43rd IEEE Conference on Decision and Control
December 14-17, 2004
Atlantis, Paradise Island, Bahamas

Nash Strategies for Large Scale Interconnected Systems

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Abstract— **In this paper, the linear quadratic N–players Nash games for infinite horizon large scale interconnected systems are discussed. The main contribution in this paper is that a new algorithm for solving the cross–coupled algebraic Riccati equations (CARE) is proposed. In order to improve the convergence rate and reduce the computing workspace, Newton's method and the fixed point algorithm are combined. As a result, it is newly proved that Nash equilibrium strategies obtained achieve a high–order approximation of the exact equilibrium. Furthermore, when the weak coupling parameter is unknown, it is shown that the proposed parameter independent Nash strategies are equivalent to the classical linear quadratic approximate controllers.**

I. INTRODUCTION

The stability analysis and control for large scale systems have been investigated extensively (see e.g., [1]). These control problems in practice are illustrated by the multiarea power systems [2], [3]. The control problem of the large scale interconnected systems which is parameterized by the small positive weak coupling parameter ε have been widely studied in [2]–[6].

The linear quadratic Nash games and their applications have been studied widely in many literatures (see e.g., [8]). It is well–known that in order to obtain Nash strategies, the cross–coupled algebraic Riccati equations (CARE) must be solved. In [9], the Newton–type algorithm for solving the CARE has been applied. However, the computing workspace needs quite large matrix dimensions. Therefore, the reduction of matrix dimensions is very important problem because the weakly coupled systems include numerous subsystems. On the other hand, in [4], the recursive algorithms for solving the CARE of the weakly coupled systems have been developed. Recently, an algorithm which is based on the fixed point algorithm for solving the CARE has been introduced [6]. Although such algorithms can be computed by the dimension of each subsystem, the convergence rate is the linear convergence.

In view of these studies above, one natural question here is whether there exists a new algorithm which is based on the Newton's method and can be computed by the reduced–order dimensions. This study will be given one method of the solution for the above question. This paper investigates the linear quadratic N–players Nash games for the infinite horizon large scale interconnected systems. After establishing the asymptotic structure for the CARE, a new algorithm for solving the CARE is proposed via Newton's method. Moreover, a new reduction algorithm for computing Newton's iterations is proposed. As a result, it is shown that the new algorithm has the quadratic convergence property and can be computed by the dimension of each subsystem. As another important feature, it is newly shown that the resulting strategies achieve the high–order approximation of the exact equilibrium. Therefore, even if the coupling parameter ε is not small, the new high–order approximate strategies can be used reliably for the large scale interconnected systems. So far, it should be noted that there is no result of the loss of the cost performance. Furthermore, if the weak coupling parameter is unknown, it is also shown that the proposed approximate Nash equilibrium strategies are equivalent to the classical linear quadratic approximate controllers. Finally, in order to demonstrate the efficiency of the algorithm, a numerical example is given for the practical power systems [2].

Notation: The notations used in this paper are fairly standard. The superscript T denotes matrix transpose. I_n denotes the $n \times n$ identity matrix. $\|\cdot\|$ denotes its Euclidean norm for a matrix. $\mathrm{vec}M$ denotes the column vector of the matrix M [13]. $\det M$ denotes the determinant of the matrix M. $\otimes$ denotes Kronecker product. δ_{ij} denotes the Kroneker delta. $\mathrm{Re}\lambda\ M$ denotes the real part of the eigenvalue of the matrix M.

II. PROBLEM FORMULATION

Consider large–scale interconnected systems with N–players

$$\begin{aligned}\dot{x}_i(t) &= A_{ii}x_i(t) + B_{ii}u_i(t) + \varepsilon \sum_{j=1,\ j\neq i}^{N} A_{ij}x_j(t) \\ &\quad + \varepsilon \sum_{j=1,\ j\neq i}^{N} B_{ij}u_j(t),\ x_i(0) = x_i^0, \qquad (1)\end{aligned}$$

where $x_i \in \mathbf{R}^{n_i},\ i = 1,\ \dots,N$ represents the i th state vector. $u_i \in \mathbf{R}^{m_i},\ i = 1,\ \dots,N$ represents the i th control input. ε denotes a small positive weak coupling parameter which connects the other subsystems. Each player is trying to minimize its own cost performance subject to (1). The cost performance for each player is defined by

$$J_i(u_1,\ \dots,u_N) = \int_0^\infty \left[x^T Q_{i\varepsilon} x + u_i^T R_{ii} u_i\right] dt, \qquad (2)$$

This work was supported in part by the Electric Technology Research Foundation of Chugoku of Japan.

