal{F}$  be a filtration of sub  $\sigma$ -fields.

$\mathfrak{Mart}_2(\mathcal{F})$  stands for the space of all square integrable  $\mathcal{F}$ -martingales with almost sure right continuous sample path.

**20.1 Lemma.** *Suppose that  $M$  is an  $\mathcal{F}$ -adapted process whose almost every sample path is right continuous. If  $T \in \mathbb{R}_{>0}$ ,  $E[|M_\sigma|] < +\infty$  and  $E[M_\sigma] = E[M_0]$  for any  $\mathcal{F}$ -stopping time  $\sigma$  with  $\sigma \leq T$  then  $M$  is an  $\mathcal{F}$ -martingale up to  $T$ .*

*Proof.* Suppose that  $s, t \in \mathbb{R}$ ,  $0 \leq s < t \leq T$  and  $A \in \mathcal{F}_s$ . We see that

$$\{\sigma \leq u\} = \begin{cases} \emptyset & \text{if } u < s \\ A & \text{if } s \leq u < t \text{ where } \sigma := \begin{cases} s & \text{on } A \\ t & \text{off } A \end{cases} \\ \Omega & \text{if } t \leq u \end{cases}.$$

It follows that  $\sigma$  is an  $\mathcal{F}$ -stopping time with  $\sigma \leq T$ . The constant function  $\omega \mapsto t$  is also an  $\mathcal{F}$ -stopping time. The relation  $E[M_\sigma] = E[M_0] = E[M_t]$  reads  $E[M_s; A] = E[M_t; A]$ .  $\square$

**20.2 Corollary.** *Let  $M, N \in \mathfrak{Mart}_2(\mathcal{F})$  and  $T \in \mathbb{R}_{>0}$ . If  $E[M_{T \wedge \sigma} N_T] = E[M_0 N_0]$  for any  $\mathcal{F}$ -stopping time  $\sigma$  then  $t \mapsto M_{t \wedge T} N_{t \wedge T}$  is an  $\mathcal{F}$ -martingale.*

*Proof.* Suppose that  $T \in \mathbb{R}_{>0}$  and  $\sigma$  is an  $\mathcal{F}$ -stopping time. We see that

$$M_{T \wedge \sigma} \text{ is } (\mathcal{F} \vee \text{Null}(P))_{(T \wedge \sigma)_+} \text{-measurable and } N_{T \wedge \sigma} \in E[N_T | (\mathcal{F} \vee \text{Null}(P))_{(T \wedge \sigma)_+}]$$

by Corollary 3.17. It follows that

$$E[M_{T \wedge \sigma} N_{T \wedge \sigma}] = E[M_{T \wedge \sigma} N_T] = E[M_0 N_0] \text{ for all } \mathcal{F}\text{-stopping times } \sigma.$$

Lemma 20.1 shows that  $t \mapsto M_t N_t$  is an  $\mathcal{F}$ -martingale up to  $T$ .  $\square$

**20.3 Definition.** A set  $\mathcal{M}$  of  $\mathcal{F}$ -martingales whose almost all sample paths are right continuous is said to be  $\mathcal{F}$ -stable if for each  $M \in \mathcal{M}$  and an  $\mathcal{F}$ -stopping time  $\sigma$  there exists  $\tilde{M} \in \mathcal{M}$  such that  $M_{t \wedge \sigma} = \tilde{M}_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s.

**20.4 Example.** All of the followings are  $\mathcal{F}_+ \vee \text{Null}(P)$ -stable.

(i)  $\mathfrak{Mart}_2(\mathcal{F}_+)$  as well as the set  $\mathfrak{Mart}_b(\mathcal{F}_+)$  of all bounded  $\mathcal{F}_+$ -martingales whose almost all sample paths are right continuous.

(ii)  $\mathfrak{Mart}_2^{\text{fv}}(\mathcal{F}_+) := \{M \in \mathfrak{Mart}_2(\mathcal{F}_+) : t \mapsto M_t \text{ of finite variation a.s.}\}$ .

(iii)  $\mathfrak{Mart}_2^c(\mathcal{F}) := \{M \in \mathfrak{Mart}_2(\mathcal{F}) : t \mapsto M_t \text{ continuous a.s.}\}$ .

*Proof.* Let  $M$  be an  $\mathcal{F}_+$ -martingale whose almost every sample path is right continuous and  $\sigma$  be an  $\mathcal{F}_+ \vee \text{Null}(P)$ -stopping time. Since  $\mathcal{F}_+ \vee \text{Null}(P)$ -stopping times are  $\mathcal{F} \vee \text{Null}(P)$ -optional times, according to Lemma 15.17, there exists an  $\mathcal{F}_+$ -martingale  $\tilde{M}$  such that  $M_{t \wedge \sigma} = \tilde{M}_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. Since  $M_{t \wedge \sigma} \in E[M_t | (\mathcal{F} \vee \text{Null}(P))_{\sigma_+}]$  by Corollary 3.17(ii), Theorem 1.13(i) shows that

$$E[|\tilde{M}_t|^2] = E[|M_{t \wedge \sigma}|^2] \leq E[|M_t|^2] < +\infty.$$

On the other hand  $\mathfrak{Mart}_2^c(\mathcal{F}) \subset \mathfrak{Mart}_2(\mathcal{F}) \subset \mathfrak{Mart}_2(\mathcal{F}_+)$ . If almost every sample path of  $M$  is continuous then Lemma 10.2 shows the existence of an  $\mathcal{F}$ -adapted process  $\tilde{M}$  such that  $M_{t \wedge \sigma} = \tilde{M}_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s.  $\square$

**20.5 Example.** Let  $M. \in \mathfrak{Mart}^c(\mathcal{F})$  and  $A. \in \text{Qvar}[M.; \mathcal{F}]$ . Then

$$\bigcup \{ \text{It}\hat{o}[fdM; \mathcal{F}]; f \in \mathfrak{L}, E[R[|f|^2 dA]_t] < +\infty \forall t \in \mathbb{R}_{\geq 0} \} \text{ is } \mathcal{F}_+ \vee \text{Null}(P)\text{-stable}$$

where  $\mathfrak{L}$  denotes the set of all  $\mathcal{F}$ -adapted process whose almost all sample paths are locally bounded and admit left hand limits everywhere.

*Proof.* Suppose that  $f \in \mathfrak{L}$ ,  $E[R[|f|^2 dA]_t] < +\infty$  for all  $t \in \mathbb{R}_{\geq 0}$  and  $I. \in \text{It}\hat{o}[fdM; \mathcal{F}]$ . Given an  $\mathcal{F}_+ \vee \text{Null}(P)$ -stopping time  $\sigma$ , being an  $\mathcal{F} \vee \text{Null}(P)$ -optional time as well, there exists  $\tilde{\sigma} \in \text{Time}(\mathcal{F})$  such that  $\sigma = \tilde{\sigma}$  a.s. by Lemma 4.8(iii). Then we have that  $1_{(0, \tilde{\sigma}]} f \in \mathfrak{L}$ ,  $E[R[1_{(0, \tilde{\sigma}]} |f|^2 dA]_t] \leq E[R[|f|^2 dA]_t] < +\infty$  and

$$\text{if } J. \in \text{It}\hat{o}[1_{(0, \tilde{\sigma}]} fdM; \mathcal{F}] \text{ then } J_t = I_{t \wedge \tilde{\sigma}} = I_{t \wedge \sigma} \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

by Lemma 11.6(v). Thus the set in question is  $\mathcal{F}_+ \vee \text{Null}(P)$ -stable.  $\square$

**20.6 Lemma.** Let  $\mathcal{M}$  be an  $\mathcal{F}$ -stable subset of  $\mathfrak{Mart}_2(\mathcal{F})$ ,  $T \in \mathbb{R}_{> 0}$  and  $N. \in \mathfrak{Mart}_2(\mathcal{F})$ . If  $E[M_T N_T] = 0$  for all  $M. \in \mathcal{M}$  then  $t \mapsto M_{t \wedge T} N_{t \wedge T}$  is an  $\mathcal{F}$ -martingale for all  $M. \in \mathcal{M}$ .

*Proof.* Let  $M. \in \mathcal{M}$ . Given an  $\mathcal{F}$ -stopping time  $\sigma$ , the stopped process  $t \mapsto M_{t \wedge \sigma}$  is indistinguishable from a process  $\tilde{M}$  in  $\mathcal{M}$ . It follows that  $E[M_{T \wedge \sigma} N_T] = E[\tilde{M}_T N_T] = 0$ . This is valid for the constant stopping time 0. Since  $N_0 \in E[N_T | \mathcal{F}_0]$ , we have that

$$E[M_0 N_0] = E[M_{T \wedge 0} N_T] = 0 = E[M_{T \wedge \sigma} N_T] \text{ for all } \mathcal{F}\text{-stopping times } \sigma.$$

According to Corollary 20.2, this implies that  $t \mapsto N_{t \wedge T} M_{t \wedge T}$  is an  $\mathcal{F}$ -martingale.  $\square$

We introduce the following family of semi-norms to the space  $\mathfrak{Mart}_2(\mathcal{F})$ :

$$M. \mapsto E[|M_T|^2] \text{ where the parameter } T \text{ runs through } \mathbb{R}_{> 0}.$$

**20.7 Lemma.** (i) Let  $M., N. \in \mathfrak{Mart}_2(\mathcal{F})$ . Then they are indistinguishable from each other if and only if  $E[|M_T - N_T|^2] = 0$  for all  $T \in \mathbb{N}$ .

(ii) If  $M^n$  is a sequence of  $\mathfrak{Mart}_2(\mathcal{F})$  and  $\limsup_{m, n \rightarrow \infty} E[|M_T^m - M_T^n|^2] = 0$  for all  $T \in \mathbb{N}$  then there exists  $M. \in \mathfrak{Mart}_2(\mathcal{F})$  such that  $E[|M_t - M_t^n|^2]$  converges to 0 for all  $t \in \mathbb{R}_{\geq 0}$ .

*Proof.* (ii) We can find  $\phi \in \text{Seq}(\mathbb{N}, \uparrow)$  such that  $E[|M_T^m - M_T^n|^2] \leq 1/2^{3T}$  for all  $T \in \mathbb{N}$  and  $m, n \in \mathbb{N}_{\geq \phi(T)}$ . Let  $T \in \mathbb{N}$ . It then follows by Example 6.4 that

$$P(\sup_{t: t \leq T} |M_t^m - M_t^n| \geq 1/2^T) \leq 2^{2T} E[|M_T^m - M_T^n|^2] \leq 1/2^T \text{ for all } m, n \in \mathbb{N}_{\geq \phi(T)}.$$

The Borel-Cantelli lemma shows that there exists  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and

$$t \mapsto M_t^n(\omega) \text{ is right continuous for all } n \in \mathbb{N} \text{ and} \\ \sup_{t: t \leq n} |M_t^{\phi(n+1)}(\omega) - M_t^{\phi(n)}(\omega)| < 1/2^n \text{ except for finitely many } n \in \mathbb{N}$$

for all  $\omega \in \Omega_0$ . We set

$$M_t(\omega) := \liminf_{n \rightarrow \infty} M_t^{\phi(n)}(\omega) \text{ for } t \in \mathbb{R}_{\geq 0} \text{ and } \omega \in \Omega.$$

The process  $t \mapsto M_t$  is  $\mathcal{F}$ -adapted. Let  $\omega \in \Omega_0$  and  $T \in \mathbb{N}$ . Then, since

$$\sum_{k=n}^{\infty} \sup_{t:t \leq T} |M_t^{\phi(k+1)}(\omega) - M_t^{\phi(k)}(\omega)| \leq \sum_{k=n}^{\infty} \sup_{t:t \leq k} |M_t^{\phi(k+1)}(\omega) - M_t^{\phi(k)}(\omega)| \leq 1/2^{n-1}$$

for all  $n \in \mathbb{N}_{\geq T}$ , it follows that

$$M_t(\omega) - 1/2^{n-1} \leq M_t^{\phi(n)}(\omega) \leq M_t(\omega) + 1/2^{n-1} \text{ for } t \in \mathbb{R}_{[0,T]} \text{ and } n \in \mathbb{N}_{\geq T}.$$

Therefore the right continuity of  $t \mapsto M_t(\omega)$  derives from that of  $t \mapsto M_t^n(\omega)$ . Moreover  $M_t^{\phi(n)}(\omega)$  converges to  $M_t(\omega)$  uniformly on  $\mathbb{R}_{[0,T]}$  as  $n$  tends to  $\infty$ . Exploiting the submartingale property of  $t \mapsto |M_t^{\phi(k)} - M_t^n|^2$ , we see that

$$E[|M_t^{\phi(k)} - M_t^n|^2] \leq E[|M_T^{\phi(k)} - M_T^n|^2] \leq 1/2^{3T} \text{ for } T \in \mathbb{N}_{\geq t}, k \in \mathbb{N}_{\geq T} \text{ and } n \in \mathbb{N}_{\geq \phi(T)}.$$

Let  $t \in \mathbb{R}_{\geq 0}$ . It then follows by Fatou's lemma that

$$E[|M_t - M_t^n|^2] \leq \liminf_{k \rightarrow \infty} E[|M_t^{\phi(k)} - M_t^n|^2] \leq 1/2^{3T} \text{ for all } T \in \mathbb{N}_{\geq t} \text{ and } n \in \mathbb{N}_{\geq \phi(T)}.$$

This implies that  $E[|M_t - M_t^n|^2]$  converges to 0 for all  $t \in \mathbb{R}_{\geq 0}$ . Taking the  $\mathcal{F}$ -adaptedness into account we infer that  $t \mapsto M_t$  is an  $\mathcal{F}$ -martingale.  $\square$

**20.8 Definition.** A subset  $\mathcal{M}$  of  $\mathfrak{Mart}_2(\mathcal{F})$  is said to be  $L_2$ -closed if the statement (ii) of Lemma 20.7 holds for  $\mathcal{M}$  in place of  $\mathfrak{Mart}_2(\mathcal{F})$ .

**20.9 Lemma.** Suppose that  $\mathcal{M}$  is an  $\mathcal{F}_+$ -stable subset of  $\mathfrak{Mart}_2(\mathcal{F}_+)$ . Set

$$\mathcal{M}^\perp := \{N \in \mathfrak{Mart}_2(\mathcal{F}_+) : E[M_T N_T] = 0 \text{ for all } M \in \mathcal{M}, T \in \mathbb{N}\}.$$

- (i) If  $M \in \mathcal{M}$  and  $N \in \mathcal{M}^\perp$  then  $t \mapsto M_t N_t$  is an  $\mathcal{F}_+$ -martingale with  $E[M_0 N_0] = 0$ .
- (ii)  $\mathcal{M}^\perp$  is an  $\mathcal{F}_+$ -stable and  $L_2$ -closed linear subspace.

*Proof.* (i) If  $M \in \mathcal{M}$  and  $N \in \mathcal{M}^\perp$  then,  $t \mapsto M_{t \wedge T} N_{t \wedge T}$  being an  $\mathcal{F}_+$ -martingale for all  $T \in \mathbb{N}$  by Lemma 20.6,  $t \mapsto M_t N_t$  is an  $\mathcal{F}_+$ -martingale.

(ii) Given  $N \in \mathcal{M}^\perp$  and an  $\mathcal{F}_+$ -stopping time  $\sigma$ , there exists  $\tilde{N} \in \mathfrak{Mart}_2(\mathcal{F}_+)$  such that  $N_{t \wedge \sigma} = \tilde{N}_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. by Lemma 15.17. Since  $M_{T \wedge \sigma} \in E[M_T | \mathcal{G}]$  and, by (i),  $M_{T \wedge \sigma} N_{T \wedge \sigma} \in E[M_T N_T | \mathcal{G}]$  where  $\mathcal{G} := (\mathcal{F} \vee \text{Null}(P))_{(T \wedge \sigma)_+}$ , it follows that

$$E[M_T \tilde{N}_T] = E[M_T N_{T \wedge \sigma}] = E[M_{T \wedge \sigma} N_{T \wedge \sigma}] = E[M_T N_T] = 0 \text{ for all } M \in \mathcal{M} \text{ and } T \in \mathbb{N}.$$

This means that  $\tilde{N} \in \mathcal{M}^\perp$ . Consequently  $\mathcal{M}^\perp$  is  $\mathcal{F}_+$ -stable.  $\square$

**20.10 Lemma.** Suppose that  $\mathcal{M}$  is an  $\mathcal{F}_+$ -stable subset of  $\mathfrak{Mart}_2(\mathcal{F}_+)$ . Then the closure  $\overline{\mathcal{M}}$  of  $\mathcal{M}$  in  $\mathfrak{Mart}_2(\mathcal{F}_+)$  remains to be  $\mathcal{F}_+$ -stable.

*Proof.* Suppose that  $M. \in \overline{\mathcal{M}}$ , that is,  $M. \in \mathfrak{Mart}_2(\mathcal{F}_+)$  and there exists a sequence  $M^n$  of  $\mathcal{M}$  such that  $E[|M_T^n - M_T|^2]$  converges to 0 as  $n$  tends to  $\infty$  for all  $T \in \mathbb{N}$ . Let  $\sigma$  be an  $\mathcal{F}_+$ -stopping time. Since  $\mathcal{M}$  is stable, there exists a sequence  $\tilde{M}^n$  of  $\mathcal{M}$  such that  $\tilde{M}^n$  is indistinguishable from  $M_{\wedge\sigma}^n$  for all  $n \in \mathbb{N}$ . We see by the submartingale property that

$$E[|\tilde{M}_T^n - M_{T \wedge \sigma}|^2] = E[|M_{T \wedge \sigma}^n - M_{T \wedge \sigma}|^2] \leq E[|M_T^n - M_T|^2] \text{ for all } n, T \in \mathbb{N}.$$

On the other hand there exists  $\tilde{M} \in \mathfrak{Mart}_2(\mathcal{F}_+)$  such that  $M_{t \wedge \sigma} = \tilde{M}_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. by Lemma 15.17. It follows that  $\tilde{M}. \in \overline{\mathcal{M}}$  and hence  $\overline{\mathcal{M}}$  is  $\mathcal{F}_+$ -stable.  $\square$

**20.11 Lemma.** *Suppose that  $\mathcal{M}$  is an  $\mathcal{F}_+$ -stable and  $L_2$ -closed subset of  $\mathfrak{Mart}_2(\mathcal{F})$ . Then  $\{[M_\sigma]; M. \in \mathcal{M}\}$  is a closed subset of  $L_2(\mathcal{F}, P)$  for all bounded  $\mathcal{F}_+$ -stopping times  $\sigma$ .*

*Proof.* Suppose that  $Y$  is a square integrable random variable,  $M^n$  is a sequence of  $\mathcal{M}$ ,  $\sigma$  is an  $\mathcal{F}_+$ -stopping time and  $E[|Y - M_\sigma^n|^2]$  converges to 0 as  $n$  tends to  $\infty$ . Since  $\mathcal{M}$  is stable, there exists a sequence  $\tilde{M}^n$  of  $\mathcal{M}$  such that  $\tilde{M}^n$  is indistinguishable from  $M_{\wedge\sigma}^n$  for all  $n \in \mathbb{N}$ . We see by the submartingale property that

$$E[|\tilde{M}_t^m - \tilde{M}_t^n|^2] = E[|M_{t \wedge \sigma}^m - M_{t \wedge \sigma}^n|^2] \leq E[|M_\sigma^m - M_\sigma^n|^2] \text{ for all } m, n \in \mathbb{N} \text{ and } t \in \mathbb{R}_{\geq 0}$$

Since  $\mathcal{M}$  is  $L_2$ -closed, there exists  $M. \in \mathcal{M}$  such that  $E[|M_t - \tilde{M}_t^n|^2]$  converges to 0 for all  $t \in \mathbb{R}_{\geq 0}$ . We have that  $\tilde{M}_\sigma^n = M_\sigma^n$  a.s. for all  $n \in \mathbb{N}$ . It follows by Example 6.4 that

$$E[|M_\sigma - M_\sigma^n|^2] = E[|M_\sigma - \tilde{M}_\sigma^n|^2] \leq E[\sup_{t \leq T} |M_t - \tilde{M}_t^n|^2] \leq 4E[|M_T - \tilde{M}_T^n|^2]$$

where  $T \in \mathbb{R}_{> 0}$  satisfies  $\sigma \leq T$  a.s. Thus we get  $Y \in [M_\sigma]$ .  $\square$

**20.12 Theorem.** *Let  $\mathcal{M}$  be an  $\mathcal{F}_+$ -stable linear subspace of  $\mathfrak{Mart}_2(\mathcal{F}_+)$ . Then*

$$\{[M_T]; M. \in \overline{\mathcal{M}}\} + \{[N_T]; N. \in \mathcal{M}^\perp\} = L_2(\mathcal{F}_{T+}, P) \text{ for all } T \in \mathbb{R}_{\geq 0}.$$

where  $\overline{\mathcal{M}}$  the closure of  $\mathcal{M}$ ,  $\mathcal{M}^\perp := \{N \in \mathfrak{Mart}_2(\mathcal{F}_+) : E[M_T N_T] = 0 \forall M \in \mathcal{M} \forall T \in \mathbb{N}\}$ .

*Proof.* We see that  $\{[M_T]; M. \in \overline{\mathcal{M}}\}$  is an closed linear subspace of  $L_2(\mathcal{F}_{T+}, P)$  by invoking Lemma 20.11 and Lemma 20.10. On the other hand  $E[M_T Y] = 0 \forall M. \in \mathcal{M}$  implies the stronger condition  $E[M_T Y] = 0 \forall M. \in \overline{\mathcal{M}}$ . Therefore it suffices to show that

$$\{Y \in \mathcal{L}_2(\mathcal{F}_{T+}, P) : E[M_T Y] = 0 \forall M. \in \mathcal{M}\} = \{N_T; N. \in \mathcal{M}^\perp\}.$$

Suppose that  $Y \in \mathcal{L}_2(\mathcal{F}_{T+}, P)$  and  $E[M_T Y] = 0$  for all  $M. \in \mathcal{M}$ . Corollary 14.12 shows the existence of  $N. \in \mathfrak{Mart}_2(\mathcal{F}_+)$  such that  $N_t \in E[Y | \mathcal{F}_{t+}]$  for all  $t \in \mathbb{R}_{\geq 0}$ . Since  $Y$  is  $\mathcal{F}_{T+}$ -measurable, we may assume that  $N_t = Y$  for all  $t \in \mathbb{R}_{\geq T}$ . In particular  $N_T = Y$ . We shall verify that  $N. \in \mathcal{M}^\perp$ . Indeed

$$E[M_t N_t] = E[M_t Y] = E[M_T Y] = 0 \text{ for all } M. \in \mathcal{M} \text{ and } t \in \mathbb{R}_{\geq T}.$$

According to Lemma 20.6 this implies that  $t \mapsto M_{t \wedge T} N_{t \wedge T}$  is an  $\mathcal{F}_+$ -martingale. Hence

$$E[M_t N_t] = E[M_T N_T] = 0 \text{ for all } M. \in \mathcal{M} \text{ and } t \in \mathbb{R}_{[0, T]}.$$

Consequently  $N. \in \mathcal{M}^\perp$  and  $Y \in [N_T]$ . We thus get the relation  $\subset$ . The converse relation derives by Lemma 20.9(i).  $\square$

**20.13 Theorem.** Suppose that  $\mathcal{M}$  is an  $\mathcal{F}_{\cdot+}$ -stable linear subspace of  $\mathfrak{Mart}_2(\mathcal{F}_{\cdot+})$ . Then

$\overline{\mathcal{M}} + \mathcal{M}^\perp = \mathfrak{Mart}_2(\mathcal{F}_{\cdot+})$  and  $t \mapsto M_t N_t$  is an  $\mathcal{F}_{\cdot+}$ -martingale for all  $M \in \overline{\mathcal{M}}, N \in \mathcal{M}^\perp$  where  $\overline{\mathcal{M}}$  the closure of  $\mathcal{M}$ ,  $\mathcal{M}^\perp := \{N \in \mathfrak{Mart}_2(\mathcal{F}_{\cdot+}) : E[M_T N_T] = 0 \forall M \in \mathcal{M} \forall T \in \mathbb{N}\}$ .

*Proof.* Let  $L \in \mathfrak{Mart}_2(\mathcal{F}_{\cdot+})$ . Since

$$\mathcal{M}^\perp = \{N \in \mathfrak{Mart}_2(\mathcal{F}_{\cdot+}) : E[M_T N_T] = 0 \forall M \in \overline{\mathcal{M}} \forall T \in \mathbb{N}\},$$

there exist a sequence  $\mathbb{N} \rightarrow \overline{\mathcal{M}}, T \mapsto M^T$  and a sequence  $\mathbb{N} \rightarrow \mathcal{M}^\perp, T \mapsto N^T$  such that

$$L_T = M_T^T + N_T^T \text{ a.s. for all } T \in \mathbb{N}$$

by Theorem 20.12. We see by Lemma 20.9(i) that  $t \mapsto (M_t^m - M_t^n)(N_t^n - N_t^m)$  is an  $\mathcal{F}_{\cdot+}$ -martingale with vanishing expectation with the help of Lemma 20.10. In particular

$$(\star) \quad E[(M_{t \wedge T}^m - M_{t \wedge T}^n)(N_{t \wedge T}^n - N_{t \wedge T}^m)] = 0 \text{ for all } m, n, T \in \mathbb{N} \text{ and } t \in \mathbb{R}_{\geq 0}.$$

Since  $L_t \in E[L_T | \mathcal{F}_{t+}]$ ,  $M_t^T \in E[M_T^T | \mathcal{F}_{t+}]$  and  $N_t^T \in E[N_T^T | \mathcal{F}_{t+}]$  for all  $t \in \mathbb{R}_{[0, T]}$ ,

$$L_t = M_t^T + N_t^T \text{ for all } t \in \mathbb{R}_{[0, T]} \text{ a.s. for all } T \in \mathbb{N}.$$

Let  $T \in \mathbb{N}$ . It then follows that

$$M_{t \wedge T}^m + N_{t \wedge T}^m = M_{t \wedge T}^n + N_{t \wedge T}^n \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s. for all } m, n \in \mathbb{N}_{\geq T}.$$

Two processes  $t \mapsto M_{t \wedge T}^m - M_{t \wedge T}^n$  and  $t \mapsto N_{t \wedge T}^n - N_{t \wedge T}^m$  being indistinguishable, we get

$$E[|M_t^m - M_t^n|^2] = 0 \text{ and } E[|N_t^n - N_t^m|^2] = 0 \text{ for all } t \in \mathbb{R}_{[0, T]} \text{ and } m, n \in \mathbb{N}_{\geq T}$$

by  $(\star)$ . Since  $\overline{\mathcal{M}}$  is  $L_2$ -closed, this implies that there exists  $M \in \overline{\mathcal{M}}$  such that

$$E[|M_t - M_t^T|^2] = 0 \text{ for all } t \in \mathbb{R}_{[0, T]} \text{ and } T \in \mathbb{N}.$$

We have that  $M_t = M_t^T$  for all  $t \in \mathbb{R}_{[0, T]}$  a.s. for all  $T \in \mathbb{N}$ . Similarly we deduce the existence of  $N \in \mathcal{M}^\perp$  such that  $N_t = N_t^T$  for all  $t \in \mathbb{R}_{[0, T]}$  a.s. for all  $T \in \mathbb{N}$ . Thus we reach that  $L_t = M_t + N_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s.  $\square$

**20.14 Example.** (i)  $\mathfrak{Mart}_2^c(\mathcal{F}) + \overline{\mathfrak{Mart}_2^{\text{fv}}(\mathcal{F}_{\cdot+})} = \mathfrak{Mart}_2(\mathcal{F}_{\cdot+})$  where  $\overline{\quad}$  denotes the closure. If  $M \in \mathfrak{Mart}_2^c(\mathcal{F})$  and  $N \in \overline{\mathfrak{Mart}_2^{\text{fv}}(\mathcal{F}_{\cdot+})}$  then  $t \mapsto M_t N_t$  is an  $\mathcal{F}_{\cdot+}$ -martingale.

(ii)  $\{M_T; M \in \mathfrak{Mart}_2^c(\mathcal{F})\} + \{N_T; N \in \overline{\mathfrak{Mart}_2^{\text{fv}}(\mathcal{F}_{\cdot+})}\} = \mathcal{L}_2(\mathcal{F}_{T+})$  for all  $T \in \mathbb{R}_{\geq 0}$ .

*Proof.* According to Lemma 20.9(i) and Theorem 19.6, we have that

$$\begin{aligned} & \{N \in \mathfrak{Mart}_2(\mathcal{F}_{\cdot+}) : E[M_T N_T] = 0 \text{ for all } M \in \mathfrak{Mart}_2^{\text{fv}}(\mathcal{F}_{\cdot+}) \text{ and } T \in \mathbb{N}\} \\ &= \{N \in \mathfrak{Mart}_2(\mathcal{F}_{\cdot+}) : N_0 = 0 \text{ a.s., } \exists \tilde{N} \in \mathfrak{Mart}_2^c(\mathcal{F}) \text{ s.t. } N_t = \tilde{N}_t \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}\}. \end{aligned}$$

On the other hand  $\mathfrak{Mart}_2^{\text{fv}}(\mathcal{F}_{\cdot+})$  is an  $\mathcal{F}_{\cdot+}$ -stable linear subspace of  $\mathfrak{Mart}_2(\mathcal{F}_{\cdot+})$  by Example 20.4(ii). Thus we get (i) by Theorem 20.13 and (ii) by Theorem 20.12. Note that

$$\{M_T; M \in \mathfrak{Mart}_2^c(\mathcal{F})\} \cap \{N_T; N \in \overline{\mathfrak{Mart}_2^{\text{fv}}(\mathcal{F}_{\cdot+})}\} = \mathcal{L}_2(\mathcal{F}_{0+}) \text{ for all } T \in \mathbb{R}_{\geq 0}$$

and  $\mathfrak{Mart}_2^c(\mathcal{F}) \cap \overline{\mathfrak{Mart}_2^{\text{fv}}(\mathcal{F}_{\cdot+})} \cap \{M_0 = 0 \text{ a.s.}\}$  consists of evanescent processes.  $\square$

## 21 General square integrable martingale

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\mathcal{F}$  be a filtration of sub  $\sigma$ -fields.

$\mathfrak{Mart}_2(\mathcal{F})$  stands for the space of all square integrable  $\mathcal{F}$ -martingales with almost sure right continuous sample path.

$$\mathfrak{Mart}_2^{\text{fv}}(\mathcal{F}) := \{M \in \mathfrak{Mart}_2(\mathcal{F}) : t \mapsto M_t \text{ of finite variation a.s.}\}$$

$$\mathfrak{Mart}_2^{\text{c}}(\mathcal{F}) := \{M \in \mathfrak{Mart}_2(\mathcal{F}) : t \mapsto M_t \text{ continuous a.s.}\}$$

**21.1 Lemma.** *Let  $f_n : T \rightarrow \mathbb{R}$  be a sequence of functions such that  $f_n(t) \geq 0$  for all  $t \in T$  and  $\sum_{t \in T} f_n(t) := \sup\{\sum_{t \in F} f_n(t) ; F \text{ finite subsets of } T\} < +\infty$  for all  $n \in \mathbb{N}$ . Then*

$$\sum_{t \in T} \liminf_{n \rightarrow \infty} f_n(t) \leq \liminf_{n \rightarrow \infty} \sum_{t \in T} f_n(t).$$

*Proof.* We see that  $D := \bigcup_{n=1}^{\infty} \{t \in T : f_n(t) > 0\}$  is an countable set. If  $t \notin D$  then  $f_n(t) = 0$  for all  $n \in \mathbb{N}$  and hence  $\liminf_{n \rightarrow \infty} f_n(t) = 0$ . It follows that

$$\sum_{t \in T} \liminf_{n \rightarrow \infty} f_n(t) = \sum_{t \in D} \liminf_{n \rightarrow \infty} f_n(t) \leq \liminf_{n \rightarrow \infty} \sum_{t \in D} f_n(t) = \liminf_{n \rightarrow \infty} \sum_{t \in T} f_n(t)$$

where we applied Fatou's lemma to deduce the inequality. □

**21.2 Theorem.** *Let  $M \in \overline{\mathfrak{Mart}_2^{\text{fv}}(\mathcal{F})}$ . Then there exists an  $\mathcal{F}$ -increasing process  $Q$  such that  $t \mapsto |M_t|^2 - Q_t$  is an  $\mathcal{F}$ -martingale and it is indistinguishable from  $t \mapsto \text{jmpv}(M, M)_t$ .*

*Proof.* We select and fix a sequence  $M^n \in \mathfrak{Mart}_2^{\text{fv}}(\mathcal{F})$  such that  $E[|M_T^n - M_T|^2]$  converges to 0 for all  $T \in \mathbb{N}$ . The argument of Lemma 20.7 shows that there exist  $\phi \in \text{Seq}(\mathbb{N}, \uparrow)$  and  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and for each  $\omega \in \Omega_0$  the following holds:

- $t \mapsto M_t^n(\omega)$  is right continuous and of finite variation for all  $n \in \mathbb{N}$ ,
- $t \mapsto M_t(\omega)$  is right continuous and admits left-hand limits everywhere, and
- $\sup_{t:t \leq T} |M_t^{\phi(n)}(\omega) - M_t(\omega)|$  converges to 0 for all  $T \in \mathbb{N}$ .

