Doléans-Dade exponential

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Local martingale and quadratic variation

$(\Omega, \mathcal{F}, P)$ complete prob. sp., $(\mathcal{F}_t)_{t\geq 0}$ with usual condition.

Processes are assumed to be adapted.

- For càdlàg processes X and Y their cross variation $[X, Y]$ is a finite variation càdlàg process q s.t. $\sum_i (X_{t_{i+1}\wedge t} - X_{t_i\wedge t})(Y_{t_{i+1}\wedge t} - Y_{t_i\wedge t})$ converges to q_t in probability as $\max|t_{i+1} - t_i| \to 0$.
- We say a stopping time τ deduces a càdlàg process X if $X_{t\wedge\tau}\mathbf{1}_{\tau>0}$ is an martingale. We call X a local martingale if $\exists\tau(n)$ reducing sequence s.t. $\tau(n) \nearrow +\infty$.

If M is a continuous local martingale then quadratic variation $[M] := [M, M]$ exists and it is an increasing continuous process.

Cross-variation as compensator

- $X = Y$ means they are indistinguishable ($X_t = Y_t$ for all t a.s.).

Polarization If M and N are continuous local martingales then $[M, N]$ exists and $[M \pm N] = [M] \pm 2[M, N] + [N]$.

Let M and N be cont. loc. mart. and q cont. finite variation process. Then $q = [M, N]$ iff $q_0 = 0$ and $M_t N_t - q_t$ is a local martingale.

For continuous local martingales M and N,

1. If $[M] = 0$ then $M_t = M_0$ for all $t \geq 0$ a.s.
2. If M is of finite variation then $M_t = M_0$ for all $t \geq 0$ a.s.
3. If $[M, X] = [N, X]$ for all bdd. cont. mart. X and $M_0 = N_0$ then $M = N$.

Cross-variation and Itô integral

- $\phi : [0, +\infty) \times \Omega \to \mathbb{R}$ is said to be progressive if for all $t \geq 0$ it is $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$-measurable on $[0, t] \times \Omega$.

Adapted càdlàg process and left continuous process are progressive.

Let M cont. loc. mart. and ϕ progressive. If $\int_{(0,t]} |\phi|^2 \, d[M] < +\infty$ for all $t \geq 0$ a.s. then $\exists!$ cont. loc. mart. I ($\int \phi \, dM$) s.t. $I_0 = 0$ and $[I, N]_t = \int_0^t \phi \, d[M, N]$ for all cont. loc. mart. N.

- For general local martingales we consider σ-field on $[0, \infty) \times \Omega$ generated by all adapted left continuous processes. We say ϕ predictable if it is measurable with respect to the σ-filed above.

Path-wise integral and Itô integral

If A and B are càdlàg processes of finite variation then

$$A_tB_t - \sum_{0<s\leq t}(A_s - A_{s-})(B_s - B_{s-})$$
$$= A_0B_0 + \int_{(0,t]} A_{s-}\, dB_s + \int_{(0,t]} B_{s-}\, dA_s.$$

For continuous local martingales M and N,

1. $M_tN_t - [M,N]_t = M_0N_0 + \int_0^t M\, dN + \int_0^t N\, dM.$
2. If A is an adapted càdlàg process of finite variation then $\int_0^t A\, dM = M_tA_t - \int_{(0,t]} M\, dA - M_0A_0.$

Brownian motion and Poisson process

- d-dimensional $(\mathcal{F}_t)$-Brownian motion $(W_t)_{t\geq 0}$:
  adapted d-dimensional continuous process, $W_0 = 0$,
  $W_t - W_s$ obeys $N(0, t-s)$, independent of $\mathcal{F}_s$ for $0 \leq s < t$

W^i continuous local martingale and $[W^i, W^j]_t = \delta_{ij} t$.

- $(\mathcal{F}_t)$-Poisson process $(N_t)_{t\geq 0}$:
  adapted càdlàg process, $N_0 = 0$, $N_t - N_s$ obeys Poisson with mean $t - s$, independent of $\mathcal{F}_s$ for $0 \leq s < t$

$N_t - t$ finite variation local martingale.

Purely discontinuous martingales

- A local martingale X is said to be purely discontinuous if XY is a local martingale for any continuous local martingale Y.

If cont. loc. mar. M is purely discontinuous then $M_t = M_0$ for all t.

Recall for adapted finite variation càdlàg A and cont. loc. mar. Y

$$A_t Y_t = A_0 Y_0 + \int_0^t A\, dY + \int_{(0,t]} Y\, dA$$

The above suggests if A is a local martingale in addition then A is purely discontinuous (see next sheet).

