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Alternative proofs of four stability properties of rigid-link manipulators under PID position control

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SUMMARY
This paper presents new proofs of four stability properties (semiglobal strict passivity, semiglobal asymptotic stability, semiglobal input-to-state stability, and semiglobal uniform ultimate boundedness with an arbitrarily reducible ultimate bound) of a rigid-link manipulator under proportional-integral-derivative (PID) position control. The proofs employ a strict Lyapunov function and a novel parameterization to provide four inequality conditions for the stability properties. In those inequalities, arithmetic operations on physical quantities are physically consistent if the joints are all revolute or all prismatic. A gain selection procedure is presented by which the ultimate bounds of velocity error, position error, and its integral can be independently designed.

KEYWORDS: Robot dynamics; Control of robotic systems; PID control; Stability analysis; Passivity.

1. Introduction

Proportional-integral-derivative (PID) position control is a widely accepted method especially for industrial manipulators. Its control law is very simple and is robust against disturbances and uncertainties. In spite of its prevalence in practical applications, its theoretical foundation remains inconclusive. When it is applied to set-point control of a robotic manipulator, the global asymptotic stability is not guaranteed due to the existence of the centrifugal and Coriolis terms. Lyapunov functions for stability analysis and inequality conditions for guaranteeing stability are generally very complicated in the literature. In addition, there is no standard way to deal with a variable desired position in the trajectory-tracking control. Such features of PID control attract the attention of academic researchers in the fields of robotics and control theory.

This paper presents a new approach for the stability analysis of the system composed of a rigid-link manipulator and a PID trajectory-tracking controller. This approach is based on a strict Lyapunov function inspired by the one by Wen and Murphy[19] and that by Pervozvanski and Freidovich.[15] The function is parameterized in a novel way to provide inequality conditions leading to the proofs of four stability properties, including the semiglobal uniform ultimate boundedness with an arbitrarily reducible ultimate bound. It also provides a procedure for selecting the controller gains to meet the presented inequality conditions. The gravity is not explicitly considered in the Lyapunov function but is treated as a disturbance. Given an upper bound of the magnitude of the disturbance including the gravity, the gains can be chosen to achieve arbitrarily small residual tracking error.

Main features of the presented proofs are that (i) in the inequality conditions, the arithmetic combinations among physical quantities are consistent with respect to physical dimensions if the joints are either of all revolute or all prismatic; and that (ii) the ultimate bound of the velocity error, position error, and the position-error integral can be independently designed according to given bounds of the disturbance, the desired velocity, and the desired acceleration. The combination of these features is in contrast to previous works. In particular, the feature (i) contrasts to most of previous works, which employ inconsistent combinations of quantities with different physical units, such as the addition of position and velocity. One exception is Pervozvanski and Freidovich's work,[15] but their approach does not share the feature (ii) because, in their approach, a given bound of the residual tracking error is achieved only when the desired trajectory is slow enough.

This paper is organized as follows. Section 2 provides some preliminaries and related works. Section 3 describes a Lyapunov function and provides three matrices (P, Q, and Π) that represent important properties of the Lyapunov function. Section 4 provides three lemmas regarding the matrices P, Q, and Π. Section 5 presents four theorems, which are the main results of the work: the semiglobal strict passivity,[†] the semiglobal asymptotic stability, the semiglobal input-to-state stability, and the uniform ultimate boundedness with the ultimate bound arbitrarily reducible by an appropriate choice of controller parameters.[‡] Section 6 shows illustrative numerical examples and Section 7 provides the concluding remarks. In Fig. 1, the relations among the three lemmas, the four theorems, and some key inequality conditions in Sections 4 and 5 are illustrated.

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[†]The definition of the strict passivity is provided in Definition 6.3, p. 236, of Khalil.[11]

[‡]This property is termed as a "semiglobal practical stability" by Cervantes and Alverez-Ramirez.[5] The paper avoids using the term "practical stability" because it has been used with different definitions by several authors. For example, Chaillet *et al.*'s[6] definition is more strict in that it demands not only the boundedness but also the stability in the sense of Lyapunov.

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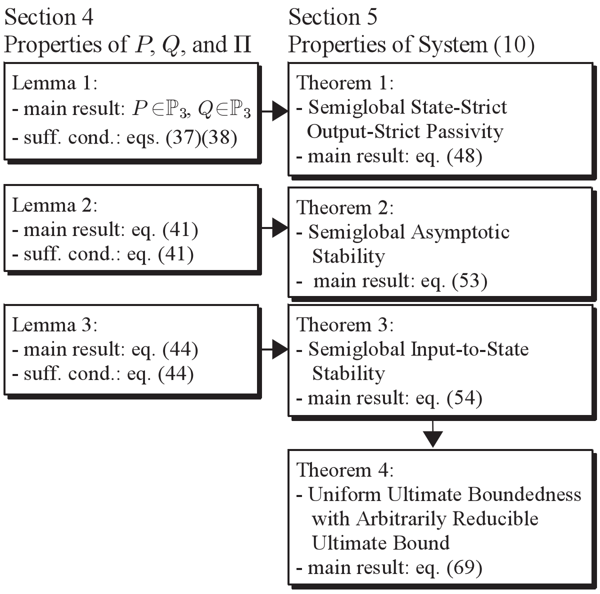

Fig. 1. Relations among lemmas, theorems, and some representative equations in this paper.

2. Preliminaries

2.1. Mathematical preliminaries

In the rest of this paper, $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{R}_+$ denotes the set of all nonnegative real numbers, $\mathbb{D}_n$ denotes the set of all $n \times n$ diagonal matrices whose diagonal elements are all strictly positive, $\mathbb{P}_n$ denotes the set of all $n \times n$ symmetric positive definite matrices, and $\mathbb{S}_n$ denotes the set of all $n \times n$ symmetric matrices. Clearly, they are related as $\mathbb{D}_n \subset \mathbb{P}_n \subset \mathbb{S}_n \subset \mathbb{R}^{n\times n}$ and $\mathbb{R}_+ \subset \mathbb{R}$. The symbol 0 denotes the zero vector or the zero matrix of appropriate dimensions and I denotes the identity matrix of an appropriate dimension. The symbol $\| * \|$ denotes the vector 2-norm or the corresponding induced matrix norm. With a positive integer k and a positive scalar a, $\mathbb{B}_k(a)$ denotes the k-dimensional ball with the radius a, which is defined as follows:

$$\mathbb{B}_k(a) \triangleq \{z \in \mathbb{R}^k |\ \|z\| \le a\}. \tag{1}$$

The maximum and minimum singular values of a matrix (equivalently, eigenvalues of positive definite matrices) are denoted by $\sigma_{\max}(*)$ and $\sigma_{\min}(*)$, respectively. In some specified cases, $\sigma_{\max}(*)$ or its upper bound is denoted in short by γ_*. Likewise, $\sigma_{\min}(*)$ or its lower bound is denoted in short by λ_*.

Throughout this paper, the symbol z is used as a versatile symbol to represent a scalar or a vector whose dimension is specified in each case.

2.2. Rigid-link manipulator and PID control

This paper considers a class of n-dimensional nonlinear systems that can be described in the following form:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = f + h, \tag{2}$$

where $q, f, h \in \mathbb{R}^n$, $M(q) \in \mathbb{P}_n$, and $C(q,\dot{q}) \in \mathbb{R}^{n\times n}$. It is assumed that the maps $M(q)$ and $C(q,\dot{q})$ satisfy

$$\dot{M}(q) = C(q,\dot{q}) + C(q,\dot{q})^T, \tag{3}$$

and there exist scalars $\lambda_M, \gamma_M, \kappa_C \in \mathbb{R}_+$ satisfying

$$\gamma_M \triangleq \sup_{q\in\mathbb{R}^n} \sigma_{\max}(M(q)),\ \lambda_M \triangleq \inf_{q\in\mathbb{R}^n} \sigma_{\min}(M(q)),$$
$$\kappa_C \triangleq \sup_{q\in\mathbb{R}^n,\,\dot{q}\in\mathbb{R}^n} \frac{\|C(q,\dot{q})\|}{\|\dot{q}\|}. \tag{4}$$

This class of systems includes n-dimensional rigid-link manipulators with nonelastic joints.[3,7,10,14] In this case, q denotes the vector of joint variables (angles for revolute joints and displacements for prismatic joints), f denotes the generalized force (torques for revolute joints and forces for prismatic joints) produced by the joint actuators, and $h \in \mathbb{R}^n$ denotes the sum of forces from all external sources. The gravitational force is included in h. The matrix $M(q)$ denotes the inertia matrix and $C(q,\dot{q})\dot{q}$ denotes the centrifugal and Coriolis forces.