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respectively, where

$$\begin{aligned} Q_{i\varepsilon} &= \begin{bmatrix} \varepsilon^{1-\delta_{i1}}Q_{i1} & \varepsilon Q_{i12} & \cdots & \varepsilon Q_{i1N} \\ \varepsilon Q_{i12}^T & \varepsilon^{1-\delta_{i2}}Q_{i2} & \cdots & \varepsilon Q_{i2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon Q_{i1N}^T & \varepsilon Q_{i2N}^T & \cdots & \varepsilon^{1-\delta_{iN}}Q_{iN} \end{bmatrix} \\ &\in \mathbf{R}^{\bar{n}\times\bar{n}}, \\ R_{ii} &= R_{ii}^T > 0 \in \mathbf{R}^{m_i\times m_i},\ i=1,\ \dots,N, \\ x(t)^T &:= \begin{bmatrix} x_1(t)^T & \cdots & x_N(t)^T \end{bmatrix}^T \in \mathbf{R}^{\bar{n}}, \\ \bar{n} &:= \sum_{i=1}^N n_i. \end{aligned}$$

It should be noted that there is no $u_j^T R_{ij} u_j$ in the cost (2) because it is assumed that each player will make effort such that own cost minimize. It is easy to extend the considered cost in this paper to the general cost function.

The Nash equilibrium strategies $(u_1^*,\ \dots,u_N^*)$ are defined as satisfying the following conditions

$$\begin{aligned} & J_i(u_1^*,\ \dots,\ u_{i-1}^*,\ u_i^*,\ u_{i+1}^*,\ \dots,\ u_N^*) \\ \le\ & J_i(u_1^*,\ \dots,\ u_{i-1}^*,\ u_i,\ u_{i+1}^*,\ \dots,\ u_N^*), \\ i\ =\ & 1,\ \dots,N. \end{aligned} \tag{3}$$

It should be noted that the following assumption guarantees the existence of the admissible strategies.

Assumption 1: There exist linear feedback strategies $u_i(t) = K_{i\varepsilon}x(t),\ i=1,\ \dots,N$ such that the closed–loop system is asymptotically stable for sufficiently small ε.

The linear feedback Nash strategies are given by

$$u_i^*(t) = -R_{ii}^{-1}B_{i\varepsilon}^T P_{i\varepsilon}x(t),\ i=1,\ \dots,N, \tag{4}$$

respectively, where $P_{i\varepsilon}$ are the positive semidefinite solutions of the following N–cross–coupled algebraic Riccati equations (CARE)

$$\begin{aligned} & \mathcal{F}_i(P_{1\varepsilon},\ \dots,P_{N\varepsilon}) \\ &:= P_{i\varepsilon}\mathcal{A}_\varepsilon + \mathcal{A}_\varepsilon^T P_{i\varepsilon} + P_{i\varepsilon}S_{i\varepsilon}P_{i\varepsilon} + Q_{i\varepsilon} = 0, \end{aligned} \tag{5}$$

with $\mathcal{A}_\varepsilon := A_\varepsilon - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}$,

$$A_\varepsilon := \begin{bmatrix} A_{11} & \varepsilon A_{12} & \cdots & \varepsilon A_{1N} \\ \varepsilon A_{21} & A_{22} & \cdots & \varepsilon A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon A_{N1} & \varepsilon A_{N2} & \cdots & A_{NN} \end{bmatrix},$$

$$B_{i\varepsilon} := \begin{bmatrix} \varepsilon^{1-\delta_{1i}}B_{1i} \\ \varepsilon^{1-\delta_{2i}}B_{2i} \\ \vdots \\ \varepsilon^{1-\delta_{Ni}}B_{Ni} \end{bmatrix},\ S_{i\varepsilon} := B_{i\varepsilon}R_{ii}^{-1}B_{i\varepsilon}^T.$$

Since A_ε and $S_{i\varepsilon}$ include the term of the small weak coupling parameter ε, the solution $P_{i\varepsilon}$ of the CARE (5), if it exists, must contain terms of order ε. Taking this fact into account, the solution $P_{i\varepsilon}$ of the CARE (5) with the following structure is considered [3].

$$\begin{aligned} P_{i\varepsilon} &:= \begin{bmatrix} \varepsilon^{1-\delta_{i1}}P_{i1} & \varepsilon P_{i12} & \cdots & \varepsilon P_{i1N} \\ \varepsilon P_{i12}^T & \varepsilon^{1-\delta_{i2}}P_{i2} & \cdots & \varepsilon P_{i2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon P_{i1N}^T & \varepsilon P_{i2N}^T & \cdots & \varepsilon^{1-\delta_{iN}}P_{iN} \end{bmatrix} \\ &\in \mathbf{R}^{\bar{n}\times\bar{n}}. \end{aligned}$$

In the following analysis, the basic assumption is needed.

Assumption 2: The triples $(A_{ii},\ B_{ii},\ \sqrt{Q_{ii}}),\ i = 1,\ \dots,N$ are stabilizable and detectable.

III. ASYMPTOTIC STRUCTURE OF THE CARE

Firstly, in order to obtain the strategies, the asymptotic structure of the CARE (5) is established. Substituting the matrices A_ε, $S_{i\varepsilon}$, $Q_{i\varepsilon}$ and $P_{i\varepsilon}$ into the CARE (5), setting $\varepsilon = 0$ and partitioning the CARE (5), the following reduced–order AREs are obtained, where $\bar{P}_{ii},\ i = 1,\ \cdots,N$ be the limiting solutions of the CARE (5) as $\varepsilon \to +0$.

$$\bar{P}_{ii}A_{ii} + A_{ii}^T\bar{P}_{ii} - \bar{P}_{ii}S_{ii}\bar{P}_{ii} + Q_{ii} = 0, \tag{6}$$

where $S_{ii} := B_{ii}R_{ii}^{-1}B_{ii}^T$.