- A process ϕ is said to be locally bounded if $\exists \tau(n)\ \exists K_n$ s.t. $\tau(n) \nearrow \infty$ and $|\phi_{t\wedge\tau(n)} 1_{\tau(n)>0}| \leq K_n$.

If Y adapted càdlàg process then Y_{t-} locally bounded predictable.

Finite variation martingale is purely discontinuous

Suppose that A is a finite variation local martingale.

Let ϕ locally bounded and predictable.

1. $\int_{(0,t]} \phi \, dA$ is a local martingale of finite variation.
2. $I_t = \int_{(0,t]} \phi \, dA$ iff I is a purely discontinuous loc. mart., $I_0 = 0$ and $I_t - I_{t-} = \phi_t (A_t - A_{t-})$ for all $t > 0$ a.s.

In particular local martingale A is purely discontinuous.

1. $A_t^2 - \sum_{0<s\leq t} (A_s - A_{s-})^2$ is a purely discontinuous loc. mart.
2. If A is a square integrable martingale then $E[A_t^2] = E[A_0^2] + E[\sum_{s\leq t} (A_s - A_{s-})^2]$

Square integrable martingale

Suppose that X is a square integrable martingale.

1. Continuous part $\exists!$ continuous square integrable martingale M such that $X - M$ is purely discontinuous.
2. If X is purely discontinuous then $\exists A^n$ square integrable martingales such that each A^n is of finite variation and $\lim_{n\to\infty} E[\sup_{s\le t} |X_s - A^n_s|^2] = 0$ for all $t \ge 0$.

Suppose that X is a purely discontinuous square integrable mart.

1. $E[\sum_{0<s\le t}(X_s - X_{s-})^2] < +\infty$.
2. $X_t^2 - \sum_{0<s\le t}(X_s - X_{s-})^2$ is a purely discontinuous mart.
3. $E[X_t^2] = E[X_0^2] + E[\sum_{0<s\le t}(X_s - X_{s-})^2]$.

Locally square integrable martingale

- We call an adapted càdlàg process X locally square integrable martingale if $\exists\tau(n)$ such that $X_{t\wedge\tau(n)}\mathbf{1}_{\tau(n)>0}$ is a square integrable martingale and $\tau(n)\nearrow\infty$.

Any continuous loc. mart. is a locally square integrable martingale.

Let X purely discont. loc. square integrable mart., ϕ loc. bdd. pred.

1. $\exists!$ purely discontinuous loc. mart. $I=\int\phi\,dX$ s.t. $I_0=0$ and $I_t-I_{t-}=\phi_t(X_t-X_{t-})$ for all $t>0$ a.s.
2. $\int\phi\,dX$ is a locally square integrable martingale.

Integration by parts

Suppose that X purely discontinuous locally square integrable mart.

1. If A is an adapted càdlàg process of finite variation then
$\int_0^t A_{s-}\, dX_s = X_t A_t - \int_{(0,t]} X\, dA - X_0 A_0$.
2. If M is a continuous local martingale then
$\int_0^t M\, dX = X_t M_t - \int_0^t X\, dM - X_0 M_0$.
3. $X_t^2 - \sum_{0<s\leq t}(X_s - X_{s-})^2 = X_0^2 + 2\int_0^t X_{s-}\, dX_s$.

Semimartingale and σ-martingale

- A càdlàg process X is called a (classical) semimartingale if there exist a local martingale M and an adapted càdlàg process F of finite variation such that $X = M + F$.

Suppose that X is a finite variation local martingale.

If ϕ locally bounded and predictable then $\int_0^t \phi \, dX$ defined as purely discont. loc. mart. coincides with path-wise integral $\int_{(0,t]} \phi \, dX$.

What if ϕ is not locally bounded but still path-wise integral survives?
$\int_{(0,t]} \phi \, dX$ is adapted and of finite variation, so a semimartingale.
M. Emery gave example that $\int_{(0,t]} \phi \, dX$ is not a local martingale.
Such semimartingale is a σ-martingale.

Fundamental theorem of local martingales

- A càdlàg process X is said to be decomposable if there exist a locally square integrable martingale M and an adapted càdlàg process F of finite variation such that $X = M + F$.

Given loc. mart. X and $\varepsilon > 0$, $\exists$ finite variation loc. mart. F s.t.

$$|(X-F)_t - (X-F)_{t-}| \leq \varepsilon \text{ for all } t > 0 \text{ a.s.}$$

The above means any local martingale is decomposable.

A semimartingale means a decomposable càdlàg process.