A PID controller applied to the system (2) can be described as follows:

$$\dot{a} = \dot{q}_d - \dot{q}, \tag{5a}$$
$$f = B\ddot{a} + K\dot{a} + La. \tag{5b}$$

Here, f is the actuator force that appeared in Eq. (2), $q_d \in \mathbb{R}^n$ is the desired position provided to the controller, $K, B, L \in \mathbb{D}_n$ are gain matrices, and $a \in \mathbb{R}^n$ is a state vector stored in the controller. The set-point control is a special case of the control law (5) with $\dot{q}_d$ being substituted by $\dot{q}_d \equiv 0$.

2.3. Stability analyses in the literature

The literature includes several reports on the stability properties of the system described by the combination of Eqs. (2) and (5). Leaving the gravity out of consideration, Lyapunov functions for analyzing such a system in many of previous works can be described in the following form:

$$V(x) = \frac{1}{2}x^T\mathcal{P}(q)x, \tag{6}$$

where $\mathcal{P}(q) \in \mathbb{S}_{3n}$ and $x = [\ddot{a}^T, \dot{a}^T, a^T]^T \in \mathbb{R}^{3n}$. Many variations of the matrix $\mathcal{P}(q)$ have been reported in the literature. Some of them, such as those by Arimoto and Miyazaki,[4] Kelly,[9] Meza *et al.*,[13] and Kelly *et al.*,[10] are sparse (including zero block matrices), and thus the analytical derivations are rather easy. However, the resultant $\dot{V}$ becomes negative semidefinite (instead of negative definite), and thus, the stability proofs cannot use Lyapunov's direct method but require LaSalle's Invariance Theorem.[11]

Some researchers proposed strict Lyapunov functions, which are rather complicated but yield negative definite time

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derivatives. For example, Wen and Murphy[19] used

$$\mathcal{P}(q) \triangleq \begin{bmatrix} M(q) & \alpha M(q) & \alpha\beta M(q) \\ & K + \alpha B & \alpha\beta B \\ \text{sym.} & & \alpha(L + \beta K) \end{bmatrix}, \tag{7}$$

where α and β are positive constants appropriately chosen. They discussed the local asymptotic stability of trajectory-tracking control but their analysis was restricted to the case where the desired trajectory converges to a fixed position. They provided a set of many inequality conditions, some of which are implicit, to assure the asymptotic stability. Pervozvanski and Freidovich[15] used

$$\mathcal{P}(q) \triangleq \begin{bmatrix} M(q) & \alpha M(q) & \alpha\beta M(q) \\ & K + \alpha B - \alpha\beta M(q) & L + \alpha\beta B \\ \text{sym.} & & \alpha(L + \beta K) \end{bmatrix}, \tag{8}$$

to show that the tracking error can be bounded as long as the desired trajectory is slow enough. Choi and Chung[7] presented a very complicated strict Lyapunov function, in which each block of $\mathcal{P}(q)$ is a product of the gain matrices. They defined an extended disturbance, which is a quantity depending on many quantities such as the desired acceleration and velocity, and they discussed the input-to-state stability of the system by considering the extended disturbance as an input.

By imposing the assumption that the norms of $\ddot{q}_d$ and $\dot{q}_d$ are upper-bounded, one can discuss the behavior of the system (2)(5) in terms of the uniform ultimate boundedness.[5,6,17,18] To show the uniform ultimate boundedness, Rocco[18] used a strict Lyapunov function, which is defined by using a solution of a Riccati algebraic equation and is not described explicitly. Qu and Dorsey's analysis[17] is based on the assumption that equal gains were chosen for all the joints. Cervantes and Alverez-Ramirez[5] have shown the semiglobal uniform ultimate boundedness and also have shown that the ultimate bound can be arbitrarily set smaller by an appropriate choice of controller parameters. It must be noted, however, that their controller includes an additional feedforward term proportional to the desired acceleration $\ddot{q}_d$. Chaillet *et al.*[6] have shown a similar but stronger property (they termed it as "the uniform semiglobal practical asymptotic stability") of a rigid-link manipulator under the standard PID controller, with taking actuator dynamics and joint friction into account. Their analysis used a "sufficiently small" positive parameter ε_1 to make the integral gain sufficiently small in comparison to the proportional gain. Besides, all of the above analyses except Pervozvanski and Freidovich's one[15] are based on physically inconsistent arithmetic, such as addition of position and velocity.

Some other variants of PID controller have been analyzed in the literature. Those variants involve torque saturation (e.g., Section 3.2 in Arimoto[3] and others[1,2]), low-pass filtered velocity measurements,[12,16] and delay in the integrator.[12] Because they do not fall within the class of controllers described by Eq. (5), the present paper does not consider them any further.

3. Lyapunov Function Candidate and Its Parameterization

For the analysis of the system composed of the robotic manipulator (2) and the controller (5), this section provides a candidate of a strict Lyapunov function. The proposed Lyapunov function candidate is reparameterized in a new way for the convenience of the analysis in the subsequent sections.

3.1. Lyapunov function candidate

Let us define the state vector of the system (2)(5) as follows:

$$x \triangleq [\ddot{a}^T, \dot{a}^T, a^T]^T \in \mathbb{R}^{3n}. \tag{9}$$

By using this state vector, the system (2)(5) can be described in the following state-space representation:

$$\dot{x} = \mathcal{A}(q, \dot{q})x + \mathcal{B}(q)\hat{h}, \tag{10a}$$

$$y = \mathcal{C}x, \tag{10b}$$

where

$$\hat{h} \triangleq h - M(q)\ddot{q}_d - C(q, \dot{q})\dot{q}_d \in \mathbb{R}^n, \tag{11}$$

$$\mathcal{A}(q, \dot{q}) \triangleq \begin{bmatrix} -M(q)^{-1}(C(q, \dot{q}) + B) & -M(q)^{-1}K & -M(q)^{-1}L \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \in \mathbb{R}^{3n\times 3n}, \tag{12}$$

$$\mathcal{B}(q) \triangleq \begin{bmatrix} -M(q)^{-1} \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^{3n\times n}, \tag{13}$$

$$\mathcal{C} \triangleq \begin{bmatrix} -I & -\alpha I & -\alpha\beta I \end{bmatrix} \in \mathbb{R}^{n\times 3n}. \tag{14}$$

Here, α and β are positive constant scalars and $y \in \mathbb{R}^n$ is an output that is intended to be power-conjugate to the extended disturbance $\hat{h}$.

For the analysis of the system (10), this paper uses the following Lyapunov function candidate:

$$V(x, q) \triangleq \frac{1}{2}x^T\mathcal{P}(q)x\ , \tag{15}$$

where $\mathcal{P} : \mathbb{R}^n \to \mathbb{S}_{3n}$ is the map defined as follows:

$$\mathcal{P}(q) \triangleq \begin{bmatrix} M(q) & \alpha M(q) & \alpha\beta M(q) \\ & K + \alpha B & L + \alpha\beta B \\ \text{sym.} & & \alpha(L + \beta K) \end{bmatrix}. \tag{16}$$

This function is similar to but different from previously proposed functions, such as Wen and Murphy's one (7) and Pervozvanski and Freidovich's one (8).