The limiting behavior of $P_{i\varepsilon}$ as the parameter $\varepsilon \to +0$ is described by the following lemma.

Lemma 1: Under Assumption 2, there exists a small σ^* such that for all $\varepsilon \in (0,\ \sigma^*)$ the CARE (5) admits a positive semidefinite solution $P_{i\varepsilon}$ which can be written as

$$\begin{aligned} P_{i\varepsilon} &= \bar{P}_i + O(\varepsilon) \\ &= \mathbf{block\ diag}\begin{pmatrix} 0 & \cdots & \bar{P}_{ii} & \cdots & 0 \end{pmatrix} + O(\varepsilon). \end{aligned} \tag{7}$$

Proof: The proof can be done by using the implicit function theorem [10] to the CARE (5). Since the proof of Lemma 1 proceeds by similar argument of the references [6], it is omitted. ■

IV. NEWTON'S METHOD FOR SOLVING THE CARE

In order to obtain the optimal strategies, the following useful algorithm which is based on Newton's method is given.

$$\begin{aligned} & P_{i\varepsilon}^{(k+1)}\mathcal{A}_\varepsilon + \mathcal{A}_\varepsilon^T P_{i\varepsilon}^{(k+1)} \\ & \quad - \sum_{j=1,\ j\neq i}^N P_{j\varepsilon}^{(k+1)}S_{j\varepsilon}P_{i\varepsilon}^{(k)} - \sum_{j=1,\ j\neq i}^N P_{i\varepsilon}^{(k)}S_{j\varepsilon}P_{j\varepsilon}^{(k+1)} \\ & \quad + \sum_{j=1,\ j\neq i}^N P_{i\varepsilon}^{(k)}S_{j\varepsilon}P_{j\varepsilon}^{(k)} + \sum_{j=1,\ j\neq i}^N P_{j\varepsilon}^{(k)}S_{j\varepsilon}P_{i\varepsilon}^{(k)} \\ & \quad + P_{i\varepsilon}^{(k)}S_{i\varepsilon}P_{i\varepsilon}^{(k)} + Q_{i\varepsilon} = 0,\quad k = 0,\ 1,\ \dots, \end{aligned} \tag{8a}$$

$$P_{i\varepsilon}^{(k)} := \begin{bmatrix} \varepsilon^{1-\delta_{i1}}P_{i1}^{(k)} & \varepsilon P_{i12}^{(k)} & \cdots & \varepsilon P_{i1N}^{(k)} \\ \varepsilon P_{i12}^{(k)T} & \varepsilon^{1-\delta_{i2}}P_{i2}^{(k)} & \cdots & \varepsilon P_{i2N}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon P_{i1N}^{(k)T} & \varepsilon P_{i2N}^{(k)T} & \cdots & \varepsilon^{1-\delta_{iN}}P_{iN}^{(k)} \end{bmatrix} \tag{8b}$$

with the initial condition

$$P_{i\varepsilon}^{(0)} = \bar{P}_i = \textbf{block diag}\left(\begin{array}{ccccc} 0 & \cdots & \bar{P}_{ii} & \cdots & 0 \end{array} \right). \quad (9)$$

The new algorithm (8) can be constructed by setting $P_{i\varepsilon}^{(k+1)} = P_{i\varepsilon}^{(k)} + \Delta P_{i\varepsilon}^{(k)}$ and neglecting $O(\Delta P_{i\varepsilon}^{(k)T} \Delta P_{i\varepsilon}^{(k)})$ term. Newton's method is well–known and is widely used to find a solution of the algebraic equations, and its local convergence properties are well understood. The algorithm (8) has the feature given in the following theorem.

Theorem 1: Under Assumptions 1 and 2, there exists a small $\bar{\sigma}$ such that for all $\varepsilon \in (0,\ \bar{\sigma}),\ \bar{\sigma} \leq \sigma^*$ Newton's method (8) converges to the exact solution of $P_{i\varepsilon}$ with the rate of the quadratic convergence, where $P_{i\varepsilon}^{(k)}$ is positive semidefinite and $A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon} P_{j\varepsilon}^{(k)}$ is stable. That is, the following conditions are satisfied.