In what follows we use the notation  $M^n$  in place of  $M^{\phi(n)}$ . For each  $n \in \mathbb{N}$  introduce

$$Q^n := t \mapsto R\{M^n dM^n\}_t - R[M^n dM^n]_t,$$

which is an  $\mathcal{F}$ -finite variation process. Observe that

$$Q_t^n(\omega) = \sum_{s:s \leq t} |M_s^n(\omega) - M_{s-}^n(\omega)|^2 \text{ for all } \omega \in \Omega_0 \text{ and } t \in \mathbb{R}_{\geq 0}.$$

Since  $t \mapsto |M_t^n|^2 - Q_t^n$  is an  $\mathcal{F}$ -martingale by Corollary 16.9(iv), it follows that

$$(\star) \quad E[\liminf_{n \rightarrow \infty} Q_t^n] \leq \liminf_{n \rightarrow \infty} E[Q_t^n] = \liminf_{n \rightarrow \infty} E[|M_t^n|^2] = E[|M_t|^2]$$

by Fatou's lemma. Let  $\omega \in \Omega_0$  and  $T \in \mathbb{N}$ . Then we have that

$$|M_{s-}^n(\omega) - M_{s-}(\omega)| \leq \sup_{t:t < T} |M_t^n(\omega) - M_t(\omega)| \text{ for all } s \in \mathbb{R}_{(0,T]} \text{ and } n \in \mathbb{N},$$

and hence  $|M_s(\omega) - M_{s-}(\omega)|^2 \leq \liminf_{n \rightarrow \infty} |M_s^n(\omega) - M_{s-}^n(\omega)|^2$ . Therefore

$$(\star\star) \quad \sum_{s:s \leq t} |M_s(\omega) - M_{s-}(\omega)|^2 \leq \liminf_{n \rightarrow \infty} \sum_{s:s \leq t} |M_s^n(\omega) - M_{s-}^n(\omega)|^2 = \liminf_{n \rightarrow \infty} Q_t^n(\omega)$$

for all  $t \in \mathbb{R}_{\geq 0}$  by Lemma 21.1. Since  $\liminf_{n \rightarrow \infty} Q_t^n$  is integrable, we may assume that

$$\text{jmpv}(M, M)_t(\omega) = \sum_{s:s \leq t} |M_s(\omega) - M_{s-}(\omega)|^2 < +\infty \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ and } \omega \in \Omega_0$$

with suitable shrink for  $\Omega_0$ . Let  $n \in \mathbb{N}$  and write  $f_s := M_s - M_s^n - (M_{s-} - M_{s-}^n)$ . Then

$$\sum_{s:s \leq t} |M_s(\omega) - M_{s-}(\omega)|^2 - Q_t^n(\omega) = \sum_{s:s \leq t} f_s(\omega)(M_s(\omega) - M_{s-}(\omega) + M_s^n(\omega) - M_{s-}^n(\omega))$$

for all  $t \in \mathbb{R}_{\geq 0}$  and  $\omega \in \Omega_0$ . The absolute value of the right hand side is dominated by

$$\sqrt{\text{jmpv}(M - M^n, M - M^n)_t(\omega)}(\sqrt{\text{jmpv}(M, M)_t(\omega)} + \sqrt{Q_t^n(\omega)})$$

as Schwarz inequality shows. According to  $(\star\star)$ , the above is further dominated by

$$g_t^n(\omega) := \sqrt{\liminf_{m \rightarrow \infty} Q_t^{nm}(\omega)}(\sqrt{\liminf_{m \rightarrow \infty} Q_t^m(\omega)} + \sqrt{Q_t^n(\omega)})$$

where  $Q_t^{nm} := t \mapsto R\{NdN\}_t - R[NdN]_t$  with  $N_t := M_t^m - M_t^n$ . Consequently

$$|\text{jmpv}(M, M)_t(\omega) - Q_t^n(\omega)| \leq g_t^n(\omega) \text{ for all } t \in \mathbb{R}_{\geq 0}, \omega \in \Omega_0 \text{ and } n \in \mathbb{N}.$$

Let  $t \in \mathbb{R}_{\geq 0}$ . Invoking Schwarz inequality we get

$$E[g_t^n] \leq \sqrt{E[\liminf_{m \rightarrow \infty} Q_t^{nm}]}(\sqrt{E[\liminf_{m \rightarrow \infty} Q_t^m]} + \sqrt{E[Q_t^n]}) \text{ for all } n \in \mathbb{N}.$$

Due to  $(\star)$ , the right hand side is dominated by

$$\sqrt{E[|M_t - M_t^n|^2]}(\sqrt{E[|M_t|^2]} + \sqrt{E[|M_t^n|^2]}),$$

which converges to 0. Note that  $t \mapsto g_t^n$  is an  $\mathcal{F}$ -increasing process. Thus, choosing a suitable subsequence and a suitable subset, we may realize an almost sure convergence

$$\limsup_{n \rightarrow \infty} g_t^n(\omega) = 0 \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ and } \omega \in \Omega_0.$$

Then  $t \mapsto \liminf_{n \rightarrow \infty} Q_t^n$  is an  $\mathcal{F}$ -increasing process. Indeed it is  $\mathcal{F}$ -adapted and

$$Q_t(\omega) := \liminf_{n \rightarrow \infty} Q_t^n(\omega) = \text{jmpv}(M, M)_t(\omega) \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ and } \omega \in \Omega_0.$$

This also implies that  $|Q_t(\omega) - Q_t^n(\omega)| \leq g_t^n(\omega)$  for all  $t \in \mathbb{R}_{\geq 0}$ ,  $\omega \in \Omega_0$  and  $n \in \mathbb{N}$ . Let  $t \in \mathbb{R}_{\geq 0}$ . As we have seen  $g_t^n$  converges to 0 in  $L^1$  and hence  $Q_t^n$  converges to  $Q_t$  in  $L^1$ . Since  $t \mapsto |M_t^n|^2 - Q_t^n$  is an  $\mathcal{F}$ -martingale, it follows that  $t \mapsto |M_t|^2 - Q_t$  is an  $\mathcal{F}$ -martingale.  $\square$

**21.3 Corollary.** *Let  $M, N \in \overline{\text{Mart}}_2^{\text{fv}}(\mathcal{F})$ .*

(i)  $\text{var}(\text{jmpv}(M, N))_t \leq \sqrt{\text{jmpv}(M, M)_t \text{jmpv}(N, N)_t}$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s.

(ii) *There exists an  $\mathcal{F}$ -finite variation process  $Q$  such that  $t \mapsto M_t N_t - Q_t$  is an  $\mathcal{F}$ -martingale and it is indistinguishable from  $t \mapsto \text{jmpv}(M, N)_t$ .*

*Proof.* The existence derives by Theorem 21.2 with the help of polarization trick.  $\square$

Given  $X. \in \mathfrak{Mart}_2(\mathcal{F}.)$  we denote the following set by  $\text{Mart}^c[X.; \mathcal{F}.]$ :

$$\{M. \in \mathfrak{Mart}_2^c(\mathcal{F}.): M_0 = 0 \text{ a.s.}, t \mapsto X_t - M_t \in \overline{\mathfrak{Mart}_2^{\text{fv}}(\mathcal{F}_{.+})}\}.$$

The requirement reads *not*  $\in \overline{\mathfrak{Mart}_2^{\text{fv}}(\mathcal{F}.)}$  but  $\in \overline{\mathfrak{Mart}_2^{\text{fv}}(\mathcal{F}_{.+})}$ . Since  $\mathfrak{Mart}_2(\mathcal{F}.) \subset \mathfrak{Mart}_2(\mathcal{F}_{.+})$ ,  $\text{Mart}^c[X.; \mathcal{F}.] \neq \emptyset$  for all  $X. \in \mathfrak{Mart}_2(\mathcal{F}.)$  according to Example 20.14.

Given  $X., Y. \in \mathfrak{Mart}_2(\mathcal{F}.)$ , we introduce the following spaces:

$$[X.; \mathcal{F}.] := \{A. : \mathcal{F}. \text{-increasing, } t \mapsto |X_t|^2 - A_t \text{ } \mathcal{F}. \text{-martingale,}$$

$$A_t - A_{t-} = |X_t - X_{t-}|^2 \forall t \in \mathbb{R}_{\geq 0} \text{ a.s.}\}$$

$$[X, Y; \mathcal{F}.] := \{A. : \mathcal{F}. \text{-finite variation, } A_0 = 0 \text{ a.s.}, t \mapsto X_t Y_t - A_t \text{ } \mathcal{F}. \text{-martingale,}$$

$$A_t - A_{t-} = (X_t - X_{t-})(Y_t - Y_{t-}) \forall t \in \mathbb{R}_{\geq 0} \text{ a.s.}\}.$$

**21.4 Theorem.** *Suppose that  $X., Y. \in \mathfrak{Mart}_2(\mathcal{F}.)$ . Then  $[X, Y; \mathcal{F}_{.+}] \neq \emptyset$ .*

- (i) *If  $M. \in \text{Mart}^c[X.; \mathcal{F}.]$ ,  $N. \in \text{Mart}^c[Y.; \mathcal{F}.]$  and  $C. \in \text{Crv}[M, N; \mathcal{F}.]$  then  $Q. \in [X, Y; \mathcal{F}_{.+}]$  is indistinguishable from  $t \mapsto C_t + \text{jmpv}(X - M, Y - N)_t$ . In particular  $[X, X; \mathcal{F}_{.+}] = [X; \mathcal{F}_{.+}]$ .*
- (ii) *If  $Q. \in [X, Y; \mathcal{F}_{.+}]$ ,  $A. \in [X; \mathcal{F}_{.+}]$  and  $B. \in [Y; \mathcal{F}_{.+}]$  then*

$$(\text{var}(Q.)_t - \text{var}(Q.)_s)^2 \leq (A_t - A_s)(B_t - B_s) \text{ for all } t, s \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

*Proof.* (i) We see that  $M_t N_t - C_t$  is an  $\mathcal{F}_{.+}$ -martingale. According to Example 20.14 both  $t \mapsto M_t(Y_t - N_t)$  and  $t \mapsto N_t(X_t - M_t)$  are  $\mathcal{F}_{.+}$ -martingales. Corollary 21.3 shows the existence of  $\mathcal{F}_{.+}$ -finite variation process  $D.$  such that

$$t \mapsto (X_t - M_t)(Y_t - N_t) - D_t \text{ is an } \mathcal{F}_{.+} \text{-martingale and}$$

$$D_t = \text{jmpv}(X - M, Y - N)_t \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

Since sample paths of  $M$  and  $N$  are continuous a.s.,  $D_t - D_{t-} = (X_t - X_{t-})(Y_t - Y_{t-})$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. Define  $Q_t := C_t + D_t$ . Then it follows that

$$t \mapsto X_t Y_t - Q_t \text{ is an } \mathcal{F}_{.+} \text{-martingale and}$$

$$Q_t - Q_{t-} = D_t - D_{t-} = (X_t - X_{t-})(Y_t - Y_{t-}) \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

Suppose that  $R.$  shares the same property as  $Q.$ . Their difference is continuous a.s. because

$$Q_t - Q_{t-} = (X_t - X_{t-})(Y_t - Y_{t-}) = R_t - R_{t-} \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

The difference being an  $\mathcal{F}_{.+}$ -martingale with almost sure continuous and of finite variation sample path,  $Q_t - R_t = Q_0 - R_0$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. by Corollary 8.17.

- (ii) Let  $c \in \mathbb{R}$ . We see by the bilinearity of the condition that

$$c(cA + Q) + cQ + B \in [cX + Y, cX + Y; \mathcal{F}_{.+}], \text{ i.e., } c^2 A + 2cQ + B \in [cM + N; \mathcal{F}_{.+}].$$

There exists  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) =$  and

$$t \mapsto c^2 A_t(\omega) + 2cQ_t(\omega) + B_t(\omega) \text{ is finite valued, right continuous and non-decreasing}$$

for all  $c \in \mathbb{Q}$  and  $\omega \in \Omega_0$ . Since  $c \in \mathbb{Q}$  is arbitrary, we infer that

$$(Q_t(\omega) - Q_s(\omega))^2 \leq (A_t(\omega) - A_s(\omega))(B_t(\omega) - B_s(\omega)) \text{ for all } t, s \in \mathbb{R}_{\geq 0} \text{ and } \omega \in \Omega_0.$$

Let  $t, s \in \mathbb{R}_{\geq 0}$  with  $s < t$  and  $\omega \in \Omega_0$ . The Schwarz inequality shows

$$\left( \sum_{J \in \Delta} |dQ(J; \omega)| \right)^2 \leq \left( \sum_{J \in \Delta} \sqrt{dA(J; \omega)} \sqrt{dB(J; \omega)} \right)^2 \leq \left( \sum_{J \in \Delta} dA(J; \omega) \right) \left( \sum_{J \in \Delta} dB(J; \omega) \right).$$

for any finite partition  $\Delta$  of  $\mathbb{R}_{(s,t]}$ . The right hand side equals  $(A_t(\omega) - A_s(\omega))(B_t(\omega) - B_s(\omega))$  and this dominates  $(\text{var}(Q.)_t(\omega) - \text{var}(Q.)_s(\omega))^2$ .  $\square$

**21.5 Lemma.** *Suppose that  $\mathcal{C}$  is a pre  $\sigma$ -field on a set  $S$  and  $m, \mu, \nu$  are finite measures on  $\sigma(\mathcal{C})$ . If  $m(J)^2 \leq \mu(J)\nu(J)$  for all  $J \in \mathcal{C}$  then*

$$\left( \int_S fg m \right)^2 \leq \int_S f^2 \mu \int_S g^2 \nu$$

for all non-negative  $\sigma(\mathcal{C})$ -measurable functions  $f, g : S \rightarrow \overline{\mathbb{R}}$ . Here we interpret  $0\infty = 0$ .

*Proof.* Let  $A \in \sigma(\mathcal{C})_{\neq \emptyset}$ . Given  $\varepsilon \in \mathbb{R}_{>0}$ , there exist countable disjoint  $\mathcal{C}$ -coverings  $\Phi(1)$  and  $\Phi(2)$  of  $A$  such that  $\sum_{J \in \Phi(1)} \mu(J) < \mu(A) + \varepsilon$  and  $\sum_{J \in \Phi(2)} \nu(J) < \nu(A) + \varepsilon$ . We set

$$\Delta := \{J \cap K; (J, K) \in \Phi(1) \times \Phi(2), J \cap K \neq \emptyset\}.$$

Then  $\Delta$  is a countable disjoint  $\mathcal{C}$ -covering of  $A$  and

$$\sum_{I \in \Delta} \mu(I) = \sum_{J \in \Phi(1)} \sum_{K \in \Phi(2)} \mu(J \cap K) \leq \sum_{J \in \Phi(1)} \mu(J) \leq \mu(A) + \varepsilon.$$

We also have that  $\sum_{I \in \Delta} \nu(I) \leq \nu(A) + \varepsilon$ . Invoking the Schwarz inequality we see that

$$m(A)^2 \leq \left( \sum_{I \in \Delta} m(I) \right)^2 \leq \left( \sum_{I \in \Delta} \sqrt{\mu(I)} \sqrt{\nu(I)} \right)^2 \leq \left( \sum_{I \in \Delta} \mu(I) \right) \left( \sum_{I \in \Delta} \nu(I) \right)$$

The right hand side is dominated by  $(\mu(A) + \varepsilon)(\nu(A) + \varepsilon)$ . Consequently

$$m(A)^2 \leq \mu(A)\nu(A) \text{ for all } A \in \sigma(\mathcal{C}).$$

We next prove the inequality for simple functions. Suppose that  $f, g$  are non-negative  $\sigma(\mathcal{C})$ -simple functions. Write  $F := \text{Image } f, G := \text{Image } g$  and  $E(y, z) := f^{-1}(\{y\}) \cap g^{-1}(\{z\})$  for  $(y, z) \in F \times G$  to save the space. It follows that

$$\int_S fg m = \sum_{y \in F, z \in G} yz m(E(y, z)) \leq \sum_{y \in F, z \in G} \sqrt{y^2 \mu(E(y, z))} \sqrt{z^2 \nu(E(y, z))}.$$

The Schwarz inequality shows that the right hand side is dominated by the square root of

$$\sum_{y \in F, z \in G} y^2 \mu(E(y, z)) \sum_{y \in F, z \in G} z^2 \nu(E(y, z)) = \int_S f^2 \mu \int_X g^2 \nu.$$

Thus the inequality in question is shown for non-negative  $\sigma(\mathcal{C})$ -simple functions. Finally general non-negative  $\sigma(\mathcal{C})$ -measurable functions are monotone increasing limits of non-negative  $\sigma(\mathcal{C})$ -simple functions.  $\square$

**21.6 Theorem.** Suppose that  $X, Y \in \mathfrak{Mart}_2(\mathcal{F})$ ,  $Q \in [X, Y; \mathcal{F}_+]$ ,  $A \in [X; \mathcal{F}_+]$  and  $B \in [X; \mathcal{F}_+]$ . Then there exists  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and the following holds on  $\Omega_0$ :

$$\int_{(0,+\infty)} f(s)g(s) \, d\text{var}(Q)_s \leq \left( \int_{(0,+\infty)} f(s)^2 \, dA_s \right)^{1/2} \left( \int_{(0,+\infty)} g(s)^2 \, dB_s \right)^{1/2}$$

for all non-negative Borel measurable functions  $f, g : \mathbb{R}_{>0} \rightarrow \overline{\mathbb{R}}$ .

*Proof.* Combine Corollary 21.4 and Lemma 21.5. □

**21.7 Lemma.** Suppose that  $M$  and  $N$  are square integrable  $\mathcal{F}$ -martingales with stochastic right continuity. Then there exists a natural  $\mathcal{F}_+$ -finite variation process  $A$  such that  $A_0 = 0$  a.s. and  $t \mapsto M_t N_t - A_t$  is an  $\mathcal{F}_+$ -martingale.

*Proof.* Clearly  $t \mapsto M_t + N_t$  as well as  $t \mapsto M_t - N_t$  are right continuous in probability and square integrable  $\mathcal{F}$ -martingales. So their squares  $t \mapsto |M_t + N_t|^2$  and  $t \mapsto |M_t - N_t|^2$  are  $\mathcal{F}$ -submartingales, which are of class DL over  $\mathbb{Q}$  by Lemma 7.12. We select and fix  $B \in \text{DM}[t \mapsto |M_t + N_t|^2, \mathcal{F}_+]$  and  $C \in \text{DM}[t \mapsto |M_t - N_t|^2, \mathcal{F}_+]$ . Since

$$4M_t N_t - B_t + C_t = (M_t + N_t)^2 - B_t - (M_t - N_t)^2 + C_t,$$

we see that  $t \mapsto M_t N_t - (B_t - C_t)/4$  is an  $\mathcal{F}_+$ -martingale. Moreover  $t \mapsto B_t - C_t$  is a natural  $\mathcal{F}_+$ -finite variation process by Lemma 15.15. □

Given square integrable  $\mathcal{F}$ -martingales  $M$  and  $N$  with stochastic right continuity,

$$\langle M, N; \mathcal{F} \rangle := \{A : \text{natural } \mathcal{F}\text{-finite variation process, } A_0 = 0 \text{ a.s.,} \\ t \mapsto M_t N_t - A_t \text{ } \mathcal{F}\text{-martingale with stochastic right continuity}\}.$$

We also introduce  $\langle M; \mathcal{F} \rangle := \text{DM}[t \mapsto |M_t|^2, \mathcal{F}]$ .

*21.8 Remark.* Suppose that  $X, Y \in \mathfrak{Mart}_2(\mathcal{F}_+)$ . If  $Q \in [X, Y; \mathcal{F}_+]$  and  $A \in \langle X, Y; \mathcal{F}_+ \rangle$  then  $A$  is an  $\mathcal{F}_+$ -natural projection of  $Q$ .

**21.9 Lemma.** Suppose that  $M$  and  $N$  are square integrable  $\mathcal{F}$ -martingales with stochastic right continuity. Here  $\langle \dots \rangle := \langle \dots; \mathcal{F} \rangle \subset \langle \dots; \mathcal{F}_+ \rangle$ .

(i)  $\langle M, N; \mathcal{F}_+ \rangle \neq \emptyset$ .  $\langle M, N \rangle = \langle N, M \rangle$ .  $\langle M, M \rangle = \langle M \rangle = \langle M - X \rangle$  where  $X \in \mathcal{L}_2(\mathcal{F}_0)$ . If  $M_t = M_0$  a.s.  $\forall t \in \mathbb{R}_{\geq 0}$  then  $0 \in \langle M, N \rangle$ . If  $0 \in \langle M \rangle$  then  $M_t = M_0$  a.s.  $\forall t \in \mathbb{R}_{\geq 0}$ .

(ii) If  $A \in \langle M, N \rangle$ ,  $B$  is an  $\mathcal{F}$ -adapted process with a.s.-right continuous path and  $A_t = B_t$  a.s.  $\forall t \in \mathbb{R}_{\geq 0}$  then  $B \in \langle M, N \rangle$ . If  $A, B \in \langle M, N \rangle$  then  $A_t = B_t \forall t \in \mathbb{R}_{\geq 0}$  a.s.

*Proof.* (i) Lemma 21.7 shows  $\langle M, N; \mathcal{F}_+ \rangle \neq \emptyset$ . Clearly  $\langle M, N \rangle = \langle N, M \rangle$ . The relation  $\langle M \rangle \subset \langle M, M \rangle$  is obvious. We note that  $\langle M; \mathcal{F}_+ \rangle \cup \langle M, M \rangle \subset \langle M, M; \mathcal{F}_+ \rangle$ . Since  $\langle M; \mathcal{F}_+ \rangle \neq \emptyset$ , each  $A$  is indistinguishable from an element in  $\langle M; \mathcal{F}_+ \rangle$  by Theorem 15.19, that is,  $A$  is an increasing process and hence  $A \in \langle M \rangle$ . Since  $t \mapsto 2M_t X - X^2$  is an  $\mathcal{F}$ -martingale, Theorem 18.7(v) shows that  $\text{DM}[(M - X)^2, \mathcal{F}] = \text{DM}[M^2, \mathcal{F}]$ . The last claim is by Theorem 18.7(vi). (ii) The second claim is by Theorem 15.19. □

**21.10 Lemma.** Suppose  $M, N$  and  $L$  are square integrable  $\mathcal{F}$ -martingales with stochastic right continuity. Here  $\langle \dots \rangle := \langle \dots; \mathcal{F} \rangle \subset \langle \dots; \mathcal{F}_+ \rangle$ .

- (iii) If  $A \in \langle M, L \rangle$ ,  $B \in \langle N, L \rangle$  and  $c \in \mathbb{R}$  then  $A + B \in \langle M + N, L \rangle$  and  $cA \in \langle cM, N \rangle$ .
- (iv) If  $\langle M, L \rangle \cap \langle N, L' \rangle \neq \emptyset$  then  $\langle M, L \rangle = \langle N, L' \rangle$ . If  $0 \in \langle M - N, L \rangle$  then  $\langle M, L \rangle = \langle N, L \rangle$ . If  $\langle M, L; \mathcal{F}_+ \rangle \cap \langle N, L; \mathcal{F}_+ \rangle \neq \emptyset$  then  $0 \in \langle M - N, L \rangle$ .
- (v) If  $\langle M, L; \mathcal{F}_+ \rangle \cap \langle N, L; \mathcal{F}_+ \rangle \neq \emptyset$  for all  $L$  then  $M_t - M_0 = N_t - N_0$  a.s. for all  $t \in \mathbb{R}_{\geq 0}$ .
- (vi) If  $A \in \langle M, N \rangle$ ,  $B \in \langle M \rangle$  and  $C \in \langle N \rangle$  then

$$(\text{var}(A)_t - \text{var}(A)_s)^2 \leq (B_t - B_s)(C_t - C_s) \text{ for all } t, s \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

$$E[\text{var}(A)_t - \text{var}(A)_s] \leq \sqrt{E[|M_t|^2 - |M_s|^2]} \sqrt{E[|N_t|^2 - |N_s|^2]} \text{ for all } t, s \in \mathbb{R}_{\geq 0} \text{ with } s < t.$$

*Proof.* (iv) If  $0 \in \langle M - N, L \rangle$  then  $A + 0 \in \langle N, L \rangle$  for all  $A \in \langle M, L \rangle$ . Conversely if  $\langle M, L; \mathcal{F}_+ \rangle \cap \langle N, L; \mathcal{F}_+ \rangle \neq \emptyset$  then  $0 \in \langle M - N, L; \mathcal{F}_+ \rangle$ , that is,  $t \mapsto (M_t - N_t)L_t$  is an  $\mathcal{F}_+$ -martingale and, being  $\mathcal{F}$ -adapted, it is an  $\mathcal{F}$ -martingale.

(v) Combine Lemma 21.9(i) and (iv). □

## 22 Predictable process and square integrable martingale

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\mathcal{F}$  be a filtration of sub  $\sigma$ -fields.

$\mathfrak{Mart}_2(\mathcal{F})$  (resp.  $\mathfrak{Mart}_b(\mathcal{F})$ ) stands for the space of all square integrable (bounded)  $\mathcal{F}$ -martingales with almost sure right continuous sample path.

$$\mathfrak{Mart}_2^{\text{fv}}(\mathcal{F}) := \{M \in \mathfrak{Mart}_2(\mathcal{F}) : t \mapsto M_t \text{ of finite variation a.s.}\}$$

$$\mathfrak{Mart}_2^c(\mathcal{F}) := \{M \in \mathfrak{Mart}_2(\mathcal{F}) : t \mapsto M_t \text{ continuous a.s.}\}$$

**22.1 Lemma.** Let  $M \in \overline{\mathfrak{Mart}_2^{\text{fv}}(\mathcal{F})}$  and  $A$  a bounded  $\mathcal{F}$ -finite variation process.

- (i)  $I : t \mapsto R[\text{Ad}M]_t \in \mathfrak{Mart}_2^{\text{fv}}(\mathcal{F})$ , and  $t \mapsto \text{jmpv}(I, N)_t$  is indistinguishable from  $t \mapsto R[A \text{djmpv}(M, N)]_t$  for all  $N \in \mathfrak{Mart}_2^{\text{fv}}(\mathcal{F})$ .
- (ii) If  $A$  is  $\mathcal{F} \vee \text{Null}(P)$ -predictable modulo evanescence then  $J := R\{\text{Ad}M\} \in \overline{\mathfrak{Mart}_2^{\text{fv}}(\mathcal{F})}$ , and  $\text{jmpv}(J, N)$  is indistinguishable from  $t \mapsto R\{A \text{djmpv}(M, N)\}_t$  for all  $N \in \mathfrak{Mart}_2^{\text{fv}}(\mathcal{F})$ .

*Proof.* We select and fix a sequence  $M^n \in \mathfrak{Mart}_2^{\text{fv}}(\mathcal{F})$  such that  $E[|M_T^n - M_T|^2]$  converges to 0 for all  $T \in \mathbb{N}$ . We may assume that  $M_0^n = M_0$ . Corollary 15.11(i) shows that

$$I_t^n := R[\text{Ad}M^n]_t = M_t^n A_t - M_0^n A_0 - R\{M^n dA\}_t \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

$$J_t^n := R\{\text{Ad}M^n\}_t = M_t^n A_t - M_0^n A_0 - R[M^n dA]_t \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

The argument of Lemma 20.7 shows that there exist  $\phi \in \text{Seq}(\mathbb{N}, \uparrow)$  and  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and for each  $\omega \in \Omega_0$  the following holds:

- $t \mapsto M_t^n(\omega)$  is of finite variation and right continuous for all  $n \in \mathbb{N}$ ,
- $t \mapsto M_t(\omega)$  is right continuous and admits left-hand limits everywhere
- $\sup_{t:t \leq T} |M_t^{\phi(n)}(\omega) - M_t(\omega)|$  converges to 0 for all  $T \in \mathbb{N}$ , and
- $t \mapsto A_t(\omega)$  is of finite variation and right continuous

In what follows we use the notation  $M^n$  in place of  $M^{\phi(n)}$ . We see that

$$|R[M^n dA]_t(\omega) - R[M dA]_t(\omega)| \leq \int_{(0,t]} |M^n_-(\omega) - M_-(\omega)| d\text{var}(A)_t(\omega)$$

for all  $t \in \mathbb{R}_{\geq 0}$  and  $\omega \in \Omega_0$ . Therefore

$$\begin{aligned} J_t^n &\text{ converges to } M_t A_t - M_0 A_0 - R\{M dA\}_t \text{ in probability for all } t \in \mathbb{R}_{\geq 0}. \\ J_t^n &\text{ converges to } M_t A_t - M_0 A_0 - R[M dA]_t \text{ in probability for all } t \in \mathbb{R}_{\geq 0}. \end{aligned}$$

The argument being parallel, we concentrate on the statement (ii). So we further assume that  $A$  is indistinguishable from an  $\mathcal{F} \vee \text{Null}(P)$ -predictable process. According to Corollary 17.17

$$\begin{aligned} J^n : t &\mapsto R\{\text{Ad}M^n\}_t \in \mathfrak{Mart}^{\text{fv}}(\mathcal{F}), \\ \text{jmpv}(J^n, J^n)_t &= R\{|A|^2 \text{djmpv}(M^n, M^n)\}_t \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s. and} \\ \text{jmpv}(J^n - J^m, \dots)_t &= R\{|A|^2 \text{djmpv}(M^n - M^m, \dots)\}_t \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.} \end{aligned}$$

There exists  $K \in \mathbb{R}_{\geq 0}$  such that  $|A_t| \leq K$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. Since

$$E[\text{jmpv}(M^n, M^n)_t] = E[|M_t^n|^2] - E[|M_0^n|^2] < +\infty$$

by Theorem 21.2, it follows that  $E[\text{jmpv}(J^n, J^n)_t] < +\infty$ , which implies  $J^n \in \mathfrak{Mart}_2(\mathcal{F})$  by Theorem 16.8(ii) and Theorem 7.17(ii). Invoking Theorem 21.2 again

$$\begin{aligned} E[|J_T^n - J_T^m|^2] &= E[\text{jmpv}(I^n - I^m, \dots)_T] \\ &\leq K^2 E[\text{jmpv}(M^n - M^m, \dots)_T] = K^2 E[|M_T^n - M_T^m|^2] \text{ for all } T \in \mathbb{N}. \end{aligned}$$

The right hand side converges to 0 as  $m, n$  tends to  $\infty$ . Thus we infer that  $J_t^n$  converges to  $M_t A_t - M_0 A_0 - R[M dA]_t$  in  $L_2$  for all  $t \in \mathbb{R}_{\geq 0}$ . Lemma 20.7(ii) shows that the limit is a square integrable martingale. Finally we observe that

$$J_t - J_{t-} = M_t A_t - M_{t-} A_{t-} - M_{t-} (A_t - A_{t-}) = (M_t - M_{t-}) A_t.$$

This shows that  $\text{jmpv}(J, N)_t = R\{A \text{djmpv}(M, N)\}_t$ . □

**22.2 Lemma.** *Suppose that  $A$  is a natural  $\mathcal{F}$ -finite variation process.*

(i) *If  $f$  is an  $\mathcal{F}_+$ -adapted process whose almost every sample path is locally bounded and admits left hand limits everywhere and  $E[R\{|f|d\text{var}(A)\}_t] < +\infty$  for all  $t \in \mathbb{R}_{\geq 0}$  then  $t \mapsto R\{fdA\}_t$  is a natural  $\mathcal{F}$ -finite variation process.*

(ii) *If  $f$  is an  $\mathcal{F}$ -adapted process with almost sure locally bounded and right continuous sample path, indistinguishable from an  $\mathcal{F} \vee \text{Null}(P)$ -predictable process and  $E[R\{|f|d\text{var}(A)\}_t] < +\infty$  for all  $t \in \mathbb{R}_{\geq 0}$  then  $t \mapsto R\{fdA\}_t$  is a natural  $\mathcal{F}$ -finite variation process.*

*Proof.* Let  $M \in \mathfrak{Mart}_b(\mathcal{F}_+)$ . By Lemma 15.18(ii)

$$E[R\{M dA\}_\sigma - R[M dA]_\sigma] = 0 \text{ for all bounded } \mathcal{F} \vee \text{Null}(P)\text{-optional times } \sigma.$$

Since all  $\mathcal{F}_+$ -stopping times are  $\mathcal{F}$ -optional times, according to Lemma 20.1, this implies that  $t \mapsto R\{M dA\}_t - R[M dA]_t$  is an  $\mathcal{F}_+$ -martingale. In view of Lemma 15.12(i)

$$t \mapsto \text{jmpv}(M, A)_t \text{ is an } \mathcal{F}_+ \vee \text{Null}(P)\text{-martingale.}$$

Clearly almost every sample is of finite variation. Actually we have that

$$|\text{jmpv}(M, A)_t - \text{jmpv}(M, A)_s| \leq 2 \sup_{u:u \geq 0} |M_u|(\text{var}(A)_t - \text{var}(A)_s) \forall s < \forall t \text{ a.s.}$$

(i) Suppose that  $f$  is  $\mathcal{F}_+$ -adapted, its almost every sample path is locally bounded and admits left hand limits everywhere. According to Corollary 9.7,  $t \mapsto R[f dA]_t$  is an  $\mathcal{F}$ -finite variation process. We see by (i) and (ii) of Lemma 15.12 that

$$R\{MdR[f dA]\}_t - R[MdR[f dA]]_t = \text{jmpv}(M, R[f dA])_t = R[f d \text{jmpv}(M, A)]_t$$

for all  $t \in \mathbb{R}_{\geq 0}$  a.s. On the other hand, since

$$R[|f| d \text{var}(\text{jmpv}(M, A))]_t \leq 2 \sup_{u:u \geq 0} |M_u| R[|f| d \text{var}(A)]_t \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.,}$$

it follows that  $t \mapsto R[f d \text{jmpv}(M, A)]_t$  is an  $\mathcal{F}_+ \vee \text{Null}(P)$ -martingale by Theorem 17.18(i). Thus we get  $E[R\{MdR[f dA]\}_t] = E[R[MdR[f dA]]_t]$ .

(ii) Suppose that  $f$  is indistinguishable from an  $\mathcal{F} \vee \text{Null}(P)$ -predictable process and its almost every sample path is locally bounded and right continuous. According to Corollary 15.11(i),  $t \mapsto R\{f dA\}_t$  is an  $\mathcal{F}$ -finite variation process. We see by (i) and (ii) of Lemma 15.12 that

$$R\{MdR\{f dA\}\}_t - R[MdR\{f dA\}]_t = \text{jmpv}(M, R\{f dA\})_t = R\{f d \text{jmpv}(M, A)\}_t$$

for all  $t \in \mathbb{R}_{\geq 0}$  a.s. Of course we have to modify Lemma 15.12(ii). Theorem 17.18(ii) shows that  $t \mapsto R\{f d \text{jmpv}(M, A)\}_t$  is an  $\mathcal{F}_+ \vee \text{Null}(P)$ -martingale.  $\square$

**22.3 Theorem.** *Suppose that  $X \in \mathfrak{Mart}_2(\mathcal{F})$ ,  $Y \in \mathfrak{Mart}_2(\mathcal{F}_+)$ ,  $Q \in [X, Y; \mathcal{F}_+]$ ,  $C \in \langle X, Y; \mathcal{F}_+ \rangle$  and  $f$  is a bounded  $\mathcal{F}$ -finite variation process.*

(i)  $I : t \mapsto R[f dX]_t \in \mathfrak{Mart}_2(\mathcal{F})$ ,  $R[f dQ] \in [I, Y; \mathcal{F}_+]$  and  $R[f dC] \in \langle I, Y; \mathcal{F}_+ \rangle$ .