The time derivative of Eq. (15) can be described as follows:

$$\dot{V}(x, q) = -W(x, q, \dot{q}) + y^T\hat{h}, \tag{17}$$

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where $W : \mathbb{R}^{3n} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined as

$$W(x, q, \dot{q}) \triangleq \frac{1}{2} x^T \mathcal{Q}(q, \dot{q}) x, \tag{18}$$

$$\mathcal{Q}(q, \dot{q}) \triangleq \begin{bmatrix} 2(B - \alpha M(q)) & -\alpha(C(q,\dot{q}) + \beta M(q)) & -\alpha\beta C(q,\dot{q}) \\ & 2\alpha(K - \beta B) - 2L & 0 \\ \text{sym.} & & 2\alpha\beta L \end{bmatrix} \in \mathbb{S}_{3n}. \tag{19}$$

The function V is said to be a Lyapunov function if both V and W are positive definite with respect to their first arguments.

3.2. New parameterization

Now, let us define a blockwise norm function $\psi : \mathbb{R}^{3n} \to \mathbb{R}^3_+$ as follows:

$$\psi\left(\left[z_1^T, z_2^T, z_3^T\right]^T\right) \triangleq [\|z_1\|, \|z_2\|, \|z_3\|]^T, \tag{20}$$

where $z_1, z_2, z_3 \in \mathbb{R}^n$. In addition, let γ_* and λ_* ($* \in \{K, B, L\}$) be the maximum and the minimum singular values, respectively, of the subscripted matrix. Besides, let us define the following parameters:

$$\hat{\gamma}_K \triangleq \frac{\gamma_K}{\lambda_K},\ \hat{\gamma}_B \triangleq \frac{\gamma_B}{\lambda_K},\ \hat{\lambda}_B \triangleq \frac{\lambda_B}{\lambda_K},\ \hat{\gamma}_L \triangleq \frac{\gamma_L}{\alpha\beta\lambda_K},$$
$$\hat{\lambda}_L \triangleq \frac{\lambda_L}{\alpha\beta\lambda_K}. \tag{21}$$

Then, by using γ_M, λ_M, and κ_C defined in Eq. (4), one can see that the followings are satisfied for all q and $\dot{q}$:

$$\frac{\lambda_K}{2}\psi(x)^T P \psi(x) < V(x, q) < \frac{\lambda_K}{2}\psi(x)^T \Pi \psi(x), \tag{22}$$

$$\frac{\lambda_K}{2}\psi(x)^T \tilde{Q}(\|\dot{q}\|, 0, 0)\psi(x) < W(x, q, \dot{q}), \tag{23}$$

where

$$P \triangleq \begin{bmatrix} \lambda_M/\lambda_K & -\alpha\gamma_M/\lambda_K & -\alpha\beta\gamma_M/\lambda_K \\ & 1 + \alpha\hat{\lambda}_B & -\alpha\beta(\hat{\gamma}_B + \hat{\gamma}_L) \\ \text{sym.} & & \alpha\beta(1 + \alpha\hat{\lambda}_L) \end{bmatrix} \in \mathbb{S}_3, \tag{24}$$

$$\Pi \triangleq 3\begin{bmatrix} \gamma_M/\lambda_K & 0 & 0 \\ & \hat{\gamma}_K + \alpha\hat{\gamma}_B & 0 \\ \text{sym.} & & \alpha\beta(\hat{\gamma}_K + \alpha\hat{\gamma}_L) \end{bmatrix} \in \mathbb{P}_3, \tag{25}$$

$$\tilde{Q}(z_1, z_2, z_3) \triangleq 2\begin{bmatrix} \hat{\lambda}_B - \frac{\alpha(\gamma_M + \kappa_C(z_2 + \beta z_3))}{\lambda_K} & -\frac{\alpha(\kappa_C z_1 + \beta\gamma_M)}{2\lambda_K} & -\frac{\alpha\beta\kappa_C z_1}{2\lambda_K} \\ & \alpha(1 - \beta(\hat{\gamma}_B + \hat{\gamma}_L)) & 0 \\ \text{sym.} & & \alpha^2\beta^2\hat{\lambda}_L \end{bmatrix} \in \mathbb{S}_3. \tag{26}$$

Here, the condition (23) can be further rewritten by using the following relations:

$$\begin{aligned}\psi(x)^T \tilde{Q}(\|\dot{q}\|, 0, 0)\psi(x) &= \psi(x)^T \tilde{Q}(\|\dot{q}\| - \|\ddot{a}\|, \\ &\qquad \|\dot{a}\|, \|a\|)\psi(x) \\ &> \psi(x)^T Q \psi(x),\end{aligned} \tag{27}$$

where

$$Q \triangleq \tilde{Q}(\xi, \eta_2, \eta_3), \tag{28}$$

if $|\|\dot{q}\| - \|\ddot{a}\|| < \xi$, $\|\dot{a}\| < \eta_2$, and $\|a\| < \eta_3$, because all entries of $\psi(x)$ are positive. Here, one can see that the first condition can be replaced by $\|\dot{q}_d\| < \xi$ because of

$$|\|\dot{q}\| - \|\ddot{a}\|| = |\|\dot{q}\| - \|\dot{q}_d - \dot{q}\|| \le \|\dot{q}_d\|. \tag{29}$$

By using Eqs. (23) and (27), one can see that

$$\frac{\lambda_K}{2}\psi(x)^T Q \psi(x) < W(x, q, \dot{q}) \tag{30}$$

is satisfied if $\dot{a} \in \mathbb{B}_3(\eta_2)$, $a \in \mathbb{B}_3(\eta_3)$, and $\dot{q}_d \in \mathbb{B}_3(\xi)$.

The conditions (22) and (30) indicate that the matrices P, Q, and Π can be used to investigate upper and lower bounds of $V(x, q)$ and $W(x, q, \dot{q})$ in a lower dimension than $\mathcal{P}$ and $\mathcal{Q}(q, \dot{q})$.

4. Properties of P, Q, and Π

This section provides three important properties of the matrices P, Q, and Π, which were introduced in the previous section.

4.1. Positive definiteness of P and Q

The following lemma provides a pair of explicit inequalities that guarantee the positive definiteness of P and Q. Its proof provides a procedure for selecting the parameter values satisfying the positive definiteness of the matrices.

Lemma 1. Consider the matrices $P \in \mathbb{S}_3$ and $Q \in \mathbb{S}_3$ defined as Eqs.(24) and (28), respectively. Then, for any sets of positive scalars

$$\mathcal{I}_1 \triangleq \{\gamma_M, \lambda_M, \kappa_C, \xi\} \cup \{\eta_2, \eta_3\}, \tag{31}$$

there exists another set of positive scalars

$$\mathcal{O}_1 \triangleq \{\lambda_K, \hat{\gamma}_K, \hat{\gamma}_B, \hat{\lambda}_B, \hat{\gamma}_L, \hat{\lambda}_L, \alpha, \beta\}, \tag{32}$$

that satisfies $P \in \mathbb{P}_3$ and $Q \in \mathbb{P}_3$.

Proof. The theorem is proven by showing that the determinants of all their diagonal submatrices can be positive. Let Λ_{pi} and Λ_{qi} ($i \in \{1, 2, 3\}$) denote the determinants of the lower-right $i \times i$ submatrices of P and Q, respectively. Those for $i = 1$ are trivially obtained as $\Lambda_{p1} = \alpha\beta$ and $\Lambda_{q1} = 2\alpha^2\beta^2\hat{\lambda}_L$, which are always positive. Those for $i > 1$