$$\|P_{i\varepsilon}^{(k)} - P_{i\varepsilon}\| \leq \frac{O(\varepsilon^{2^k})}{2^k \beta \mathcal{L}},\quad k = 0,\ 1,\ \ldots, \quad (10a)$$

$$\mathrm{Re}\lambda \left[A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon} P_{j\varepsilon}^{(k)} \right] = \mathrm{Re}\lambda \mathcal{A}_\varepsilon < 0, \quad (10b)$$

where

$$\begin{aligned} \mathcal{L} &:= 2\sqrt{N}(2N-1) \sum_{j=1}^{N} \|S_{j\varepsilon}\|, \\ \beta &:= \|[\nabla \mathcal{F}(P_{1\varepsilon}^{(0)},\ \ldots,\ P_{N\varepsilon}^{(0)})]^{-1}\|, \\ \nabla \mathcal{F}(P_{1\varepsilon},\ \ldots,\ P_{N\varepsilon}) &:= \begin{bmatrix} \frac{\partial \mathrm{vec}\mathcal{F}_1}{\partial \mathrm{vec} P_{1\varepsilon}} & \cdots & \frac{\partial \mathrm{vec}\mathcal{F}_1}{\partial \mathrm{vec} P_{N\varepsilon}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathrm{vec}\mathcal{F}_N}{\partial \mathrm{vec} P_{1\varepsilon}} & \cdots & \frac{\partial \mathrm{vec}\mathcal{F}_N}{\partial \mathrm{vec} P_{N\varepsilon}} \end{bmatrix}. \end{aligned}$$

Proof: The proof of this theorem can be done by using Newton–Kantorovich theorem [12]. Taking the partial derivative of the function $\mathcal{F}_i(P_{1\varepsilon},\ \ldots,\ P_{N\varepsilon})$ with respect to $P_{i\varepsilon}$ and $P_{j\varepsilon}$ results in (11).

$$\frac{\partial \mathrm{vec}\mathcal{F}_i}{\partial \mathrm{vec} P_{i\varepsilon}} = I_{\bar{n}} \otimes \mathcal{A}_\varepsilon^T + \mathcal{A}_\varepsilon^T \otimes I_{\bar{n}}, \quad (11a)$$

$$\frac{\partial \mathrm{vec}\mathcal{F}_i}{\partial \mathrm{vec} P_{j\varepsilon}} = -I_{\bar{n}} \otimes (S_{j\varepsilon} P_{i\varepsilon})^T - (S_{j\varepsilon} P_{j\varepsilon})^T \otimes I_{\bar{n}}. \quad (11b)$$

Since the function $\mathcal{F}_i(P_{1\varepsilon},\ \ldots,\ P_{N\varepsilon})$ is continuous at any $P_{i\varepsilon}$, it is immediately obtained from the equation (5) that

$$\begin{aligned} &\|\nabla \mathcal{F}(P_{1\varepsilon}^a,\ \ldots,\ P_{N\varepsilon}^a) - \nabla \mathcal{F}(P_{1\varepsilon}^b,\ \ldots,\ P_{N\varepsilon}^b)\| \\ &\leq \mathcal{L} \|\mathrm{vec}[P_{1\varepsilon}^a\ \ldots\ P_{N\varepsilon}^a] - \mathrm{vec}[P_{1\varepsilon}^b\ \ldots\ P_{N\varepsilon}^b]\|. \end{aligned} \quad (12)$$

Moreover, it is easy to derive that

$$\nabla \mathcal{F}(P_{1\varepsilon}^{(0)},\ \ldots,\ P_{N\varepsilon}^{(0)}) = \begin{bmatrix} \mathbf{J}_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{J}_0 \end{bmatrix}, \quad (13)$$

where

$$\begin{aligned} \mathbf{J}_0 &= \textbf{block diag}\left(\begin{array}{ccc} \mathbf{D}_{11} & \cdots & \mathbf{D}_{NN} \end{array} \right), \\ \mathbf{D}_{ii} &:= (A_{ii} - S_{ii}\bar{P}_{ii})^T \otimes I_{n_i} + I_{n_i} \otimes (A_{ii} - S_{ii}\bar{P}_{ii})^T. \end{aligned}$$

Obviously, $D_{ii} := A_{ii} - S_{ii}\bar{P}_{ii}$ is nonsingular because the ARE (6) has the positive semidefinite stabilizing solution under Assumption 2. Therefore, there exists β such that $\beta = \|[\nabla \mathcal{F}(P_{1\varepsilon}^{(0)},\ \ldots,\ P_{N\varepsilon}^{(0)})]^{-1}\|$ for sufficiently small ε. On the other hand, since $\mathcal{F}_i(P_{1\varepsilon}^{(0)},\ \ldots,\ P_{N\varepsilon}^{(0)}) = O(\varepsilon)$, there exists η such that $\eta = \|[\nabla \mathcal{F}(P_{1\varepsilon}^{(0)},\ \ldots,\ P_{N\varepsilon}^{(0)})]^{-1}\| \cdot \|\mathcal{F}(P_{1\varepsilon}^{(0)},\ \ldots,\ P_{N\varepsilon}^{(0)})\| = O(\varepsilon)$, where $\mathcal{F}(P_{1\varepsilon},\ \ldots,\ P_{N\varepsilon}) := [\mathcal{F}_1(P_{1\varepsilon},\ \ldots,\ P_{N\varepsilon}),\ \ldots,\ \mathcal{F}_N(P_{1\varepsilon},\ \ldots,\ P_{N\varepsilon})]$. Thus, there exists θ such that $\theta = \beta\eta\mathcal{L} < 2^{-1}$ because of $\eta = O(\varepsilon)$. Using the Newton–Kantorovich theorem, the error estimate is given by

$$\|P_{i\varepsilon}^{(k)} - P_{i\varepsilon}\| \leq \frac{(2\theta)^{2^k}}{2^k \beta \mathcal{L}},\ k = 0,\ 1,\ \cdots. \quad (14)$$