(ii) *If  $f$  is  $\mathcal{F} \vee \text{Null}(P)$ -predictable modulo evanescence then  $J : t \mapsto R\{f dX\}_t \in \mathfrak{Mart}_2(\mathcal{F})$ ,  $R\{f dQ\} \in [J, Y; \mathcal{F}_+]$  and  $R\{f dC\} \in \langle J, Y; \mathcal{F}_+ \rangle$ .  $t \mapsto R\{X df\}_t - R[X df]_t \in \mathfrak{Mart}_2(\mathcal{F})$*

*Proof.* Let  $M \in \overline{\mathfrak{Mart}_2^{\text{fv}}(\mathcal{F}_+)}$  such that  $t \mapsto X_t - M_t \in \text{Mart}^c[X; \mathcal{F}]$ . A posteriori  $M$  is  $\mathcal{F}$ -adapted. The argument being parallel for (i) and (ii), we concentrate on the latter. So we further assume that  $f$  is indistinguishable from an  $\mathcal{F} \vee \text{Null}(P)$ -predictable process. We see by Lemma 22.1(ii) that

$$L : t \mapsto R\{f dM\}_t \in \overline{\mathfrak{Mart}_2^{\text{fv}}(\mathcal{F}_+)}.$$

According to Corollary 15.11(ii) this is indistinguishable from

$$t \mapsto M_t f_t - X_0 f_0 - R[M df]_t.$$

Being  $\mathcal{F}$ -adapted, the above belongs to  $\mathfrak{Mart}_2(\mathcal{F})$ . Corollary 15.11(i) shows that

$$(X_t - M_t) f_t - R\{(X - M) df\}_t = R[f d(X - M)]_t \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

Due to the sample path continuity of  $X - M$  the process in the left hand side is indistinguishable from  $t \mapsto (X_t - M_t) f_t - R[(X - M) df]_t$ . Therefore

$$J_t - L_t = (X_t - M_t) f_t - R[(X - M) df]_t = R[f d(X - M)]_t \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

The right hand side belongs to  $\mathfrak{Mart}_2^c(\mathcal{F})$ . Indeed let  $B. \in \text{Qvar}[X. - M.; \mathcal{F}]$ . Then

$$E[B_t] = E[|X_t - M_t|^2] < +\infty$$

by Theorem 8.15. We see by Theorem 9.10 that

$$t \mapsto R[f d(X - M)]_t \in \mathfrak{Mart}^c(\mathcal{F}) \text{ and } R[|f|^2 dB]. \in \text{Qvar}[L.; \mathcal{F}].$$

Since there exists  $K \in \mathbb{R}_{\geq 0}$  such that  $|f_t| \leq K$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s., we have that

$$E[R[|f|^2 dB]_t] \leq K^2 E[B_t] < +\infty.$$

Therefore Theorem 8.15 and Theorem 7.17(ii) shows that

$$t \mapsto J_t - L_t \in \mathfrak{Mart}_2^c(\mathcal{F}).$$

We next select and fix  $N. \in \overline{\mathfrak{Mart}_2^{\text{fv}}(\mathcal{F}_+)}$  such that  $t \mapsto Y_t - N_t \in \text{Mart}^c[Y; \mathcal{F}]$ . Let  $A. \in \text{Crv}[X - M, Y - N; \mathcal{F}]$ . It follows by Theorem 21.4 that

$$Q. \text{ is indistinguishable from } t \mapsto A_t + \text{jmpv}(M, N)_t.$$

We see that  $R[f dA]. \in \text{Crv}[J - L, Y - N; \mathcal{F}]$  by Theorem 10.10. Since  $A$  has continuous sample path and the discontinuous points of  $f$  is countable almost surely, it follows that

$$R\{f dA\}. \in \text{Crv}[J - L, Y - N; \mathcal{F}].$$

On the other hand according to Lemma 22.1(ii)

$$R\{f d(Q - A)\}_t = R\{f d\text{jmpv}(M, N)\}_t = \text{jmpv}(L, N)_t \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

Thus we conclude by invoking Theorem 21.4 that

$$R\{f dQ\}. \in [J, Y; \mathcal{F}_+].$$

Observe that  $t \mapsto Q_t - C_t$  is an  $\mathcal{F}_+$ -martingale. Since  $f$  is bounded and indistinguishable from an  $\mathcal{F} \vee \text{Null}(P)$ -predictable process, it follows by Theorem 17.18(ii) that

$$t \mapsto R\{f d(Q - C)\}_t \text{ is an } \mathcal{F}_+\text{-martingale.}$$

This implies that  $t \mapsto J_t Y_t - R\{f dC\}_t$  is an  $\mathcal{F}_+$ -martingale. Finally, since  $f$  is bounded and indistinguishable from an  $\mathcal{F} \vee \text{Null}(P)$ -predictable process, the  $\mathcal{F}_+$ -increasing process  $t \mapsto R\{f dC\}_t$  is natural by Lemma 22.2(ii).  $\square$

**22.4 Theorem.** *Suppose that  $A.$  is an  $\mathcal{F}$ -finite variation process indistinguishable from an  $\mathcal{F} \vee \text{Null}(P)$ -predictable process.*

(i) *Let  $X. \in \mathfrak{Mart}_2(\mathcal{F})$  and  $\sigma \in \text{Time}(\mathcal{F} \vee \text{Null}(P))$ . If*

$$E[\sup_{s:s \leq t \wedge \sigma} |X_s| (|A_0| + \text{var}(A)_{t \wedge \sigma}); \sigma > 0] < +\infty \text{ for all } t \in \mathbb{R}_{\geq 0}$$

*then  $t \mapsto X_{t \wedge \sigma} A_{t \wedge \sigma} - X_0 A_0 - R[X dA]_{t \wedge \sigma}$  is an  $\mathcal{F} \vee \text{Null}(P)$ -martingale.*

(ii) *If  $E[\text{var}(A)_t] < +\infty$  for all  $t \in \mathbb{R}_{\geq 0}$  then  $t \mapsto R\{X dA\}_t - R[X dA]_t$  is an  $\mathcal{F}_+$ -martingale for all  $X. \in \mathfrak{Mart}_b(\mathcal{F}_+)$ , and hence  $A.$  is  $\mathcal{F}$ -natural.*

*Proof.* (i) We introduce a sequence of functions  $\phi_n : \mathbb{R} \rightarrow \mathbb{R}$ ,  $x \mapsto \max\{\min\{x, n\}, -n\}$ . The composition process  $t \mapsto \max\{\min\{A_t, n\}, -n\}$  is of finite variation. Indeed we have that

$$\text{var}(\phi_n(A.))_t - \text{var}(\phi_n(A.))_s \leq \text{var}(A.)_t - \text{var}(A.)_s \text{ for all } s, t \in \mathbb{R}_{\geq 0} \text{ with } s < t \text{ a.s.}$$

Clearly  $|\phi_n(A_t)| \leq n$  for all  $t \in \mathbb{R}_{\geq 0}$  and the  $\mathcal{F} \vee \text{Null}(P)$ -predictability modulo evanescence is preserved. We see that

$$I^n : t \mapsto X_t \phi_n(A_t) - X_0 \phi_n(A_0) - R[X d \phi_n(A.)]_t \in \mathfrak{Mart}_2(\mathcal{F})$$

by Corollary 22.3(ii). In view of Example 20.4 we have that that

$$t \mapsto I^n_{t \wedge \sigma} \in \mathfrak{Mart}_2(\mathcal{F} \vee \text{Null}(P)).$$

If  $T \in \mathbb{N}$ ,  $n \in \mathbb{N}$  and  $|A_0| + \text{var}(A)_T \leq n$  then  $\phi_n(A_t) = A_t$  for all  $t \in \mathbb{R}_{[0, T]}$ . Therefore

$$\sup_{t: t \leq T} |I_t^n - I_t| \text{ converges to } 0 \text{ almost surely for all } T \in \mathbb{N},$$

where we write  $I : t \mapsto X_t A_t - X_0 A_0 - R[X d A]_t$ . On the other hand

$$|X_{t \wedge \sigma} \phi_n(A_{t \wedge \sigma}) - X_0 \phi_n(A_0) - R[X d \phi_n(A.)]_{t \wedge \sigma}| \leq 2 \sup_{s: s \leq t \wedge \sigma} |X_s| (|A_0| + \text{var}(A)_{t \wedge \sigma}) \text{ a.s.}$$

Thus we deduce the  $\mathcal{F} \vee \text{Null}(P)$ -martingale property of the stopped process  $t \mapsto I_{t \wedge \sigma}$  by applying the dominated convergence theorem.

(ii) We may assume that  $A_0 = 0$  a.s. Observe that  $A.$  is an  $\mathcal{F}_{.+}$ -finite variation process  $\mathcal{F}_{.+} \vee \text{Null}(P)$ -predictable modulo evanescence as well. Let  $X. \in \mathfrak{Mart}_b(\mathcal{F}_{.+})$ . Thus (i) shows that  $t \mapsto X_t A_t - R[X d A]_t$  is an  $\mathcal{F}_{.+}$ -martingale. The  $\mathcal{F}_{.+}$ -martingale property of  $t \mapsto X_t A_t - R\{X d A\}_t$  is discussed in Lemma 15.8.  $\square$

## 23 Square integrable martingale as integrator

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\mathcal{F}.$  be a filtration of sub  $\sigma$ -fields.

$\mathfrak{Mart}_2(\mathcal{F}.)$  stands for the space of all square integrable  $\mathcal{F}.$ -martingales with almost sure right continuous sample path.

**23.1 Lemma.** *Let  $X. \in \mathfrak{Mart}_2(\mathcal{F}.)$ ,  $Q. \in [X; \mathcal{F}_{.+}]$ ,  $Y. \in \mathfrak{Mart}_2(\mathcal{F}_{.+})$  and  $B. \in [X, Y; \mathcal{F}_{.+}]$ .*

(i) *If almost every sample path of  $f.$  is locally bounded and admits left-hand limits everywhere then*

$$E[R\{|f| d \text{var}(B.)\}_t] \leq \sqrt{E[R\{|f|^2 d Q\}_t]} \sqrt{E[|Y_t|^2] - E[|Y_0|^2]} \text{ for all } t \in \mathbb{R}_{\geq 0}.$$

(ii) *If almost every sample path of  $f.$  is locally bounded and right continuous then*

$$E[R\{|f| d \text{var}(B.)\}_t] \leq \sqrt{E[R\{|f|^2 d Q\}_t]} \sqrt{E[|Y_t|^2] - E[|Y_0|^2]} \text{ for all } t \in \mathbb{R}_{\geq 0}.$$

*Proof.* Let  $C. \in [Y; \mathcal{F}_{.+}]$ . Note that  $E[Y_t] < +\infty$  for all  $t \in \mathbb{R}_{\geq 0}$ . We have that

$$R\{|f| d \text{var}(B.)\}_t \leq \sqrt{R\{|f|^2 d Q\}_t} C_t \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

by Theorem 21.6. Thus we get the claim by invoking Schwarz' inequality.  $\square$

**23.2 Theorem.** Suppose that  $X. \in \mathfrak{Mart}_2(\mathcal{F})$ ,  $Q. \in [X; \mathcal{F}_+]$ ,  $f$  is an  $\mathcal{F}$ -finite variation process,  $Y. \in \mathfrak{Mart}_2(\mathcal{F}_+)$  and  $B. \in [X, Y; \mathcal{F}_+]$ .

- (i)  $E[R\{|f|^2 dQ\}_t] < +\infty \forall t \in \mathbb{R}_{\geq 0} \Rightarrow R\{fdX\}. \in \mathfrak{Mart}_2(\mathcal{F})$  and  $R\{fdB\}. \in [R\{fdX\}, Y; \mathcal{F}_+]$ .
- (ii) If  $f$  is  $\mathcal{F} \vee \text{Null}(P)$ -predictable modulo evanescence and  $E[R\{|f|^2 dQ\}_t] < +\infty$  for all  $t \in \mathbb{R}_{\geq 0}$  then  $J. : t \mapsto R\{fdX\}_t \in \mathfrak{Mart}_2(\mathcal{F})$  and  $R\{fdB\}. \in [J, Y; \mathcal{F}_+]$ .
- (iii) If  $f$  is  $\mathcal{F} \vee \text{Null}(P)$ -predictable modulo evanescence,  $E[R\{|f|^2 dQ\}_t] < +\infty$  for all  $t \in \mathbb{R}_{\geq 0}$  and  $E[R\{|f|^2 dQ\}_t] < +\infty$  for all  $t \in \mathbb{R}_{\geq 0}$  then  $t \mapsto R\{Xdf\}_t - R[Xdf]_t \in \mathfrak{Mart}_2(\mathcal{F})$ .

*Proof.* (ii) We partly repeat the proof of Lemma 22.1 and Theorem 22.4. As before we introduce a sequence of functions  $\phi_n : \mathbb{R} \rightarrow \mathbb{R}$ ,  $x \mapsto \max\{\min\{x, n\}, -n\}$ . Then

$$\begin{aligned} J^n : t \mapsto X_t \phi_n(f_t) - X_0 \phi_n(f_0) - R[Xd\phi_n(f)]_t &\in \mathfrak{Mart}_2(\mathcal{F}), \\ \sup_{t:t \leq T} |J_t^n - J_t| &\text{converges to 0 almost surely for all } T \in \mathbb{N} \text{ and} \\ t \mapsto R\{|\phi_n(f) - \phi_m(f)|^2 dQ\}_t &\in [J^n - J^m; \mathcal{F}_+]. \end{aligned}$$

The third line drives by Theorem 22.3(ii). Invoking Theorem 21.4(i), we get

$$\begin{aligned} \sqrt{E[|J_T^n - J_T^m|^2]} &= \sqrt{E[R\{|\phi_n(f) - \phi_m(f)|^2 dQ\}_T]} \\ &\leq \sqrt{E[R\{|\phi_n(f) - f|^2 dQ\}_T]} + \sqrt{E[R\{|f - \phi_m(f)|^2 dQ\}_T]} \text{ for all } T \in \mathbb{N}. \end{aligned}$$

The right hand side converges to 0 as  $m, n$  tends to  $\infty$ . Indeed

$$R\{|\phi_n(f) - f|^2 dQ\}_T \leq R\{|f|^2 dQ\}_T \text{ and } \phi_n(f) \text{ converges to } f \text{ point wise.}$$

Thus we infer that  $J_t^n$  converges to  $X_t f_t - X_0 f_0 - R[Xdf]_t$  in  $L_2$  for all  $t \in \mathbb{R}_{\geq 0}$ . Finally Lemma 20.7(ii) shows that the limit is a square integrable martingale. By Theorem 22.3(ii),

$$J_t^n Y_t - R\{\phi_n(f) dB\}_t \text{ is an } \mathcal{F}_+ \text{-martingale.}$$

The first term converges to  $J_t Y_t$  in  $L_1$ . On the other hand Lemma 23.1 shows that

$$E[|R\{\phi_n(f) dB\}_t - R\{fdB\}_t|] \leq \sqrt{E[R\{|\phi_n(f) - f|^2 dA\}_t]} \sqrt{E[|Y_t|^2] - E[|Y_0|^2]}$$

Consequently  $R\{\phi_n(f) dB\}_t$  converges to  $R\{fdB\}_t$  in  $L_1$ . It follows that

$$J_t Y_t - R\{fdB\}_t \text{ is an } \mathcal{F}_+ \text{-martingale.}$$

Finally we see that

$$(J_t - J_{t-})(Y_t - Y_{t-}) = f_t(X_t - X_{t-})(Y_t - Y_{t-}) = f_t(B_t - B_{t-})$$

for all  $t \in \mathbb{R}_{\geq 0}$  a.s. Consequently  $t \mapsto R\{fdB\}_t \in [J, Y; \mathcal{F}_+]$ . □

**23.3 Lemma.** Let  $X. \in \mathfrak{Mart}_2(\mathcal{F})$ ,  $A. \in \langle X; \mathcal{F}_+ \rangle$ ,  $Y. \in \mathfrak{Mart}_2(\mathcal{F}_+)$  and  $B. \in \langle X, Y; \mathcal{F}_+ \rangle$ .

- (i) If almost every sample path of  $f.$  is locally bounded and admits left-hand limits everywhere then

$$E[R\{|f| d\text{var}(B.)\}_t] \leq \sqrt{E[R\{|f|^2 dA\}_t]} \sqrt{E[|Y_t|^2] - E[|Y_0|^2]} \text{ for all } t \in \mathbb{R}_{\geq 0}.$$

- (ii) If almost every sample path of  $f.$  is locally bounded and right continuous then

$$E[R\{|f| d\text{var}(B.)\}_t] \leq \sqrt{E[R\{|f|^2 dA\}_t]} \sqrt{E[|Y_t|^2] - E[|Y_0|^2]} \text{ for all } t \in \mathbb{R}_{\geq 0}.$$

*Proof.* The proof is a repetition of that of Lemma 23.1 with square brackets replaced by corresponding angle brackets.  $\square$

**23.4 Theorem.** *Suppose that  $X. \in \mathfrak{Mart}_2(\mathcal{F})$ ,  $A. \in \langle X; \mathcal{F}_+ \rangle$ ,  $f$  is an  $\mathcal{F}$ -finite variation process,  $Y. \in \mathfrak{Mart}_2(\mathcal{F}_+)$  and  $B. \in \langle X, Y; \mathcal{F}_+ \rangle$ .*

- (i)  $E[R[|f|^2 dA]_t] < +\infty \forall t \in \mathbb{R}_{\geq 0} \Rightarrow R[f dX]. \in \mathfrak{Mart}_2(\mathcal{F})$  and  $R[f dB]. \in \langle R[f dX], Y; \mathcal{F}_+ \rangle$ .
- (ii) *If  $f$  is  $\mathcal{F} \vee \text{Null}(P)$ -predictable modulo evanescence and  $E[R\{|f|^2 dA\}_t] < +\infty$  for all  $t \in \mathbb{R}_{\geq 0}$  then  $J. : t \mapsto R\{f dX\}_t \in \mathfrak{Mart}_2(\mathcal{F})$  and  $R\{f dB\}. \in \langle J, Y; \mathcal{F}_+ \rangle$ .*
- (iii) *If  $f$  is  $\mathcal{F} \vee \text{Null}(P)$ -predictable modulo evanescence,  $E[R[|f|^2 dA]_t] < +\infty$  for all  $t \in \mathbb{R}_{\geq 0}$  and  $E[R\{|f|^2 dA\}_t] < +\infty$  for all  $t \in \mathbb{R}_{\geq 0}$  then  $t \mapsto R\{X df\}_t - R[X df]_t \in \mathfrak{Mart}_2(\mathcal{F})$ .*

*Proof.* The proof is a repetition of that of Theorem 23.2 with square brackets replaced by corresponding angle brackets.  $\square$

**23.5 Lemma.** *Suppose that  $Y.$  and  $Z.$  are  $\mathcal{F}$ -adapted processes whose almost all sample paths are right continuous and locally bounded,  $F.$  and  $G.$  are  $\mathcal{F}$ -finite variation processes,  $F_0 = 0$  a.s.,  $G_0 = 0$  a.s.,  $M. : t \mapsto Y_t - F_t$ ,  $N. : t \mapsto Z_t - G_t \in \mathfrak{Mart}_2(\mathcal{F})$ ,  $A. \in \langle M; \mathcal{F}_+ \rangle$ ,  $B. \in \langle N; \mathcal{F}_+ \rangle$ ,  $E[R[|G|^2 dA]_t] < \infty \forall t \in \mathbb{R}_{\geq 0}$ ,  $E[R[|F|^2 dB]_t] < \infty \forall t \in \mathbb{R}_{\geq 0}$ ,  $C.$  is a natural  $\mathcal{F}_+$ -finite variation process and  $E[|C_0|] < +\infty$ . Then*

$$t \mapsto Y_t Z_t - M_t N_t - R\{Y dG\}_t - R\{Z dF\}_t + \text{jmpv}(F, G)_t \in \mathfrak{Mart}_2(\mathcal{F}),$$

moreover  $C. - C_0 \in \langle M, N; \mathcal{F}_+ \rangle$  if and only if is the following is an  $\mathcal{F}_+$ -martingale:

$$t \mapsto Y_t Z_t - R\{Y dG\}_t - R\{Z dF\}_t + \text{jmpv}(F, G)_t - C_t$$

*Proof.*  $R\{M dG\}_t + R\{N dF\}_t = M_t G_t - R[G dM]_t + F_t N_t - R[F dN]_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. by Corollary 15.11(i).  $R\{F dG\}_t + R\{G dF\}_t - \text{jmpv}(F, G)_t = F_t G_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. by (i) and (ii) of Lemma 15.12. Thus the process in question is indistinguishable from  $t \mapsto R[G dM]_t + R[F dN]_t$ , which belongs to  $\mathfrak{Mart}_2(\mathcal{F})$  by Theorem 23.4.  $\square$

We will get the next statement with the help of Lemma 20.7.

**23.6 Theorem.** *Suppose that  $M. \in \mathfrak{Mart}_2(\mathcal{F})$ ,  $Q. \in [M; \mathcal{F}_+]$ ,  $A. \in \langle M; \mathcal{F}_+ \rangle$  and  $f.$  is an  $\mathcal{F}$ -adapted process with almost sure locally bounded and right continuous sample path and indistinguishable from an  $\mathcal{F} \vee \text{Null}(P)$ -predictable process.*

- (i) *If  $E[R\{|f|^2 dQ\}_t] < +\infty$  for all  $t \in \mathbb{R}_{\geq 0}$  then there exists  $I. \in \mathfrak{Mart}_2(\mathcal{F})$  such that  $I_0 = 0$  a.s. and if  $N. \in \mathfrak{Mart}_2(\mathcal{F}_+)$  and  $B. \in [M, N; \mathcal{F}_+]$  then  $R\{f dB\}. \in [I., N; \mathcal{F}_+]$ .*
- (ii) *If  $E[R\{|f|^2 dA\}_t] < +\infty$  for all  $t \in \mathbb{R}_{\geq 0}$  then there exists  $I. \in \mathfrak{Mart}_2(\mathcal{F})$  such that  $I_0 = 0$  a.s. and if  $N. \in \mathfrak{Mart}_2(\mathcal{F}_+)$  and  $B. \in \langle M, N; \mathcal{F}_+ \rangle$  then  $R\{f dB\}. \in \langle I., N; \mathcal{F}_+ \rangle$ .*

Given  $M. \in \mathfrak{Mart}_2(\mathcal{F})$ ,  $A. \in [M; \mathcal{F}_+]$  and an  $\mathcal{F}$ -predictable process  $f.$  with almost sure locally bounded and right continuous sample path, and  $E[R\{|f|^2 dA\}_t] < +\infty$  for all  $t \in \mathbb{R}_{\geq 0}$ , we set

$$\text{It}\hat{o}[f dM; \mathcal{F}] := \{I. \in \mathfrak{Mart}_2(\mathcal{F}) : I_0 = 0 \text{ a.s., and } R\{f dB\}. \in [I., N; \mathcal{F}_+]\}$$

$$\text{for any } N. \in \mathfrak{Mart}_2(\mathcal{F}) \text{ and } B. \in [M, N; \mathcal{F}_+].$$

## 24 Predictable processes

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\mathcal{F}$  be a filtration of sub  $\sigma$ -fields. The index set is  $\mathbb{R}_{\geq 0}$ .

**24.1 Lemma.** *Suppose that  $\tau$  is an  $\mathcal{F}$ -optional time. If there exists a sequence  $\sigma(\cdot)$  of  $\mathcal{F}$ -optional times such that  $\sigma(n) \leq \sigma(n+1)$ ,  $\sup_{n \in \mathbb{N}} \sigma(n) = \tau$  and  $\{\tau > 0\} \subset \{\sigma(n) < \tau\}$  then  $\{(t, \omega) : t > 0, \tau(\omega) \leq t\} \in \text{Pred}(\mathcal{F})$ .*

*Proof.* We observe that  $\{(t, \omega) : t > 0, \tau(\omega) \leq t\} = \bigcap_{n=1}^{\infty} \{(t, \omega) : \sigma(n, \omega) < t\}$ , which belongs to  $\text{Pred}(\mathcal{F})$  by Lemma 17.2(iii). Incidentally  $\{\omega : \tau(\omega) \leq t\} = \bigcap_{n=1}^{\infty} \{\omega : \sigma(n, \omega) < t\}$  for all  $t \in \mathbb{R}_{> 0}$ , whence  $\tau$  is an  $\mathcal{F}$ -stopping time provided  $\{\tau = 0\} \in \mathcal{F}_0$ .  $\square$

**24.2 Definition.** An  $\mathcal{F}$ -optional time  $\tau$  is said to be predictable if  $\{\tau = 0\} \in \mathcal{F}_0$  and there exist a sequence  $\sigma(\cdot)$  of  $\mathcal{F}$ -optional times such that  $\sigma(n) \leq \sigma(n+1)$ ,  $\sup_{n \in \mathbb{N}} \sigma(n) = \tau$  and  $\{\tau > 0\} \subset \{\sigma(n) < \tau\}$  (a posteriori  $\tau$  is a stopping time). Such sequence is called an announcing sequence.

**24.3 Example.** Let  $s, t \in \mathbb{R}_{\geq 0}$ ,  $s < t$ ,  $A \in \mathcal{F}_0$ ,  $B \in \mathcal{F}_s$  and  $A \cap B = \emptyset$ . We set

$$\tau := \begin{cases} 0 & \text{on } A \\ t & \text{on } B \\ +\infty & \text{off } A \cup B \end{cases} \quad \sigma(n) := \begin{cases} 0 & \text{on } A \\ s/(n+1) + nt/(n+1) & \text{on } B \\ t+n & \text{off } A \cup B \end{cases}$$

Then  $\tau$  is a predictable  $\mathcal{F}$ -optional time, which is announced by the sequence  $\sigma(\cdot)$ . Indeed  $\{\tau = 0\} = A \in \mathcal{F}_0$ , each  $\sigma(n)$  is an  $\mathcal{F}$ -stopping time and  $\sigma(n) \uparrow \tau$  on  $\{\tau > 0\} = A^c$ .

**24.4 Lemma.** *If  $\mathcal{F}$ -optional times  $\sigma$  and  $\tau$  are predictable then so are  $\sigma \wedge \tau$  and  $\sigma \vee \tau$ .*

*Proof.* Let  $\sigma(\cdot)$  respectively  $\tau(\cdot)$  be announcing sequences for  $\sigma$  and  $\tau$ . We see that

$$\begin{aligned} \sigma(k) \wedge \tau(k) &\leq \sigma(k+1) \wedge \tau(k) \leq \sigma(k+1) \wedge \tau(k+1) \leq \sigma \wedge \tau \\ \sigma \wedge \tau(k) &= \sup_{n \geq k} \sigma(n) \wedge \tau(k) \leq \sup_{n \geq k} \sigma(n) \wedge \tau(n) = \sup_{n \in \mathbb{N}} \sigma(n) \wedge \tau(n) \leq \sigma \wedge \tau \end{aligned}$$

because  $(x, y) \mapsto x \wedge y$  is continuous and separately non-decreasing. Moreover

$$\{\sigma \wedge \tau > 0\} = \{\sigma > 0\} \cap \{\tau > 0\} \subset \{\sigma(k) < \sigma\} \cap \{\tau(k) < \tau\} \subset \{\sigma(k) \wedge \tau(k) < \sigma \wedge \tau\}.$$

On the other hand if  $\sigma = 0$  and  $\tau > 0$  then  $\sigma(k) = 0$  and  $\tau(k) < \tau \leq \sigma \vee \tau$ .  $\square$

**24.5 Lemma.** *Suppose that  $\sigma$  is an  $\mathcal{F}$ -optional time,  $Q \subset \mathbb{R}_{> 0}$  and for  $t \in \mathbb{R}_{\geq 0}$  with  $Q_{>t} := \{q \in Q : q > t\} \neq \emptyset$  there exists  $r \in Q_{>t}$  such that  $r \leq q$  for all  $q \in Q_{>t}$ .*

(i)  $\tau := \min\{q \in Q : q > \sigma\}$  is an  $\mathcal{F}$ -stopping time.

(ii) If  $\#\{q \in Q : q \leq t\} < \infty$  for all  $t \in \mathbb{R}_{\geq 0}$  then  $\tau$  is predictable.

*Proof.* (i)  $\{\tau \leq t\} = \{\sigma < \sup\{q \in Q : q \leq t\}\} \in \mathcal{F}_t$  for all  $t \in \mathbb{R}_{\geq 0}$ .

(ii) For each  $n \in \mathbb{N}$  the following set satisfies that whenever  $\{q \in Q_n : q > t\}$  is not void it has a minimum element:

$$Q_n := \{2^{-k} \max\{i \in Q \cup \{0\} : i < q\} + (1 - 2^{-k})q ; k \in \mathbb{Z}_{\geq n}, q \in Q\} \cup Q.$$

It follows by (i) that  $\tau(n) := \min\{q \in Q_n : q > \sigma\} \wedge n$  is an  $\mathcal{F}$ -stopping time. Clearly  $Q_{n+1} \subset Q_n$  implies that  $\tau(n) \leq \tau(n+1)$ . Observe that if  $q \in Q$  and  $t < q$  then there exists  $r \in Q_n$  such that  $t < r < q$ , which means  $\tau(n) < \tau$ . Moreover  $\sup_{n \in \mathbb{N}} \tau(n) = \tau$ .  $\square$

**24.6 Corollary.** (i) For each  $\mathcal{F}$ -optional time  $\sigma$  there exists a sequence  $\tau(\cdot)$  of predictable  $\mathcal{F}$ -optional times such that  $\tau(n) \geq \tau(n+1)$ ,  $\inf_{n \in \mathbb{N}} \tau(n) = \sigma$  and  $\{\sigma < +\infty\} \subset \{\sigma < \tau(n)\}$ .  
(ii)  $\text{Pred}(\mathcal{F})$  is generated by  $\{\{(t, \omega) : \tau(\omega) \leq t\}; \tau \text{ predictable } \mathcal{F}\text{-optional time}\}$ .

*Proof.* (i)  $\tau(n) := \min\{q \in 2^{-n}\mathbb{Z} : q > \sigma\}$  provides us with a desired sequence.

(ii) We have that  $\mathcal{C} := \{\{(t, \omega) : \tau(\omega) \leq t\}; \tau \text{ predictable } \mathcal{F}\text{-optional time}\} \subset \text{Pred}(\mathcal{F})$ . Indeed, according to Lemma 24.1, if  $\tau$  is a predictable  $\mathcal{F}$ -optional time then

$$\{(t, \omega) : \tau(\omega) \leq t\} = \{(t, \omega) : t > 0, \tau(\omega) \leq t\} \cup (\{0\} \times \{\tau = 0\}) \in \text{Pred}(\mathcal{F}).$$

Lemma 17.2 shows that  $\text{Pred}(\mathcal{F})$  is generated by

$$\{\{(t, \omega) : \sigma(\omega) < t\}; \sigma \in \text{Time}(\mathcal{F})\} \cup \{\{0\} \times A; A \in \mathcal{F}_0\}.$$

Let  $\sigma \in \text{Time}(\mathcal{F})$ . There exists a sequence  $\tau(\cdot)$  of predictable  $\mathcal{F}$ -optional times such that  $\tau(n) \geq \tau(n+1)$ ,  $\inf_{n \in \mathbb{N}} \tau(n) = \sigma$  and  $\{\sigma < +\infty\} \subset \{\sigma < \tau(n)\}$  by (i). It follows that

$$\{(t, \omega) : \sigma(\omega) < t\} = \bigcup_{n=1}^{\infty} \{(t, \omega) : \tau(n, \omega) \leq t\} \in \sigma(\mathcal{C}).$$

Let  $A \in \mathcal{F}_0$ . Example 24.3 shows the existence of an predictable  $\mathcal{F}$ -optional time  $\tau$  such that  $A = \{\tau = 0\} = \{\tau < +\infty\}$ . Since

$$\mathbb{R}_{>0} \times A = \{(t, \omega) : \tau(\omega) < t\} \in \sigma(\mathcal{C}) \text{ and } \mathbb{R}_{\geq 0} \times A = \{(t, \omega) : \tau(\omega) \leq t\},$$

we see that  $\{0\} \times A \in \sigma(\mathcal{C})$ . Thus we get the claim.  $\square$

*24.7 Question.* Suppose that  $\tau$  is an  $\mathcal{F}$ -optional time and  $\{(t, \omega) : \tau(\omega) \leq t\} \in \text{Pred}(\mathcal{F})$  ( $t \mapsto 1_{[\tau, +\infty)}(t)$  is a predictable and right continuous increasing process). Is  $\tau$  predictable?

**24.8 Lemma.** Suppose that  $\mathcal{N}$  is a  $\sigma$ -consistent exceptional family on  $\Omega$  with  $\mathcal{N} \subset \mathcal{F}$ . If  $\tau$  is an  $\mathcal{F}$ -optional time,  $\{\tau = 0\} \in \mathcal{F}_0 \vee \mathcal{N}$  and there exist a sequence  $\sigma(n)$  of  $\mathcal{F}$ -optional times such that  $\{\sup_{n \in \mathbb{N}} \sigma(n) \neq \tau\} \in \mathcal{N}$  and  $\{\tau > 0, \sigma(n) \geq \tau\} \in \mathcal{N}$  then there exists a predictable  $\mathcal{F}$ -optional time  $\tilde{\tau}$  such that  $\{\tau \neq \tilde{\tau}\} \in \mathcal{N}$ .

*Proof.* We write  $S(n) := \max_{k \in \mathbb{N}; k \leq n} \sigma(k)$  and  $S := \sup_{n \in \mathbb{N}} \sigma(n)$ . All of them are  $\mathcal{F}$ -optional times. Clearly  $S(n) \leq S(n+1)$  and  $\sup_{n \in \mathbb{N}} S(n) = S$ . We see that

$$\{S > 0, \sigma(k) = S\} \subset \{S \neq \tau\} \cup \{S = \tau, \tau > 0, \sigma(k) = \tau\} \subset \{S \neq \tau\} \cup \{\tau > 0, \sigma(k) \geq \tau\}.$$

Since the right hand side belongs to  $\mathcal{N}$ , it follows that  $\{S > 0, \sigma(k) = S\} \in \mathcal{N}$ . Therefore

$$\bigcup_{n=1}^{\infty} \{S(n) = S > 0\} = \bigcup_{k=1}^{\infty} \{S > 0, \sigma(k) = S\} \in \mathcal{N}.$$

Observe that  $\{S(n) < S\} \in \mathcal{F}_{S^+}$  by Lemma 3.15(iii). Hence  $\{S < t\} \cap \bigcap_{n=1}^{\infty} \{S(n) < S\}$  belongs to  $\mathcal{F}_t$  for all  $t \in \mathbb{R}_{>0}$ . Select and fix  $A \in \mathcal{F}_0$  with  $\{\tau = 0\} \times A \in \mathcal{N}$ . We introduce the following  $\mathcal{F}$ -optional time

$$\tilde{\tau} := \begin{cases} 0 & \text{on } A \\ S & \text{on } \bigcap_{n=1}^{\infty} \{S(n) < S\} \cap A^c \\ +\infty & \text{on } \bigcup_{n=1}^{\infty} \{S(n) = S\} \cap A^c \end{cases}$$

so that we have  $\{\tilde{\tau} = 0\} = A \in \mathcal{F}_0$  and  $\{\tau \neq \tilde{\tau}\} \in \mathcal{N}$ . Indeed

$$\{\tau \neq \tilde{\tau}\} \subset \{S \neq \tau\} \cup \{S = \tau, S \neq \tilde{\tau}\} \subset \{S \neq \tau\} \cup \bigcup_{n=1}^{\infty} \{S(n) = S > 0\} \cup (\{S = 0\} \times A).$$

To show the predictability of  $\tilde{\tau}$  we further introduce for each  $n \in \mathbb{N}$

$$\tilde{\sigma}(n) := \begin{cases} 0 & \text{on } A \\ S(n) \wedge n & \text{on } \{S(n) < S\} \cap A^c \\ n & \text{on } \{S(n) = S\} \cap A^c \end{cases}$$

This is an  $\mathcal{F}$ -optional time because  $\{S(n) < S\} \in \mathcal{F}_{S(n)+}$  and hence

$$\{\tilde{\sigma}(n) < t\} = \begin{cases} (\{S(n) < t\} \cap \{S(n) < S\}) \cup A \in \mathcal{F}_t & 0 < t \leq n \\ \Omega & t > n \end{cases}$$

On  $\bigcap_{n=1}^{\infty} \{S(n) < S\} \cap A^c$  we have that

$$\begin{aligned} \tilde{\sigma}(k) &= S(k) \wedge k \leq S(k+1) \wedge k \leq S(k+1) \wedge (k+1) \leq S(k+1) < S = \tilde{\tau} \\ \tilde{\tau} \wedge k &= S \wedge k = \sup_{n \geq k} S(n) \wedge k \leq \sup_{n \geq k} \tilde{\sigma}(n) = \sup_{n \in \mathbb{N}} \tilde{\sigma}(n) \leq \tilde{\tau} \end{aligned}$$

While on  $\{S(n) = S\} \cap A^c$  we have that

$$\tilde{\sigma}(k) = k < +\infty = \tilde{\tau} \text{ for all } k \in \mathbb{N}_{\geq n}.$$

Thus  $\tilde{\sigma}(k) \leq \tilde{\sigma}(k+1)$  and  $\sup_{n \in \mathbb{N}} \tilde{\sigma}(n) = \tilde{\tau}$ . Moreover  $\tilde{\sigma}(k) < \tilde{\tau}$  off  $\{\tilde{\tau} = 0\} = A$ .  $\square$

**24.9 Definition.** Let  $X$  be an  $\mathcal{F}$ -adapted process with almost sure right continuous path. It is said to be  $\mathcal{F}$ -quasi left continuous if almost every path admits left hand limits everywhere and for any sequence  $\tau(n)$  of  $\mathcal{F}$ -optional times such that  $\tau(n) \leq \tau(n+1)$  a.s. and  $\tau(n) \leq T$  a.s. for some  $T \in \mathbb{R}_{>0}$  the sequence  $X_{\tau(n)}$  converges to  $X_{\sigma}$  a.s. where  $\sigma := \sup_{n \in \mathbb{N}} \tau(n)$ .