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are obtained as follows:

$$\Lambda_{p2} = \alpha\beta\left(\alpha^2\hat{\lambda}_B\hat{\lambda}_L + \alpha((\hat{\lambda}_B + \hat{\lambda}_L) - \beta(\hat{\gamma}_B + \hat{\gamma}_L)^2) + 1\right), \tag{33}$$

$$\Lambda_{p3} = \frac{\Lambda_{p2}\lambda_M}{\lambda_K^2}(\lambda_K - \Lambda_A), \tag{34}$$

$$\Lambda_{q2} = 4\alpha^3\beta^2\hat{\lambda}_L(1 - \beta(\hat{\gamma}_B + \hat{\gamma}_L)), \tag{35}$$

$$\Lambda_{q3} = \frac{2\Lambda_{q2}\hat{\lambda}_B}{\lambda_K^2}\left(\lambda_K^2 - 2\Lambda_B\lambda_K - \Lambda_C\right), \tag{36}$$

where Λ_A, Λ_B, and Λ_C are scalars independent from λ_K. Now, it is easy to see that Λ_{p2} and Λ_{q2} are positive if

$$\beta < \frac{\hat{\lambda}_B + \hat{\lambda}_L}{(\hat{\gamma}_B + \hat{\gamma}_L)^2}\left(\leq \frac{1}{\hat{\gamma}_B + \hat{\gamma}_L}\right) \tag{37}$$

is satisfied, irrespective of the value of α. Here, Λ_A and Λ_C are also positive if Eq. (37) is satisfied and Λ_B is always positive. Thus, one can see that Λ_{p3} and Λ_{q3} are positive if the following condition is satisfied:

$$\lambda_K > \max\left(\Lambda_A, \Lambda_B + \sqrt{\Lambda_B^2 + \Lambda_C}\right). \tag{38}$$

Based on the discussion above, for an arbitrary set $\mathcal{I}_1$, one can choose a set $\mathcal{O}_1$ in the following procedure. First, choose $\{\hat{\gamma}_K, \hat{\gamma}_B, \hat{\lambda}_B, \hat{\gamma}_L, \hat{\lambda}_L, \alpha\}$ arbitrarily, except for $\hat{\gamma}_B \geq \hat{\lambda}_B$ and $\hat{\gamma}_L \geq \hat{\lambda}_L$. Next, choose a small enough β to satisfy Eq. (37). Finally, choose a large enough λ_K so that Eq. (38) is satisfied. □

Wen and Murphy[19] investigated the positive definiteness of the matrices correspondent to P and Q based on their eigenvalues, while the presented proof focuses on the determinants. One advantage of the use of determinants lies in the fact that the determinants of P and Q can have physical units while the eigenvalues cannot because the entries of P and Q have different physical dimensions from one another. Besides, Wen and Murphy used the assumption that α and β are so small that the higher order terms of α and β can be ignored. The presented proof is free from such an assumption.

4.2. *Involving* Π *and* $\{\theta_1, \theta_2, \theta_3\}$

The following lemma is to impose another restriction on the choice of the parameter λ_K to satisfy another condition with another set of given parameters $\{\theta_1, \theta_2, \theta_3\}$. In the upcoming Section 5.2, this lemma will be used to yield Eq. (53) in the proof of Theorem 2 to show that the region of attraction can be arbitrarily enlarged.

Lemma 2. Consider the matrices $P \in \mathbb{S}_3$, $\Pi \in \mathbb{P}_3$, and $Q \in \mathbb{S}_3$ that are defined as Eqs. (24), (25), and (28), respectively. Then, for any sets of positive scalars

$$\mathcal{I}_2 \triangleq \{\gamma_M, \lambda_M, \kappa_C, \xi\} \cup \{\theta_1, \theta_2, \theta_3\}, \tag{39}$$

there exists another set of positive scalars

$$\mathcal{O}_2 \triangleq \{\lambda_K, \hat{\gamma}_K, \hat{\gamma}_B, \hat{\lambda}_B, \hat{\gamma}_L, \hat{\lambda}_L, \alpha, \beta\} \cup \{\eta_2, \eta_3\}, \tag{40}$$

that satisfies $P \in \mathbb{P}_3$, $Q \in \mathbb{P}_3$, and

$$\Omega_{pi}[\theta_1, \theta_2, \theta_3]\Pi[\theta_1, \theta_2, \theta_3]^T < \eta_i^2, \quad \forall i \in \{2, 3\}, \tag{41}$$

where Ω_{pi} ($i \in \{2, 3\}$) is the (i,i)th entry of P^{-1}.

Proof. This lemma is proven by showing the existence of a set $\mathcal{O}_2$ satisfying Eqs. (37), (38), and (41). It is easily shown that the left-hand side of Eq. (41) is a monotonously decreasing function of λ_K. Therefore, one can choose $\mathcal{O}_2$ satisfying only Eqs. (37) and (41) with large enough values being chosen for λ_K, η_2, and η_3. After that, one can increase λ_K to satisfy Eq. (38) because the right-hand side of Eq. (38) is independent from λ_K.

In conclusions, $\mathcal{O}_2$ can be chosen in the following procedure. First, choose $\{\hat{\gamma}_K, \hat{\gamma}_B, \hat{\lambda}_B, \hat{\gamma}_L, \hat{\lambda}_L, \alpha\}$ arbitrarily, except for $\hat{\gamma}_B \geq \hat{\lambda}_B$ and $\hat{\gamma}_L \geq \hat{\lambda}_L$. Second, choose a small enough β to satisfy Eq. (37). Third, choose large enough values for $\{\lambda_K, \eta_2, \eta_3\}$ so that Eq. (41) is satisfied. Finally, increase the value of λ_K so that Eq. (38) is satisfied. □

4.3. *Another restriction on* Q

The following Lemma 3 is to further refine the value λ_K to satisfy another condition with another set of given parameters $\{\phi_1, \phi_2, \phi_3\}$. In the upcoming Sections 5.3 and 5.4, this lemma will be used to yield Eq. (60) in the proof of Theorem 3 and to show that the ultimate bound can be arbitrarily made small in the proof of Theorem 4.

Lemma 3. Consider the matrices $P \in \mathbb{S}_3$, $\Pi \in \mathbb{P}_3$, and $Q \in \mathbb{S}_3$ that are defined as Eqs. (24), (25), and (28), respectively. Then, for any sets

$$\mathcal{I}_3 \triangleq \{\gamma_M, \lambda_M, \kappa_C, \xi\} \cup \{\theta_1, \theta_2, \theta_3, \phi_1, \phi_2, \phi_3\}, \tag{42}$$

there exists another set of positive scalars

$$\mathcal{O}_3 \triangleq \mathcal{O}_2 = \{\lambda_K, \hat{\gamma}_K, \hat{\gamma}_B, \hat{\lambda}_B, \hat{\gamma}_L, \hat{\lambda}_L, \alpha, \beta\} \cup \{\eta_2, \eta_3\}, \tag{43}$$

with which $P \in \mathbb{P}_3$, $Q \in \mathbb{P}_3$, Eq. (41), and

$$\Psi_{qi}[1, \alpha, \alpha\beta]Q^{-1}[1, \alpha, \alpha\beta]^T < \lambda_K^2\phi_i^2, \quad \forall i \in \{1, 2, 3\} \tag{44}$$

are satisfied where Ψ_{qi} ($i \in \{1, 2, 3\}$) is the (i, i)th entry of Q^{-1}.

Proof. This lemma is proven by showing the existence of a set $\mathcal{O}_3$ satisfying Eqs. (37), (38), (41), and (44). It is easy to see that the left-hand side of Eq. (44) is a monotonously decreasing function of λ_K for every $i \in \{1, 2, 3\}$. This implies that Eq. (44) is satisfied with any λ_K larger than a particular value, as is the case with Eqs. (38) and (41). Therefore, once the set $\mathcal{O}_3$ is chosen so that Eqs. (37), (38), and (41) hold,

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one can increase λ_K so that Eqs. (37), (38), (41), and (44) are satisfied simultaneously.

In conclusions, $\mathcal{O}_3$ can be chosen in the following procedure. First, choose $\{\hat{\gamma}_K, \hat{\gamma}_B, \hat{\lambda}_B, \hat{\gamma}_L, \hat{\lambda}_L, \alpha\}$ arbitrarily except that $\hat{\gamma}_B \geq \hat{\lambda}_B$ and $\hat{\gamma}_L \geq \hat{\lambda}_L$. Second, choose a small enough β to satisfy Eq. (37). Third, choose large enough $\{\lambda_K, \eta_2, \eta_3\}$ to satisfy Eq. (41). Finally, increase λ_K to satisfy both Eqs. (38) and (44). □

5. Main Results

We are now in position to present the main results.

5.1. Semiglobal strict passivity

First, the systems (10), which is composed of a rigid-link manipulator and a PID controller, is shown to be strictly passive in the semiglobal sense. This is a direct consequence of Lemma 1.