Substituting $2\theta = O(\varepsilon)$ into (14), the equation (10a) holds. Since the remainder of the proof for the equation (10b) is similar to the proof which is given in [6], it is omitted. ■

When the cross–coupled algebraic Lyapunov equation (CALE) (8a) is solved, the dimension $\bar{n} := \sum_{i=1}^{N} n_i$ which is larger than the dimensions $n_i,\ i = 0,\ 1, \ldots, N$ is needed. Moreover, since there exists the cross–coupled term

$$-\sum_{j=1,\ j\neq i}^{N} P_{j\varepsilon}^{(k+1)} S_{j\varepsilon} P_{i\varepsilon}^{(k)} - \sum_{j=1,\ j\neq i}^{N} P_{i\varepsilon}^{(k)} S_{j\varepsilon} P_{j\varepsilon}^{(k+1)}$$

in the CALE (8a), it is difficult to solve these equations. Thus, in order to reduce the dimension of the workspace and avoid the cross–coupled term, the new algorithm which is based on the fixed point algorithm is established. Taking the fact $S_{j\varepsilon} P_{i\varepsilon}^{(k)} = O(\varepsilon),\ i \neq j$ into account, let us consider the following CALE (15), in a general form.

$$\begin{aligned} &X_{i\varepsilon}\Lambda_\varepsilon + \Lambda_\varepsilon^T X_{i\varepsilon} + \varepsilon \sum_{j=1,\ j\neq i}^{N} (X_{j\varepsilon}\Phi_{j\varepsilon} + \Phi_{j\varepsilon}^T X_{j\varepsilon}) \\ &\quad + U_\varepsilon = 0,\ i = 1,\ \ldots, N. \end{aligned} \quad (15)$$

where

$$X_{i\varepsilon} := \begin{bmatrix} \varepsilon^{1-\delta_{i1}} X_{i1} & \varepsilon X_{i12} & \cdots & \varepsilon X_{i1N} \\ \varepsilon X_{i12}^T & \varepsilon^{1-\delta_{i2}} X_{i2} & \cdots & \varepsilon X_{i2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon X_{i1N}^T & \varepsilon X_{i2N}^T & \cdots & \varepsilon^{1-\delta_{iN}} X_{iN} \end{bmatrix},$$

$$\Lambda_\varepsilon := \begin{bmatrix} \Lambda_{11} & \varepsilon\Lambda_{12} & \cdots & \varepsilon\Lambda_{1N} \\ \varepsilon\Lambda_{21} & \Lambda_{22} & \cdots & \varepsilon\Lambda_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon\Lambda_{N1} & \varepsilon\Lambda_{N2} & \cdots & \Lambda_{NN} \end{bmatrix},$$

$$\Phi_{i\varepsilon} := \begin{bmatrix} \Phi_{i11} & \varepsilon\Phi_{i12} & \cdots & \varepsilon\Phi_{i1N} \\ \varepsilon\Phi_{i21} & \Phi_{i22} & \cdots & \varepsilon\Phi_{i2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon\Phi_{iN1} & \varepsilon\Phi_{iN2} & \cdots & \Phi_{iNN} \end{bmatrix},$$

$$U_\varepsilon := \begin{bmatrix} \varepsilon^{1-\delta_{i1}} U_{i1} & \varepsilon U_{i12} & \cdots & \varepsilon U_{i1N} \\ \varepsilon U_{i12}^T & \varepsilon^{1-\delta_{i2}} U_{i2} & \cdots & \varepsilon U_{i2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon U_{i1N}^T & \varepsilon U_{i2N}^T & \cdots & \varepsilon^{1-\delta_{iN}} U_{iN} \end{bmatrix}.$$

It should be noted that

$$\begin{aligned} & P_{i\varepsilon}^{(k+1)} \Rightarrow X_{i\varepsilon},\ P_{j\varepsilon}^{(k+1)} \Rightarrow X_{j\varepsilon}, \\ & A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon} P_{j\varepsilon}^{(k)} \Rightarrow \Lambda_\varepsilon,\ -S_{j\varepsilon} P_{i\varepsilon}^{(k)} \Rightarrow \varepsilon \Phi_{j\varepsilon}, \\ & \sum_{j=1,\ j\neq i}^{N} P_{i\varepsilon}^{(k)} S_{j\varepsilon} P_{j\varepsilon}^{(k)} + \sum_{j=1,\ j\neq i}^{N} P_{j\varepsilon}^{(k)} S_{j\varepsilon} P_{i\varepsilon}^{(k)} \\ & \qquad + P_{i\varepsilon}^{(k)} S_{i\varepsilon} P_{i\varepsilon}^{(k)} + Q_{i\varepsilon} \Rightarrow U_\varepsilon \end{aligned}$$

where $\Rightarrow$ stands for the replacement.

Without loss of generality, the following condition for the CALE (15) is assumed.

Assumption 3: $\Lambda_{11},\ \cdots,\ \Lambda_{NN}$ are stable.