**24.10 Lemma.** Suppose that  $E$  is a metrizable topological space and  $X$  is an  $\mathcal{F}$ -adapted process with almost sure  $D(\mathbb{R}_{>0}, E)$ -sample path. Then  $X$  is  $\mathcal{F}$ -quasi left continuous if and only if  $P(X_{\tau} \neq X_{\tau-}, 0 < \tau < +\infty) = 0$  for all predictable  $\mathcal{F}$ -optional times  $\tau$ .

*Proof.* Suppose that  $X$  is  $\mathcal{F}$ -quasi left continuous and  $\tau$  is a predictable  $\mathcal{F}$ -optional time with an announcing sequence  $\sigma(\cdot)$ . Fix  $T \in \mathbb{N}$ . Then  $X_{\sigma(n) \wedge T}$  converges to  $X_{\tau \wedge T}$  a.s. Since  $\{0 < \tau \leq T\} \subset \{\sigma(n) \wedge T = \sigma(n), \tau \wedge T = \tau, \sigma(n) < \tau\}$ , it follows that

$$P(X_{\tau} \neq X_{\tau-}, 0 < \tau \leq T) = 0 \text{ for all } T \in \mathbb{N}$$

Conversely suppose that  $P(X_{\tau} \neq X_{\tau-}, 0 < \tau < +\infty) = 0$  for all predictable  $\mathcal{F}$ -optional times  $\tau$ . Given a sequence  $\tau(\cdot)$  of  $\mathcal{F}$ -optional times such that  $\tau(n) \leq \tau(n+1)$  a.s. and  $\sigma := \sup_{n \in \mathbb{N}} \tau(n) < +\infty$  a.s., we introduce

$$\tau := \begin{cases} \sigma & \text{on } \bigcap_{n=1}^{\infty} \{\tau(n) < \sigma\} \\ +\infty & \text{on } \bigcup_{n=1}^{\infty} \{\tau(n) = \sigma\} \end{cases}$$

This is a predictable  $\mathcal{F}$ -optional time according to the proof of Lemma 24.8. Taking

$$\{0 < \tau < +\infty\} = \{\sigma < +\infty, \tau(n) < \sigma \forall n\} \subset \{\tau = \sigma\}$$

into account we infer that

$$P(X_\sigma \neq X_{\sigma-}, \sigma < +\infty, \tau(n) < \sigma \forall n) = 0.$$

Since  $P(\sigma < +\infty) = 1$ , the following set has full measure

$$\{X_\sigma = X_{\sigma-}, \sigma < +\infty, \tau(n) < \sigma \forall n\} \cup \{\sigma < +\infty, \exists n \text{ s.t. } \tau(n) = \sigma\}.$$

Moreover on this set  $X_{\sigma(n)}$  converges to  $X_\sigma$ . □

**24.11 Lemma.** *Suppose that  $\tau$  is an  $\mathcal{F}$ -optional time and  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  is bounded, strictly increasing and continuous. If  $M \in \mathfrak{Mart}_b(\mathcal{F}_+)$  and  $M_t \in E[f(\tau) | \mathcal{F}_{t+}]$  for all  $t \in \mathbb{R}_{\geq 0}$  with  $f(\infty) = \sup_{t \geq 0} f(t)$  then  $\sup_{n \in \mathbb{N}} \inf\{t \in \mathbb{R}_{\geq 0} : M_t - f(\tau \wedge t) < 1/n\} = \tau$  a.s.*

*Proof.* According to Corollary 14.12, we have that  $M_\infty = f(\tau)$  a.s. and

$$M_\sigma \in E[f(\tau) | (\mathcal{F} \vee \text{Null}(P))_{\sigma+}] \text{ for all } \mathcal{F} \vee \text{Null}(P)\text{-optional times } \sigma.$$

Since  $f$  is continuous,  $t \mapsto f(\tau \wedge t)$  is a continuous  $\mathcal{F}$ -adapted process. Set

$$\zeta := \inf\{t \in \mathbb{R}_{\geq 0} : M_t - f(\tau \wedge t) \leq 0\} \wedge \inf\{t \in \mathbb{R}_{> 0} : M_{t-} - f(\tau \wedge t) \leq 0\}.$$

Observe that  $M_\tau = f(\tau)$  a.s. and  $M_t > f(\tau \wedge t)$  if  $t < \zeta$ . This implies that

$$\zeta \leq \tau \text{ a.s.}$$

For each  $n \in \mathbb{N}$  denote the  $\mathcal{F} \vee \text{Null}(P)$ -optional time  $\inf\{t \in \mathbb{R}_{\geq 0} : M_t - f(\tau \wedge t) < 1/n\}$  by  $\tau(n)$ . Then  $\sup_{n \in \mathbb{N}} \tau(n) = \zeta$  a.s. by Lemma 4.5. Recall that  $M_\infty = f(\tau)$  a.s. Therefore

$$E[f(\tau)] = E[M_{\tau(n)}] = E[f(\tau); \tau(n) = +\infty] + E[M_{\tau(n)}; \tau(n) < +\infty].$$

Since  $M_{\tau(n)} - f(\tau \wedge \tau(n)) \leq 1/n$  on  $\{\tau(n) < +\infty\}$ , the right hand side is dominated by

$$E[f(\tau \wedge \tau(n)); \tau(n) = +\infty] + E[f(\tau \wedge \tau(n)) + 1/n; \tau(n) < +\infty].$$

This reads  $E[f(\tau \wedge \tau(n))] + P(\tau(n) < +\infty)/n$ . It follows that

$$E[f(\tau)] \leq E[f(\tau \wedge \tau(n))] + 1/n \leq E[f(\zeta)] + 1/n \text{ for all } n \in \mathbb{N}$$

where we used that  $\tau(n) \leq \zeta$  a.s. and  $f$  is non-decreasing. Tending  $n$  to  $\infty$ , we get

$$E[f(\tau)] \leq E[f(\zeta)].$$

Since  $f$  is strictly increasing and  $\zeta \leq \tau$  a.s., we reach that  $\sup_{n \in \mathbb{N}} \tau(n) = \zeta = \tau$  a.s. □

**24.12 Lemma.** *Suppose that  $\tau$  is an  $\mathcal{F}$ -optional time,  $\{\tau = 0\} \in \mathcal{F}_0 \vee \text{Null}(P)$  and*

$$E[M_\tau; \tau > 0] = E[M_{\tau-}; \tau > 0] \text{ for all } M \in \mathfrak{Mart}_b(\mathcal{F}_+).$$

*Then there exists a predictable  $\mathcal{F}$ -optional time  $\tilde{\tau}$  such that  $\tau = \tilde{\tau}$  a.s.*

*Proof.* Set  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ ,  $t \mapsto t/(1+t)$ , which is bounded, strictly increasing and continuous. We select and fix  $M. \in \mathfrak{Mart}_b(\mathcal{F}_{\cdot+})$  such that  $M_t \in E[f(\tau)|\mathcal{F}_{t+}]$  for all  $t \in \mathbb{R}_{\geq 0}$ . Then

$$\sup_{n \in \mathbb{N}} \tau(n) = \tau \text{ a.s. where } \tau(n) := \inf\{t \in \mathbb{R}_{\geq 0} : M_t - f(\tau \wedge t) < 1/n\}$$

by Lemma 24.11. Since  $M_\tau \in E[f(\tau)|(\mathcal{F} \vee \text{Null}(P))_{\tau+}]$  by Corollary 14.12,

$$M_\tau = f(\tau) \text{ a.s.}$$

On the other hand  $t \mapsto \tau \wedge t$  is  $\mathcal{F}$ -adapted and  $f(\tau \wedge t) \leq f(\tau)$ , which implies that

$$f(\tau \wedge t) \leq M_t \text{ a.s. for all } t \in \mathbb{R}_{\geq 0}.$$

With the help of the sample path right continuity, we see that

$$f(\tau \wedge t) \leq M_t \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s. and hence } f(\tau) \leq M_{\tau-} \text{ a.s. on } \{\tau > 0\}.$$

Consequently, due to  $E[M_\tau; \tau > 0] = E[M_{\tau-}; \tau > 0]$ , we get

$$f(\tau) = M_{\tau-} \text{ a.s. on } \{\tau > 0\}.$$

Let  $n \in \mathbb{N}$ . Since  $M_{\tau(n)-} - f(\tau \wedge \tau(n)) \geq 1/n$  a.s. on  $\{\tau(n) > 0\}$ , it follows that

$$\begin{aligned} \frac{1}{n} P(0 < \tau(n) = \tau) &\leq E[M_{\tau(n)-} - f(\tau \wedge \tau(n)); 0 < \tau(n) = \tau] \\ &= E[M_{\tau-} - f(\tau); 0 < \tau(n) = \tau] = 0. \end{aligned}$$

Thus we conclude that  $P(0 < \tau(n) = \tau) = 0$  for each  $n \in \mathbb{N}$ . Now apply Lemma 24.8 together with Lemma 4.8(iii).  $\square$

**24.13 Theorem.** *Let  $\sigma \in \text{Time}(\mathcal{F})$ . Then the followings are equivalent to each other:*

- (i) *There exists a predictable  $\mathcal{F}$ -optional time  $\tilde{\sigma}$  such that  $\sigma = \tilde{\sigma}$  a.s.*
- (ii)  *$t \mapsto 1_{[\sigma, +\infty)}(t)$  is indistinguishable from an  $\mathcal{F} \vee \text{Null}(P)$ -predictable process.*
- (iii) *The  $\mathcal{F}$ -finite variation process  $t \mapsto 1_{[\sigma, +\infty)}$  is natural and  $\{\sigma = 0\} \in \mathcal{F}_0 \vee \text{Null}(P)$ .*

*Proof.* (i)  $\Rightarrow$  (ii) by Corollary 24.6(ii). (ii)  $\Rightarrow$  (iii) by Theorem 22.4(ii). (iii)  $\Rightarrow$  (i) by Lemma 24.12.  $\square$

**24.14 Lemma.** *Suppose that  $E$  is a metrizable topological space and  $X$  is an  $E$ -valued process with almost sure right continuous sample path and it is indistinguishable from an  $\mathcal{F} \vee \text{Null}(P)$ -predictable process. Then for each closed subset  $F$  there exists a predictable  $\mathcal{F}$ -optional time  $\tau$  such that  $\inf\{t \in \mathbb{R}_{\geq 0} : X_t \in F\} = \tau$  a.s.*

*Proof.* We see by Lemma 17.16 that there exists  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and

$t \mapsto X_t(\omega)$  is an  $(\mathcal{F} \vee \text{Null}(P)) \cap \Omega_0$ -predictable process with right continuous sample path on  $(\Omega_0, \mathcal{F} \cap \Omega_0, P|_{\Omega_0})$ .

We work on the probability space  $(\Omega_0, \mathcal{F} \cap \Omega_0, P|_{\Omega_0})$  with filtration  $\tilde{\mathcal{F}} := (\mathcal{F} \vee \text{Null}(P)) \cap \Omega_0$ . Then  $\sigma := \inf\{t \in \mathbb{R}_{\geq 0} : X_t \in F\}$  is an  $\tilde{\mathcal{F}}$ -optional time by Corollary 19.5. It follows by Lemma 17.2 that

$$\{(t, \omega) : \sigma(\omega) < t\} \in \text{Pred}(\tilde{\mathcal{F}}).$$

Due to the sample path right continuity of  $X$ .

$$\{(t, \omega) : t = \sigma(\omega)\} \subset \{(t, \omega) : X_t(\omega) \in F\} \subset \{(t, \omega) : \sigma(\omega) \leq t\}.$$

The set in the middle belongs to  $\text{Pred}(\tilde{\mathcal{F}})$  due to the predictability. Therefore

$$\{(t, \omega) : \sigma(\omega) \leq t\} = \{(t, \omega) : X_t(\omega) \in F\} \cup \{(t, \omega) : \sigma(\omega) < t\} \in \text{Pred}(\tilde{\mathcal{F}}).$$

This means that  $t \mapsto 1_{[\sigma, +\infty)}(t)$  is an  $\tilde{\mathcal{F}}$ -predictable process. Thus, invoking Theorem 24.13 together with Lemma 4.8(iii), we get the statement.  $\square$

**24.15 Corollary.** *Suppose that  $X$  is an  $\mathcal{F}$ -adapted process with almost sure right continuous and locally bounded sample path and indistinguishable from an  $\mathcal{F} \vee \text{Null}(P)$ -predictable process. Then  $X$  is  $\mathcal{F}$ -locally bounded.*

*Proof.* There exists a sequence  $\tau(\cdot)$  of predictable  $\mathcal{F}$ -optional times such that

$$\inf\{t \in \mathbb{R}_{\geq 0} : |X_t| \geq k\} = \tau(k) \text{ a.s. for all } k \in \mathbb{N}$$

by Lemma 24.14. We have that  $\tau(k) \leq \tau(k+1)$  a.s. and, since almost every sample path is locally bounded,  $\sup_{k \in \mathbb{N}} \tau(k) = +\infty$  a.s. We select and fix  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and

$$\inf\{t \in \mathbb{R}_{\geq 0} : |X_t| \geq k\} = \tau(k), \tau(k) \leq \tau(k+1) \text{ and } \sup_{k \in \mathbb{N}} \tau(k) = +\infty \text{ on } \Omega_0.$$

In particular we that

$$|X_t(\omega)| < k \text{ for all } t \in \mathbb{R}_{[0, \tau(k, \omega))} \text{ for all } \omega \in \{\tau(k) > 0\} \cap \Omega_0.$$

For each  $k \in \mathbb{N}$  there exists a sequence  $\sigma(k, \cdot)$  of  $\mathcal{F}$ -optional times such that

$$\sigma(k, l) \leq \sigma(k, l+1), \sup_{l \in \mathbb{N}} \sigma(k, l) = \tau(k) \text{ and } \{\tau(k) > 0\} \subset \{\sigma(k, l) < \tau(k)\}.$$

We introduce the following  $\mathcal{F}$ -optional times:

$$S(n) := \max\{\sigma(k, n) ; k \in \mathbb{N}_{\leq n}\}.$$

Since  $\sigma(k, n) \leq \sigma(k, n+1)$  for all  $k \in \mathbb{N}$ , we have that  $S(n) \leq S(n+1)$ . On the other hand  $\tau(k) = \sup_{n \in \mathbb{N}} \sigma(k, n) = \sup_{n \in \mathbb{N}: n \geq k} \sigma(k, n) \leq \sup_{n \in \mathbb{N}: n \geq k} S(n) = \sup_{n \in \mathbb{N}} S(n)$ . Therefore

$$S(n) \leq S(n+1) \text{ and } \sup_{n \in \mathbb{N}} S(n) = +\infty \text{ on } \Omega_0.$$

Let  $\omega \in \{S(n) > 0\} \cap \Omega_0$ . There exists  $k \in \mathbb{N}_{\leq n}$  with  $S(n, \omega) = \sigma(k, n, \omega)$ . It follows that

$$0 < S(n, \omega) = \sigma(k, n, \omega) \leq \tau(k, \omega) \text{ and hence } \sigma(k, n, \omega) < \tau(k, \omega) \leq \tau(n, \omega).$$

We thus get  $S(n) < \tau(n)$  on  $\{S(n) > 0\} \cap \Omega_0$ . This implies that

$$|X_t(\omega)| < n \text{ for all } t \in \mathbb{R}_{[0, S(n, \omega)]} \text{ for all } \omega \in \{S(n) > 0\} \cap \Omega_0.$$

Consequently the pair  $S(\cdot)$  and  $n \mapsto n$  is a reducing sequence.  $\square$

Given an  $\mathcal{F}$ -stopping time  $\tau$  we set  $\mathcal{F}_{\tau-} := \sigma(\{\{\tau > t\} \cap A; t \in \mathbb{R}_{>0}, A \in \mathcal{F}_t\}) \vee \mathcal{F}_0$ .

**24.16 Lemma.** *Suppose that  $\tau$  is an  $\mathcal{F}$ -stopping time.*

(i)  $\mathcal{F}_{\tau-} \subset \mathcal{F}_\tau$ . If  $t \in \mathbb{R}_{>0}$  and  $\tau(\Omega) = \{t\}$  then  $\mathcal{F}_{\tau-} = \bigvee_{s:s < t} \mathcal{F}_s$ .

(ii) If  $\sigma$  is an  $\mathcal{F}$ -stopping time and  $\sigma \leq \tau$  then  $\mathcal{F}_{\sigma-} \subset \mathcal{F}_{\tau-}$ .

(iii) Suppose that  $\sigma$  is an  $\mathcal{F}$ -optional time,  $\sigma \leq \tau$  and  $\{\sigma < +\infty, \tau > 0\} \subset \{\sigma < \tau\}$ . If  $A \in \mathcal{F}_{\sigma+}$  (and  $A \cap \{\sigma = 0\} \in \mathcal{F}_0$ ) then  $A \cap \{\sigma < +\infty, \tau > 0\} \in \mathcal{F}_{\tau-}$  ( $A \cap \{\sigma < +\infty\} \in \mathcal{F}_{\tau-}$ ).

(iv) Suppose that there exist a sequence  $\sigma(n)$  of  $\mathcal{F}$ -optional times ( $\mathcal{F}$ -stopping times) such that  $\sigma(n) \leq \sigma(n+1)$ ,  $\sup_{n \in \mathbb{N}} \sigma(n) = \tau$  and  $\{\tau > 0\} \subset \{\sigma(n) < \tau\}$ . Then  $\mathcal{F}_{\tau-} \subset \bigvee_{n=1}^{\infty} \mathcal{F}_{\sigma(n)+}$  and  $A \cap \{\tau > 0\} \in \mathcal{F}_{\tau-}$  for  $A \in \bigvee_{n=1}^{\infty} \mathcal{F}_{\sigma(n)+}$  ( $\mathcal{F}_{\tau-} = \bigvee_{n=1}^{\infty} \mathcal{F}_{\sigma(n)}$ ).

*Proof.* (i) If  $t \in \mathbb{R}_{>0}$  and  $A \in \mathcal{F}_t$  then  $\{\tau > t\} \cap A \in \mathcal{F}_\tau$  by Lemma 3.15(iv).

(ii) If  $t \in \mathbb{R}_{>0}$  and  $A \in \mathcal{F}_t$  then  $\{\sigma > t\} \cap A = \{\tau > t\} \cap (\{\sigma > t\} \cap A) \in \mathcal{F}_{\tau-}$ .

(iii) We have that  $\Omega = \{\sigma < \tau\} \cup \{\sigma = \tau = 0\} \cup \{\sigma = \tau = +\infty\}$ . Let  $A \in \mathcal{F}_{\sigma+}$ . Then  $A \cap \{\sigma < t\} \in \mathcal{F}_t$  for all  $t \in \mathbb{R}_{>0}$ .

$$A \cap \{\sigma < +\infty, \tau > 0\} = A \cap \{\sigma < \tau\} = \bigcup_{r \in \mathbb{Q}: r > 0} \{\tau > r\} \cap (A \cap \{\sigma < r\})$$

On the other hand  $A \cap \{\sigma < +\infty, \tau = 0\} = (A \cap \{\sigma = 0\}) \cap \{\tau = 0\}$ .

(iv) Since  $B \cap \{\sigma(n) \geq t\} \in (\mathcal{F}_{\cdot+})_{\sigma(n)} = \mathcal{F}_{\sigma(n)+}$  for all  $B \in \mathcal{F}_{t+}$  by Lemma 3.15(iv),

$$\{\tau > t\} \cap A = \bigcup_{n=1}^{\infty} ((\{\tau > t\} \cap A) \cap \{\sigma(n) \geq t\}) \in \bigvee_{n=1}^{\infty} \mathcal{F}_{\sigma(n)+} \text{ for all } t \in \mathbb{R}_{>0} \text{ and } A \in \mathcal{F}_t.$$

It follows that  $\mathcal{F}_{\tau-} \subset \bigvee_{n=1}^{\infty} \mathcal{F}_{\sigma(n)+}$ . This time we have that  $\{\sigma(n) = +\infty\} = \emptyset$ . Therefore if  $A \in \mathcal{F}_{\sigma(n)+}$  then  $A \cap \{\tau > 0\} = A \cap \{\sigma(n) < +\infty, \tau > 0\} \in \mathcal{F}_{\tau-}$  by (iii).  $\square$

**24.17 Lemma.** *Suppose that  $\tau$  is a predictable  $\mathcal{F}$ -optional time and  $f : \Omega \rightarrow \mathbb{R}$  is  $\mathcal{F}_{\tau-}$ -measurable. Then the process with sample path  $t \mapsto f(\omega)1_{[\tau(\omega), +\infty)}(t)$  is  $\mathcal{F}$ -predictable.*

*Proof.* The optional time  $\tau$  being predictable, there exist a sequence  $\sigma(n)$  of  $\mathcal{F}$ -optional times such that  $\sigma(n) \leq \sigma(n+1)$ ,  $\sup_{n \in \mathbb{N}} \sigma(n) = \tau$  and  $\{\tau > 0\} \subset \{\sigma(n) < \tau\}$ . Given  $B \in \bigcup_{k=1}^{\infty} \mathcal{F}_{\sigma(k)+}$ , there exists  $n \in \mathbb{N}$  such that  $B \in \mathcal{F}_{\sigma(n)+}$ . The process with sample path  $t \mapsto 1_B(\omega)1_{(\sigma(k,\omega), +\infty)}(t)$  is left continuous and  $\mathcal{F}$ -adapted provided  $k \geq n$ . As  $k$  tends to  $\infty$  this sequence of processes point wise converges to the process with sample path

$$t \mapsto 1_B(\omega)1_{[\tau(\omega), +\infty) \cap (0, +\infty)}(t),$$

which is  $\mathcal{F}$ -predictable. The  $\mathcal{F}$ -predictability is preserved for  $B \in \bigvee_{n=1}^{\infty} \mathcal{F}_{\sigma(n)+}$  as the monotone class theorem shows. Let  $A \in \mathcal{F}_{\tau-}$ . We infer by Lemma 24.16(iv) that

the process with sample path  $t \mapsto 1_A(\omega)1_{[\tau(\omega), +\infty) \cap (0, +\infty)}(t)$  is  $\mathcal{F}$ -predictable.

On the other hand, since  $\tau$  is an  $\mathcal{F}$ -stopping time and, by Lemma 24.16(i),  $\mathcal{F}_{\tau-} \subset \mathcal{F}_\tau$ , we have that  $A \cap \{\tau = 0\} \in \mathcal{F}_0$ . This means that

the process with sample path  $t \mapsto 1_{A \cap \{\tau = 0\}}(\omega)1_{\{0\}}(t)$  is  $\mathcal{F}$ -predictable.

Being the sum of two predictable processes,  $t \mapsto 1_A(\omega)1_{[\tau(\omega), +\infty)}(t)$  is  $\mathcal{F}$ -predictable.  $\square$

**24.18 Lemma.** *Suppose that  $M$  is an  $\mathcal{F}_+$ -martingale with almost sure right continuous path and  $\tau$  is a predictable  $\mathcal{F}$ -optional time. Then*

$$E[M_{\tau-}; A \cap \{0 < \tau \leq t\}] = E[M_{\tau}; A \cap \{0 < \tau \leq t\}]$$

for all  $A \in \mathcal{F}_{\tau-}$  and  $t \in \mathbb{R}_{>0}$ .

*Proof.* Since  $t \mapsto 1_A 1_{[\tau, +\infty)}(t)$  is an  $\mathcal{F}$ -predictable finite variation process by Lemma 24.17, we see by Theorem 22.4(ii) that  $t \mapsto 1_A 1_{[\tau, +\infty)}(t)$  is natural. This is exactly what we want. However we give a more straightforward discussion. The optional time  $\tau$  being predictable, there exist a sequence  $\sigma(n)$  of  $\mathcal{F}$ -optional times such that  $\sigma(n) \leq \sigma(n+1)$ ,  $\sup_{n \in \mathbb{N}} \sigma(n) = \tau$  and  $\{\tau > 0\} \subset \{\sigma(n) < \tau\}$ . Recall that  $\text{Time}(\mathcal{F}) = \text{Time}(\mathcal{F}_+)$ . Let  $t \in \mathbb{R}_{>0}$ . According to Corollary 3.17(i), we have that

$$M_{\sigma(k) \wedge t} \in E[M_{\tau \wedge t} | (\mathcal{F}_+ \vee \text{Null}(P))_{\sigma(k)+}] \text{ for all } k \in \mathbb{N}.$$

Let  $B \in \bigcup_{k=1}^{\infty} \mathcal{F}_{\sigma(k)+}$ , that is,  $B \in \mathcal{F}_{\sigma(n)+}$  for some  $n \in \mathbb{N}$ . Since  $\{\sigma(k) < t\} \in \mathcal{F}_{\sigma(k)+}$  for all  $k \in \mathbb{N}$  and  $\mathcal{F}_{\sigma(n)+} \subset \mathcal{F}_{\sigma(k)+} \subset (\mathcal{F}_+ \vee \text{Null}(P))_{\sigma(k)+}$  for all  $k \in \mathbb{N}_{\geq n}$ , it follows that

$$E[M_{\sigma(k) \wedge t}; B \cap \{\sigma(k) < t\}] = E[M_{\tau \wedge t}; B \cap \{\sigma(k) < t\}] \text{ for all } k \in \mathbb{N}_{\geq n}.$$

We see that  $\{\sigma(k) < t\} = \{\tau = 0, \sigma(k) = 0\} \cup \{\tau > 0, \sigma(k) < t\}$ . Therefore

$$E[M_{\sigma(k)}; B \cap \{\tau > 0, \sigma(k) < t\}] = E[M_{\tau \wedge t}; B \cap \{\tau > 0, \sigma(k) < t\}] \text{ for all } k \in \mathbb{N}_{\geq n}.$$

Since  $\{\sigma(k+1) < t\} \subset \{\sigma(k) < t\}$  and, by Lemma 4.3,  $\bigcap_{k=1}^{\infty} \{\sigma(k) < t\} = \{\tau \leq t\}$ , the right hand side converges to  $E[M_{\tau \wedge t}; B \cap \{0 < \tau \leq t\}]$  as  $k$  tends to  $\infty$ . On the other hand there exists  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and  $M(\omega)$  admits left-hand limits everywhere for all  $\omega \in \Omega_0$ . Since  $\sigma(k) < \tau$  on  $\{\tau > 0\}$ , the sequence  $M_{\sigma(k)} 1_{\{\tau > 0, \sigma(k) < t\}}$  converges to  $M_{\tau-} 1_{\{0 < \tau \leq t\}}$  on  $\Omega_0$ . Taking the uniform integrability of  $M_{\sigma(k) \wedge t}$  into account, we infer that

$$E[M_{\tau-}; B \cap \{0 < \tau \leq t\}] = E[M_{\tau \wedge t}; B \cap \{0 < \tau \leq t\}] = E[M_{\tau}; B \cap \{0 < \tau \leq t\}].$$

The above extends to  $\bigvee_{n=1}^{\infty} \mathcal{F}_{\sigma(n)+}$  automatically. Since  $\mathcal{F}_{\tau-} \subset \bigvee_{n=1}^{\infty} \mathcal{F}_{\sigma(n)+}$  by Lemma 24.16, we reach the statement.  $\square$

## 25 Semigroup of measure kernel and Markov process

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\mathcal{F}$  be a filtration of sub  $\sigma$ -fields.

$S$  is a locally compact Hausdorff space with countable base and  $\kappa_t(\cdot; \star)$  is a one parameter family of probability measure kernels on  $S \times \text{Borel}(S)$  such that

$$\int_S \kappa_t(\cdot; A) \kappa_s(x; \cdot) = \kappa_{t+s}(x; A) \text{ for all } t, s \in \mathbb{R}_{\geq 0}, x \in S, A \in \text{Borel}(S).$$

We additionally assume that  $\kappa_0(x; \cdot) = \delta_x$  for all  $x \in S$  and

$$t \mapsto \int_S f \kappa_t(x; \cdot) \text{ is right continuous at } 0 \text{ for all } x \in S, f \in C_\infty(S).$$

Then, given  $t(1), \dots, t(n) \in \mathbb{R}_{\geq 0}$  with  $t(1) < \dots < t(n)$  and  $A_1, \dots, A_n \in \text{Borel}(S)$ , we can inductively define a  $\text{Borel}(S)$ -measurable function by  $\kappa(x; t; A) = \kappa_t(x; A)$  and

$$\kappa(x; t(1), \dots, t(n); A_1, \dots, A_n) := \int_{A_1} \kappa(\cdot; t(2) - t(1), \dots, t(n) - t(1); A_2, \dots, A_n) \kappa_{t(1)}(x; \cdot).$$

**25.1 Lemma.** *Suppose that  $0 \leq s(1) < \dots < s(m-1) < s(m)$  and  $A_1, \dots, A_{m-1} \in \text{Borel}(S)$ . Then  $\kappa(\cdot; s(1), \dots, s(m); A_1, \dots, A_{m-1}, \cdot)$  is a probability measure kernel on  $S \times \text{Borel}(S)$  and*

$$\begin{aligned} & \int_A \kappa(\cdot; t(1), \dots, t(n); B_1, \dots, B_n) \kappa(x; s(1), \dots, s(m); A_1, \dots, A_{m-1}, \cdot) \\ &= \kappa(x; s(1), \dots, s(m), t(1) + s(m), \dots, t(n) + s(m); A_1, \dots, A_{m-1}, A, B_1, \dots, B_n) \end{aligned}$$

for all  $t(1), \dots, t(n) \in \mathbb{R}_{\geq 0}$  with  $0 < t(1) < \dots < t(n)$  and  $A, B_1, \dots, B_n \in \text{Borel}(S)$ .

Given  $t_1, \dots, t_n \in \mathbb{R}_{\geq 0}$ ,  $A_1, \dots, A_n \in \text{Sbset}(S)$ , and  $f_1, \dots, f_n \in \text{Map}(S, \mathbb{R})$ , we set

$$\begin{aligned} \text{cylldr}(t_1, \dots, t_n; A_1, \dots, A_n) &:= \{w \in \text{Map}(\mathbb{R}_{\geq 0}, S) : w(t_1) \in A_1, \dots, w(t_n) \in A_n\}, \\ \text{cylldr}[t_1, \dots, t_n; f_1, \dots, f_n] &: \text{Map}(\mathbb{R}_{\geq 0}, S) \rightarrow \mathbb{R}, w \mapsto f_1(w(t_1)) \cdots f_n(w(t_n)). \end{aligned}$$

Given a subinterval  $I$  of  $\mathbb{R}_{\geq 0}$ , denote by  $\text{Cylldr}(I, S)$  the collection of all the sets  $\text{cylldr}(t_1, \dots, t_n; A_1, \dots, A_n)$  with  $n \in \mathbb{N}$ ,  $t_1, \dots, t_n \in I$  and  $A_1, \dots, A_n \in \text{Borel}(S)$ .

**25.2 Theorem.** *There exists a unique measure kernel  $\mu(\cdot; \star)$  on  $S \times \sigma(\text{Cylldr}(\mathbb{R}_{\geq 0}, S))$  such that*

$$\mu(x; \text{cylldr}(t(1), \dots, t(n); A_1, \dots, A_n)) = \kappa(x; t(1), \dots, t(n); A_1, \dots, A_n)$$

for all  $t(1), \dots, t(n) \in \mathbb{R}_{\geq 0}$  with  $t(1) < \dots < t(n)$  and  $A_1, \dots, A_n \in \text{Borel}(S)$ .

*Proof.* We can apply the Kolmogorov extension theorem to show the existence and the uniqueness by virtue of Lemma 25.1. One deduces the  $\text{Borel}(S)$ -measurability of  $x \mapsto \mu(x; A)$  for each  $A \in \sigma(\text{Cylldr}(\mathbb{R}_{\geq 0}, S))$  from that of the functions  $x \mapsto \kappa(x; t(1), \dots, t(n); A_1, \dots, A_n)$  by invoking the monotone class theorem.  $\square$

We denote by  $\kappa^\circ$  the probability measure kernel described in Theorem 25.2.

**25.3 Corollary.** For all  $x \in S, r \in \mathbb{R}_{\geq 0}$ ,  $A \in \sigma(\text{Cyldr}(\mathbb{R}_{[0,r]}, S))$  and  $B \in \sigma(\text{Cyldr}(\mathbb{R}_{\geq 0}, S))$

$$\int_A \kappa^\circ(\text{eval}_r; B) \kappa^\circ(x; \cdot) = \kappa^\circ(x; \{\text{eval}_{+r} \in B\} \cap A)$$

where  $\text{eval}_t : \text{Map}(\mathbb{R}_{\geq 0}, S) \rightarrow S$  is the evaluation  $w \rightarrow w(t)$  for each  $t \in \mathbb{R}_{\geq 0}$ . In particular

$$T_t f(\text{eval}_r) \in E[f(\text{eval}_{t+r}) | \sigma(\text{Cyldr}(\mathbb{R}_{[0,r]}, S))] \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ and } f \in C_\infty(S)$$

where the conditional expectation refers to the probability measure  $\kappa^\circ(x; \cdot)$ .

*Proof.* Theorem 25.2 and Lemma 25.1. □

Given  $t \in \mathbb{R}_{\geq 0}$ ,  $f \in \text{bBorel}(S, \mathbb{R})$  denote by  $T_t f$  the function  $x \mapsto \int_S f \kappa_t(x; \cdot)$  on  $S$ .

**25.4 Lemma.** Suppose that  $s \in \mathbb{R}_{\geq 0}$  and  $X_\cdot$  is an  $\mathcal{F}$ -adapted  $S$ -valued process, and

$$T_t f(X_r) \in E[f(X_{t+r}) | \mathcal{F}_r] \text{ for all } t \in \mathbb{R}_{\geq 0}, r \in \mathbb{R}_{\geq s} \text{ and } f \in C_\infty(S).$$

- (i)  $\kappa^\circ(X_r; B) \in P(\{X_{+r} \in B\} | \mathcal{F}_r)$  for all  $r \in \mathbb{R}_{\geq s}$  and  $B \in \sigma(\text{Cyldr}(\mathbb{R}_{\geq 0}, S))$ .
- (ii)  $t \mapsto X_t$  and  $t \mapsto f(X_t)$  with  $f \in C(S)$  are right continuous in probability on  $\mathbb{R}_{\geq s}$ .