For ϕ locally bounded predictable and semimartingale $X = M + F$

$$\int_0^t \phi\, dX = \int_0^t \phi\, dM + \int_{(0,t]} \phi\, dF.$$

Bichteler–Dellacherie Theorem: measure free characterization of semimartingale in terms of integrand and integrator.

Product of semimartingales

- X is called a quadratic pure-jump semimartingale if $X =$ purely discont. loc. mart. $+$ finite variation.

Continuous martingale part $\forall$ semimartingale X $\exists!$ continuous local martingale M s.t. $X - M$ quadratic pure-jump semimartingale.

Let X and Y be semimartingales.

1. $\sum_{0<s\leq t} |(X_s - X_{s-})(Y_s - Y_{s-})| < +\infty$ for all $t \geq 0$ a.s.
2. $M :=$ cont. mart. part of X, $N :=$ cont. mart. part of Y. Then $X_t Y_t - X_0 Y_0 - [M, N]_t - \sum_{0<s\leq t} (X_s - X_{s-})(Y_s - Y_{s-})$ $= \int_0^t X_{s-}\, dY_s + \int_0^t Y_{s-}\, dX_s$.
3. $X_t Y_t$ is a semimartingale.

Cross-variation and Itô's formula

Suppose that X and Y are semimartingales.
$M :=$ cont. mart. part of X, $N :=$ cont. mart. part of Y.

1. Cross-variation $[X, Y]$ exists and $[X, Y]_t = [M, N]_t + \sum_{0<s\le t}(X_s - X_{s-})(Y_s - Y_{s-})$.
2. $[X, Y]^c$ (continuous part) coincides with $[M, N]$.

Why do they write as follows?
$[X, Y]_t = X_0 Y_0 + [M, N]_t + \sum_{0<s\le t}(X_s - X_{s-})(Y_s - Y_{s-})$,

If X d-dimensional semimartingale and $f \in C^2(\mathbb{R}^d)$ then

$$f(X_t) = f(X_0) + \int_0^t f'(X_{\cdot-})\, dX + \frac{1}{2}\int_{(0,t]} f''(X)\, d[X]^c$$
$$+ \sum_{0<s\le t}\{f(X_s) - f(X_{s-}) - f'(X_{s-})(X_s - X_{s-})\}.$$

Itô's formula – example

If M continuous local martingale and $X_t := M_t - M_0 - [M]_t/2$ then $\exp\{X_t\} = 1 + \int_0^t \exp\{X\}\, dM$.

Given a quadratic pure-jump semimartingale L s.t. $L_0 = 0$ and $1 + L_s - L_{s-} > 0$ for all $s > 0$ a.s.

1. $\sum_{s \le t} |\log(1 + L_s - L_{s-}) - L_s + L_{s-}| < \infty$ for all t a.s.
2. $X_t := L_t + \sum_{s \le t}\{\log(1 + L_s - L_{s-}) - L_s + L_{s-}\}$ is quadratic pure-jump, $\exp\{X_t\} = 1 + \int_0^t \exp\{X_{s-}\}\, dL_s$ and $\exp\{X_t\} = \exp\{L_t\} \prod_{s \le t}(1 + L_s - L_{s-}) e^{-L_s + L_{s-}}$.

$$\because \exp\{X_s\} - \exp\{X_{s-}\} - \exp\{X_{s-}\}(X_s - X_{s-})$$
$$= -\exp\{X_{s-}\}\{\log(1 + L_s - L_{s-}) - L_s + L_{s-}\}$$
$$= \exp\{X_{s-}\}(L_s - L_{s-}) - \exp\{X_{s-}\}(X_s - X_{s-})$$

Continuous martingale part of Itô integral

Let M continuous martingale part of X and ϕ loc. bdd. predictable.

1. $\int \phi \, d(X - M)$ quadratic pure-jump semimartingale.
2. $\int \phi \, dM$ continuous martingale part of $\int \phi \, dX$

Suppose that X and Y are semimartingales. We have

$$X_t Y_t = X_0 Y_0 + \int_0^t X_{s-} \, dY_s + \int_0^t Y_{s-} \, dX_s + [X, Y]_t.$$

Consider the situation that $[X, Y] = 0$.

If X cont. loc. mart. and Y quadratic pure-jump semimartingale then

1. $X_t Y_t = X_0 Y_0 + \int_0^t X_s \, dY_s + \int_0^t Y_{s-} \, dX_s$.
2. $\int_0^t Y_{s-} \, dX_s$ continuous martingale part of $X_t Y_t$.