The algorithm (16) for solving the CALE (15) is given.

$$\begin{aligned} & X_{i\varepsilon}^{(n+1)} \Lambda_\varepsilon + \Lambda_\varepsilon^T X_{i\varepsilon}^{(n+1)} + \varepsilon \sum_{j=1,\ j\neq i}^{N} (X_{j\varepsilon}^{(n)} \Phi_{j\varepsilon} + \Phi_{j\varepsilon}^T X_{j\varepsilon}^{(n)}) \\ & \quad + U_\varepsilon = 0,\ i = 1,\ \ldots,N,\ n = 0,\ 1,\ \ldots . \end{aligned} \tag{16}$$

where $X_{i\varepsilon}^{(0)} = 0,\ i = 1,\ \ldots,N$.

It should be noted that since the equations (15) are partitioned into (16), the numerical computation can be done by each subsystems dimension n_i. The following theorem indicates the convergence of the algorithm (16).

Theorem 2: Under Assumption 3, the fixed point algorithm (16) converges to the exact solution $X_{i\varepsilon}$ with the rate of

$$\|X_{i\varepsilon}^{(n)} - X_{i\varepsilon}\| = O(\varepsilon^n),\ n = 1,\ 2,\ \ldots . \tag{17}$$

Proof: The proof of Theorem 2 can be done by using mathematical induction. When $n = 0$ for the algorithms (16), it is easy to verify that the first order approximations $X_{i\varepsilon}$ corresponding to the small parameter ε are $X_{i\varepsilon}^{(1)}$. When $n = h,\ h \geq 2$, it is assumed that

$$\|X_{i\varepsilon}^{(h)} - X_{i\varepsilon}\| = O(\varepsilon^h). \tag{18}$$

Subtracting (15) from (16) and substituting h into n, the following equations hold.

$$\begin{aligned} & (X_{i\varepsilon}^{(h+1)} - X_{i\varepsilon})\Lambda_\varepsilon + \Lambda_\varepsilon^T (X_{i\varepsilon}^{(h+1)} - X_{i\varepsilon}) \\ & \quad = -\varepsilon \sum_{j=1,\ j\neq i}^{N} \left[(X_{j\varepsilon}^{(h)} - X_{j\varepsilon})\Phi_{j\varepsilon} + \Phi_{j\varepsilon}^T (X_{j\varepsilon}^{(h)} - X_{j\varepsilon}) \right] \\ & \quad i = 1,\ \ldots,N. \end{aligned} \tag{19}$$

Using the assumption (18), the following equations are satisfied.

$$(X_{i\varepsilon}^{(h+1)} - X_{i\varepsilon})\Lambda_\varepsilon + \Lambda_\varepsilon^T (X_{i\varepsilon}^{(h+1)} - X_{i\varepsilon}) = O(\varepsilon^{h+1}). \tag{20}$$

Since $\Lambda_{ii},\ i = 1,\ 2, \ldots,N$ are stable from Assumption 3, Λ_ε is stable. Using the standard properties of the algebraic Lyapunov equation (ALE) [11], it is easy to verify that

$$\|X_{i\varepsilon}^{(h+1)} - X_{i\varepsilon}\| = O(\varepsilon^{h+1}). \tag{21}$$

Consequently, the error equations (17) hold for all $n \in \mathbf{N}$. This completes the proof of Theorem 2. ■

It is well–known that it is very hard to solve the CARE. For example, when Newton's method is applied to the CARE, although the convergence guarantees the quadratic convergence, the derived algorithm depend on the other solutions. In this paper, the reduction algorithm which combine Newton's method with the fixed point algorithm has been newly given. The novel idea is based on the property of $P_{j\varepsilon}^{(k+1)} S_{i\varepsilon} P_{i\varepsilon}^{(k)} = O(\varepsilon)$, where ε is the weakly–coupled perturbation parameter. As a result, we succeed in avoiding the large dimension for solving the linear equations. Namely, since the solutions $P_{1\varepsilon}^{(k+1)}$, $P_{2\varepsilon}^{(k+1)}$, ... do not depend on the other solutions, we can solve each solution independently. It should be noted that the methodology for the proposed algorithm and the ordinary Newton's method [5] is quite different.

V. HIGH–ORDER NASH STRATEGIES

In this section, the high–order Nash strategies which are based on the proposed algorithm (8a) are established. Such strategy is obtained by solving the algorithm (8a).

$$u_i^{(k)*}(t) = -R_{ii}^{-1} B_{i\varepsilon}^T P_{i\varepsilon}^{(k)} x(t),\ i = 1,\ \ldots,N, \tag{22}$$

The degradation of the cost performance via the new high–order Nash strategies (22) is given as follows.