*Proof.* (i) It suffices to show the following claim: if  $0 \leq t(1) < t(2) < \dots < t(n)$ ,  $r \in \mathbb{R}_{\geq s}$ ,  $B_1, B_2, \dots, B_n \in \text{Borel}(S)$  and  $A \in \mathcal{F}_r$  then

$$\begin{aligned} & E[\kappa^\circ(X_r; \text{cyldr}(t(1), \dots, t(n); B_1, \dots, B_n)); A] \\ &= P(\{X_{+r} \in \text{cyldr}(t(1), \dots, t(n); B_1, \dots, B_n)\} \cap A). \end{aligned}$$

This reads by Theorem 25.2 that

$$\begin{aligned} & E[\kappa(X_r; t(1), \dots, t(n); B_1, \dots, B_n); A] \\ &= P(\{X_{+r} \in \text{cyldr}(t(1), \dots, t(n); B_1, \dots, B_n)\} \cap A). \end{aligned}$$

We carry out the proof by using the induction. The assumption implies that

$$E\left[\int_S f \kappa_t(X_r \cdot); A\right] = E[f(X_{t+r}); A] \text{ for all } f \in C_\infty(S).$$

Since the indicator function of any open subset of  $S$  is a monotone increasing limit of functions in  $C_\infty(S)$ , we get the equality in question for  $n = 1$  by the Dynkin class theorem:

$$E[\kappa_{t(1)}(X_r; B_1); A] = P(\{X_{t(1)+r} \in B_1\} \cap A),$$

which is a special case of  $E[\int_S f \kappa_t(X_r \cdot); A] = E[f(X_{t+r}); A]$  where  $f \in \text{bBorel}(S, \mathbb{R})$ . Suppose that  $k \in \mathbb{N}$  and the equality in question holds for all  $n \in \mathbb{N}_{\leq k}$ . We set  $g := \kappa(\cdot; t(2) - t(1), \dots, t(n+1) - t(1); B_2, \dots, B_{n+1})$ . Then it follows that

$$\begin{aligned} & E[\kappa(X_r; t(1), \dots, t(n+1); B_1, \dots, B_{n+1}); A] \\ &= E\left[\int_{B_1} g \kappa_{t(1)}(X_r; \cdot); A\right] = E[(1_{B_1} g)(X_{t(1)+r}); A]. \end{aligned}$$

The right hand side equals

$$E[\kappa(X_{t(1)+r}; t(2) - t(1), \dots, t(n+1) - t(1); B_2, \dots, B_{n+1}); \{X_{t(1)+r} \in B_1\} \cap A].$$

The induction assumption with  $r$  replaced by  $t(1) + r$  shows that the above coincides with

$$\begin{aligned} &P(\{X_{. + t(1)+r} \in \text{cyldr}(t(2) - t(1), \dots, t(n+1) - t(1); B_2, \dots, B_{n+1}), X_{t(1)+r} \in B_1\} \cap A) \\ &= P(\{X_{. + r} \in \text{cyldr}(t(1), \dots, t(n+1); B_1, \dots, B_{n+1})\} \cap A). \end{aligned}$$

Thus the equality in question also holds for  $n = k + 1$ .

(ii) We exploit the additional property of the measure kernels:  $\kappa_t(x; \cdot)$  converges to  $\delta_x$  in  $C_b(S)$ -weak topology as  $t \downarrow 0$ , which so far played no role. Suppose that  $r, t \in \mathbb{R}_{\geq 0}$ ,  $s \leq r < t$  and  $g : S \times S \rightarrow \mathbb{R}$  is bounded and continuous. Then we have that

$$E[g(X_r, X_t)] = E\left[\int_S g(X_r, \cdot) \kappa_{t-r}(X_r; \cdot)\right]$$

according to (i). Since  $\int_S g(x, \cdot) \kappa_{t-r}(x; \cdot)$  converges to  $g(x, x)$  as  $t \downarrow r$  for all  $x \in S$ , the right hand side above converges to  $E[g(X_r, X_r)]$  by the dominated convergence theorem. Specifying  $g$  as  $\min\{\text{dist}(\cdot, \star), 1\}$  respectively  $\min\{|f(\cdot) - f(\star)|, 1\}$ , we get the claim.  $\square$

**25.5 Lemma.** (i) *If  $f \in C_b(S)$  then  $t \mapsto T_t f(x)$  is right continuous for all  $x \in S$ .*  
(ii) *If  $f \in \text{bBorel}(S, \mathbb{R})$  then  $(t, x) \mapsto T_t f(x)$  is  $\text{Borel}(\mathbb{R}_{\geq 0}) \otimes \text{Borel}(S)$ -measurable.*

*Proof.* Let  $f \in C_b(S)$ . Since  $T_t f(x) = \int f(\text{eval}_t) \kappa^\circ(x; \cdot)$ , it follows by Corollary 25.3 and Lemma 25.4 that  $t \mapsto T_t f(x)$  is right continuous for all  $x \in S$ . Taking into account that  $x \mapsto T_t f(x)$  is  $\text{Borel}(S)$ -measurable for all  $t \in \mathbb{R}_{\geq 0}$ , we see that  $(t, x) \mapsto T_t f(x)$  is measurable. The indicator function of any open subset of  $S$  is a non-decreasing limit of functions in  $C_\infty(S)$ . We therefore get (ii) by the usual monotone class argument.  $\square$

Given  $\alpha \in \mathbb{R}_{>0}$ ,  $f \in \text{bBorel}(S, \mathbb{R})$ , set  $U^\alpha f : S \rightarrow \mathbb{R}$ ,  $x \mapsto \int_{(0, +\infty)} e^{-\alpha t} T_t f(x) \lambda(dt)$ .

**25.6 Lemma.** (i) *If  $f \in \text{bBorel}(S, \mathbb{R})$  then  $U^\alpha f$  is  $\text{Borel}(S)$ -measurable for all  $\alpha \in \mathbb{R}_{>0}$ . If  $f \in C_b(S)$  then  $\alpha U^\alpha f(x)$  converges to  $f(x)$  as  $\alpha$  tends to  $+\infty$  for all  $x \in S$ .*  
(ii) *If  $T_t$  sends  $C_\infty(S)$  into itself for all  $t \in \mathbb{R}_{>0}$  then so does  $U^\alpha$  for all  $\alpha \in \mathbb{R}_{>0}$ .*  
(iii) *If  $U^\alpha$  sends  $C_\infty(S)$  into itself for all  $\alpha \in \mathbb{R}_{>0}$  then  $\alpha U^\alpha f$  converges to  $f$  uniformly as  $\alpha$  tends to  $+\infty$  for all  $f \in C_\infty(S)$ .*

*Proof.* (i) We get the measurability of  $U^\alpha f$  by Lemma 25.5(i) and Fubini's theorem. Let  $f \in C_b(S)$ . Since  $T_t f(x)$  converges to  $f(x)$  for all  $x \in S$  as  $t$  tends to 0 by Lemma 25.5(i), it therefore follows by the dominated convergence theorem that

$$\alpha U^\alpha f(x) = \int_0^\infty T_{t/\alpha} f(x) e^{-t} dt \text{ converges to } f(x) \text{ for all } x \in S \text{ as } \alpha \text{ tends to } +\infty.$$

(ii) Suppose that  $T_t f \in C_\infty(S)$  for all  $t \in \mathbb{R}_{>0}$  and  $f \in C_\infty(S)$ . Since  $|T_t f(x)| \leq \|f\|$  for all  $t \in \mathbb{R}_{>0}$  and  $x \in S$  where  $\|f\| := \sup_{x \in S} |f(x)|$ , we infer by the dominated convergence theorem that  $U^\alpha f \in C_\infty(S)$ .

(iii) Fubini's theorem shows that

$$\int_S U^\alpha f \kappa_t(x; \cdot) = \int_{(0, \infty) \times S} T_s f(y) e^{-\alpha s} \lambda(ds) \kappa_t(x; dy) = \int_{(0, \infty)} T_{s+t} f(x) e^{-\alpha s} \lambda(ds)$$

for  $\alpha \in \mathbb{R}_{>0}$ ,  $f \in \text{bBorel}(S, \mathbb{R})$ ,  $t \in \mathbb{R}_{\geq 0}$  and  $x \in S$ , which reads

$$T_t U^\alpha f(x) = \int_{(t, \infty)} T_s f(x) e^{-\alpha(s-t)} \lambda(ds).$$

We first deduce the resolvent equation. We see that

$$\int_{(0, \infty)} T_t U^\alpha f(x) e^{-\beta t} \lambda(dt) = \int_{(s, t): 0 < t < s} T_s f(x) e^{-\alpha(s-t)} e^{-\beta t} \lambda(ds) \lambda(dt)$$

for  $\alpha, \beta \in \mathbb{R}_{>0}$ ,  $f \in \text{bBorel}(S, \mathbb{R})$  and  $x \in S$ . If  $\alpha \neq \beta$  then the right hand equals

$$\int_{(0, \infty)} T_s f(x) e^{-\alpha s} \frac{e^{(\alpha-\beta)s} - 1}{\alpha - \beta} \lambda(ds) = \frac{U^\beta f(x) - U^\alpha f(x)}{\alpha - \beta}.$$

Thus we get the resolvent equation  $(\alpha - \beta)U^\beta U^\alpha f = U^\beta f - U^\alpha f$ . Since  $U^\alpha$  sends  $C_\infty(S)$  into itself for all  $\alpha \in \mathbb{R}_{>0}$ , this implies that

$$U^\alpha f = U^\beta(f - (\alpha - \beta)U^\alpha f) \in U^\beta C_\infty(S) \text{ for all } f \in C_\infty(S).$$

The role of parameters  $\alpha, \beta$  being symmetric, we see that  $U^\alpha C_\infty(S) = U^\beta C_\infty(S)$ . Denote this common linear subspace of  $C_\infty(S)$  by  $\mathcal{R}$ . Let  $g \in \mathcal{R}$ . Then  $g = U^1 f$  for some  $f \in C_\infty(S)$ . Since  $\beta U^\beta U^1 f - U^1 f = U^\beta(U^1 f - f)$  by the resolvent equation, it follows that

$$|\beta U^\beta g(x) - g(x)| \leq \int_0^\infty |T_t(U^1 f - f)(x)| e^{-\beta t} dt \leq \|U^1 f - f\|/\beta \text{ for all } x \in S.$$

The right hand side converges to 0 as  $\beta$  tends to  $+\infty$ . Hence

$$\|\beta U^\beta g - g\| \text{ converges to 0 as } \beta \text{ tends to } +\infty \text{ for all } g \in \mathcal{R}.$$

We identify the norm closure of  $\mathcal{R}$ . Being a convex subset, its norm closure coincides with the weak closure. Let  $f \in C_\infty(S)$ . According to (i),

$$\beta U^\beta f(x) \text{ converges to } f(x) \text{ for all } x \in S \text{ as } \beta \text{ tends to } +\infty.$$

The dual space of  $C_\infty(S)$  is canonically identified with the set of all finite signed Borel measures on  $S$ . Invoking the dominated convergence theorem, we infer that

$$\beta U^\beta f \text{ weakly converges to } f \text{ as } \beta \text{ tends to } +\infty.$$

Consequently the weak closure of  $\mathcal{R}$  coincides with  $C_\infty(S)$  and so does the norm closure. Finally, given  $\varepsilon \in \mathbb{R}_{>0}$ , choose  $g \in \mathcal{R}$  such that  $\|f - g\| < \varepsilon$ . Taking into account that  $\|U^\beta(f - g)\| \leq \|f - g\|/\beta$  we reach that

$$\|\beta U^\beta f - f\| \leq \|\beta U^\beta g - g\| + \|\beta U^\beta(f - g)\| + \|f - g\| \leq \|\beta U^\beta g - g\| + 2\varepsilon.$$

The first term of the right hand side converges to 0 as  $\beta$  tends to  $+\infty$ . □

We characterize the Markov property in terms of Laplace transform.

**25.7 Lemma.** *Suppose that  $r \in \mathbb{R}_{\geq 0}$ ,  $f \in C_b(S)$ ,  $X$  is an  $\mathcal{F}$ -adapted  $S$ -valued process, and  $t \mapsto f(X_t)$  is right continuous in probability. Then*

$$\begin{aligned} T_t f(X_r) &\in E[f(X_{t+r})|\mathcal{F}_r] \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ if and only if} \\ E[U^\alpha f(X_r); A]e^{-\alpha r} &= \int_r^\infty E[f(X_t); A]e^{-\alpha t} dt \text{ for all } \alpha \in \mathbb{R}_{> 0} \text{ and } A \in \mathcal{F}_r. \end{aligned}$$

Let  $s \in \mathbb{R}_{\geq 0}$ . If  $f(x) \geq 0$  for all  $x \in S$  and the above condition holds for all  $r \in \mathbb{R}_{\geq s}$  then  $t \mapsto U^\alpha f(X_t)e^{-\alpha t}$  is an  $\mathcal{F}$ -supermartingale after time  $s$ .

*Proof.* Let  $\alpha \in \mathbb{R}_{> 0}$  and  $A \in \mathcal{F}_r$ . Since  $(t, x) \mapsto T_t f(x)$  is measurable by Lemma 25.5(ii), invoking Fubini's theorem, we see that

$$E[U^\alpha f(X_r); A] = \int_0^{+\infty} E[T_t f(X_r); A]e^{-\alpha t} dt.$$

Note that  $t \mapsto E[T_t f(X_r); A]$  is right continuous by Lemma 25.5(i). Since  $t \mapsto f(X_t)$  is right continuous in probability,  $t \mapsto E[f(X_{t+r}); A]$  is also right continuous. We see that

$$\int_0^\infty E[f(X_{t+r}); A]e^{-\alpha t} dt = e^{\alpha r} \int_r^\infty E[f(X_t); A]e^{-\alpha t} dt$$

by the translation invariance of the Lebesgue measure. It follows that

$$\begin{aligned} \int_0^{+\infty} E[T_t f(X_r); A]e^{-\alpha t} dt &= \int_0^{+\infty} E[f(X_{t+r}); A]e^{-\alpha t} dt \\ \text{if and only if } E[U^\alpha f(X_r); A]e^{-\alpha r} &= \int_r^\infty E[f(X_t); A]e^{-\alpha t} dt. \end{aligned}$$

Thus we get the equivalence in question by applying the uniqueness of Laplace transform.  $\square$

**25.8 Definition.** Suppose that  $X$  is an  $S$ -valued process. A subset  $\Theta$  of  $\Omega$  is called an  $X$ -cylinder set based at time parameters up to  $t \in \mathbb{R}_{\geq 0}$  if there exists  $B \in \text{Cyldr}(\mathbb{R}_{[0,t]}, S)$  such that  $\Theta = \{\omega \in \Omega : X(\omega) \in B\}$ .

Let  $t \in \mathbb{R}_{\geq 0}$ .  $\text{Cyldr}\{X\}_{\leq t}$  stands for the family of all  $X$ -cylinder sets based at time parameters up to  $t$  and  $\sigma\{X\}_{\leq t}$  the  $\sigma$ -field over  $\Omega$  generated by  $\text{Cyldr}\{X\}_{\leq t}$ .

**25.9 Theorem.** *Suppose that  $X$  is an  $\mathcal{F}$ -adapted  $S$ -valued process,  $t \mapsto U^\alpha f(X_t)$  is right continuous in probability for all  $\alpha \in \mathbb{R}_{> 0}$  and  $f \in C_\infty(S)$ , and*

$$T_t f(X_r) \in E[f(X_{t+r})|\mathcal{F}_r] \text{ for all } t \in \mathbb{R}_{\geq 0}, r \in \mathbb{R}_{\geq 0} \text{ and } f \in C_\infty(S).$$

- (i) For each lower semi-integrable and  $\sigma\{X\} \vee \text{Null}(P)$ -measurable random variable  $Y$  the class  $E[Y|\mathcal{F}_{t+} \vee \text{Null}(P)]$  contains a  $\sigma\{X\}_{\leq t}$ -measurable representative for all  $t \in \mathbb{R}_{\geq 0}$ .
- (ii)  $(\mathcal{F}_{t+} \vee \text{Null}(P)) \cap (\sigma\{X\} \vee \text{Null}(P)) = \sigma\{X\}_{\leq t} \vee \text{Null}(P)$  for all  $t \in \mathbb{R}_{\geq 0}$ .
- (iii) A process  $M$  is a  $\sigma\{X\}_{\leq \cdot} \vee \text{Null}(P)$ -martingale if and only if it is an  $\mathcal{F}_{\cdot+} \vee \text{Null}(P)$ -martingale and  $M_t$  is  $\sigma\{X\}_{\leq t} \vee \text{Null}(P)$ -measurable for each  $t \in \mathbb{R}_{\geq 0}$ .

*Proof.* (i) Let  $f \in C_\infty(S)$ ,  $s \in \mathbb{R}_{\geq 0}$  and  $A \in \mathcal{F}_{s+}$ . According to Lemma 25.4(ii),  $t \mapsto f(X_t)$  is right continuous in probability. Since  $A \in \mathcal{F}_r$  for all  $r \in \mathbb{R}_{> s}$ , we see by Lemma 25.7 that

$$E[U^\alpha f(X_r); A]e^{-\alpha r} = \int_r^\infty E[f(X_t); A]e^{-\alpha t} dt \text{ for all } r \in \mathbb{R}_{> s} \text{ and } \alpha \in \mathbb{R}_{> 0}.$$

Since  $t \mapsto U^\alpha f(X_t)$  is right continuous in probability, tending  $r$  to  $s$ , we get

$$E[U^\alpha f(X_s); A]e^{-\alpha s} = \int_s^\infty E[f(X_t); A]e^{-\alpha t} dt \text{ for all } \alpha \in \mathbb{R}_{> 0} \text{ and } A \in \mathcal{F}_{s+}.$$

Applying Lemma 25.7 with Lemma 25.4(ii) taking into account, we infer that

$$T_t f(X_s) \in E[f(X_{t+s}) | \mathcal{F}_{s+} \vee \mathcal{N}] \text{ for all } t \in \mathbb{R}_{\geq 0}, s \in \mathbb{R}_{\geq 0} \text{ and } f \in C_\infty(S).$$

Here  $\mathcal{N} := \text{Null}(P)$ . Let  $t \in \mathbb{R}_{\geq 0}$ . Given  $\Theta \in \sigma\{X.\}$ , there exists  $A \in \text{Cyldr}(\mathbb{R}_{[0,t]}, S)$  and  $B \in \text{Cyldr}(\mathbb{R}_{\geq 0}, S)$  such that  $\Theta = \{X. \in A, X_{.+t} \in B\}$ . It follows by Lemma 25.4(i) that

$$\kappa^\circ(X_t; B) \in E[1_B(X_{.+t}) | \mathcal{F}_{t+} \vee \mathcal{N}].$$

Since  $\{X. \in A\} \in \sigma\{X.\}_{\leq t} \subset \mathcal{F}_{t+}$ , we see that

$$1_A(X.)\kappa^\circ(X_t; B) \in E[1_A(X.)1_B(X_{.+t}) | \mathcal{F}_{t+} \vee \mathcal{N}] = E[1_\Theta | \mathcal{F}_{t+} \vee \mathcal{N}].$$

Observe that  $1_A(X.)\kappa^\circ(X_t; B)$  is  $\sigma\{X.\}_{\leq t}$ -measurable. We introduce the following family:

$$\mathcal{D} := \{\Theta \in \sigma\{X.\} : \exists Y \text{ } \sigma\{X.\}_{\leq t}\text{-measurable, non-negative and } Y \in E[1_\Theta | \mathcal{F}_{t+} \vee \mathcal{N}]\},$$

which is a Dynkin system by Theorem 1.10. Since  $\text{Cyldr}\{X.\} \subset \mathcal{D}$ , we reach the statement by invoking the monotone class theorem.

(ii) Let  $A \in (\mathcal{F}_{t+} \vee \mathcal{N}) \cap (\sigma\{X.\} \vee \mathcal{N})$ . Since  $1_A \in E[1_A | \mathcal{F}_{t+} \vee \mathcal{N}]$ , there exists a  $\sigma\{X.\}_{\leq t}$ -measurable modification  $Y$  of  $1_A$  by (i), which means  $A \in \sigma\{X.\}_{\leq t} \vee \mathcal{N}$ .

(iii) Suppose that  $M.$  is a  $\sigma\{X.\}_{\leq} \vee \mathcal{N}$ -martingale,  $s, t \in \mathbb{R}_{\geq 0}$  and  $s < t$ . Then there exists a  $\sigma\{X.\}_{\leq s}$ -measurable random variable  $Y$  such that  $Y \in E[M_t | \mathcal{F}_{s+} \vee \mathcal{N}]$  by (i). Since  $\sigma\{X.\}_{\leq s} \subset \mathcal{F}_s \subset \mathcal{F}_{s+}$ , we have that  $Y \in E[M_t | \sigma\{X.\}_{\leq s} \vee \mathcal{N}]$ . The right hand contains  $M_s$  as well. It follows that  $M_s = Y$  a.s. and hence  $M_s \in E[M_t | \mathcal{F}_{s+} \vee \mathcal{N}]$ .  $\square$

**25.10 Theorem.** *Suppose that  $X.$  is an  $\mathcal{F}$ -adapted  $S$ -valued process such that*

$$T_t f(X_r) \in E[f(X_{t+r}) | \mathcal{F}_r] \text{ for all } t \in \mathbb{R}_{\geq 0}, r \in \mathbb{R}_{\geq 0} \text{ and } f \in C_\infty(S).$$

(i) *If  $\alpha U^\alpha f$  converges to  $f$  uniformly as  $\alpha$  tends to  $+\infty$  for all  $f \in C_\infty(S)$  then  $X.$  has a modification with  $D(\mathbb{R}_{\geq 0}, \tilde{S})$  sample path where  $\tilde{S}$  is a one point compactification of  $S$ .*

(ii) *If  $U^\alpha$  sends  $C_\infty(S)$  into itself for all  $\alpha \in \mathbb{R}_{> 0}$  then  $X.$  has a modification with  $D(\mathbb{R}_{\geq 0}, S)$  sample path.*

*Proof.* (i) Suppose that  $f \in C_\infty(S)$ ,  $f \geq 0$  and  $\alpha \in \mathbb{R}_{> 0}$ . Then it follows by Lemma 25.7 that  $t \mapsto U^\alpha f(X_t)e^{-\alpha t}$  is an  $\mathcal{F}$ -supermartingale. Invoking Lemma 14.10, we see that

$$\begin{aligned} \liminf_{s \downarrow t: s \in \mathbb{Q}} U^\alpha f(X_s) &= \limsup_{s \downarrow t: s \in \mathbb{Q}} U^\alpha f(X_s) \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.} \\ \liminf_{s \uparrow t: s \in \mathbb{Q}} U^\alpha f(X_s) &= \limsup_{s \uparrow t: s \in \mathbb{Q}} U^\alpha f(X_s) \text{ for all } t \in \mathbb{R}_{> 0} \text{ a.s.} \end{aligned}$$

Due to the linearity of  $U^\alpha$  the above holds for any  $f \in C_\infty(S)$ . Let  $R$  be a countable dense subset of  $C_\infty(S)$ . We select and fix  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and

$$\begin{aligned} \liminf_{s \downarrow t: s \in \mathbb{Q}} U^n f(X_s(\omega)) &= \limsup_{s \downarrow t: s \in \mathbb{Q}} U^n f(X_s(\omega)) \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ and} \\ \liminf_{s \uparrow t: s \in \mathbb{Q}} U^n f(X_s(\omega)) &= \limsup_{s \uparrow t: s \in \mathbb{Q}} U^n f(X_s(\omega)) \text{ for all } t \in \mathbb{R}_{> 0} \end{aligned}$$

for all  $\omega \in \Omega_0$ ,  $n \in \mathbb{N}$  and  $f \in R$ . Let  $\omega \in \Omega_0$ ,  $t \in \mathbb{R}_{\geq 0}$  and  $f \in R$ . Given  $\varepsilon \in \mathbb{R}_{> 0}$ , there exists  $n \in \mathbb{N}$  such that  $|nU^n f(x) - f(x)| < \varepsilon$  for all  $x \in S$ . It follows that

$$\begin{aligned} \limsup_{s \downarrow t: s \in \mathbb{Q}} f(X_s(\omega)) &\leq \limsup_{s \downarrow t: s \in \mathbb{Q}} nU^n f(X_s(\omega)) + \varepsilon \\ &= \liminf_{s \downarrow t: s \in \mathbb{Q}} nU^n f(X_s(\omega)) + \varepsilon \leq \liminf_{s \downarrow t: s \in \mathbb{Q}} f(X_s(\omega)) + 2\varepsilon. \end{aligned}$$

The same argument works for the left hand limits. Thus the following holds for all  $\omega \in \Omega_0$ :

$$\begin{aligned} \liminf_{s \downarrow t: s \in \mathbb{Q}} f(X_s(\omega)) &= \limsup_{s \downarrow t: s \in \mathbb{Q}} f(X_s(\omega)) \text{ for all } t \in \mathbb{R}_{\geq 0}, f \in R \text{ and} \\ \liminf_{s \uparrow t: s \in \mathbb{Q}} f(X_s(\omega)) &= \limsup_{s \uparrow t: s \in \mathbb{Q}} f(X_s(\omega)) \text{ for all } t \in \mathbb{R}_{> 0}, f \in R. \end{aligned}$$

Since  $R$  is dense in  $C_\infty(S)$ , this implies that if  $\omega \in \Omega_0$  then

$$\lim_{s \downarrow t: s \in \mathbb{Q}} X_s(\omega) \neq \emptyset \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ and } \lim_{s \uparrow t: s \in \mathbb{Q}} X_s(\omega) \neq \emptyset \text{ for all } t \in \mathbb{R}_{> 0}.$$

Here the limit is considered in the one point compactification  $\tilde{S}$ . Consequently there exists an  $\tilde{S}$ -valued process  $Y$  such that its every sample path is right continuous and admits left hand limits everywhere, and  $\lim_{s \downarrow t: s \in \mathbb{Q}} X_s(\omega) = \{Y_t(\omega)\}$  for all  $\omega \in \Omega_0$  and  $t \in \mathbb{R}_{\geq 0}$ . On the other hand, according to Lemma 25.4(ii),  $t \mapsto X_t$  is right continuous in probability. This implies that  $Y_t = X_t$  a.s. for all  $t \in \mathbb{R}_{\geq 0}$ .

(ii) Suppose that  $U^\alpha f \in C_\infty(S)$  for all  $\alpha \in \mathbb{R}_{> 0}$  and  $f \in C_\infty(S)$ . We regard functions in  $C_\infty(S)$  take the value 0 at the infinity of the compactification  $\tilde{S}$ . According to Lemma 25.6(iii), the premise in (i) is fulfilled. Let  $Y$  be as in the discussion for (i). Its every sample path is right continuous and admits left hand limits everywhere. We select and fix  $f \in C_\infty(S)$  such that  $f(x) > 0$  for all  $x \in S$ . Note that  $U^1 f(x) > 0$  for all  $x \in S$ . Set

$$\zeta := \inf\{t \in \mathbb{R}_{\geq 0} : U^1 f(Y_t)e^{-t} \leq 0\} \wedge \inf\{t \in \mathbb{R}_{> 0} : U^1 f(Y_{t-0})e^{-t} \leq 0\}.$$

For each  $n \in \mathbb{N}$  introduce the following  $\mathcal{F} \vee \text{Null}(P)$ -optional time:

$$\tau(n) := \inf\{t \in \mathbb{R}_{\geq 0} : Z_t < 1/n\} \text{ where } Z_t : t \mapsto U^1 f(Y_t)e^{-t}.$$

Since  $\sup_{n \in \mathbb{N}} \tau(n) = \zeta$  by Lemma 4.5,  $\zeta$  is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time. On the other hand  $Z_t : t \mapsto U^1 f(Y_t)e^{-t}$  is an  $\mathcal{F} \vee \text{Null}(P)$ -supermartingale by Lemma 25.7. Moreover its sample path is right continuous and  $Z_t \geq 0$  for all  $t \in \mathbb{R}_{\geq 0}$ . Invoking Lemma 7.7(i), we see that

$$E[Z_\zeta; \zeta < +\infty] \leq E[Z_{\tau(n)}; \tau(n) < +\infty] \leq P(\tau(n) < +\infty)/n \leq 1/n \text{ for all } n \in \mathbb{N}.$$

The second inequality derives by  $Z_{\tau(n)} \leq 1/n$  on  $\{\tau(n) < +\infty\}$ . Tending  $n$  to  $\infty$ , we reach that  $E[Z_\zeta; \zeta < +\infty] = 0$  and, due to the non-negativity,  $Z_\zeta = 0$  a.s. on  $\{\zeta < +\infty\}$ . This argument works for any optional time  $\sigma$  with  $\zeta \leq \sigma$ . Taking the right continuity into account we infer that there exists  $\Omega_1 \in \mathcal{F}$  such that  $\Omega_1 \subset \Omega_0$ ,  $P(\Omega_1) = 1$  and  $Z_{\zeta+t} = 0$  for all  $t \in \mathbb{R}_{\geq 0}$  on  $\{\zeta < +\infty\} \cap \Omega_1$ . Since  $U^1 f(x) > 0$  for all  $x \in S$ , this reads as follows:

if  $\omega \in \Omega_1$  and  $\zeta(\omega) < +\infty$  then  $Y_t(\omega) \notin S$  for all  $t \in \mathbb{R}_{\geq \zeta(\omega)}$ .

On the other hand if  $\omega \in \Omega_1$  and  $T < \zeta(\omega)$  then  $T < \tau(n, \omega)$  for some  $n \in \mathbb{N}$ , which means  $U^1 f(Y_t(\omega)) \geq Z_t(\omega) \geq 1/n$  for all  $t \in \mathbb{R}_{[0, T]}$ . Since  $\{x \in S : U^1 f(x) \geq 1/n\}$  is compact,

if  $\omega \in \Omega_1$  and  $T < \zeta(\omega)$  then  $\{Y_t(\omega); t \in \mathbb{R}_{[0, T]}\}$  is a relatively compact subset of  $S$ .

Let  $t \in \mathbb{R}_{\geq 0}$ . Then we see that  $\{t < \zeta\} \cap \Omega_1 = \{Y_t \in S\} \cap \Omega_1$  and hence

$$P(t < \zeta) = P(Y_t \in S) = P(X_t \in S) = E[\kappa_t(X_0; S)].$$

The right hand side equals 1. Therefore, tending  $t$  to  $+\infty$ , we conclude that  $\zeta = +\infty$  a.s.  $\square$

## 26 Strong Markov property and quasi left continuity

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\mathcal{F}$  be a filtration of sub  $\sigma$ -fields.

$S$  is a locally compact Hausdorff space with countable base and  $\kappa_t(\cdot; \star)$  is a semigroup of probability measure kernels on  $S \times \text{Borel}(S)$  such that  $\kappa_0(x; \cdot) = \delta_x$  and  $t \mapsto \kappa_t(x; \cdot)$  is  $C_\infty(S)$ -weak right continuous at 0 for all  $x \in S$ .

We still keep the same scheme of notations as in the previous section.

**26.1 Lemma.** *Suppose that  $s \in \mathbb{R}_{\geq 0}$ ,  $f \in \text{bBorel}(S, \mathbb{R})$ ,  $X$  is an  $\mathcal{F}$ -adapted  $S$ -valued process,  $t \mapsto f(X_t)$  is right continuous a.s., and*

$$T_t f(X_r) \in E[f(X_{t+r}) | \mathcal{F}_r] \text{ for all } t \in \mathbb{R}_{\geq 0}, r \in \mathbb{R}_{\geq s}.$$

- (i)  $t \mapsto U^\alpha f(X_t) e^{-\alpha t} + R\{e^{-\alpha} f(X) d\lambda\}_t$  is an  $\mathcal{F}$ -martingale after time  $s$  for all  $\alpha \in \mathbb{R}_{> 0}$ .
- (ii) Let  $\alpha \in \mathbb{R}_{> 0}$ . If  $t \mapsto U^\alpha f(X_t)$  is right continuous a.s. then

$$E[U^\alpha f(X_\sigma) e^{-\alpha \sigma}; A \cap \{\sigma < +\infty\}] = \int_0^\infty E[f(X_t); A \cap \{\sigma \leq t\}] e^{-\alpha t} dt$$

for all  $\mathcal{F} \vee \text{Null}(P)$ -optional times  $\sigma$  with  $\sigma \geq s$  and  $A \in (\mathcal{F} \vee \text{Null}(P))_{\sigma+}$ .

*Proof.* (i) There exists  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and  $t \mapsto f(X_t(\omega))$  is right continuous for every  $\omega \in \Omega_0$ . Let  $\alpha \in \mathbb{R}_{> 0}$ ,  $r \in \mathbb{R}_{\geq s}$  and  $A \in \mathcal{F}_r$ . According to Fubini's theorem,

$$\int_r^\infty E[f(X_t); A] e^{-\alpha t} dt = E\left[\int_{(0, +\infty)} f(X) e^{-\alpha \lambda} - \int_{(0, r)} f(X) e^{-\alpha \lambda}; A \cap \Omega_0\right]$$

If  $\omega \in \Omega_0$  then  $\int_{(0, t)} e^{-\alpha} f(X(\omega)) \lambda = R\{e^{-\alpha} f(X) d\lambda\}_t(\omega)$  for all  $t \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$ . Therefore it follows by Lemma 25.7 that

$$E[U^\alpha f(X_r) e^{-\alpha r}; A] = E[R\{e^{-\alpha} f(X) d\lambda\}_\infty; A] - E[R\{e^{-\alpha} f(X) d\lambda\}_r; A] \text{ for all } A \in \mathcal{F}_r.$$

This implies that  $U^\alpha f(X_r) e^{-\alpha r} + R\{e^{-\alpha} f(X) d\lambda\}_r \in E[R\{e^{-\alpha} f(X) d\lambda\}_\infty | \mathcal{F}_r]$ .

(ii) Suppose that  $t \mapsto U^\alpha f(X_t)$  is right continuous a.s. Then so is the martingale discussed in (i). We also verified its uniform integrability. Let  $\sigma$  be an  $\mathcal{F} \vee \text{Null}(P)$ -optional time with  $\sigma \geq s$ . It follows by Theorem 14.8(iv) that

$$U^\alpha f(X_\sigma) e^{-\alpha\sigma} 1_{\sigma < +\infty} + R\{e^{-\alpha\cdot} f(X) d\lambda\}_\sigma \in E[R\{e^{-\alpha\cdot} f(X) d\lambda\}_\infty | (\mathcal{F} \vee \text{Null}(P))_{\sigma+}].$$

Let  $A \in (\mathcal{F} \vee \text{Null}(P))_{\sigma+}$ . The notation  $\Omega_0$  in (i) is still in force. We infer that

$$\begin{aligned} E[U^\alpha f(X_\sigma) e^{-\alpha\sigma}; A \cap \{\sigma < +\infty\}] &= E[R\{e^{-\alpha\cdot} f(X) d\lambda\}_\infty - R\{\dots\}_\sigma; A \cap \{\sigma < +\infty\}] \\ &= E\left[\int_{[\sigma, +\infty)} f(X_t) e^{-\alpha t} \lambda; A \cap \{\sigma < +\infty\} \cap \Omega_0\right] = \int_0^\infty E[f(X_t); A \cap \{\sigma \leq t\}] e^{-\alpha t} dt. \end{aligned}$$

The last equality derives by Fubini's theorem. □

$X$  is an  $\mathcal{F}$ -adapted  $S$ -valued process whose almost every sample path is right continuous, and  $\sigma$  is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time unless otherwise stated.

**26.2 Lemma.** *Suppose that  $f \in C_b(S)$ , and for all  $\alpha \in \mathbb{R}_{>0}$  and  $A \in (\mathcal{F} \vee \text{Null}(P))_{\sigma+}$*

$$E[U^\alpha f(X_\sigma) e^{-\alpha\sigma}; A \cap \{\sigma < +\infty\}] = \int_0^\infty E[f(X_t); A \cap \{\sigma \leq t\}] e^{-\alpha t} dt.$$

(i)  $T_{t-\sigma} f(X_\sigma) 1_{\sigma \leq t} \in E[f(X_t) 1_{\sigma \leq t} | (\mathcal{F} \vee \text{Null}(P))_{\sigma+}]$  for all  $t \in \mathbb{R}_{\geq 0}$ .

(ii) If  $f \geq 0$  then  $T_t f(X_\sigma) 1_{\sigma < +\infty} \in E[f(X_{t+\sigma}) 1_{\sigma < +\infty} | (\mathcal{F} \vee \text{Null}(P))_{\sigma+}]$  for all  $t \in \mathbb{R}_{\geq 0}$ .