Theorem 1. The system (10) is semiglobally strictly passive.

Proof. Lemma 1 implies that, for any given $\{M, C, \xi, \eta_2, \eta_3\}$, there exists a set $\{K, B, L, \alpha, \beta\}$ that guarantees $P \in \mathbb{P}_3$ and $Q \in \mathbb{P}_3$. Assume that such a set of parameters are chosen. Then, the following inequalities are satisfied:

$$V(x, q) > \frac{\lambda_K}{2}\psi(x)^T P\psi(x) > 0, \ \forall x \in \mathbb{R}^{3n}, \ \forall q \in \mathbb{R}^n, \tag{45}$$

$$\dot{V}(x, q) < y^T\hat{h} - \frac{\lambda_K}{2}\psi(x)^T Q\psi(x), \ \forall x \in \mathbb{C}(\infty, \eta_2, \eta_3),$$
$$\forall \dot{q}_d \in \mathbb{B}_n(\xi), \ \forall q \in \mathbb{R}^n, \tag{46}$$

where

$$\mathbb{C}(z_1, z_2, z_3) \triangleq \left\{ x \in \mathbb{R}^3 \;\middle|\; \max\left(\frac{\|\ddot{a}\|}{z_1}, \frac{\|\dot{a}\|}{z_2}, \frac{\|a\|}{z_3}\right) \leq 1 \right\}. \tag{47}$$

Here, $\mathbb{C}(\infty, \eta_2, \eta_3)$ is the region in which $W(x, q, \dot{q}) \geq 0$. Because $Q \in \mathbb{P}_3$, Eq. (46) implies the following:*

$$\dot{V}(x, q) < y^T\hat{h} - \frac{\lambda_K\sigma_{\min}(Q)}{2}\|x\|^2, \ \forall x \in \mathbb{C}(\infty, \eta_2, \eta_3),$$
$$\forall \dot{q}_d \in \mathbb{B}_n(\xi), \ \forall q \in \mathbb{R}^n. \tag{48}$$

Therefore, for all $\{M, C, \xi, \eta_2, \eta_3\}$, there exists a set $\{K, B, L, \alpha, \beta\}$ with which Eq. (48) is satisfied. Thus, one can conclude that the system (10) is strictly passive in the semiglobal sense. □

*The inequality (48) includes the quantity $\|x\|$, which cannot have a consistent physical unit. In this paper, such quantities are avoided in inequalities and equations to be numerically evaluated, but not in analytical proofs. The inequality (54) also follows this rule.

5.2. Semiglobal asymptotic stability

Second, Lemma 2 leads to the following Theorem 2, which states that a rigid-link manipulator under PID set-point control ($\dot{q}_d \equiv 0$) under no disturbance or no gravity ($\hat{h} \equiv 0$) can be asymptotically stable and that the region of attraction can be arbitrarily enlarged. When there is gravity, the proof does not apply but the upcoming Theorem 4 will show that the tracking error can be arbitrarily reduced.

Theorem 2. The system (10a) with $\dot{q}_d \equiv 0$ and $\hat{h} \equiv 0$ is semiglobally asymptotically stable.

Proof. The theorem can be proven by showing that, for all sets $\{M, C, \theta_1, \theta_2, \theta_3\}$, there exists a set $\{K, B, L, \alpha, \beta\}$ with which $x \to 0$ as $t \to \infty$ for all $x(0) \in \mathbb{C}(\theta_1, \theta_2, \theta_3)$. Here, $\mathbb{C}(\theta_1, \theta_2, \theta_3)$ is the set termed as the region of attraction.

The proof of Lemma 2 implies that, for any sets $\{M, C, \theta_1, \theta_2, \theta_3\}$, one can choose $\{K, B, L, \alpha, \beta, \eta_2, \eta_3\}$ so that Eqs. (37), (38), and (41) with $\xi = 0$ are satisfied. Assume that such values are chosen. Then, $P \in \mathbb{P}_3$, $Q \in \mathbb{P}_3$, and

$$\sup_{\substack{z \in \mathbb{R}^3_+ \\ z_1 \leq \theta_1 \wedge z_2 \leq \theta_2 \wedge z_3 \leq \theta_3}} z^T \Pi z < \inf_{\substack{z \in \mathbb{R}^3_+ \\ z_2 \geq \eta_2 \vee z_3 \geq \eta_3}} z^T P z \tag{49}$$

are satisfied, and thus,

$$V(x, q) > 0, \forall x \in \mathbb{R}^{3n}, \forall q \in \mathbb{R}^n, \tag{50}$$

$$\dot{V}(x, q) < 0, \ \forall x \in \mathbb{C}(\infty, \eta_2, \eta_3), \forall q \in \mathbb{R}^n, \tag{51}$$

$$0 \in \mathbb{C}(\theta_1, \theta_2, \theta_3) \subset \mathbb{C}(\infty, \eta_2, \eta_3) \subset \mathbb{R}^{3n}, \tag{52}$$

$$\sup_{\substack{x \in \mathbb{R}^{3n}, \ q \in \mathbb{R}^n \\ x \in \mathbb{C}(\theta_1, \theta_2, \theta_3)}} V(x, q) < \inf_{\substack{x \in \mathbb{R}^{3n}, \ q \in \mathbb{R}^n \\ x \notin \mathbb{C}(\infty, \eta_2, \eta_3)}} V(x, q) \tag{53}$$

are also satisfied. Here, Eq. (52) is easily proven by showing $\theta_2 < \eta_2$ and $\theta_3 < \eta_3$, and Eq. (53) is the direct consequence of Eq. (49). Therefore, one can conclude that, if $x(0) \in \mathbb{C}(\theta_1, \theta_2, \theta_3)$, x does not deviate from $\mathbb{C}(\infty, \eta_2, \eta_3)$ and does converge to the origin as $t \to \infty$. □

Taking the gravity out of consideration, the presented proof can be viewed as an alternative to the previous proofs shown by, e.g., Meza *et al.*[13] One advantage of the new proof is that, in the proof, the asymptotic stability requires only three inequalities (37), (38), and (41), which are much simpler than implicit inequality conditions in the literature.[17,19] Another important point is that the presented proof provides a closed-form gain-selection procedure to achieve desired dimensions $\{\theta_1, \theta_2, \theta_3\}$ of the region of attraction as suggested in the proof of Lemma 2. This feature is in contrast to previous procedures such as Kelly's,[9] Meza *et al.*'s,[13] and Hernández-Guzmán *et al.*'s.[8] One limitation of the presented procedure is that it leaves many parameters unconstrained, which implies the need for additional optimization criteria.

5.3. Semiglobal input-to-state stability

Third, it is shown that a PID-controlled rigid-link manipulator is semiglobally input-to-state stable, where the input is $\hat{h}$ defined in Eq. (11).

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Theorem 3. The system (10a) is semiglobally input-to-state stable.

Proof. The theorem is proven by showing that, for any sets $\{M, C, \xi, \theta_1, \theta_2, \theta_3, \gamma_h\}$, there exists a set $\{K, B, L, \alpha, \beta, \eta_2, \eta_3\}$ with which there exist a class-$\mathcal{KL}$ function Γ_1 and a class-$\mathcal{K}$ function* Γ_2 with which

$$\|x(t)\| < \max\left(\Gamma_1(\|x(0)\|, t)\,, \Gamma_2\left(\sup_{\tau\in[0,t]} \|\hat{h}(\tau)\|\right)\right) \quad (54)$$

is satisfied if the followings are satisfied:

$$x(0) \in \mathbb{C}(\theta_1, \theta_2, \theta_3),\ \sup_{\tau\in[0,t]} \|\hat{h}(\tau)\| \le \gamma_h,\ \text{and } \dot{q}_d \in \mathbb{B}_n(\xi). \quad (55)$$