Doléans-Dade exponential

Given semimartingale X with $X_0 = 0$ consider

- $Z_t^X := \exp\{X_t - [X]_t^c/2\} \prod_{s \le t} (1 + X_s - X_{s-}) e^{-X_s + X_{s-}}$.

Let M be continuous martingale part of X and $L := X - M$.

1. $Z_t^M = \exp\{M_t - [M]_t/2\}$.
2. $Z_t^L = \exp\{L_t\} \prod_{s \le t} (1 + L_s - L_{s-}) e^{-L_s + L_{s-}}$.
3. $Z_t^M Z_t^L = Z_t^X$ and $Z_t^X = 1 + \int_0^t Z_{s-}^X \, dX_s$.

$\because$ $[M] = [X]^c$ and $L_s - L_{s-} = X_s - X_{s-}$.
Z^M continuous local martingale and $Z_t^M = 1 + \int_0^t Z_{s-}^M \, dM_s$.
Z^L quadratic pure-jump and $Z_t^L = 1 + \int_0^t Z_{s-}^L \, dL_s$.
$Z_t^M Z_t^L = 1 + \int_0^t Z_{s-}^M \, dZ_s^L + \int_0^t Z_{s-}^L \, dZ_s^M$.

Non-vanishing semimartingale

- We say càdlàg process Z non-vanishing (positive) if $Z_t \neq 0$ $(Z_t > 0)$ for all $t \geq 0$ and $Z_{t-} \neq 0$ $(Z_{t-} > 0)$ for all $t > 0$.

Let X semimart. s.t. $X_0 = 0$, $1 + X_s - X_{s-} \neq 0$ for all $s > 0$.

1. Z^X non-vanishing semimartingale and $X_t = \int_0^t (1/Z^X_{s-})\, dZ^X_s$.
2. Let $X^\star_t := -X_t + \int_{(0,t]} (Z^X_{s-}/Z^X_s)\, d[X]_s$. Then $1 + X^\star_s - X^\star_{s-} \neq 0$ for all $s > 0$ and $Z^X_t Z^{X^\star}_t = 1$.

For non-vanishing semimartingale Z let $L_t := \int_0^t (1/Z_{s-})\, dZ_s$.

1. $1 + L_s - L_{s-} \neq 0$ for all $s > 0$ and $Z_t = Z_0 Z^L_t$.
2. $1/Z_t$ is a semimartingale and $1/Z_t = Z^{L^\star}_t / Z_0$.

Yor's formula

X, Y semimartingales, $X_0 = 0$, $1 + X_s - X_{s-} \neq 0$ for all $s > 0$ and $Y_0 = 0$, $1 + Y_s - Y_{s-} \neq 0$ for all $s > 0$.

1. $L_t := X_t + Y_t + [X, Y]_t$ is a semimartingale, $L_0 = 0$, $1 + L_s - L_{s-} \neq 0$ for all $s > 0$ and $Z^X Z^Y = Z^L$.
2. $[X, Y] = 0$ if and only if $Z^X Z^Y = Z^{X+Y}$.

Deflator

Let X semimart. s.t. $X_0 = 0$, $1 + X_s - X_{s-} \neq 0$ for all $s > 0$.

1. Z^X is positive if and only if $1 + X_s - X_{s-} > 0$ for all $s > 0$.
2. Z^X local martingale if and only if X local martingale.

- We call Z a deflator if it is a positive semimartingale.

Recall that non-negative local martingale is a supermartingale.

Suppose that Z càdlàg supermartingale and $Z_t \geq 0$ a.s.

1. Let $\zeta := \min\{\inf\{t \geq 0 : Z_t = 0\}, \inf\{t > 0 : Z_{t-} = 0\}\}$ (approach time to $\{0\}$) and $Z_\infty := 0$. Then $Z_{t \vee \zeta} = 0$.
2. If $Z_t > 0$ a.s. for all $t \geq 0$ then Z is a deflator.

References

- K. Bichteler, Stochastic Integration with Jumps, Cambridge UP 2002
- I. Karatzas S. E. Shreve, Brownian Motion and Stochastic Calculus, Springer 1991
- P. E. Protter, Stochastic Integration and Differential Equations. 2nd ed corr 3rd pr, Springer 2005
- L. C. G. Rogers D. Williams, Diffusions, Markov Processes, and Martingales : Foundations Vol. 1, Cambridge Univ. Press, 2000
- L. C. G. Rogers D. Williams, Diffusions, Markov Processes, and Martingales : Ito Calculus Vol. 2, Cambridge Univ. Press, 2000
- D, Revuz, M. Yor, Continuous martingales and Brownian motion, Springer 1999