*Proof.* (i) Let  $\alpha \in \mathbb{R}_{>0}$ . Due to the translation invariance of the Lebesgue measure,

$$U^\alpha f(x) e^{-\alpha s} = \int_{[0, +\infty)} T_t f(x) e^{-\alpha(t+s)} \lambda(dt) = \int_{[s, +\infty)} T_{t-s} f(x) e^{-\alpha t} \lambda(dt) \text{ for all } s \in \mathbb{R}_{\geq 0}.$$

Let  $A \in (\mathcal{F} \vee \text{Null}(P))_{\sigma+}$ . Since  $(t, x) \mapsto T_t f(x)$  is measurable by Lemma 25.5, we see that

$$E[U^\alpha f(X_\sigma) e^{-\alpha\sigma}; A \cap \{\sigma < +\infty\}] = \int_0^{+\infty} E[T_{t-\sigma} f(X_\sigma); A \cap \{\sigma \leq t\}] e^{-\alpha t} dt$$

by Fubini's theorem. The right continuity of  $t \mapsto T_t f(x)$  for each  $x \in S$  implies the right continuity of  $t \mapsto E[T_{t-\sigma} f(X_\sigma); A \cap \{\sigma \leq t\}]$ . While, due to the sample path right continuity,  $t \mapsto E[f(X_t); A \cap \{\sigma \leq t\}]$  is also right continuous. Thus we conclude that

$$E[T_{t-\sigma} f(X_\sigma); A \cap \{\sigma \leq t\}] = E[f(X_t); A \cap \{\sigma \leq t\}] \text{ for all } t \in \mathbb{R}_{\geq 0}$$

by applying the uniqueness of Laplace transform.

(ii) Suppose that  $\alpha \in \mathbb{R}_{>0}$  and  $A \in (\mathcal{F} \vee \text{Null}(P))_{\sigma+}$ . Observe that  $e^{\alpha\sigma}$  is non-negative and  $(\mathcal{F} \vee \text{Null}(P))_{\sigma+}$ -measurable. The integrands being non-negative, we infer that

$$E[U^\alpha f(X_\sigma); A \cap \{\sigma < +\infty\}] = \int_0^{+\infty} E[f(X_t) e^{-\alpha(t-\sigma)}; A \cap \{\sigma \leq t\}] dt$$

by the monotone convergence theorem. There exists  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and  $t \mapsto f(X_t(\omega))$  is right continuous for every  $\omega \in \Omega_0$ . The right hand side equals

$$E\left[\int_{[\sigma, +\infty)} f(X_\cdot) e^{-\alpha(\cdot - \sigma)} \lambda; \Omega_0 \cap A \cap \{\sigma < +\infty\}\right] = \int_0^{+\infty} E[f(X_{t+\sigma}) e^{-\alpha t}; A \cap \{\sigma < +\infty\}] dt$$

by Fubini's theorem. While,  $(t, x) \mapsto T_t f(x)$  being measurable, the left hand side is the Laplace transform of  $t \mapsto E[T_t f(X_\sigma); A \cap \{\sigma < +\infty\}]$ . Therefore it follows that

$$E[T_t f(X_\sigma); A \cap \{\sigma < +\infty\}] = E[f(X_{t+\sigma}); A \cap \{\sigma < +\infty\}] \text{ for all } t \in \mathbb{R}_{\geq 0}$$

by applying the uniqueness of Laplace transforms.  $\square$

**26.3 Lemma.** (i) Suppose that  $s \in \mathbb{R}_{\geq 0}$  and the followings hold for all  $f \in C_\infty(S)$ :

$$\begin{aligned} T_t f(X_r) &\in E[f(X_{t+r}) | \mathcal{F}_r] \text{ for all } t \in \mathbb{R}_{\geq 0}, r \in \mathbb{R}_{> s}, \\ T_t f(X_s) &\in E[f(X_{t+s}) | \mathcal{F}_{s+}] \text{ for all } t \in \mathbb{R}_{\geq 0}, \text{ and} \\ T_{s-\sigma} f(X_\sigma) 1_{\sigma \leq s} &\in E[f(X_s) 1_{\sigma \leq s} | (\mathcal{F} \vee \text{Null}(P))_{\sigma+}]. \end{aligned}$$

Then, given  $B \in \sigma(\text{Cylldr}(\mathbb{R}_{\geq s}, S))$ , there exists  $g \in \text{bBorel}(S, \mathbb{R})$  such that

$$T_{s-\sigma} g(X_\sigma) 1_{\sigma \leq s} \in P(X \in B, \sigma \leq s | (\mathcal{F} \vee \text{Null}(P))_{\sigma+}).$$

(ii) If  $T_t f(X_{\sigma+r}) 1_{\sigma < +\infty} \in E[f(X_{t+\sigma+r}) 1_{\sigma < +\infty} | (\mathcal{F} \vee \text{Null}(P))_{(\sigma+r)+}]$  for all  $t, r \in \mathbb{R}_{\geq 0}$ ,  $f \in C_\infty(S)$  then  $\kappa^\circ(X_\sigma; B) \in P(\{X_{\cdot+\sigma} \in B\} | (\mathcal{F} \vee \text{Null}(P))_{\sigma+})$  for all  $B \in \sigma(\text{Cylldr}(\mathbb{R}_{\geq 0}, S))$ .

*Proof.* (i) Let  $A \in (\mathcal{F} \vee \text{Null}(P))_{\sigma+}$  and  $C \in \sigma(\text{Cylldr}(\mathbb{R}_{\geq 0}, S))$ . Set  $\mathcal{G}_t := \mathcal{F}_t$  for  $t \in \mathbb{R}_{> s}$  and  $\mathcal{G}_s := \mathcal{F}_{s+}$ . Applying Lemma 25.4(i) with the filtration  $\mathcal{G}$  we get

$$\kappa^*(X_s; C) \in P(X_{\cdot+s} \in C | \mathcal{F}_{s+}).$$

Since  $A \cap \{\sigma \leq s\} \in (\mathcal{F} \vee \text{Null}(P))_{s+} = \mathcal{F}_{s+} \vee \text{Null}(P)$  by Lemma 4.8(i), it follows that

$$P(X_{\cdot+s} \in C, \sigma \leq s, A) = E[\kappa^*(X_s; C); A \cap \{\sigma \leq s\}] \text{ for all } A \in (\mathcal{F} \vee \text{Null}(P))_{\sigma+}.$$

Thus the statement holds for  $B := \theta_s^{-1}(C)$  and  $g := \kappa^*(\cdot; C)$ .

(ii) The discussion is the same as that for Lemma 25.4(i). Consider the process  $t \mapsto X_{t+\sigma}$  with the filtration  $t \mapsto (\mathcal{F} \vee \text{Null}(P))_{(\sigma+t)+}$  under the measure  $P(\cdot | \sigma < +\infty)$ .  $\square$

**26.4 Theorem.** Suppose that  $s \in \mathbb{R}_{\geq 0}$ ,  $\sigma \geq s$ ,

$$\begin{aligned} T_t f(X_r) &\in E[f(X_{t+r}) | \mathcal{F}_r] \text{ for all } t \in \mathbb{R}_{\geq 0}, r \in \mathbb{R}_{\geq s} \text{ and } f \in C_\infty(S), \\ t \mapsto U^\alpha f(X_t) &\text{ is right continuous a.s. for all } \alpha \in \mathbb{R}_{> 0}, f \in C_\infty(S). \end{aligned}$$

If  $t \in \mathbb{R}_{\geq s}$  and  $A \in \sigma(\text{Cylldr}(\mathbb{R}_{\geq t}, S))$  then there exists  $g \in \text{bBorel}(S, \mathbb{R})$  such that

$$T_{t-\sigma} g(X_\sigma) 1_{\sigma \leq t} \in P(X \in A, \sigma \leq t | (\mathcal{F} \vee \text{Null}(P))_{\sigma+}).$$

If  $B \in \sigma(\text{Cylldr}(\mathbb{R}_{\geq 0}, S))$  then  $\kappa^\circ(X_\sigma; B) \in P(\{X_{\cdot+\sigma} \in B\} | (\mathcal{F} \vee \text{Null}(P))_{\sigma+})$ .

*Proof.* Lemma 26.1, Lemma 26.2 and Lemma 26.3.  $\square$

**26.5 Lemma.** Suppose that  $\lim_{s \uparrow t} X_s \neq \emptyset$  for all  $t \in \mathbb{R}_{>0}$  a.s.,  $f \in C_b(S)$ ,  $f \geq 0$ ,

$$\begin{aligned} T_t f(X_r) &\in E[f(X_{t+r}) | \mathcal{F}_r] \text{ for all } t \in \mathbb{R}_{\geq 0}, r \in \mathbb{R}_{\geq 0}, \\ U^\alpha f(\lim_{s \downarrow t} X_s) &= \lim_{s \downarrow t} U^\alpha f(X_s) \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s. for all } \alpha \in \mathbb{R}_{>0}, \\ U^\alpha f(\lim_{s \uparrow t} X_s) &= \lim_{s \uparrow t} U^\alpha f(X_s) \text{ for all } t \in \mathbb{R}_{>0} \text{ a.s. for all } \alpha \in \mathbb{R}_{>0}. \end{aligned}$$

If  $\tau(\cdot)$  is a non-decreasing sequence of  $\mathcal{F} \vee \text{Null}(P)$ -optional times then

$$\liminf_{n \rightarrow \infty} f(X_{\tau(n)}) 1_{\tau < +\infty} \in E[f(X_\tau) 1_{\tau < +\infty} | \bigvee_{n=1}^{\infty} (\mathcal{F} \vee \text{Null}(P))_{\tau(n)+}] \text{ where } \tau := \sup_{n \in \mathbb{N}} \tau(n).$$

*Proof.* Suppose that  $\alpha \in \mathbb{R}_{>0}$  and  $\sigma$  is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time. Observe that  $U^\alpha f \geq 0$  and  $e^{\alpha \sigma}$  is a non-negative  $(\mathcal{F} \vee \text{Null}(P))_{\sigma+}$ -measurable function. Since  $t \mapsto U^\alpha f(X_t)$  is right continuous a.s., it follows by Lemma 26.1(ii) and the proof of Lemma 26.2 that

$$E[U^\alpha f(X_\sigma); A \cap \{\sigma < +\infty\}] = \int_0^\infty E[f(X_{t+\sigma}); A \cap \{\sigma < +\infty\}] e^{-\alpha t} dt$$

for all  $A \in (\mathcal{F} \vee \text{Null}(P))_{\sigma+}$ . Suppose that  $\tau(\cdot)$  is a sequence of  $\mathcal{F} \vee \text{Null}(P)$ -optional times,  $\tau(n) \leq \tau(n+1)$  for all  $n \in \mathbb{N}$ . Clearly

$$\liminf_{n \rightarrow \infty} f(X_{\tau(n)}) 1_{\tau < +\infty} \text{ is } \bigvee_{n=1}^{\infty} (\mathcal{F} \vee \text{Null}(P))_{\tau(n)+}\text{-measurable.}$$

Let  $k, m \in \mathbb{N}$  and  $A \in (\mathcal{F} \vee \text{Null}(P))_{\tau(m)+}$ . Then

$$E[U^\alpha f(X_{\tau(n)}); A, \tau(n) \leq k] = \int_0^\infty E[f(X_{t+\tau(n)}); A, \tau(n) \leq k] e^{-\alpha t} dt$$

for  $\alpha \in \mathbb{R}_{>0}$  and  $n \in \mathbb{N}_{\geq m}$ . Observe that  $\{\tau(n+1) \leq k\} \subset \{\tau(n) \leq k\}$  for all  $n \in \mathbb{N}$  and  $\bigcap_{n=1}^{\infty} \{\tau(n) \leq k\} = \{\tau \leq k\}$ . There exists  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and for every  $\omega \in \Omega_0$  the sample path  $t \mapsto X_t(\omega)$  is right continuous and admits left hand limits everywhere, and  $U^\alpha f(X_{t-o}(\omega)) \in \lim_{s \uparrow t} U^\alpha f(X_s(\omega))$  for all  $t \in \mathbb{R}_{>0}$ . We tend  $n$  to  $\infty$ . The dominated convergence theorem shows that the left hand side converges to

$$E[U^\alpha f(X_{\tau-o}); A \cap \Omega_0, \tau(n) < \tau \forall n, \tau \leq k] + E[U^\alpha f(X_\tau); A, \exists n : \tau(n) = \tau, \tau \leq k]$$

while the right hand side converges to

$$\begin{aligned} &\int_0^\infty E[f(X_{(t+\tau)-o}); A \cap \Omega_0, \tau(n) < \tau \forall n, \tau \leq k] e^{-\alpha t} dt \\ &+ \int_0^\infty E[f(X_{t+\tau}); A, \exists n : \tau(n) = \tau, \tau \leq k] e^{-\alpha t} dt. \end{aligned}$$

We see that  $\tau$  is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time and the event  $\{\exists n : \tau(n) = \tau, \tau \leq k\}$  belongs to  $(\mathcal{F} \vee \text{Null}(P))_{\tau+}$ . Therefore the second terms of the two limits coincide. Thus we get

$$\begin{aligned} &E[\alpha U^\alpha f(X_{\tau-o}); A \cap \Omega_0, \tau(n) < \tau \forall n, \tau \leq k] \\ &= \int_0^\infty \alpha e^{-\alpha t} E[f(X_{(t+\tau)-o}); A \cap \Omega_0, \tau(n) < \tau \forall n, \tau \leq k] dt \end{aligned}$$

for all  $\alpha \in \mathbb{R}_{>0}$  and  $k \in \mathbb{N}$ , where we multiplied both sides by  $\alpha$ . Since  $\alpha U^\alpha f(x)$  converges to  $f(x)$  as  $\alpha$  tends to  $+\infty$  for all  $x \in S$  by lemma 25.6(i), we infer that

$$E[f(X_{\tau-o}); A \cap \Omega_0, \tau(n) < \tau \forall n, \tau \leq k] = E[f(X_\tau); A \cap \Omega_0, \tau(n) < \tau \forall n, \tau \leq k]$$

by the dominated convergence theorem. We add  $E[f(X_\tau); A, \exists n : \tau(n) = \tau, \tau \leq k]$  to the both sides. Since  $f(X_{\tau-o}(\omega)) \in \lim_{n \rightarrow \infty} f(X_{\tau(n)}(\omega))$  for  $\omega \in \Omega_0 \cap \{\tau(n) < \tau \forall n, \tau < +\infty\}$ , tending  $k$  to  $\infty$ , we reach that

$$E[\liminf_{n \rightarrow \infty} f(X_{\tau(n)}); A \cap \{\tau < +\infty\}] = E[f(X_\tau); A \cap \{\tau < +\infty\}].$$

This holds for all  $A \in \bigcup_{m=1}^{\infty} (\mathcal{F} \vee \text{Null}(P))_{\tau(m)+}$ . □

**26.6 Theorem.** *Suppose that  $\lim_{s \uparrow t} X_s \neq \emptyset$  for all  $t \in \mathbb{R}_{>0}$  a.s.,*

$$T_t f(X_r) \in E[f(X_{t+r}) | \mathcal{F}_r] \text{ for all } t \in \mathbb{R}_{\geq 0}, r \in \mathbb{R}_{\geq 0}, f \in C_\infty(S),$$

$$U^\alpha f(\lim_{s \downarrow t} X_s) = \lim_{s \downarrow t} U^\alpha f(X_s) \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s. for all } \alpha \in \mathbb{R}_{>0}, f \in C_\infty(S),$$

$$U^\alpha f(\lim_{s \uparrow t} X_s) = \lim_{s \uparrow t} U^\alpha f(X_s) \text{ for all } t \in \mathbb{R}_{>0} \text{ a.s. for all } \alpha \in \mathbb{R}_{>0}, f \in C_\infty(S),$$

$\tau(\cdot)$  is a non-decreasing sequence of  $\mathcal{F} \vee \text{Null}(P)$ -optional times and  $\tau := \sup_{n \in \mathbb{N}} \tau(n)$ .

(i)  $P(X_\tau \in \lim_{n \rightarrow \infty} X_{\tau(n)}, \tau < +\infty) = P(\tau < +\infty)$ .

(ii) For each lower semi-integrable and  $\sigma\{X_\cdot\} \vee \text{Null}(P)$ -measurable random variable  $Y$  the class  $E[Y | (\mathcal{F} \vee \text{Null}(P))_{\tau+}]$  contains a  $\bigvee_{n=1}^{\infty} (\mathcal{F} \vee \text{Null}(P))_{\tau(n)+}$ -measurable representative.

(iii) If  $\mathcal{F} \subset \sigma\{X_\cdot\} \vee \text{Null}(P)$  then  $(\mathcal{F} \vee \text{Null}(P))_{\tau+} = \bigvee_{n=1}^{\infty} (\mathcal{F} \vee \text{Null}(P))_{\tau(n)+}$ .

*Proof.* (i) We may assume that  $P(\tau(n) < \tau \forall n, \tau < +\infty) > 0$ . Otherwise the statement becomes trivial. There exists  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and  $t \mapsto X_t(\omega)$  admits left hand limits everywhere for every  $\omega \in \Omega_0$ . We make use of the following fact: if  $\mu, \nu$  is a finite measure on  $S \times S$  and  $\int_{S \times S} f \otimes g \mu = \int_{S \times S} f \otimes g \nu$  for all  $f, g \in C_\infty(S)$  then  $\mu = \nu$ . It then follows by Lemma 26.5 that  $(X_\tau, X_{\tau-o})$  induces the same law as  $(X_{\tau-o}, X_{\tau-o})$  under  $P(\cdot | \Omega_0, \tau(n) < \tau \forall n, \tau < +\infty)$ . Thus  $P(X_{\tau-o} = X_\tau | \Omega_0, \tau(n) < \tau \forall n, \tau < +\infty) = 1$ .

(ii) Suppose that  $t \in \mathbb{R}_{\geq 0}$  and  $A \in \sigma(\text{Cyldr}(\mathbb{R}_{\geq t}, S))$ . According to Corollary 26.4, there exists  $g \in \text{bBorel}(S, \mathbb{R})$  such that

$$T_{t-\tau} g(X_\tau) 1_{\tau \leq t} \in P(X_\cdot \in A, \tau \leq t | (\mathcal{F} \vee \text{Null}(P))_{\tau+}).$$

Clearly  $\tau$  is  $\mathcal{G} := \bigvee_{n=1}^{\infty} (\mathcal{F} \vee \text{Null}(P))_{\tau(n)+}$ -measurable. We verified in (i) that the same measurability holds for  $X_\tau 1_{\tau < +\infty}$ . Thus, due to the joint measurability  $(t, x) \mapsto T_t g(x)$ , we infer that  $T_{t-\tau} g(X_\tau) 1_{\tau \leq t}$  is  $\mathcal{G}$ -measurable. Consequently

$$P(X_\cdot \in A, \tau \leq t | (\mathcal{F} \vee \text{Null}(P))_{\tau+}) \text{ contains a } \mathcal{G}\text{-measurable representative.}$$

We next suppose that  $0 \leq s(1) < \dots < s(m)$  and  $A_1, \dots, A_m \in \text{Borel}(S)$ . Set

$$A := \text{cyldr}(s(1), \dots, s(m); A_1, \dots, A_m),$$

$$B(i) := \text{cyldr}(s(1), \dots, s(i-1); A_1, \dots, A_{i-1}) \text{ for } i \in \mathbb{N}_{[2, m+1]} \text{ and}$$

$$C(i) := \text{cyldr}(s(i), \dots, s(m); A_i, \dots, A_m) \text{ for } i \in \mathbb{N}_{[1, m]}$$

so that we have  $A = C(1)$ ,  $A = B(i) \cap C(i)$  for  $i \in \mathbb{N}_{[2,m]}$ , and  $A = B(m+1)$ . We make use of the following decomposition:

$$1_A(X_\cdot) = 1_{C(1)}(X_\cdot)1_{\tau \leq s(1)} + \sum_{i=2}^m 1_{B(i)}(X_\cdot)1_{s(i-1) < \tau \leq s(i)}1_{C(i)}(X_\cdot) + 1_{B(m+1)}(X_\cdot)1_{s(m) < \tau}$$

Observe that  $\{X_\cdot \in B(i)\} \in \mathcal{F}_{\leq s(i-1)}$  for  $i \in \mathbb{N}_{[2,m+1]}$  and  $C(i) \in \sigma(\text{Cyldr}(\mathbb{R}_{\geq s(i)}, S))$  for  $i \in \mathbb{N}_{[1,m]}$ . We see by Lemma 3.15(iv) that

$$\{X_\cdot \in B(i)\} \cap \{\tau(n) > s(i-1)\} \in (\mathcal{F} \vee \text{Null}(P))_{\tau(n)_+} \text{ for all } n \in \mathbb{N}$$

Since  $\bigcup_{n=1}^{\infty} \{\tau(n) > s(i-1)\} = \{\sup_{n \in \mathbb{N}} \tau(n) > s(i-1)\}$ , we get

$$\{X_\cdot \in B(i)\} \cap \{s(i-1) < \tau\} \in \bigvee_{n=1}^{\infty} (\mathcal{F} \vee \text{Null}(P))_{\tau(n)_+} = \mathcal{G}.$$

On the other hand there exist  $\mathcal{G}$ -measurable functions  $g_i$  such that

$$g_i \in P(X_\cdot \in C(i), \tau \leq s(i) | (\mathcal{F} \vee \text{Null}(P))_{\tau_+})$$

by Corollary 26.6. It follows that

$$g_1 + \sum_{i=2}^m 1_{B(i)}(X_\cdot)1_{s(i-1) < \tau}g_i + 1_{B(m+1)}(X_\cdot)1_{s(m) < \tau} \in P(X_\cdot \in A | (\mathcal{F} \vee \text{Null}(P))_{\tau_+}).$$

Observe that  $g_1 + \sum_{i=2}^m 1_{B(i)}(X_\cdot)1_{s(i-1) < \tau}g_i + 1_{B(m+1)}(X_\cdot)1_{s(m) < \tau}$  is  $\mathcal{G}$ -measurable. We introduce the following family:

$$\mathcal{D} := \{\Theta \in \sigma\{X_\cdot\} : \exists Y \text{ } \mathcal{G}\text{-measurable, non-negative and } Y \in E[1_\Theta | (\mathcal{F} \vee \text{Null}(P))_{\tau_+}]\}.$$

We see that  $\mathcal{D}$  is a Dynkin system by Theorem 1.10 and  $\text{Cyldr}\{X_\cdot\} \subset \mathcal{D}$ . Thus, invoking the monotone class theorem, we get the statement.  $\square$

## 27 orthogonality of martingales

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\mathcal{F}$  be a filtration of sub  $\sigma$ -fields.

$S$  is a locally compact Hausdorff space with countable base,  $X_\cdot$  is an  $\mathcal{F}$ -adapted  $S$ -valued process whose almost every sample path is right continuous, and  $L$  is a linear mapping from a subspace of  $C_\infty(S)$  into  $C_\infty(S)$ .

**27.1 Definition.** Let  $X_\cdot$  be an  $S$ -valued process. We say a triplet  $(X_\cdot, \mathcal{F}, P)$  solves the  $L$ -martingale problem if  $X_\cdot$  is  $\mathcal{F}$ -adapted and its almost every sample path is right continuous, and  $t \mapsto f(X_t) - R\{Lf(X_\cdot)d\lambda\}_t$  is an  $(\mathcal{F}, P)$ -martingale for every  $f \in \text{Dom}(L)$ .

$\mathfrak{Mart}_2(\mathcal{F})$  stands for the space of all square integrable  $\mathcal{F}$ -martingales with almost sure right continuous sample path.

**27.2 Lemma.** (i) Let  $f \in \text{Dom}(L)$ ,  $\alpha \in \mathbb{R}_{>0}$  and  $s \in \mathbb{R}_{\geq 0}$ . Then  $t \mapsto f(X_t) - R\{Lf(X_\cdot)d\lambda\}_t$  is an  $\mathcal{F}$ -martingale after time  $s$  if and only if

$$\int_r^\infty E[(\alpha - L)f(X_t); A]e^{-\alpha t} dt = E[f(X_r)e^{-\alpha r}; A] \text{ for all } r \in \mathbb{R}_{\geq s} \text{ and } A \in \mathcal{F}_r.$$

(ii) Suppose that  $(X_\cdot, \mathcal{F}, P)$  solves the  $L$ -martingale problem after time  $s \in \mathbb{R}_{\geq 0}$  and  $\sigma$  is an  $\mathcal{F}$ -optional time with  $\sigma \geq s$ . Then for  $\alpha \in \mathbb{R}_{>0}$ ,  $f \in \text{Dom}(L)$  and  $A \in \mathcal{F}_{\sigma+}$

$$\int_0^\infty E[(\alpha - L)f(X_t); A \cap \{\sigma \leq t\}]e^{-\alpha t} dt = E[f(X_\sigma)e^{-\alpha\sigma}; A \cap \{\sigma < +\infty\}].$$

*Proof.* (i) To save the space we assume that  $s = 0$ , which cause no loss of generality. The latter condition reads that  $t \mapsto f(X_t)e^{-\alpha t} + R\{e^{-\alpha}(\alpha - L)f(X)d\lambda\}_t$  is an  $\mathcal{F}$ -martingale. Suppose that  $Y$  and  $h$  are  $\mathcal{F}$ -adapted processes whose almost sure sample paths are right continuous and locally bounded. Let  $\beta \in \mathbb{R}$ . We set  $F : t \mapsto R\{hd\lambda\}_t$ ,  $Z : t \mapsto e^{\beta t}$  and  $G : t \mapsto e^{\beta t} - 1$ . Since  $R\{ZdR\{hd\lambda\}\}_t = R\{Zhd\lambda\}_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s., it follows that

$$\begin{aligned} & Y_t Z_t - (Y_t - F_t)(Z_t - G_t) - R\{YdG\}_t - R\{ZdF\}_t + Y_t - F_t \\ &= Y_t e^{\beta t} - R\{Y(\beta e^{\beta \cdot})d\lambda\}_t - R\{e^{\beta \cdot}hd\lambda\}_t = Y_t e^{\beta t} - R\{e^{\beta \cdot}(\beta Y + h)d\lambda\}_t \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.} \end{aligned}$$

Clearly  $Z - G \in \mathfrak{Mart}_2(\mathcal{F})$ . Therefore if  $t \mapsto Y_t - R\{hd\lambda\}_t$  belongs to  $\mathfrak{Mart}_2(\mathcal{F})$  then so does  $t \mapsto Y_t e^{\beta t} - R\{e^{\beta \cdot}(\beta Y + h)d\lambda\}_t$  by Lemma 23.5. The correspondence of process pairs  $(Y, h) \mapsto (Y e^{\beta \cdot}, e^{\beta \cdot}(\beta Y + h))$  is inverted by changing the parameter  $\beta$  to its negative. Indeed  $e^{-\beta t}\{(-\beta)Y_t e^{\beta t} + e^{\beta t}(\beta Y_t + h_t)\} = h_t$ . Consequently we infer that

$$t \mapsto Y_t - R\{hd\lambda\}_t \in \mathfrak{Mart}_2(\mathcal{F}) \text{ if and only if } t \mapsto Y_t e^{\beta t} - R\{e^{\beta \cdot}(\beta Y + h)d\lambda\}_t \in \mathfrak{Mart}_2(\mathcal{F}).$$

(ii) The discussion is the same as that for Lemma 26.1.  $\square$

$$W := D(\mathbb{R}_{\geq 0}, S) \text{ and } \text{eval}_t : W \rightarrow S \text{ is the evaluation } w \rightarrow w(t) \text{ for each } t \in \mathbb{R}_{\geq 0}.$$

**27.3 Lemma.**  $\text{Borel}(W) = \sigma\{\text{eval}\}$ . ( $\text{eval} : W \rightarrow \text{Map}(\mathbb{R}_{\geq 0}, S)$ )

$$T \text{ is a conservative Feller semigroup on } C_\infty(S).$$

Given  $t(1), \dots, t(n) \in \mathbb{R}_{\geq 0}$  with  $t(1) < \dots < t(n)$  and  $f_1, \dots, f_n \in C_\infty(S)$ , we can inductively define a function belonging to  $C_\infty(S)$  by  $T[t; f] := T_t f$  and

$$T[t(1), \dots, t(n); f_1, \dots, f_n] := T_{t(1)}[f_1 T[t(2) - t(1), \dots, t(n) - t(1); f_2, \dots, f_n]].$$

**27.4 Theorem.** There exists a unique measure kernel  $\mu(\cdot; \star)$  on  $S \times \text{Borel}(W)$  such that

$$\int_W \text{cylldr}[t(1), \dots, t(n); f_1, \dots, f_n](\text{eval}) \mu(x; \cdot) = T[t(1), \dots, t(n); f_1, \dots, f_n](x)$$

for all  $x \in S$ ,  $t(1), \dots, t(n) \in \mathbb{R}_{\geq 0}$  with  $t(1) < \dots < t(n)$  and  $f_1, \dots, f_n \in C_\infty(S)$ .

*Proof.* We write  $\kappa_0(x; \cdot) := \delta_x$  for  $x \in S$ . According to the Riesz-Markov theorem, for each  $t \in \mathbb{R}_{>0}$  and  $x \in S$  there exists a unique probability measure  $\kappa_t(x; \cdot)$  such that

$$T_t f(x) = \int_S f \kappa_t(x; \cdot) \text{ for all } f \in C_\infty(S).$$

Let  $t \in \mathbb{R}_{>0}$ . Being continuous,  $x \mapsto \int_S f \kappa_t(x; \cdot)$  is Borel( $S$ )-measurable for each  $f \in C_\infty(S)$ . The indicator function of any open subset of  $S$  is a non-decreasing limit of functions in  $C_\infty(S)$ . Therefore, if  $A \in \text{Borel}(S)$  then  $x \mapsto \kappa_t(x; A)$  is Borel( $S$ )-measurable by the Dynkin class theorem. In other words

$$\kappa_t(\cdot; \star) \text{ is a probability measure kernel on } S \times \text{Borel}(S).$$

Let  $t, s \in \mathbb{R}_{\geq 0}$  and  $x \in S$ . The semigroup property of  $T$  reads

$$\int_S \int_S f \kappa_t(\cdot; \star) \kappa_s(x; \cdot) = \int_S f \kappa_{t+s}(x; \cdot) \text{ for all } f \in C_\infty(S).$$

This implies that  $\int_S \kappa_t(\cdot; A) \kappa_s(x; \cdot) = \kappa_{t+s}(x; A)$  for all  $A \in \text{Borel}(S)$ . We denote by  $\kappa^\circ$  the probability measure kernel described in Theorem 25.2. We have that

$$\int_\Omega f(\text{eval}_t) \kappa^\circ(x, \cdot) = T_t f(x) \text{ for all } x \in S, t \in \mathbb{R}_{\geq 0} \text{ and } f \in C_\infty(S)$$

where  $\Omega := \text{Map}(\mathbb{R}_{\geq 0}, S)$ . Suppose that  $x \in S$ ,  $t(1), \dots, t(n) \in \mathbb{R}_{\geq 0}$  with  $t(1) < \dots < t(n)$  and  $f_1, \dots, f_n \in C_\infty(S)$ . It follows by Corollary 25.3 that

$$\begin{aligned} & \int_\Omega \text{cylldr}[t(1), \dots, t(n); f_1, \dots, f_n] \kappa^\circ(x, \cdot) \\ &= \int_\Omega T_{t(n)-t(n-1)} f_n(\text{eval}_{t(n-1)}) \text{cylldr}[t(1), \dots, t(n-1); f_1, \dots, f_{n-1}] \kappa^\circ(x, \cdot). \end{aligned}$$

The integrand in the right hand side equals  $\text{cylldr}[t(1), \dots, t(n-1); f_1, \dots, f_{n-1}g]$  with  $g := T_{t(n)-t(n-1)} f_n$ . Therefore we obtain by invoking the induction that

$$\int_\Omega \text{cylldr}[t(1), \dots, t(n); f_1, \dots, f_n] \kappa^\circ(x, \cdot) = T[t(1), \dots, t(n); f_1, \dots, f_n](x).$$

We see that  $t \mapsto \int_S f \kappa_t(x; \cdot)$  is continuous for each  $x \in S$  and  $f \in C_\infty(S)$ . On the other hand the resolvent  $U^\alpha$  sends  $C_\infty(S)$  into itself for each  $\alpha \in \mathbb{R}_{>0}$  according to Lemma 25.6(ii). We fix  $x \in S$  for the time being. Invoking Theorem 25.10(ii), we infer that the canonical process  $\text{eval.}$  on  $(\Omega, \mathcal{F}, P)$ , where  $(\mathcal{F}, P)$  is the completion of the pair  $\sigma(\text{Cylldr}(\mathbb{R}_{\geq 0}, S))$  and  $\kappa^\circ(x, \cdot)$ , has a modification, say  $X.$ , with  $D(\mathbb{R}_{\geq 0}, S)$ -sample path. If  $A \in \sigma(\text{Cylldr}(\mathbb{R}_{\geq 0}, S))$  then  $\{X. \in A\} = A \text{ mod Null}(P)$ . Since  $X.(w) \in W$  for all  $w \in \Omega$ , it follows that

$$\{X. \in A \cap W\} = A \text{ mod Null}(P) \text{ for all } A \in \sigma(\text{Cylldr}(\mathbb{R}_{\geq 0}, S)).$$

Lemma 27.3 means that  $\text{Borel}(W) = \{A \cap W; A \in \sigma(\text{Cylldr}(\mathbb{R}_{\geq 0}, S))\}$ . Therefore the mapping  $X. : \Omega \rightarrow W$  is measurable with respect to the pair  $\mathcal{F}$  and  $\text{Borel}(W)$ . Denote by  $\mu(x; \cdot)$  the image measure  $(X.)_* P$ . Then

$$\kappa^\circ(x, A) = P(A) = P(\{X. \in A \cap W\}) = \mu(x; A \cap W) \text{ for all } A \in \sigma(\text{Cylldr}(\mathbb{R}_{\geq 0}, S)).$$

This also implies the measurability of  $x \mapsto \mu(x; B)$  for each  $B \in \text{Borel}(W)$ .

The uniqueness derives by Lemma 27.3 and the monotone class theorem.  $\square$

We denote by  $(T)^\circ(\cdot; \star)$  the probability measure kernel described in Theorem 27.4 Feller process measure kernel associate with  $T$ .