Theorem 3: Under Assumption 2, the use of the high–order Nash strategies (22) results in (23)

$$\begin{aligned} & J_i(u_1^{(k)*},\ \ldots,\ u_N^{(k)*}) = J_i(u_1^*,\ \ldots,\ u_N^*) + O(\varepsilon^{2^k+1}), \\ & i = 1,\ \ldots,N. \end{aligned} \tag{23}$$

Proof: When $u_i^{(k)*}(t)$ is used, the equilibrium value of the cost performances are

$$J_i(u_1^{(k)*},\ \ldots,\ u_N^{(k)*}) = x^T(0) Y_{i\varepsilon} x(0), \tag{24}$$

where $Y_{i\varepsilon}$ is a positive semidefinite solution of the following ALE

$$Y_{i\varepsilon}\mathcal{A}_\varepsilon + \mathcal{A}_\varepsilon^T Y_{i\varepsilon} + P_{i\varepsilon}^{(k)} S_{i\varepsilon} P_{i\varepsilon}^{(k)} + Q_{i\varepsilon} = 0. \tag{25}$$

Subtracting (5) from (25), the matrix $Z_{i\varepsilon} = Y_{i\varepsilon} - P_{i\varepsilon}$ is the solution of the following ALE.

$$\begin{aligned} & Z_{i\varepsilon}\mathcal{A}_\varepsilon + \mathcal{A}_\varepsilon Z_{i\varepsilon} + \sum_{j=1,\ j\neq i}^{N} P_{i\varepsilon} S_{j\varepsilon} \left(P_{j\varepsilon} - P_{j\varepsilon}^{(k)} \right) \\ & \quad + \sum_{j=1,\ j\neq i}^{N} \left(P_{j\varepsilon} - P_{j\varepsilon}^{(k)} \right) S_{j\varepsilon} P_{i\varepsilon} \\ & \quad + \left(P_{i\varepsilon}^{(k)} - P_{i\varepsilon} \right) S_{i\varepsilon} \left(P_{i\varepsilon}^{(k)} - P_{i\varepsilon} \right) = 0. \end{aligned} \tag{26}$$

By using the similar technique in the derivation of (17), it is easy to verify that

$$Z_{i\varepsilon}\mathcal{A}_\varepsilon + \mathcal{A}_\varepsilon^T Z_{i\varepsilon} + O(\varepsilon^{2^k+1}) = 0. \tag{27}$$

Therefore, $Z_{i\varepsilon} = O(\varepsilon^{2^k+1})$ because of the stability condition (10b) and the standard properties of the ALE [11]. Hence

$$\begin{aligned} x(0)^T Z_{i\varepsilon} x(0) &= J_i(u_1^{(k)*},\ \dots\ ,\ u_N^{(k)*}) \\ &\quad - J_i(u_1^*,\ \dots\ ,\ u_N^*) = O(\varepsilon^{2^k+1}) \end{aligned} \tag{28}$$

results in (23). ■

Using the similar technique of the proof of Theorem 3, the following conditions are satisfied.

Theorem 4: Under Assumptions 1 and 2, the following result holds.

$$\begin{aligned} &J_i(u_1^{(k)*},\ \dots\ ,\ u_{i-1}^{(k)*},\ u_i,\ u_{i+1}^{(k)*},\ \dots\ ,\ u_N^{(k)*}) \\ &= J_i(u_1^*,\ \dots\ ,\ u_{i-1}^*,\ u_i,\ u_{i+1}^*,\ \dots\ ,\ u_N^*) + O(\varepsilon^{2^k+1}), \\ &i = 1,\ \dots\ , N. \end{aligned} \tag{29}$$

Proof: Since this proof is done by using the similar technique in [7], it is omitted. ■

Finally, the main result is easily derived.

Theorem 5: Under Assumptions 1 and 2, the use of the high–order strategies (22) results in (30)

$$\begin{aligned} &J_i(u_1^{(k)*},\ \dots\ ,\ u_{i-1}^{(k)*},\ u_i^{(k)*},\ u_{i+1}^{(k)*},\ \dots\ ,\ u_N^{(k)*}) \\ \le\ &J_i(u_1^{(k)*},\ \dots\ ,\ u_{i-1}^{(k)*},\ u_i,\ u_{i+1}^{(k)*},\ \dots\ ,\ u_N^{(k)*}) \\ &+ O(\varepsilon^{2^k+1}),\ i = 1,\ \dots\ , N. \end{aligned} \tag{30}$$

Proof: Let us rewrite an inequality (30) as

$$\begin{aligned} &J_i(u_1^{(k)*},\ \dots\ ,\ u_N^{(k)*}) \\ &\quad - J_i(u_1^{(k)*},\ \dots\ ,\ u_{i-1}^{(k)*},\ u_i,\ u_{i+1}^{(k)*},\ \dots\ ,\ u_N^{(k)*}) \\ &= J_i(u_1^{(k)*},\ \dots\ ,\ u_N^{(k)*}) - J_i(u_1^*,\ \dots\ ,\ u_N^*) \\ &\quad + J_i(u_1^*,\ \dots\ ,\ u_{i-1}^*,\ u_i^*,\ u_{i+1}^*,\ \dots\ ,\ u_N^*) \\ &\quad - J_i(u_1^*,\ \dots\ ,\ u_{i-1}^*,\ u_i,\ u_{i+1}^*,\ \dots\ ,\ u_N^*) \\ &\quad + J_i(u_1^*,\ \dots\ ,\ u_{i-1}^*,\ u_i,\ u_{i+1}^*,\ \dots\ ,\ u_N^*) \\ &\quad - J_i(u_1^{(k)*},\ \dots\ ,\ u_{i-1}^{(k)*},\ u_i,\ u_{i+1}^{(k)*},\ \dots\ ,\ u_N^{(k)*}) \end{aligned} \tag{31}$$

Using (23), (3) and (29), the proof of (30) completes. ■

It should be noted that the inequality (3) holds as long as the optimal strategies u_i^* are used.