With the given set $\{M, C, \xi, \theta_1, \theta_2, \theta_3, \gamma_h\}$, choose another three positive scalars $\{\phi_1, \phi_2, \phi_3\}$ so that they satisfy

$$\phi_i \le \theta_i/\gamma_h\ (i \in \{1, 2, 3\}). \quad (56)$$

The proof of Lemma 3 implies that, for the set $\{M, C, \xi, \theta_1, \theta_2, \theta_3, \phi_1, \phi_2, \phi_3\}$, a set $\{K, B, L, \alpha, \beta, \eta_2, \eta_3\}$ can be chosen to guarantee the satisfaction of Eqs. (37), (38), (41), and (44). Assume that the parameters are chosen in such a manner and that the conditions (55) are satisfied. Then, the following conditions are also satisfied:

$$V(x, q) > 0,\ \forall x \in \mathbb{R}^{3n}\ \forall q \in \mathbb{R}^n, \quad (57)$$

$$\dot{V}(x, q) < (c^T\psi(x))\|\hat{h}\| - \frac{\lambda_K}{2}\psi(x)^T Q\psi(x),$$
$$\forall x \in \mathbb{C}(\infty, \eta_2, \eta_3),\ \forall q \in \mathbb{R}^n,\ \forall \dot{q}_d \in \mathbb{B}_n(\xi), \quad (58)$$

$$\sup_{\substack{x\in\mathbb{R}^{3n},\ q\in\mathbb{R}^n\\ x\in\mathbb{C}(\theta_1,\theta_2,\theta_3)}} V(x, q) < \inf_{\substack{x\in\mathbb{R}^{3n},\ q\in\mathbb{R}^n\\ x\notin\mathbb{C}(\infty,\eta_2,\eta_3)}} V(x, q), \quad (59)$$

$$0 \in \mathbb{C}(\Phi_1\|\hat{h}\|, \Phi_2\|\hat{h}\|, \Phi_3\|\hat{h}\|) \subset \mathbb{C}(\theta_1, \theta_2, \theta_3)$$
$$\subset \mathbb{C}(\infty, \eta_2, \eta_3) \subset \mathbb{R}^{3n}, \quad (60)$$

where

$$c \triangleq [1, \alpha, \alpha\beta]^T, \quad (61)$$

$$\Phi_i \triangleq \frac{\sqrt{\Psi_{qi} c^T Q^{-1} c}}{\lambda_K},\ \forall i \in \{1, 2, 3\}. \quad (62)$$

The condition (58) can be proven by using $y^T\hat{h} \le \|y\|\|\hat{h}\| \le c^T\psi(x)\|\hat{h}\|$. Besides, with regard to Eq. (60), it is easy to show that $\mathbb{C}(\Phi_1\|\hat{h}\|, \Phi_2\|\hat{h}\|, \Phi_3\|\hat{h}\|)$ is a superset of the set in which the right-hand side of Eq. (58) is positive. This means that $\dot{V}(x, q) < 0$ is satisfied if

$$x \in \neg\mathbb{C}(\Phi_1\|\hat{h}\|, \Phi_2\|\hat{h}\|, \Phi_3\|\hat{h}\|) \cap \mathbb{C}(\infty, \eta_2, \eta_3). \quad (63)$$

*The definitions of class-$\mathcal{K}$ and class-$\mathcal{KL}$ functions are provided in Definition 4.3, p. 144, of Khalil.[11]

The aforementioned discussion implies that, as long as $x \in \mathbb{C}(\infty, \eta_2, \eta_3)$, $V(x)$ decreases until x reaches the set $\mathbb{C}(\Phi_1\|\hat{h}\|, \Phi_2\|\hat{h}\|\phi_2, \Phi_3\|\hat{h}\|)$, and that x does not deviate from $\mathbb{C}(\infty, \eta_2, \eta_3)$ if $x(0) \in \mathbb{C}(\theta_1, \theta_2, \theta_3)$. Therefore, there exists a class-$\mathcal{KL}$ function $\tilde{\Gamma}_1$ that satisfies

$$V(x(t), q(t)) < \max\Bigg(\tilde{\Gamma}_1(V(x(0), q(0)), t),$$
$$\sup_{\tau\in[0,t]}\left(\sup_{z\in\mathbb{C}(\Phi_1\|\hat{h}(\tau)\|, \Phi_2\|\hat{h}(\tau)\|, \Phi_3\|\hat{h}(\tau)\|)} V(z, q(\tau))\right)\Bigg), \quad (64)$$

if $x(0) \in \mathbb{C}(\theta_1, \theta_2, \theta_3)$. Now, let $\lambda_{\mathcal{P}}$ and $\gamma_{\mathcal{P}}$ be defined as follows:

$$\lambda_{\mathcal{P}} \triangleq \inf_{q\in\mathbb{R}^3} \sigma_{\min}(\mathcal{P}(q)), \quad (65)$$

$$\gamma_{\mathcal{P}} \triangleq \sup_{q\in\mathbb{R}^3} \sigma_{\max}(\mathcal{P}(q)). \quad (66)$$

Then, because $\lambda_{\mathcal{P}}\|x\|^2/2 < V(x, q) < \gamma_{\mathcal{P}}\|x\|^2/2$, Eq. (64) leads to Eq. (54) with Γ_1 and Γ_2 being replaced by

$$\Gamma_1(z, t) \triangleq \sqrt{\frac{\tilde{\Gamma}_1(\gamma_{\mathcal{P}} z^2/2, t)}{\lambda_{\mathcal{P}}/2}}, \quad (67)$$

$$\Gamma_2(z) \triangleq z\sqrt{\frac{\gamma_{\mathcal{P}}\left(\Phi_1^2 + \Phi_2^2 + \Phi_3^2\right)}{\lambda_{\mathcal{P}}}}, \quad (68)$$

respectively. Because these definitions of $\Gamma_1(z, t)$ and $\Gamma_2(z)$ are class-$\mathcal{KL}$ and class-$\mathcal{K}$ functions, respectively, and because these functions can be found with any sets $\{M, C, \xi, \theta_1, \theta_2, \theta_3, \gamma_h\}$, one can conclude that the system is semiglobally input-to-state stable. □

Here, it must be noted that Φ_i $(i \in \{1, 2, 3\})$ can be arbitrarily made small by choosing sufficiently small values for ϕ_i because $\Phi_i < \phi_i$ from Eq. (44). This fact is not relevant to this theorem, but is important for the next Theorem 4.

Choi and Chung[7] also presented a proof of an input-to-state stability, but their definition of the input (which they named an "extended disturbance") is different from the one in the present analysis. Specifically, their "extended disturbance" is a function of a, $\dot{a}$, and $\ddot{a}$, which is not the case with $\hat{h}$ in the present analysis.

5.4. *Uniform ultimate boundedness with arbitrarily reducible ultimate bound*

Finally, it is shown that arbitrarily small residual tracking error can be achieved from arbitrarily large initial error by an appropriate choice of gains.

Theorem 4. The system (10a) with $\|\dot{q}_d\|$ and $\|\hat{h}\|$ being bounded is semiglobally uniformly ultimately bounded and the ultimate bound can be set arbitrarily small with an appropriate choice of controller parameters.

Proof. Let ξ and γ_h be the upper bound of $\|\dot{q}_d\|$ and $\|\hat{h}\|$, respectively. The theorem can be proven by showing that, for

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any sets $\{M, C, \xi, \theta_1, \theta_2, \theta_3, \phi_1, \phi_2, \phi_3, \gamma_h\}$, there exists a set $\{K, B, L, \alpha, \beta, \eta_2, \eta_3\}$ with which

$$\begin{aligned}&\text{if } x(0) \in \mathbb{C}(\theta_1, \theta_2, \theta_3),\\ &\quad \exists t_1 > t \ \text{ s.t. } \ x(t) \in \mathbb{C}(\phi_1\gamma_h, \phi_2\gamma_h, \phi_3\gamma_h), \ \forall\, t > t_1.\end{aligned} \tag{69}$$

Here, the set $\mathbb{C}(\phi_1\gamma_h, \phi_2\gamma_h, \phi_3\gamma_h)$ is the set termed as an ultimate bound.