**27.5 Lemma.** (i) *There exists a linear mapping  $L$  from a subspace of  $C_\infty(S)$  into  $C_\infty(S)$  such that  $\alpha - L$  is the inverse mapping of  $U^\alpha$  for all  $\alpha \in \mathbb{R}_{>0}$ , that is,  $\text{Dom}(L) = U^\alpha C_\infty(S)$ ,  $(\alpha - L)U^\alpha f = f$  for all  $f \in C_\infty(S)$ , and  $U^\alpha(\alpha - L)f = f$  for all  $f \in \text{Dom}(L)$ . Moreover*

$$\{f \in C_\infty(S) : \lim_{\alpha \rightarrow +\infty} \alpha(\alpha U^\alpha f - f) \neq \emptyset\} = \text{Dom}(L) = \{f \in C_\infty(S) : \lim_{t \rightarrow 0} (T_t f - f)/t \neq \emptyset\}$$

*If  $f \in \text{Dom}(L)$  then  $Lf \in \lim_{\alpha \rightarrow +\infty} \alpha(\alpha U^\alpha f - f)$  and  $Lf \in \lim_{t \rightarrow 0} (T_t f - f)/t$ .*

(ii) *Let  $L$  be as in (i),  $\alpha \in \mathbb{R}_{>0}$  and  $\mathcal{D}$  is a subspace of  $\text{Dom}(L)$ . Then  $(\alpha - L)\mathcal{D}$  is dense in  $C_\infty(S)$  if and only if  $\mathcal{D}$  is dense in  $\text{Dom}(L)$  with respect to the graph norm.*

*Proof.* (i) The resolvent  $U^\alpha$  sends  $C_\infty(S)$  into itself for each  $\alpha \in \mathbb{R}_{>0}$  by Lemma 25.6(ii) and, according to the proof of Lemma 25.6(iii),  $U^\alpha C_\infty(S) = U^1 C_\infty(S)$  for all  $\alpha \in \mathbb{R}_{>0}$  and

$$U^\alpha f - U^\beta f = (\beta - \alpha)U^\beta U^\alpha f \text{ for all } \alpha, \beta \in \mathbb{R}_{>0} \text{ and } f \in C_\infty(S).$$

Suppose that  $\alpha \in \mathbb{R}_{>0}$ ,  $f \in C_\infty(S)$  and  $U^\alpha f = 0$ . Then it follows by the resolvent equation that  $U^\beta f = 0$  for all  $\beta \in \mathbb{R}_{>0}$ . Since  $\beta U^\beta f$  converges to  $f$  uniformly as  $\beta$  tends to  $+\infty$  by Lemma 25.6(iii), we get  $f = 0$ . Thus we verified that

$$U^\alpha \text{ induces a linear bijection from } C_\infty(S) \rightarrow U^1 C_\infty(S) \text{ for all } \alpha \in \mathbb{R}_{>0}.$$

We denote by  $L$  the linear mapping  $U^1 C_\infty(S) \rightarrow C_\infty(S)$ ,  $f \mapsto f - (U^1)^{-1}f$  so that we have  $(1 - L)U^1 f = f$  for all  $f \in C_\infty(S)$ . Then the resolvent equation shows that

$$(\alpha - L)U^\alpha f = (\alpha - 1)U^\alpha f + (1 - L)U^1((1 - \alpha)U^\alpha f + f) = f \text{ for all } f \in C_\infty(S).$$

Due to the bijectivity, this also implies that  $U^\alpha(\alpha - L)f = f$  for all  $f \in U^1 C_\infty(S)$ . Thus the first part is proved. We next suppose that  $f \in U^1 C_\infty(S)$ . Then

$$\alpha(\alpha U^\alpha f - f) = \alpha(\alpha U^\alpha f - U^\alpha(\alpha - L)f) = \alpha U^\alpha Lf.$$

The right hand side converges to  $Lf$  uniformly as  $\alpha$  tends to  $+\infty$  by Lemma 25.6(iii). Conversely suppose that  $f, g \in C_\infty(S)$  and  $g \in \lim_{\alpha \rightarrow +\infty} \alpha(\alpha U^\alpha f - f)$ . We have that

$$U^1 \alpha(\alpha U^\alpha f - f) = \alpha(U^1 U^\alpha f - U^\alpha f)$$

by the resolvent equation. The left hand side converges to  $U^1 g$  uniformly as  $\alpha$  tends to  $+\infty$  while the right hand side converges to  $U^1 f - f$  by Lemma 25.6(iii). It therefore follows that  $f = U^1 f - U^1 g \in U^1 C_\infty(S)$ . Finally we directly relate  $L$  to the semigroup  $T$ . In the proof of Lemma 25.6(iii) we obtained the following:

$$T_t U^1 f = U^1 T_t f = e^t \int_t^\infty T_s f e^{-s} ds \text{ for all } t \in \mathbb{R}_{>0}, f \in C_\infty(S).$$

Therefore we see that

$$\frac{1}{t}(T_t U^1 f - U^1 f) = U^1 \frac{1}{t}(T_t f - f) = \frac{e^t - 1}{t} \int_t^\infty T_s f e^{-s} ds - \frac{1}{t} \int_0^t T_s f e^{-s} ds.$$

The right hand side converges to  $U^1 f - f = LU^1 f$  uniformly as  $t$  tends to 0, which means that if  $g \in U^1 C_\infty(S)$  then  $Lg \in \lim_{t \rightarrow 0} (T_t g - g)/t$  while if  $f, g \in C_\infty(S)$  and  $g \in \lim_{t \rightarrow 0} (T_t f - f)/t$  then  $U^1 g = U^1 f - f$  and hence  $f = U^1 f - U^1 g \in U^1 C_\infty(S)$ .

(ii) Let  $f \in \text{Dom}(L)$ . Since  $\|f\| = \|U^\alpha(\alpha - L)f\| \leq \|(\alpha - L)f\|/\alpha$ , we see that

$$\|f\| + \|Lf\| \leq (1 + \alpha)\|f\| + \|-(\alpha - L)f\| \leq (2 + 1/\alpha)\|(\alpha - L)f\|.$$

On the other hand  $\|(\alpha - L)f\|$  is dominated by  $\max\{\alpha, 1\}(\|f\| + \|Lf\|)$ .  $\square$

**27.6 Definition.** The linear mapping described in Lemma 27.5(i) is called the *generator* of the Feller semigroup  $T$ . If the condition in (ii) holds then  $\mathcal{D}$  is called a *core* of the generator.

In what follows  $L$  is a restriction of the generator of the Feller semigroup  $T$ .

We will refer to the canonical filtration  $\sigma\{\text{eval.}\}_{\leq}$  on  $(W, \text{Borel}(W))$ .

**27.7 Theorem.** Suppose that  $\text{Dom}(L)$  is a core of the generator of the Feller semigroup  $T$ .

(i) If  $(X, \mathcal{F}, P)$  solves the  $L$ -martingale problem after time  $s \in \mathbb{R}_{\geq 0}$  then  $(T)^\circ(X_r; B \cap W) \in P(\{X_{\cdot+r} \in B\} | \mathcal{F}_r)$  for all  $B \in \sigma(\text{Cylldr}(\mathbb{R}_{\geq 0}, S))$  and  $r \in \mathbb{R}_{\geq s}$ .

(ii) Let  $x \in S$ . Then  $(T)^\circ(x, \cdot)$  is the unique probability measure on  $W$  under which the pair  $(\text{eval.}, \sigma\{\text{eval.}\}_{\leq})$  solves the  $L$ -martingale problem starting from  $x$ .

*Proof.* (i) Let  $\alpha \in \mathbb{R}_{> 0}$  and  $f \in (\alpha - L)\text{Dom}(L)$ . Then  $U^\alpha f \in \text{Dom}(L)$  and  $(\alpha - L)U^\alpha f = f$  by Lemma 27.5(i). Applying Lemma 27.2(i), we see that

$$E[U^\alpha f(X_r)e^{-\alpha r}; A] = \int_r^\infty E[f(X_t); A]e^{-\alpha t} dt \text{ for all } r \in \mathbb{R}_{\geq s} \text{ and } A \in \mathcal{F}_r.$$

Since  $(\alpha - L)\text{Dom}(L)$  is dense in  $C_\infty(S)$  by Lemma 27.5(ii), the above holds for all  $f \in C_\infty(S)$  provided  $\alpha \in \mathbb{R}_{> 0}$ . It therefore follows by Lemma 25.7 that

$$T_t f(X_r) \in E[f(X_{t+r}) | \mathcal{F}_r] \text{ for all } t \in \mathbb{R}_{\geq 0}, r \in \mathbb{R}_{\geq s} \text{ and } f \in C_\infty(S).$$

Denote by  $\kappa_t(\cdot; \star)$  the family of measure kernels associate with the semigroup  $T$  and by  $\kappa^\circ$  the measure kernel described in Theorem 25.2. Then, since  $T_t f(x) = \int_S f \kappa_t(x; \cdot)$  for all  $t \in \mathbb{R}_{\geq 0}$ ,  $x \in S$  and  $f \in C_\infty(S)$ , it follows by Lemma 25.4 that

$$\kappa^\circ(X_r; B) \in P(\{X_{\cdot+r} \in B\} | \mathcal{F}_r) \text{ for all } B \in \sigma(\text{Cylldr}(\mathbb{R}_{\geq 0}, S)) \text{ and } r \in \mathbb{R}_{\geq s}.$$

Since  $(T)^\circ(\cdot; \star \cap W) = \kappa^\circ(\cdot; \star)$  according to the proof of Theorem 27.4, we get the claim.

(ii) Let  $x \in S$  and  $t, r \in \mathbb{R}_{\geq 0}$ . It follows by Corollary 25.3 that

$$\begin{aligned} \int_{B \cap W} T_t f(\text{eval}_r) (T)^\circ(x; \cdot) &= \int_B T_t f(\text{eval}_r) \kappa^\circ(x; \cdot) = \int_B f(\text{eval}_{t+r}) \kappa^\circ(x; \cdot) \\ &= \int_{B \cap W} f(\text{eval}_{t+r}) (T)^\circ(x; \cdot) \text{ for all } f \in C_\infty(S) \text{ and } B \in \sigma(\text{Cylldr}(\mathbb{R}_{[0, r]}, S)). \end{aligned}$$

Since  $\{B \cap W; B \in \sigma(\text{Cylldr}(\mathbb{R}_{[0, r]}, S))\}$  coincides with  $\sigma\{\text{eval.}\}_{\leq r}$ , we get

$$T_t f(\text{eval}_r) \in E[f(\text{eval}_{t+r}) | \sigma\{\text{eval.}\}_{\leq r}] \text{ for all } f \in C_\infty(S)$$

where the conditional expectation refers to the probability measure  $(T)^\circ(x; \cdot)$ . Reverting the argument in (i) we infer that  $(\text{eval.}, \sigma\{\text{eval.}\}_{\leq}, (T)^\circ(x; \cdot))$  solves the  $L$ -martingale problem. The uniqueness follows from (i).  $\square$

**27.8 Theorem.** Suppose that for each  $x \in S$  there exists at most one probability measure on  $W$  under which the pair  $(\text{eval.}, \sigma\{\text{eval.}\}_{\leq})$  solves the  $L$ -martingale problem starting from  $x$ . If  $(X, \mathcal{F}, P)$  solves the  $L$ -martingale problem then

$$P(X \in B) = E[(T.)^\circ(X_0; B \cap W)] \text{ for all } B \in \sigma(\text{Cylldr}(\mathbb{R}_{\geq 0}, S)).$$

**27.9 Theorem.** Suppose that  $\text{Dom}(L)$  is a core of the generator of the Feller semigroup  $T$ . and  $(X, \mathcal{F}, P)$  solves the  $L$ -martingale problem. If  $M_t \in \mathfrak{Mart}_2(\mathcal{F})$ ,  $M_t$  is  $\sigma\{X_s\} \vee \text{Null}(P)$ -measurable for all  $t \in \mathbb{R}_{\geq 0}$  and  $t \mapsto M_t(f(X_t) - R\{Lf(X)d\lambda\}_t)$  is an  $(\mathcal{F}, P)$ -martingale for every  $f \in \text{Dom}(L)$  then  $M_t = M_0$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s.

*Proof.* Let  $s \in \mathbb{R}_{\geq 0}$ ,  $A \in \mathcal{F}_s$ ,  $\alpha \in \mathbb{R}_{> 0}$  and  $g \in (\alpha - L)\text{Dom}(L)$ . Then

$$E[U^\alpha g(X_r)e^{-\alpha r} M_r; A] = \int_r^{+\infty} E[g(X_t)M_r; A]e^{-\alpha t} dt \text{ for all } r \in \mathbb{R}_{\geq s}$$

according to the proof of Theorem 27.7(i). Therefore the following holds for all  $r \in \mathbb{R}_{\geq s}$ :

$$(\star) \quad \int_s^{+\infty} E[M_{r \wedge t} g(X_t); A]e^{-\alpha t} dt = E[M_r U^\alpha g(X_r)e^{-\alpha r}; A] + \int_s^r E[M_t g(X_t); A]e^{-\alpha t} dt.$$

We set  $f := U^\alpha g \in \text{Dom}(L)$ . Since  $R\{M dR\{Lf(X)d\lambda\}\}_t = R\{MLf(X)d\lambda\}_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. and  $M_t \in \mathfrak{Mart}_2(\mathcal{F})$ , it follows by Lemma 23.5 that

$$t \mapsto M_t R\{Lf(X)d\lambda\}_t - R\{MLf(X)d\lambda\}_t \in \mathfrak{Mart}_2(\mathcal{F}).$$

Adding the martingale  $t \mapsto M_t(f(X_t) - R\{Lf(X)d\lambda\}_t)$ , we see that

$$t \mapsto M_t f(X_t) - R\{MLf(X)d\lambda\}_t \in \mathfrak{Mart}_2(\mathcal{F}).$$

Consequently we infer by the proof of Lemma 27.2(i) that

$$t \mapsto M_t f(X_t)e^{-\alpha t} + R\{e^{-\alpha \cdot} M(\alpha - L)f(X)d\lambda\}_t \text{ is an } \mathcal{F}\text{-martingale.}$$

Recall that  $f = U^\alpha g$  and  $(\alpha - L)f = g$ . Therefore this implies together with  $(\star)$  that  $r \mapsto \int_s^{+\infty} E[M_{r \wedge t} g(X_t); A]e^{-\alpha t} dt$  is constant on  $\mathbb{R}_{\geq s}$ . In particular the following holds:

$$\int_s^{+\infty} E[M_{r \wedge t} g(X_t); A]e^{-\alpha t} dt = \int_s^{+\infty} E[M_s g(X_t); A]e^{-\alpha t} dt \text{ for all } r \in \mathbb{R}_{\geq s}.$$

Since  $(\alpha - L)\text{Dom}(L)$  is dense in  $C_\infty(S)$  by Lemma 27.5(ii), the above holds for all  $g \in C_\infty(S)$  provided  $\alpha \in \mathbb{R}_{> 0}$ . Due to the sample path right continuity we conclude that

$$E[M_{r \wedge t} g(X_t); A] = E[M_s g(X_t); A] \text{ for all } r, t \in \mathbb{R}_{\geq s}$$

by the uniqueness of Laplace transforms. We may take  $r = t$ . Then, provided  $t \in \mathbb{R}_{\geq s}$ , the following holds for all  $A \in \mathcal{F}_s$  and  $g \in C_\infty(S)$ :

$$(\aleph) \quad E[M_t g(X_t); A] = E[M_s g(X_t); A] = E[M_s T_{t-s} g(X_s); A]$$

where we invoked Theorem 27.7(i) to get the second equality. Given  $0 \leq t(1) < \dots < t(n)$  and  $f_1, \dots, f_n \in C_\infty(S)$ , with the help of Theorem 27.4 and (N), we can inductively verify the following by mimicking the argument of Lemma 25.4:

$$\begin{aligned} & E[M_{t(n)\text{cyl}}[t(1), \dots, t(n); f_1, \dots, f_n](X)] \\ &= E[M_0 \int_W \text{cyl}[t(1), \dots, t(n); f_1, \dots, f_n](T)^\circ(X_0; \cdot)]. \end{aligned}$$

Let  $t \in \mathbb{R}_{\geq 0}$ . Then, applying the monotone class argument, we deduce that

$$E[M_t; X \in B] = E[M_0(T)^\circ(X_0; B \cap W)] \text{ for all } B \in \sigma(\text{Cyl}(\mathbb{R}_{[0,t]}, S)).$$

The right hand side coincides with  $E[M_0; X \in B]$  by Theorem 27.7(i). It follows that

$$E[M_t; X \in B] = E[M_0; X \in B] \text{ for all } B \in \sigma(\text{Cyl}(\mathbb{R}_{[0,t]}, S)).$$

Since  $(X)_* \sigma(\text{Cyl}(\mathbb{R}_{[0,t]}, S)) = \sigma\{X\}_{\leq t}$  and, according to Theorem 27.7(i) and Theorem 25.9,  $\mathcal{F}_t \cap (\sigma\{X\} \vee \text{Null}(P)) \subset \sigma\{X\}_{\leq t} \vee \text{Null}(P)$ , we infer that  $M_t = M_0$  a.s. Thus we reach the statement due to the sample path right continuity.  $\square$

**27.10 Theorem.** *Suppose that  $\text{Dom}(L)$  is a core of the generator of the Feller semigroup,  $T \in \mathbb{R}_{>0}$  and  $(X, \mathcal{F}, P)$  solves the  $L$ -martingale problem. If  $Y$  is a square integrable and  $(\mathcal{F}_{T+} \vee \text{Null}(P)) \cap (\sigma\{X\} \vee \text{Null}(P))$ -measurable random variable, and*

$$E[Y(f(X_{T \wedge \tau}) - R\{Lf(X)d\lambda\}_{T \wedge \tau})] = E[Yf(X_0)]$$

for every  $f \in \text{Dom}(L)$  and  $\sigma\{X\}_{\leq \cdot}$ -optional time  $\tau$  then  $Y$  is  $\sigma\{X_0\} \vee \text{Null}(P)$ -measurable.

*Proof.* We write  $\mathcal{N} := \text{Null}(P)$ . Since  $\sigma\{X\}_{\leq t+} \subset \sigma\{X\}_{\leq t} \vee \mathcal{N}$  for all  $t \in \mathbb{R}_{\geq 0}$  and  $Y$  is  $\sigma\{X\}_{\leq T} \vee \mathcal{N}$ -measurable by Theorem 27.7(i) and Theorem 25.9, it follows that

$$\text{there exists } M \in \mathfrak{Mart}_2(\sigma\{X\}_{\leq \cdot} \vee \mathcal{N}) \text{ such that } M_T = Y$$

by Theorem 14.11. Let  $f \in \text{Dom}(L)$ . Since  $\sigma\{X\}_{\leq t} \subset \mathcal{F}_t$  for all  $t \in \mathbb{R}_{\geq 0}$ , we see that

$$N : t \mapsto f(X_t) - R\{Lf(X)d\lambda\}_t \text{ is a } \sigma\{X\}_{\leq \cdot} \vee \mathcal{N}\text{-martingale.}$$

Hence  $(X, \sigma\{X\}_{\leq \cdot} \vee \mathcal{N}, P)$  solves the  $L$ -martingale problem. Let  $\sigma$  be a  $\sigma\{X\}_{\leq \cdot} \vee \mathcal{N}$ -optional time. Since there exists a  $\sigma\{X\}_{\leq \cdot}$ -optional time  $\tau$  such that  $\tau = \sigma$  a.s. by Lemma 4.8(iii),

$$E[M_T N_{T \wedge \sigma}] = E[Y(f(X_{T \wedge \tau}) - R\{Lf(X)d\lambda\}_{T \wedge \tau})] = E[Yf(X_0)] = E[M_0 N_0].$$

Corollary 20.2 shows the  $\sigma\{X\}_{\leq \cdot} \vee \mathcal{N}$ -martingale property of  $t \mapsto M_{t \wedge T} N_{t \wedge T}$ . Adding another martingale  $t \mapsto M_{t \wedge T} (N_t - N_{t \wedge T})$ , we obtain that

$$M_{t \wedge T} (f(X_t) - R\{Lf(X)d\lambda\}_t) \text{ is a } \sigma\{X\}_{\leq \cdot} \vee \mathcal{N}\text{-martingale for all } f \in \text{Dom}(L).$$

Consequently, applying Theorem 27.9, we deduce that  $M_{t \wedge T} = M_0$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s.  $\square$

**27.11 Corollary.** Suppose that  $\text{Dom}(L)$  is a core of the generator of the Feller semigroup and  $(X, \mathcal{F}, P)$  solves the  $L$ -martingale problem. Then for each  $T \in \mathbb{R}_{>0}$  the linear span of the random variables below is dense in  $\mathcal{L}_2((\mathcal{F}_{T+} \vee \mathcal{N}) \cap (\sigma\{X.\} \vee \mathcal{N}))$  where  $\mathcal{N} := \text{Null}(P)$ :

$$\{f(X_{T \wedge \tau}) - R\{Lf(X)d\lambda\}_{T \wedge \tau} + Y; f \in \text{Dom}(L), \tau \in \text{Opt}(\sigma\{X.\}_{\leq}), Y \in \mathcal{L}_2(\sigma\{X_0\})\}.$$

The linear span of the martingales below is dense in  $\mathfrak{Mart}_2(\sigma\{X.\}_{\leq} \vee \mathcal{N})$ :

$$\{f(X_{\cdot \wedge \tau}) - R\{Lf(X)d\lambda\}_{\cdot \wedge \tau} + Y; f \in \text{Dom}(L), \tau \in \text{Opt}(\sigma\{X.\}_{\leq}), Y \in \mathcal{L}_2(\sigma\{X_0\})\}.$$

**27.12 Lemma.** Suppose that  $(X, \mathcal{F}, P)$  solves the  $L$ -martingale problem and  $\text{Dom}(L)$  is a subalgebra of  $C_\infty(S)$ . Then the following is an  $\mathcal{F}$ -martingale for each  $f, g \in \text{Dom}(L)$ :

$$t \mapsto (f(X_t) - R\{Lf(X)d\lambda\}_t)(g(X_t) - R\{Lg(X)d\lambda\}_t) - R\{(L(fg) - fLg - gLf)(X)d\lambda\}_t.$$

*Proof.* Due to the bilinearity and the symmetricity it suffices to prove the statement for  $g = f$ . Set  $Y : t \mapsto f(X_t)$ ,  $F : t \mapsto R\{Lf(X)d\lambda\}_t$  and  $C : t \mapsto R\{(L(f^2) - 2fLf)(X)d\lambda\}_t$ . Since  $R\{YdR\{Lf(X)d\lambda\}\}_t = R\{YLf(X)d\lambda\}_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s., it follows that

$$\begin{aligned} Y_t^2 - 2R\{YdF\}_t - C_t \\ = f(X_t)^2 - 2R\{f(X)Lf(X)d\lambda\}_t - R\{(L(f^2) - 2fLf)(X)d\lambda\}_t \text{ for all } t \in \mathbb{R}_{\geq 0} \text{ a.s.} \end{aligned}$$

The right hand side is indistinguishable from  $t \mapsto f(X_t)^2 - R\{L(f^2)(X)d\lambda\}_t$ , which is an  $\mathcal{F}$ -martingale. Thus we obtain the statement by Lemma 23.5.  $\square$

Denote by  $\kappa^\bullet(x, \cdot)$  the outer measure induced by  $\kappa^\circ(x, \cdot)$   $\text{Mble}(\kappa^\circ(x, \cdot))$ .

## 28 Integrands for stochastic integration

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\mathcal{F}$  be a filtration of sub  $\sigma$ -fields. The index set is  $\mathbb{R}_{\geq 0}$ .

**28.1 Lemma.** Suppose that  $p \in \mathbb{R}_{\geq 1}$ ,  $T \in \mathbb{R}_{>0}$ ,  $f : \mathbb{R}_{[0,T]} \times \Omega \rightarrow \mathbb{R}$  is Borel( $\mathbb{R}_{[0,T]} \otimes \mathcal{F}$ -measurable and  $E[\int_{[0,T]} |f|^p \lambda] < +\infty$  where  $\lambda$  is the Lebesgue measure. Then

$$\limsup_{n \rightarrow \infty} \int_{[0,1]} E[\int_{[0,T]} |f([n(\cdot - s)]/n + s, \star) - f(\cdot, \star)|^p \lambda] ds = 0$$

where  $f(t, \omega) = 0$  for  $t < 0$  and  $[x] := \max\{i \in \mathbb{Z} : i < x\}$  for  $x \in \mathbb{R}$ .

*Proof.* Let  $n \in \mathbb{N}$ ,  $t \in \mathbb{R}_{[0,T]}$  and  $\omega \in \Omega$ . Since  $-1/n < t - ([nt] + 1)/n \leq 0$ , we have that

$$[0, 1) \subset \bigcup_{i=-[nt]}^{n-[nt]} [t + (i-1)/n, t + i/n) \text{ (disjoint union).}$$

Observe that  $[n(t-s)]/n + s = s - i/n$  for  $s \in [t + (i-1)/n, t + i/n)$ . It follows that

$$\begin{aligned} & \int_{[0,1]} |f([n(t-s)]/n + s, \omega) - f(t, \omega)|^p ds \\ & \leq \sum_{i=-[nt]}^{n-[nt]} \int_{[t+(i-1)/n, t+i/n)} |f(s - i/n, \omega) - f(t, \omega)|^p ds. \end{aligned}$$

The summand in the right hand side reads  $\int_{[-1/n,0]} |f(t+s, \omega) - f(t, \omega)|^p ds$ . Consequently

$$\int_{[0,1]} |f([n(t-s)]/n + s, \omega) - f(t, \omega)|^p ds \leq (n+1) \int_{[-1/n,0]} |f(t+s, \omega) - f(t, \omega)|^p ds.$$

The translation of functions  $g \mapsto g(\cdot + s, \star)$  for  $s \in \mathbb{R}$  induces a strongly continuous one parameter isometry group on  $L^p(\mathbb{R} \times \Omega, \lambda \otimes P)$ . Therefore, given  $\varepsilon \in \mathbb{R}_{>0}$ , there exists  $\delta \in \mathbb{R}_{>0}$  such that if  $-\delta < s < 0$  then

$$E\left[\int_{[0,T]} |f(\cdot + s, \star) - f(\cdot, \star)|^p \lambda\right] < \varepsilon.$$

Suppose that  $n\delta > 1$ . Then, invoking Fubini's theorem, we infer that

$$\begin{aligned} & \int_{[0,1]} E\left[\int_{[0,T]} |f([n(\cdot - s)]/n + s, \star) - f(\cdot, \star)|^p \lambda\right] ds \\ & \leq (n+1) \int_{[-1/n,0]} E\left[\int_{[0,T]} |f(\cdot + s, \star) - f(\cdot, \star)|^p \lambda\right] ds \leq (n+1)\varepsilon/n < 2\varepsilon. \end{aligned}$$

Thus we reach the statement.  $\square$

**28.2 Corollary.** *Suppose  $p \in \mathbb{R}_{\geq 1}$ ,  $f : \mathbb{R}_{\geq 0} \times \Omega \rightarrow \mathbb{R}$  is Borel( $\mathbb{R}_{\geq 0}$ )  $\otimes \mathcal{F}$ -measurable and  $E[\int_{[0,T]} |f|^p \lambda] < +\infty$  for all  $T \in \mathbb{R}_{>0}$ . Then, given  $\mathbf{n} \in \text{Seq}(\mathbb{N})$  tending to  $\infty$ , there exist  $s \in [0, 1)$  and  $\phi \in \text{Seq}(\mathbb{N}, \uparrow)$  such that*

$$\limsup_{n \rightarrow \infty} E\left[\int_{[0,T]} |f([\mathbf{n}_{\phi(n)}(\cdot - s)]/\mathbf{n}_{\phi(n)} + s, \star) - f(\cdot, \star)|^p \lambda\right] = 0 \text{ for all } T \in \mathbb{R}_{>0}.$$

where  $f(t, \omega) = 0$  for  $t < 0$  and  $[x] := \max\{i \in \mathbb{Z} : i < x\}$  for  $x \in \mathbb{R}$ .

Let  $\delta \in \mathbb{R}_{>0}$  and  $s \in \mathbb{R}$ . We see that  $t \mapsto f([(t-s)/\delta]\delta + s, \omega)$  is left continuous for each  $\omega \in \Omega$  and that  $t - \delta \leq [(t-s)/\delta]\delta + s < t$ .

**28.3 Lemma.** *Suppose that  $f : \mathbb{R}_{\geq 0} \times \Omega \rightarrow \mathbb{R}$  is Borel( $\mathbb{R}_{\geq 0}$ )  $\otimes \mathcal{F}$ -measurable. If  $f(t, \cdot)$  is  $\mathcal{F}_{t+}$ -measurable for all  $t \in \mathbb{R}_{\geq 0}$  then there exists a  $\text{Pred}(\mathcal{F})$ -measurable function  $g : \mathbb{R}_{\geq 0} \times \Omega \rightarrow \mathbb{R}$  such that  $(\lambda \otimes P)(\{f \neq g\}) = 0$ .*

*Proof.* We first assume that there exists  $K \in \mathbb{R}_{\geq 0}$  such that  $|f(t, \omega)| \leq K$  for all  $t \in \mathbb{R}_{\geq 0}$  and  $\omega \in \Omega$ . According to Corollary 28.2, there exist  $s \in \mathbb{R}$  and  $\phi \in \text{Seq}(\mathbb{N}, \uparrow)$  such that

$$\limsup_{n \rightarrow \infty} E\left[\int_{[0,T]} |f([\phi(n)(\cdot - s)]/\phi(n) + s, \star) - f(\cdot, \star)| \lambda\right] = 0 \text{ for all } T \in \mathbb{R}_{>0}.$$

By choosing a subsequence if necessary, we may assume that  $f([\phi(n)(\cdot - s)]/\phi(n) + s, \star)$  converges to  $f(\cdot, \star)$  almost everywhere with respect to  $\lambda \otimes P$ . Since the process with sample path  $t \mapsto f([\phi(n)(t - s)]/\phi(n) + s, \omega)$  is left continuous and  $\mathcal{F}$ -adapted, it follows that

$$g : \mathbb{R}_{\geq 0} \times \Omega \rightarrow \mathbb{R}, (t, \omega) \mapsto \limsup_{n \rightarrow \infty} f([\phi(n)(t - s)]/\phi(n) + s, \omega)$$

is  $\text{Pred}(\mathcal{F})$ -measurable and  $(\lambda \otimes P)(\{f \neq g\}) = 0$ . For general functions we consider the contraction  $\min\{\max\{f, -n\}, n\}$  where  $n \in \mathbb{N}$ . For each  $n \in \mathbb{N}$  there exists a  $\text{Pred}(\mathcal{F})$ -measurable function  $g_n : \mathbb{R}_{\geq 0} \times \Omega \rightarrow \mathbb{R}$  such that  $\min\{\max\{f, -n\}, n\} = g_n$  a.e. It follows that  $\limsup_{n \rightarrow \infty} g_n$  is  $\text{Pred}(\mathcal{F})$ -measurable and  $(\lambda \otimes P)(\{f \neq \limsup_{n \rightarrow \infty} g_n\}) = 0$ .  $\square$

## 29 Quasi-left continuous increasing process

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\mathcal{F}$  be a filtration of sub  $\sigma$ -fields. We discuss submartingales with continuous parameter. So the index space is  $\mathbb{R}_{\geq 0}$ .

**29.1 Lemma.** *Let  $X$  be an  $\mathcal{F}$ -supermartingale with almost sure right continuous sample path, and  $\sigma, \tau$  be  $\mathcal{F} \vee \text{Null}(P)$ -optional times with  $\sigma \leq \tau \leq T$  a.s. for some  $T \in \mathbb{R}_{>0}$ . Then*

$$P(\sup\{X_s; s \in \mathbb{R}_{\geq 0}, \sigma \leq s \leq \tau\} > \lambda, B) \leq (E[X_\sigma; B] - E[\min\{X_\tau, 0\}; B])/\lambda$$

for  $\lambda \in \mathbb{R}_{>0}$  and  $B \in (\mathcal{F} \vee \text{Null}(P))_{\sigma+}$

*Proof.* There exists a right continuous stochastic process  $Y$  such that  $X_t = Y_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. Then  $Y$  is an  $\mathcal{F} \vee \text{Null}(P)$ -supermartingale. Given  $\lambda \in \mathbb{R}_{>0}$ , we write

$$S := \inf\{s \in \mathbb{R}_{\geq 0} : s \geq \sigma \wedge T, Y_s > \lambda\},$$

which is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time. Indeed

$$\{S < t\} = \bigcup_{s \in \mathbb{Q}: 0 \leq s < t} \{\sigma \wedge T \leq s, Y_s > \lambda\} \in \mathcal{F}_t \vee \text{Null}(P) \text{ for all } t \in \mathbb{R}_{>0}$$

due to the right continuity of sample path. We have that

$$\sup\{Y_s; s \in \mathbb{R}_{\geq 0}, \sigma \wedge T \leq s \leq \tau \wedge T\} > \lambda \Leftrightarrow S < \tau \wedge T \text{ or } 'S = \tau \wedge T, Y_{\tau \wedge T} > \lambda'$$

Note that  $(\mathcal{F} \vee \text{Null}(P))_{(\sigma \wedge T)+} = (\mathcal{F} \vee \text{Null}(P))_{\sigma+}$  and  $\sigma \wedge T \leq S$ . We see that

$$E[Y_{\sigma \wedge T}; B] \geq E[Y_{S \wedge \tau \wedge T}; B] \text{ for } B \in (\mathcal{F} \vee \text{Null}(P))_{\sigma+}$$

by Theorem 3.9. The right hand side equals

$$E[Y_S; S < \tau \wedge T, B] + E[Y_{\tau \wedge T}; S = \tau \wedge T, Y_{\tau \wedge T} > \lambda, B] + E[Y_{\tau \wedge T}; S \geq \tau \wedge T, Y_{\tau \wedge T} \leq \lambda, B].$$

The second term dominates  $\lambda P(S = \tau \wedge T, Y_{\tau \wedge T} > \lambda, B)$  and the last term dominates  $E[\min\{Y_{\tau \wedge T}, 0\}; B]$ . On the other hand, since  $Y_S \geq \lambda$  on  $\{S < +\infty\}$  due to the right continuity of sample path, it follows that

$$\lambda P(S < \tau \wedge T, B) + \lambda P(S = \tau \wedge T, Y_{\tau \wedge T} > \lambda, B) \leq E[Y_{\sigma \wedge T}; B] - E[\min\{Y_{\tau \wedge T}, 0\}; B].$$

Thus we get the claim. □

Let  $A$  be an integrable  $\mathcal{F}$ -increasing process.

**29.2 Corollary.** *Suppose that  $q \in \mathbb{R}_{>0}$  and  $M$  is a stochastic process such that almost every sample path is right continuous and  $M_t \in E[A_q | \mathcal{F}_{t+}]$  for all  $t \in \mathbb{R}_{\geq 0}$ . Then*

$$P(\sup\{M_s - A_s; s \in \mathbb{R}_{(p,q)}\} > \lambda, B) \leq E[A_q - A_\sigma; \sigma < q, B]/\lambda$$

for all  $p \in \mathbb{R}_{[0,q)}$ ,  $\lambda \in \mathbb{R}_{>0}$  and  $B \in \mathcal{F}_{p+} \vee \text{Null}(P)$  where  $\sigma := \inf\{t \in \mathbb{R}_{>p}; M_t - A_t > \lambda\}$ .

*Proof.* There exists  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and for each  $\omega \in \Omega_0$

$t \mapsto A_t(\omega)$  is finite valued, right continuous and non-decreasing,  
 $t \mapsto M_t(\omega)$  is finite valued and right continuous.

Since  $A_q \in E[A_q | \mathcal{F}_{t+}]$  if  $t \geq q$ , we can choose an appropriate subset so that

$$M_t(\omega) = A_q(\omega) \leq A_t(\omega) \text{ for all } t \in \mathbb{R}_{\geq q} \text{ and } \omega \in \Omega_0.$$

Clearly  $M$  is an  $\mathcal{F}_+$ -martingale and  $A$  is an  $\mathcal{F}_+$ -increasing process. So the difference  $M - A$  is an  $\mathcal{F}_+$ -supermartingale. Note that  $M_q = A_q$  a.s. It follows by Lemma 29.1 that

$$P(\sup\{M_s - A_s; s \in \mathbb{R}_{\geq 0}, \sigma \wedge q \leq s \leq q\} > \lambda, C) \leq E[M_{\sigma \wedge q} - A_{\sigma \wedge q}; C]/\lambda$$

for all  $C \in (\mathcal{F} \vee \text{Null}(P))_{(\sigma \wedge q)+}$ . Since  $M_q(\omega) - A_q(\omega) = 0$  for all  $\omega \in \Omega_0$ ,

$$\sup\{M_s - A_s; s \in \mathbb{R}_{(p,q]}\} > \lambda \text{ if and only if } \sup\{M_s - A_s; s \in \mathbb{R}_{(p,q]}\} > \lambda$$

on  $\Omega_0$ . The latter is equivalent to  $\sigma < q$ . If  $\sigma < q$  and  $M_\sigma - A_\sigma > \lambda$  then, due to the right continuity of the sample path,  $\{t \in \mathbb{R}_{\geq 0}; \sigma < t < q, M_t - A_t > \lambda\} \neq \emptyset$  on  $\Omega_0$ . Obviously  $\sigma < q$  and  $M_\sigma - A_\sigma \leq \lambda$  imply the same conclusion. Therefore we have that

$$\{\sup\{M_s - A_s; s \in \mathbb{R}_{(p,q]}\} > \lambda\} \cap \Omega_0 \subset \{\sup\{M_s - A_s; s \in \mathbb{R}_{\geq 0}, \sigma < s < q\} > \lambda\}$$

Since  $E[M_{\sigma \wedge q}; C] = E[M_q; C]$  by Theorem 3.9 and  $M_q = A_q$  a.s., it follows that

$$P(\sup\{M_s - A_s; s \in \mathbb{R}_{(p,q]}\} > \lambda, C) \leq E[A_q - A_{\sigma \wedge q}; C]/\lambda \text{ for all } C \in (\mathcal{F} \vee \text{Null}(P))_{(\sigma \wedge q)+}.$$

Finally, taking  $p \leq \sigma$  into account, we see that  $\mathcal{F}_{p+} \vee \text{Null}(P) \subset (\mathcal{F} \vee \text{Null}(P))_{(\sigma \wedge q)+}$ .  $\square$

We write  $\mathbb{Q}^{(2)} := \bigcup_{n \in \mathbb{Z}; n \geq 0} 2^{-n}\mathbb{Z}$ . For each  $q \in \mathbb{Q}_{>0}^{(2)}$  choose a stochastic process  $N^q$  such that almost every sample path is right continuous and

$$N_t^q \in E[A_q | \mathcal{F}_{t+}] \text{ for all } t \in \mathbb{R}_{\geq 0}.$$

which exists by Corollary 14.12. We set  $A_t^n := N_t^{[2^{2^n}t]/2^n}$  for  $t \in \mathbb{R}_{>0}$  and  $n \in \mathbb{Z}_{\geq 0}$  where  $[x] := \min \mathbb{Z}_{\geq x}$ .

**29.3 Lemma.** *Let  $n \in \mathbb{Z}_{\geq 0}$ ,  $\varepsilon \in \mathbb{R}_{>0}$  and  $T \in \mathbb{N}$ . Then  $\tau := \inf\{t \in \mathbb{R}_{>0}; A_t^n - A_t > \varepsilon\} \wedge T$  is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time and  $P(\sup\{A_s^n - A_s; s \in \mathbb{R}_{(0,T]}\} > \varepsilon) \leq E[A_{\tau+2^{-n}} - A_\tau]/\varepsilon$ .*

*Proof.* There exists  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and for each  $\omega \in \Omega_0$

$t \mapsto A_t(\omega)$  is finite valued, right continuous and non-decreasing,  
 $t \mapsto N_t^q(\omega)$  is finite valued and right continuous for all  $q \in \mathbb{Q}_{>0}^{(2)}$ ,  
 $N_t^q(\omega) = A_q(\omega)$  for all  $q \in \mathbb{Q}_{>0}^{(2)}$  and  $t \in \mathbb{R}_{\geq q}$

For each  $p, q \in \mathbb{Q}_{>0}^{(2)}$  with  $p < q$  we introduce

$$\sigma(p, q) := \inf\{t \in \mathbb{R}_{>p}; N_t^q - A_t > \varepsilon\},$$

which is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time. Since  $\sigma(p, q) \in \mathbb{R}_{[p,q]} \cup \{+\infty\}$  on  $\Omega_0$ ,

$$\inf\{t \in \mathbb{R}_{>0}; A_t^n - A_t > \varepsilon\} = \min\{\sigma(q - 2^{-n}, q); q \in 2^{-n}\mathbb{Z}_{>0}\} \text{ on } \Omega_0.$$

This implies that  $\tau$  is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time. Moreover

$$\{\tau < T\} \cap \Omega_0 = \bigcup_{q \in 2^{-n}\mathbb{Z}: 0 < q \leq T} (\{\sigma(q - 2^{-n}, q) < q\} \cap \bigcap_{p \in 2^{-n}\mathbb{Z}: 0 < p < q} \{\sigma(p - 2^{-n}, p) \geq p\} \cap \Omega_0).$$

We write  $B(q) := \bigcap_{p \in 2^{-n}\mathbb{Z}: 0 < p < q} \{\sigma(p - 2^{-n}, p) \geq p\}$  to save the space. It follows that

$$P(\tau < T) = \sum_{q \in 2^{-n}\mathbb{Z}: 0 < q \leq T} P(\{\sigma(q - 2^{-n}, q) < q\} \cap B(q)).$$

Recall that  $\sup\{N_s^q - A_s; s \in \mathbb{R}_{(p,q]}\} > \varepsilon$  if and only if  $\sigma(p, q) < q$  on  $\Omega_0$ . On the other hand  $B(q) \in \mathcal{F}_{q-2^{-n}} \vee \text{Null}(P)$  for all  $q \in 2^{-n}\mathbb{Z}_{>0}$ . Thus we see by Corollary 29.2 that

$$P(\{\sigma(q - 2^{-n}, q) < q\} \cap B(q)) \leq E[A_q - A_{\sigma(q-2^{-n}, q)}; \{\sigma(q - 2^{-n}, q) < q\} \cap B(q)]/\varepsilon.$$

Since  $q \leq \sigma(q - 2^{-n}, q) + 2^{-n}$ , the right hand side is dominated by

$$E[A_{\sigma(q-2^{-n}, q)+2^{-n}} - A_{\sigma(q-2^{-n}, q)}; \{\sigma(q - 2^{-n}, q) < q\} \cap B(q)]/\varepsilon,$$

which coincides with  $E[A_{\tau+2^{-n}} - A_\tau; \{\sigma(q - 2^{-n}, q) < q\} \cap B(q)]/\varepsilon$ . Consequently

$$P(\tau < T) \leq E[A_{\tau+2^{-n}} - A_\tau; \tau < T]/\varepsilon \leq E[A_{\tau+2^{-n}} - A_\tau]/\varepsilon.$$

Finally, since  $A_\tau^n = A_\tau$  a.s., we have that  $P(\sup\{A_s^n - A_s; s \in \mathbb{R}_{(0,T]}\} > \varepsilon) = P(\tau < T)$ .  $\square$

**29.4 Corollary.** *If  $A$  is  $\mathcal{F}$ -quasi left continuous then the sequence  $A^n$  converges to  $A$  uniformly on any bounded interval as  $n$  tends to  $\infty$  a.s.*

*Proof.* We fix  $\varepsilon \in \mathbb{R}_{>0}$  and  $T \in \mathbb{N}$ . For each  $n \in \mathbb{N}$  with we introduce

$$\tau(n) := \inf\{t \in \mathbb{R}_{>0}; A_t^n - A_t > \varepsilon\},$$

which is an  $\mathcal{F} \vee \text{Null}(P)$ -optional time by Lemma 29.3. We have that  $A_t^n \geq A_t^{n+1} \geq A_t$  a.s. for all  $t \in \mathbb{R}_{>0}$  since  $A_t^n \in E[A_{\lceil 2^n t \rceil / 2^n} | \mathcal{F}_{t+}]$ ,  $A_t^{n+1} \in E[A_{\lceil 2^{n+1} t \rceil / 2^{n+1}} | \mathcal{F}_{t+}]$  and  $\lceil 2^n t \rceil / 2^n \geq \lceil 2^{n+1} t \rceil / 2^{n+1} \geq t$ . The sample paths being right continuous off  $2^{-n-1}\mathbb{N}$  a.s., it follows that

$$A_t^n \geq A_t^{n+1} \geq A_t \text{ for all } t \in \mathbb{R}_{>0} \text{ a.s.}$$

Consequently we have that  $\tau(n) \leq \tau(n+1)$  a.s. for all  $n \in \mathbb{N}$ . According to Lemma 4.8(iii), there exists a sequence  $\tilde{\tau}(\cdot)$  of  $\mathcal{F}$ -optional times such that  $\tilde{\tau}(n) = \tau(n)$  a.s. for all  $n \in \mathbb{N}$ . Since  $\sigma = \sup_{n \in \mathbb{N}} \tilde{\tau}(n)$  a.s., the quasi-left continuity reads that  $A_{\tilde{\tau}(n) \wedge T}$  converges to  $A_{\sigma \wedge T}$  as  $n$  tends to  $\infty$  a.s. It follows that

$$A_{\tau(n) \wedge T} \text{ converges to } A_{\sigma \wedge T} \text{ as } n \text{ tends to } \infty \text{ a.s.}$$

On the other hand  $A_{\tau(n) \wedge T} \leq A_{\tau(n) \wedge T + 2^{-n}} \leq A_{\sigma \wedge T + 2^{-n}}$ . Taking into account that almost every sample path is right continuous, we infer that

$$A_{\tau(n) \wedge T + 2^{-n}} \text{ also converges to } A_{\sigma \wedge T} \text{ as } n \text{ tends to } \infty \text{ a.s.}$$

Recall that  $A_s^n - A_s \geq 0$  for all  $s \in \mathbb{R}_{>0}$  a.s. Therefore Lemma 29.3 shows that

$$P(\sup\{|A_s^n - A_s|; s \in \mathbb{R}_{(0,T]}\} > \varepsilon) \leq E[A_{\tau(n) \wedge T + 2^{-n}} - A_{\tau(n) \wedge T}]/\varepsilon.$$

Since  $E[A_{T+1}] < +\infty$  and  $0 \leq A_{\tau(n) \wedge T + 2^{-n}} - A_{\tau(n) \wedge T} \leq A_{T+1}$  a.s., invoking the dominated convergence, we thus get that

$$\limsup_{n \rightarrow \infty} P(\sup\{|A_s^n - A_s|; s \in \mathbb{R}_{(0,T]}\} > \varepsilon) = 0 \text{ for all } \varepsilon \in \mathbb{R}_{>0} \text{ and } T \in \mathbb{N}.$$

The convergence in probability implies the existence of almost sure converging subsequence. There exists  $\phi \in \text{Map}(\mathbb{N}, \mathbb{Z}_{\geq 0})$  and  $\Omega_0 \in \mathcal{F}$  such that  $\phi(k) < \phi(k+1)$ ,  $P(\Omega_0) = 1$  and

$$\limsup_{k \rightarrow \infty} \sup\{|A_s^{\phi(k)}(\omega) - A_s(\omega)|; s \in \mathbb{R}_{(0,T]}\} = 0 \text{ for all } \omega \in \Omega_0.$$

Since  $A_t^n \geq A_t^{n+1}$  for all  $t \in \mathbb{R}_{>0}$  a.s., by choosing a proper subset if necessary, we see that

$$\limsup_{n \rightarrow \infty} \sup\{|A_s^n(\omega) - A_s(\omega)|; s \in \mathbb{R}_{(0,T]}\} = 0 \text{ for all } \omega \in \Omega_0.$$

Finally,  $T \in \mathbb{N}$  being arbitrary, we get the claim.  $\square$

*29.5 Remark.* Fix  $t \in \mathbb{R}_{\geq 0}$ . Then  $A_t^n$  converges to  $A_t$  as  $n$  tends to  $\infty$  a.s. Indeed we have that  $A_t^n \geq A_t^{n+1}$  a.s. and  $A_t^n \in E[A_{\lceil 2^n t \rceil / 2^n} | \mathcal{F}_{t+}]$  for all  $n \in \mathbb{N}$ . However one can *not* expect that  $t \mapsto \liminf_{n \rightarrow \infty} A_t^n$  is right continuous without extra cost like quasi left continuity.

**29.6 Lemma.** *Let  $X$  be an  $\mathcal{F}$ -finite variation process. If it is  $\mathcal{F}$ -quasi left continuous then so are  $\text{var}_+(X)$  and  $\text{var}_-(X)$  as  $\mathcal{F}_+$ -increasing processes.*

*Proof.* Given a sequence  $\tau(\cdot)$  of  $\mathcal{F}$ -optional times such that  $\tau(n) \leq \tau(n+1)$  a.s. and  $\tau(n) \leq T$  a.s. for some  $T \in \mathbb{R}_{>0}$ . Set  $\sigma := \sup_{n \in \mathbb{N}} \tau(n)$ . There exists  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and

$$t \mapsto X_t(\omega) \text{ is right continuous and of finite variation for each } \omega \in \Omega_0.$$

If  $\omega \in \Omega_0$  then  $\inf_{s < t} \text{var}_+(dX)(\omega); \mathbb{R}_{(s,t]} = \max\{X_t(\omega) - X_{t-}(\omega), 0\}$ , i.e.,

$$\text{var}_+(X)_t - \text{var}_+(X)_{t-} = \max\{X_t - X_{t-}, 0\} \text{ for all } t \in \mathbb{R}_{>0} \text{ on } \Omega_0.$$

By choosing a proper subset, we may regard that

$$\tau(n, \omega) \leq \tau(n+1, \omega) \text{ and } \tau(n, \omega) \leq T \text{ for all } \omega \in \Omega_0.$$

The quasi-left continuity of  $X$  means

$$P(\{\omega \in \Omega_0 : X_\sigma(\omega) \neq X_{\sigma-}(\omega), \tau(n, \omega) < \sigma(\omega) \text{ for all } n \in \mathbb{N}\}) = 0$$

It follows that  $\text{var}_+(X)_{\tau(n)}$  converges to  $\text{var}_+(X)_\sigma$  as  $n$  tends to  $\infty$  a.s. Consequently  $\text{var}_+(X)$  is quasi-left continuous as  $\mathcal{F}_+$ -increasing process since  $\text{Time}(\mathcal{F}) = \text{Time}(\mathcal{F}_+)$ .  $\square$

**29.7 Theorem.** *Let  $A$  and  $B$  be  $\mathcal{F}$ -finite variation processes. If  $A$  is bounded and  $\mathcal{F}$ -quasi left continuous and  $B$  is natural then  $E[R\{AdB\}_t] = E[R[AdB]_t]$  for all  $t \in \mathbb{R}_{>0}$ .*

*Proof.* In view of Lemma 29.6, we prove the statement when  $A$  is an  $\mathcal{F}_+$ -increasing process. There exist  $\Omega_0 \in \mathcal{F}$  and  $K \in \mathbb{R}_{>0}$  such that  $P(\Omega_0) = 1$  and for each  $\omega \in \Omega_0$

$t \mapsto A_t(\omega)$  is finite valued, right continuous and non-decreasing,  
 $t \mapsto B_t(\omega)$  is right continuous and of finite variation,  
 $t \mapsto N_t^q(\omega)$  is right continuous and admits left hand limits for all  $q \in \mathbb{Q}_{>0}^{(2)}$ ,  
 $0 \leq N_t^q(\omega) \leq K$  for all  $t \in \mathbb{R}_{>0}$ ,  $N_t^q(\omega) = A_q(\omega)$  for all  $t \in \mathbb{R}_{\geq q}$  for all  $q \in \mathbb{Q}_{>0}^{(2)}$ .

Here  $t \mapsto N_t^q$  are defined before Lemma 29.3. Note that  $(\mathcal{F}_{\cdot+})_{t+} = \mathcal{F}_{t+}$  for all  $t \in \mathbb{R}_{\geq 0}$ . According to Corollary 29.4, by choosing a proper subset if necessary, we have that

$$\limsup_{n \rightarrow \infty} \sup\{|A_s^n(\omega) - A_s(\omega)|; s \in \mathbb{R}_{(0,T)}\} = 0 \text{ for all } T \in \mathbb{N} \text{ and } \omega \in \Omega_0$$

where  $A_s^n := N_s^{\lceil 2^n s \rceil / 2^n}$ . Let  $t \in \mathbb{R}_{>0}$ . It then follows that  $\int_{(0,t]} A_s^n(\omega) dB_s(\omega)$  converges to  $R\{AdB\}_t(\omega)$  for all  $\omega \in \Omega_0$ . Since  $K \text{var}(B)_t$  serves as an dominating function, we have that

$$E\left[\int_{(0,t]} A_s^n dB_s : \Omega_0\right] \text{ converges to } E[R\{AdB\}_t]$$

by the dominated convergence theorem. The same reasoning works for the following:

$$E\left[\int_{(0,t]} A_{s-}^n dB_s : \Omega_0\right] \text{ converges to } E[R[AdB]_t].$$

Fix  $n \in \mathbb{N}$  for the time being. Observe that

$$\int_{(p-2^{-n},u]} A_s^n dB_s = \int_{(p-2^{-n},u]} N_s^p dB_s = \int_{(0,u]} N_s^p dB_s - \int_{(0,p-2^{-n})} N_s^p dB_s$$

for all  $p \in 2^{-n}\mathbb{N}$  and  $u \in \mathbb{R}_{(p-2^{-n},p]}$  on  $\Omega_0$ . Consequently  $E[\int_{(0,t]} A_s^n dB_s : \Omega_0]$  equals

$$\sum_{p \in 2^{-n}\mathbb{N}: p < q} (E[R\{N^p dB\}_p] - E[R\{N^p dB\}_{p-2^{-n}}]) + E[R\{N^q dB\}_t] - E[R\{N^q dB\}_{q-2^{-n}}]$$

where  $q := 2^{-n}\lceil 2^n t \rceil$ . On the other hand

$$\int_{(p-2^{-n},u]} A_{s-}^n dB_s = \int_{(p-2^{-n},u]} N_{s-}^p dB_s = \int_{(0,u]} N_{s-}^p dB_s - \int_{(0,p-2^{-n})} N_{s-}^p dB_s$$

for all  $p \in 2^{-n}\mathbb{N}$  and  $u \in \mathbb{R}_{(p-2^{-n},p]}$  on  $\Omega_0$ . Thus  $E[\int_{(0,t]} A_{s-}^n dB_s : \Omega_0]$  equals

$$\sum_{p \in 2^{-n}\mathbb{N}: p < q} (E[R[N^p dB]_p] - E[R[N^p dB]_{p-2^{-n}}]) + E[R[N^q dB]_t] - E[R[N^q dB]_{q-2^{-n}}].$$

Since  $B$  is natural and all  $N^p$  are bounded  $\mathcal{F}_{\cdot+}$ -martingales we reach that

$$E\left[\int_{(0,t]} A_s^n dB_s : \Omega_0\right] = E\left[\int_{(0,t]} A_{s-}^n dB_s : \Omega_0\right] \text{ for all } n \in \mathbb{N}.$$

Tending  $n$  to  $\infty$  we get the claim. □

**29.8 Theorem.** *Let  $A$  be an  $\mathcal{F}$ -finite variation process. If  $A$  is  $\mathcal{F}$ -quasi left continuous and natural then almost every sample path is continuous.*

*Proof.* There exists  $\Omega_0 \in \mathcal{F}$  such that  $P(\Omega_0) = 1$  and for each  $\omega \in \Omega_0$

$t \mapsto A_t(\omega)$  is right continuous and of finite variation.

Let  $n \in \mathbb{N}$  and  $T \in \mathbb{N}$ . The truncate process  $t \mapsto \text{var}_+(A.)_t \wedge n$  is a quasi-left continuous  $\mathcal{F}_+$ -increasing process by Lemma 29.6. It follows by the proof of Theorem 29.7 that

$$E\left[\int_{(0,T]} (\text{var}_+(A.)_s \wedge n) dA_s; \Omega_0\right] = E\left[\int_{(0,T]} (\text{var}_+(A.)_{s-} \wedge n) dA_s; \Omega_0\right].$$

The difference of the integrand is non-negative. Indeed for each  $\omega \in \Omega_0$

$$\begin{aligned} & \int_{(0,T]} (\text{var}_+(A.)_s(\omega) \wedge n - \text{var}_+(A.)_{s-}(\omega) \wedge n) dA_s(\omega) \\ &= \sum_{s \leq T} (\text{var}_+(A.)_s(\omega) \wedge n - \text{var}_+(A.)_{s-}(\omega) \wedge n) (\text{var}_+(A.)_s(\omega) - \text{var}_+(A.)_{s-}(\omega)). \end{aligned}$$

The right hand side converges to  $\sum_{s \in \mathbb{R}: s > 0} (\text{var}_+(A.)_s(\omega) - \text{var}_+(A.)_{s-}(\omega))^2$  as  $n$  and  $T$  tend to  $\infty$ . The convergence being monotone, it follows that

$$E\left[\sum_{s \in \mathbb{R}: s > 0} (\text{var}_+(A.)_s - \text{var}_+(A.)_{s-})^2; \Omega_0\right] = 0.$$

The same argument works for the negative variation  $\text{var}_-(A.)$ . Thus, by choosing a subset of  $\Omega_0$  if necessary, we infer that  $\sum_{s \in \mathbb{R}: s > 0} (A_s(\omega) - A_{s-}(\omega))^2 = 0$  for all  $\omega \in \Omega_0$ .  $\square$

**29.9 Corollary.** *Given a quasi left continuous  $\mathcal{F}$ -finite variation process  $X$ . such that  $E[\text{var}(X.)_t] < +\infty$  for all  $t \in \mathbb{R}_{\geq 0}$ , there exists an  $\mathcal{F}$ -finite variation process  $Y$ . such that  $Y_0 = X_0$  a.s., almost every sample path is continuous and  $t \mapsto X_t - Y_t$  is an  $\mathcal{F}$ -martingale. Any such process  $Z$ . is also an  $\mathcal{F}_+$ -natural projection of  $X$ .*

*Proof.* Let  $A$ . be an  $\mathcal{F}_+$ -natural projection of  $X$ ., which exists by Corollary 18.9(i). We verify that  $A$ . is quasi-left continuous. Suppose that  $\tau(\cdot)$  is a sequence of  $\mathcal{F}_+$ -optional times such that  $\tau(n) \leq \tau(n+1)$  a.s. and  $\tau(n) \leq T$  for some  $T \in \mathbb{R}_{>0}$ . Corollary 18.9(iii) shows that  $t \mapsto \text{var}(X.)_t - \text{var}(A.)_t$  is an  $\mathcal{F}_+$ -submartingale. According to Theorem 3.9,

$$E[\text{var}(A.)_\sigma - \text{var}(A.)_{\tau(n)}] \leq E[\text{var}(X.)_\sigma - \text{var}(X.)_{\tau(n)}] \leq E[\text{var}(X.)_T] < +\infty.$$

where  $\sigma := \sup_{n \in \mathbb{N}} \tau(n)$ , which is an  $\mathcal{F}_+$ -optional time. We see by Lemma 29.6 that  $\text{var}(X.)$  is quasi-left continuous as  $\mathcal{F}_+$ -increasing process. It then follows by the monotone convergence theorem that

$$E[\text{var}(A.)_\sigma - \sup_{n \in \mathbb{N}} \text{var}(A.)_{\tau(n)}] \leq E[\text{var}(X.)_\sigma - \sup_{n \in \mathbb{N}} \text{var}(X.)_{\tau(n)}] = 0.$$

Since  $|A_{\tau(n)} - A_\sigma| \leq \text{var}(A.)_\sigma - \text{var}(A.)_{\tau(n)}$  a.s., this means that  $A_{\tau(n)}$  converges to  $A_\sigma$  a.s. Being natural and quasi-left continuous, it follows by Theorem 29.8 that

almost every sample path of  $A$ . is continuous.

We see by Lemma 10.2 that there exists an  $\mathcal{F}$ -adapted process  $Y$  such that  $A_t = Y_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. It follows that  $Y$  is an  $\mathcal{F}$ -finite variation process whose almost every sample path is continuous. Being an  $\mathcal{F}$ -adapted modification of  $t \mapsto X_t - A_t$ , the process  $t \mapsto X_t - Y_t$  is an  $\mathcal{F}$ -martingale. Finally we see that  $t \mapsto Y_t - Z_t = X_t - Z_t - (X_t - Y_t)$  is an  $\mathcal{F}$ -martingale such that almost every path is continuous and of finite variation. It follows by Corollary 8.17 that  $Y_t - Z_t = Y_0 - Z_0 = 0$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. Therefore  $Z_t = Y_t = A_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. Being  $\mathcal{F}$ -adapted,  $Z$  is  $\mathcal{F}_+$ -adapted and hence it is a natural  $\mathcal{F}_+$ -finite variation process. Moreover  $t \mapsto X_t - Z_t$  is an  $\mathcal{F}_+$ -martingale since it is  $\mathcal{F}_+$ -adapted and a modification of  $t \mapsto X_t - A_t$ . We thus infer that  $Z$  is an  $\mathcal{F}_+$ -natural projection of  $X$ .  $\square$

**29.10 Definition.** Let  $X$  be an  $\mathcal{F}$ -submartingale whose almost every sample path is right continuous. It is said to be regular if for any sequence  $\tau(\cdot)$  of  $\mathcal{F}$ -optional times such that  $\tau(n) \leq \tau(n+1)$  a.s. and  $\tau(n) \leq T$  for some  $T \in \mathbb{R}_{>0}$  the sequence  $E[X_{\tau(\cdot)}]$  converges to  $E[X_\sigma]$  where  $\sigma := \sup_{n \in \mathbb{N}} \tau(n)$ , which is an  $\mathcal{F}$ -optional time.

**29.11 Lemma.** (i) *An integrable  $\mathcal{F}$ -increasing process is  $\mathcal{F}$ -quasi left continuous if and only if it is regular as an  $\mathcal{F}$ -submartingale.*

(ii) *Every martingale with almost sure right continuous path is regular.*

(iii) *Let  $X$  be an  $\mathcal{F}$ -submartingale whose almost every sample path is right continuous. If it is of class DL and  $\mathcal{F}$ -quasi left continuous then it is regular.*

*Proof.* Let  $\tau(\cdot)$  be a sequence of  $\mathcal{F}$ -optional times such that  $\tau(n) \leq \tau(n+1)$  a.s. and  $\tau(n) \leq T$  for some  $T \in \mathbb{R}_{>0}$ . We write  $\sigma := \sup_{n \in \mathbb{N}} \tau(n)$ . Since  $A_\sigma \leq A_T$  a.s., we have that  $E[A_\sigma] < +\infty$ . Due to the non-decreasing property of  $t \mapsto A_t$  we see that the sequence  $E[A_{\tau(\cdot)}]$  converges to  $E[\sup_{n \in \mathbb{N}} A_{\tau(n)}]$ , by the monotone convergence theorem, and  $\sup_{n \in \mathbb{N}} A_{\tau(n)} \leq A_\sigma$  a.s. Consequently the sequence  $E[A_{\tau(\cdot)}]$  converges to  $E[A_\sigma]$  if and only if  $\sup_{n \in \mathbb{N}} A_{\tau(n)} = A_\sigma$  a.s. Thus we get (i). Theorem 3.9 immediately shows (ii). (iii) is obvious.  $\square$

We prove the existence of continuous compensator for regular submartingales.

**29.12 Theorem.** *Suppose that  $X$  is an  $\mathcal{F}$ -submartingale whose almost every sample path is right continuous. If  $X$  is of class DL and regular then there exists an  $\mathcal{F}$ -increasing process  $A$  such that almost every sample path is continuous and  $t \mapsto X_t - A_t$  is an  $\mathcal{F}$ -martingale.*

*Proof.* According to Theorem 18.6, there exists  $B \in \text{DM}[X, \mathcal{F}_+]$ . Theorem 3.9 shows that  $E[X_\tau] = E[X_0] + E[B_\tau]$  for all bounded  $\tau \in \text{Time}(\mathcal{F}_+)$ . Since  $\text{Time}(\mathcal{F}_+) = \text{Time}(\mathcal{F})$ , the regularity of  $X$  implies that  $B$  is  $\mathcal{F}_+$ -quasi-left continuous by Lemma 29.11(ii). The rest of the discussion is the same as in the proof of Corollary 29.9.  $\square$

Given a stochastic process  $X$ , we set

$$\text{cDM}[X, \mathcal{F}] := \{A : \mathcal{F}\text{-increasing process, } t \mapsto A_t \text{ continuous a.s.,} \\ t \mapsto X_t - A_t \text{ } \mathcal{F}\text{-martingale with stochastic right continuity}\}.$$

**29.13 Theorem.** *Let  $X$  be an  $\mathcal{F}$ -submartingale with almost sure right continuous path.  $\text{cDM}[X, \mathcal{F}] \subset \text{DM}[X, \mathcal{F}_+]$ .  $\text{cDM}[X, \mathcal{F}] \neq \emptyset$  if and only if  $X$  is of class DL and regular.*

*Proof.* Let  $A \in \text{cDM}[X, \mathcal{F}]$ . It follows that  $t \mapsto X_t - A_t$  is an  $\mathcal{F}_+$ -martingale by Theorem 14.11(ii). Observe that  $A_t \leq |X_t| + |X_t - A_t|$ . Being an integrable increasing process with almost sure continuous path,  $A$  is natural by Lemma 15.14.  $\square$

**29.14 Lemma.** *Suppose that  $M$  is a square integrable  $\mathcal{F}$ -martingale with stochastic right continuity. If  $M$  is almost surely right continuous and  $\mathcal{F}$ -quasi left continuous then  $A$  has an  $\mathcal{F}$ -adapted and almost sure continuous modification.*

*Proof.* We see by Lemma 29.11(iii) that the  $\mathcal{F}$ -submartingale  $t \mapsto M_t^2$  with almost sure right continuous path and quasi-left continuity is regular.  $\square$

Given a square integrable  $\mathcal{F}$ -martingale  $M$  with stochastic right continuity, we set

$$c\langle M; \mathcal{F} \rangle := \text{cDM}[t \mapsto |M_t|^2, \mathcal{F}]$$

**29.15 Theorem.** *Let  $M$  be a square integrable  $\mathcal{F}$ -martingale with a.s.-right continuous path. Then  $c\langle M; \mathcal{F} \rangle \subset \langle M; \mathcal{F}_+ \rangle$ .  $c\langle M; \mathcal{F} \rangle \neq \emptyset$  if and only if  $M$  is  $\mathcal{F}$ -quasi left continuous.*

*Proof.* Theorem 29.13 claims that  $c\langle M; \mathcal{F} \rangle \subset \langle M; \mathcal{F}_+ \rangle$ . Suppose that  $c\langle M; \mathcal{F} \rangle \neq \emptyset$ . Choose  $A \in \langle M; \mathcal{F} \rangle$ . Given a sequence  $\tau(\cdot)$  of  $\mathcal{F}$ -optional times such that  $\tau(n) \leq \tau(n+1)$  a.s. and  $\tau(n) \leq T$  for some  $T \in \mathbb{R}_{>0}$ . We write  $\sigma := \sup_{n \in \mathbb{N}} \tau(n)$ , which is an  $\mathcal{F}$ -optional time. It follows by Lemma 6.1 that

$$P(\sup\{(M_s - M_{\tau(n)})^2; s \in \mathbb{R}_{\geq 0}, \tau(n) \leq s \leq \sigma\} \geq \varepsilon) \leq E[(M_\sigma - M_{\tau(n)})^2]/\varepsilon$$

for all  $n \in \mathbb{N}$  and  $\varepsilon \in \mathbb{R}_{>0}$ . Since

$$(M_\sigma - M_{\tau(n)})^2 = (M_\sigma)^2 - (M_{\tau(n)})^2 - 2(M_\sigma - M_{\tau(n)})M_{\tau(n)},$$

we have that  $E[(M_\sigma - M_{\tau(n)})^2] = E[(M_\sigma)^2 - (M_{\tau(n)})^2]$ . The right hand side coincides with  $E[A_\sigma - A_{\tau(n)}]$ . We thus infer that

$$P(\sup\{(M_s - M_{\tau(n)})^2; s \in \mathbb{R}_{\geq 0}, \tau(n) \leq s \leq \sigma\} \geq \varepsilon) \leq E[A_\sigma - A_{\tau(n)}]/\varepsilon.$$

Since  $t \mapsto A_t$  is continuous almost surely, the dominated convergence theorem shows that

$$\limsup_{n \rightarrow \infty} P(\sup\{(M_s - M_{\tau(n)})^2; s \in \mathbb{R}_{\geq 0}, \tau(n) \leq s \leq \sigma\} \geq \varepsilon) = 0 \text{ for all } \varepsilon \in \mathbb{R}_{>0}.$$

This implies that  $M_{\tau(n)}$  converges to  $M_\sigma$  a.s., that is,  $M$  is quasi left continuous. The converse implication is discussed in Lemma 29.14.  $\square$

Given square integrable  $\mathcal{F}$ -martingales  $M$  and  $N$  with stochastic right continuity,

$$c\langle M, N; \mathcal{F} \rangle := \{A : \mathcal{F}\text{-finite variation process, } A_0 = 0 \text{ a.s., } t \mapsto A_t \text{ continuous a.s.,} \\ t \mapsto M_t N_t - A_t \text{ } \mathcal{F}\text{-martingale with stochastic right continuity}\}.$$

**29.16 Corollary.** *If  $M, N$  are square integrable  $\mathcal{F}$ -martingales with a.s.-right continuous path and  $\mathcal{F}$ -quasi left continuity then  $c\langle M, N; \mathcal{F} \rangle \neq \emptyset$  and  $c\langle M, N; \mathcal{F} \rangle \subset \langle M, N; \mathcal{F}_+ \rangle$ .*

*Proof.* Let  $A \in \langle M, N; \mathcal{F}_+ \rangle$ , which exists by Lemma 21.7. We verify that  $t \mapsto A_t$  is continuous a.s. According to Theorem 29.15,  $c\langle M; \mathcal{F} \rangle \neq \emptyset$ ,  $c\langle N; \mathcal{F} \rangle \neq \emptyset$ ,  $c\langle M; \mathcal{F} \rangle \subset \langle M; \mathcal{F}_+ \rangle$  and  $c\langle N; \mathcal{F} \rangle \subset \langle N; \mathcal{F}_+ \rangle$ . Choose  $B \in c\langle M; \mathcal{F} \rangle$  and  $C \in c\langle N; \mathcal{F} \rangle$ . We see that

$$(\text{var}(A)_t - \text{var}(A)_s)^2 \leq (B_t - B_s)(C_t - C_s) \text{ for all } t, s \in \mathbb{R}_{\geq 0} \text{ a.s.}$$

by Lemma 21.9(v). It follows that almost every sample path of  $A$  is continuous. We see by Lemma 10.2 that there exists an  $\mathcal{F}$ -adapted process  $Y$  such that  $A_t = Y_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. Then  $Y$  is an  $\mathcal{F}$ -finite variation process whose almost every sample path is continuous. Being an  $\mathcal{F}$ -adapted modification of  $t \mapsto M_t N_t - A_t$ ,

the process  $t \mapsto M_t N_t - Y_t$  is an  $\mathcal{F}$ -martingale.

Finally suppose that  $Z \in c\langle M, N; \mathcal{F} \rangle$ . Then the process  $t \mapsto Y_t - Z_t$  is an  $\mathcal{F}$ -martingale such that almost every path is continuous and of finite variation. It follows by Corollary 8.17 that  $Y_t - Z_t = Y_0 - Z_0 = 0$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. Therefore  $Z_t = Y_t = A_t$  for all  $t \in \mathbb{R}_{\geq 0}$  a.s. Being  $\mathcal{F}$ -adapted,  $Z$  is  $\mathcal{F}_+$ -adapted and hence it is a natural  $\mathcal{F}_+$ -finite variation process. Moreover  $t \mapsto M_t N_t - Z_t$  is an  $\mathcal{F}_+$ -martingale since it is  $\mathcal{F}_+$ -adapted and a modification of  $t \mapsto M_t N_t - A_t$ . We thus infer that  $Z \in \langle M, N; \mathcal{F}_+ \rangle$ .  $\square$