In the middle of the proof of Theorem 3, it has been shown that, if Eq. (56) is satisfied, $\{K, B, L, \alpha, \beta, \eta_2, \eta_3\}$ can be chosen so that $V(x, q)$ decreases until x reaches the set $\mathbb{C}(\Phi_1\|\hat{h}\|, \Phi_2\|\hat{h}\|, \Phi_3\|\hat{h}\|)$. By noticing that $\|\hat{h}\| \le \gamma_h$ and $\Phi_i < \phi_i$, one can see that $\mathbb{C}(\phi_1\gamma_h, \phi_2\gamma_h, \phi_3\gamma_h)$ is a superset of the set $\mathbb{C}(\Phi_1\|\hat{h}\|, \Phi_2\|\hat{h}\|, \Phi_3\|\hat{h}\|)$, and that Eq. (69) is satisfied.

Even if Eq. (56) is not satisfied, one can use the values of θ_i/γ_h as the substitutes of ϕ_i to choose the parameters $\{K, B, L, \alpha, \beta, \eta_2, \eta_3\}$. Then

$$\Phi_i < \min(\phi_i, \theta_i/\gamma_h) \le \phi_i, \ \forall i \in \{1, 2, 3\} \tag{70}$$

is satisfied and thus $\mathbb{C}(\phi_1\gamma_h, \phi_2\gamma_h, \phi_3\gamma_h)$ is a superset of the set $\mathbb{C}(\Phi_1\|\hat{h}\|, \Phi_2\|\hat{h}\|, \Phi_3\|\hat{h}\|)$. Thus, one can conclude that Eq. (69) is satisfied. □

The proof implies that, by the procedure presented in the proof of Lemma 3, one can choose the gain matrices $\{K, B, L\}$ to achieve desired dimensions $\{\phi_1\gamma_h, \phi_2\gamma_h, \phi_3\gamma_h\}$ of the ultimate bound and desired dimensions $\{\theta_1, \theta_2, \theta_3\}$ of the region of attraction under known upper bounds γ_h and ξ of $\hat{h}$ (including the gravity) and $\dot{q}$, respectively. One limitation is that, again, the procedure does not provide any guidelines for the choice of the six parameters $\{\hat{\gamma}_K, \hat{\gamma}_B, \hat{\lambda}_B, \hat{\gamma}_L, \hat{\lambda}_L, \alpha\}$.

A result similar to this theorem has been obtained by Rocco.[18] In his analysis, however, the term $C(q, \dot{q})\dot{q}$ is excluded from the nominal system and included in the disturbance term $\hat{h}$. Cervantes and Alvarez-Ramirez's result[5] is also similar, but in their analysis, the desired jerk (the time derivative of the desired acceleration) is assumed to be bounded, and no external forces are considered except the gravity. Besides, the control law includes two additional terms, one being proportional to the desired acceleration and the other being constant. It may also be worth noting that their definition of the state vector depends on the actual and desired accelerations.

Chaillet *et al.*[6] showed a stronger result regarding the standard PID control, which satisfies not only the boundedness but also the Lyapunov stability. Their analysis also clarifies an explicit gain-selection procedure. In their analysis, however, the region of attraction and the ultimate bound are defined as balls in the state space consisting of a transformed state vector $[\ddot{a}^T, \dot{a}^T, (a + \dot{a}/\varepsilon_1)^T]^T$, where ε_1 is a "small" constant. Thus, it is unclear how an upper bound of a can be designed, although it may not be important in practice. In addition, as is the case with most previous methods, their analysis includes physically inconsistent arithmetic operations among different physical quantities, such as the definition of the norm of the state vector comprising position and velocity.

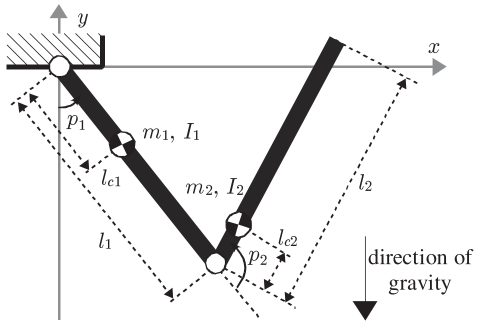

Fig. 2. 2-DOF manipulator for the simulation. ($l_1 = 0.26$ m, $l_{c1} = 0.0983$ m, $l_2 = 0.26$ m, $l_{c2} = 0.0229$ m, $m_1 = 6.5225$ kg, $m_2 = 2.0458$ kg, $I_1 = 0.1213$ kg·m^2, $I_2 = 0.0116$ kg·m^2. All of these parameters are the same as those in p. 115 of Kelly *et al.*[10])

6. Illustrative Example

6.1. Problem setting and gain selection

A set of numerical examples is now presented for the illustration of the proven properties of a PID-controlled robot. Let us consider the model of a two-DOF manipulator shown in Fig. 2. The parameters of the robot were chosen identical to the example in p. 115 of Kelly *et al.*,[10] as detailed in the caption of Fig. 2. The disturbance force h was set equal to the gravity force, which was determined as indicated in p. 214 of Kelly *et al.*[10] This means that no disturbance except the gravity was considered. The other parameters were identified as $\lambda_M = 0.011$ kg·m^2, $\gamma_M = 0.361$ kg·m^2 (from p. 215 of Kelly *et al.*[10]), and $\kappa_C = 0.0487$ kg·m^2 (from p. 126 of Kelly *et al.*[10]).

The desired dimensions of the region of attraction and the ultimate bound (i.e., the desired maximum residual tracking errors) were respectively chosen as follows:

$$\{\theta_1, \theta_2, \theta_3\} = \{5 \text{ rad/s}, 3 \text{ rad}, 1 \text{ rad·s}\},$$
$$\{\phi_1\gamma_h, \phi_2\gamma_h, \phi_3\gamma_h\} = \{3 \text{ rad/s}, 0.1 \text{ rad}, 0.6 \text{ rad·s}\}.$$

The expected maximum magnitudes of $\dot{q}_d$ and $\hat{h}$ were set as $\xi = 1.0\,\text{rad s}^{-1}$ and $\gamma_h = 14.0$ Nm, respectively.

The gains were chosen based on the procedure suggested in the proof of Lemma 3. Here, parameters that are not constrained by the procedure were chosen based on some trial and errors. First, the parameters $\{\hat{\gamma}_K, \hat{\gamma}_B, \hat{\lambda}_B, \hat{\gamma}_L, \hat{\lambda}_L, \alpha\}$ were chosen as $\hat{\gamma}_K = 1$, $\hat{\gamma}_B = \hat{\lambda}_B = 0.01$ s, $\hat{\gamma}_L = \hat{\lambda}_L = 0.5$ s, $\alpha = 4\,\text{s}^{-1}$. Second, to satisfy Eq. (37), β was chosen as $\beta = 0.5\,\text{s}^{-1}$. Third, to satisfy Eq. (41), $\{\lambda_K, \eta_2, \eta_3\}$ were chosen as $\lambda_K = 640$ Nm/rad, $\eta_2 = 9.5$ rad, and $\eta_3 = 3.4$ rad·s. Last, these parameters were confirmed to satisfy Eqs. (38) and (44). Thus, the gain matrices were chosen as $K = \text{diag}[640, 640]$ Nm/rad, $B = \text{diag}[6.4, 6.4]$ Nm·s/rad, $L = \text{diag}[640, 640]$ Nm/rad/s.

It was found that the procedure tends to result in high values for λ_K unless the first seven parameters $\{\hat{\gamma}_K, \hat{\gamma}_B, \hat{\lambda}_B, \hat{\gamma}_L, \hat{\lambda}_L, \alpha, \beta\}$ are carefully chosen although

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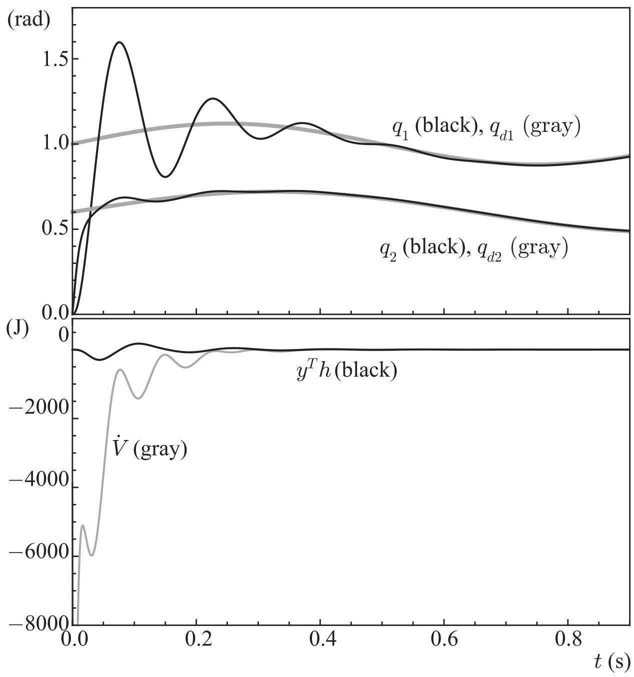

Fig. 3. Simulation results with unscaled $\hat{h}(t)$.

they are restricted only by Eq. (37) in the procedure. For example, once $\{\hat{\gamma}_K, \hat{\gamma}_B, \hat{\lambda}_B, \hat{\gamma}_L, \hat{\lambda}_L, \alpha\}$ are chosen as indicated above, Eq. (37) suggests $\beta < 1.96\,\mathrm{s}^{-1}$. If β is chosen as, e.g., $\beta = 1.5\,\mathrm{s}^{-1}$, then the conditions (41), (38) suggest, e.g., $\{\lambda_K, \eta_2, \eta_3\} = \{1000\,\mathrm{Nm/rad}, 20\,\mathrm{rad}, 5\,\mathrm{rad{\cdot}s}\}$, and the condition (44) further restricts λ_K as $\lambda_K > 1200\,\mathrm{Nm/rad}$. As another example, if the parameters are chosen identical to the aforementioned example except $\alpha = 5\,\mathrm{s}^{-1}$, then Eq. (41) suggests $\{\lambda_K, \eta_2, \eta_3\} = \{1000\,\mathrm{Nm/rad}, 52\,\mathrm{rad}, 9\,\mathrm{rad{\cdot}s}\}$, Eq. (38) suggests $\lambda_K > 1800\,\mathrm{Nm/rad}$, and Eq. (44) suggests $\lambda_K > 2100\,\mathrm{Nm/rad}$. Because very high gains cannot be implemented in practice, an improved procedure that can impose an upper bound on λ_K should be sought in a future study.

6.2. Simulation results

Simulation was performed by using the aforementioned robot model and controller gains. In the simulation, the initial state vector $x(0)$ and the desired trajectory $q_d(t)$ were respectively chosen as follows:

$$x(0) = \begin{bmatrix} \dot{q}_d(0) \\ q_d(0) \\ 0 \end{bmatrix} \in \mathbb{R}^6, \tag{71}$$

$$q_d(t) = \begin{bmatrix} 1.0 + 0.12\sin(2\pi t) \quad \mathrm{rad} \\ 0.6 + 0.12\sin(2\pi t/1.3)\mathrm{rad} \end{bmatrix} \in \mathbb{R}^2. \tag{72}$$

Thus, $x(0) \in \mathbb{C}(\theta_1, \theta_2, \theta_3)$, $\|\dot{q}_d(t)\| < \xi$, and $\|\hat{h}(t)\| < \gamma_h$ were satisfied for all $t > 0$. Besides, to show the influence of $\|\hat{h}(t)\|$, another three sets of simulation were performed with the gravitational acceleration and the desired position amplitude ("0.12" in (72)) being scaled by factors of 0.1, 0.01, and 0, so as to exactly scale the extended disturbance $\hat{h}(t)$ by those factors.

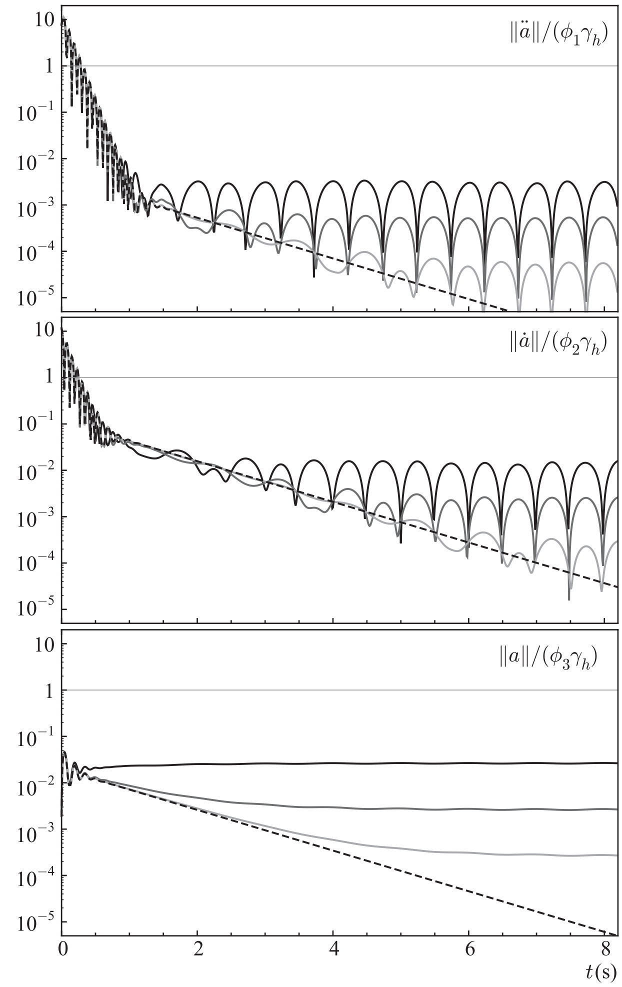

Fig. 4. Simulation results with unscaled $\hat{h}(t)$ (black solid), $\times 0.1$-scaled $\hat{h}(t)$ (dark gray solid), $\times 0.01$-scaled $\hat{h}(t)$ (light gray solid), and $\hat{h}(t) \equiv 0$ (black dashed).

Figure 3 shows the results based on the unscaled $\hat{h}(t)$. The first panel of Fig. 3 shows that $q(t)$ converges to $q_d(t)$, illustrating the validity of the trajectory-tracking control. The second panel of Fig. 3 shows that $\dot{V} < y^T h$ is always satisfied, illustrating the passivity. In Fig. 4, the solid black curves show that $x \in \mathbb{C}(\phi_1\gamma_h, \phi_2\gamma_h, \phi_3\gamma_h)$ is achieved in finite time (at $t \approx 0.3$ s), showing the ultimate boundedness.

Figure 4 also shows that the ultimate bound monotonously reduces as $\|\hat{h}(t)\|$ decreases. This can be considered as a consequence of the existence of a class-$\mathcal{K}$ function Γ_2, which is necessary for the input-to-state stability. The dotted curves show the extreme case where $\hat{h}(t) \equiv 0$, in which x decreases asymptotically to zero, exhibiting the asymptotic stability.

7. Conclusions

This paper has presented an alternative approach for analyzing four stability properties (semiglobal passivity, semiglobal asymptotic stability, semiglobal input-to-state stability, and semiglobal uniform ultimate boundedness with an arbitrarily reducible ultimate bound) of a rigid-link manipulator under PID position control. The approach employs a strict Lyapunov functions and a novel

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parameterization of the gain parameters. The stability conditions are provided as four inequalities based on some properties of three 3×3 matrices, P, Q and Π. In those inequalities, all arithmetic combinations of quantities are consistent in terms of physical dimensions. Although the analysis does not explicitly take the gravity into account, one can choose gain parameters to achieve arbitrary residual tracking errors of the velocity, position, and its integral by regarding the gravity as a disturbance.

An important point that should be addressed in a future study is that the presented gain-selection procedure leaves many parameters unconstrained. Unless these parameters are carefully chosen, the procedure tends to yield very high gains. It would be useful if the presented procedure is combined with some optimization algorithms, such as those for minimizing the time needed to achieve the ultimate bound under a given upper bound of the proportional gain. Besides, it would also be important to compare the presented inequality conditions to those in the literature in terms of the simplicity and the conservativeness. Such comparisons may lead us to a better guideline for gain selection.

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