In the rest of this section, two important implications are given. If the parameter ε is unknown, then the following corollary is easily seen in view of Theorem 5.

Corollary 1: Consider the ε–independent Nash strategies

$$\bar{u}_i^*(t) = -R_{ii}^{-1} B_i^T \tilde{P}_i x(t),\ i = 1,\ \dots\ , N, \tag{32}$$

where $\tilde{P}_i := \begin{bmatrix} 0 \cdots \bar{P}_{ii} \cdots 0 \end{bmatrix}$. $B_i^T := \begin{bmatrix} 0 \cdots B_{ii}^T \cdots 0 \end{bmatrix}$.

Under Assumptions 1 and 2, the use of the reduced–order strategies (32) results in (33)

$$\begin{aligned} &J_i(\bar{u}_1^*,\ \dots\ ,\ \bar{u}_{i-1}^*,\ \bar{u}_i^*,\ \bar{u}_{i+1}^*,\ \dots\ ,\ \bar{u}_N^*) \\ &\le J_i(\bar{u}_1^*,\ \dots\ ,\ \bar{u}_{i-1}^*,\ u_i,\ \bar{u}_{i+1}^*,\ \dots\ ,\ \bar{u}_N^*) + O(\varepsilon^2), \\ &i = 1,\ \dots\ , N. \end{aligned} \tag{33}$$

Proof: Since the result of Corollary 1 can be proved by using the similar technique in Theorem 5 under the fact that $P_{i\varepsilon} - \bar{P}_i = O(\varepsilon)$, the proof is omitted. ■

It is worth noting that the $O(\varepsilon^2)$ near–optimality of the approximate strategies (32) is proved for the first time to the Nash games problem of the large–scale systems. Secondly, the following property indicates the relation between the proposed strategies and the approximate LQR controllers in [2].

Corollary 2: The proposed approximate strategies (32) are equivalent to the first order near–optimal LQR controllers which have been established in [2].

Proof: Reviewing the previous result in [2], it is easy to show that the proposed strategies are the same as the first order near–optimal controllers. Therefore, the above mention is satisfied. ■

It is interesting to point out that the ordinary linear quadratic approximate controllers satisfy the approximate Nash equilibrium (33). That is, it has been newly shown that the approximate Nash equilibrium strategies function also as the first order near–optimal controllers which is proposed in [2].

VI. NUMERICAL EXAMPLE

In order to demonstrate the efficiency of the proposed algorithm, an illustrative example is given. Consider a practical power systems plant which are known as the large–scale system (1) composed of three four–dimensional subsystems [2]. The system matrices are given at the top of the next page.

The small parameter is chosen as $\varepsilon = 0.5065$. It should be noted that $\varepsilon A_{ij},\ i \ne j$ includes the value of the parameter $\varepsilon = 0.5065$. The weighting matrices of the cost performance are given by $R_{11} = R_{22} = R_{33} = 1$, $Q_1 = \mathbf{block\ diag}\begin{pmatrix} 0.5I_4 & O_{8\times 8} \end{pmatrix}$, $Q_2 = \mathbf{block\ diag}\begin{pmatrix} O_{4\times 4} & 0.5I_4 & O_{4\times 4} \end{pmatrix}$, $Q_3 = \mathbf{block\ diag}\begin{pmatrix} O_{8\times 8} & 0.5I_4 \end{pmatrix}$. It should be noted that the algorithm (8a) converges to the exact solution with accuracy of $\|\mathcal{F}^{(k)}(\varepsilon)\| < 1.0e-10$ after 5 iterations, where

$$\|\mathcal{F}^{(k)}(\varepsilon)\| := \sum_{h=1}^{3} \|\mathcal{F}_h(P_{1\varepsilon}^{(k)},\ P_{2\varepsilon}^{(k)},\ P_{3\varepsilon}^{(k)})\|$$

In order to verify the exactitude of the solution, the remainder per iteration by substituting $P_{i\varepsilon}^{(k)}$ into the CARE (5) is computed. In Table 1, the results for the error $\|\mathcal{F}^{(k)}(\varepsilon)\|$ per iterations are given. It can be seen that the algorithm (8a) has the quadratic convergence.

The required iterations of the proposed algorithm (8a) versus the fixed point iterations [6] are presented in Table 2. It can be seen from Table 2 that the proposed algorithm (8a) succeed in reducing the iterations compared with the fixed point iterations for different values of ε. Particularly, for large ε the required iterations are small. Hence, the resulting algorithm of this paper is very reliable.

$$A_{11} = \begin{bmatrix} 0 & 1 & -0.266 & -0.009 \\ -2.75 & -2.78 & -1.36 & -0.037 \\ 0 & 0 & 0 & 1 \\ -4.95 & 0 & -55.5 & -0.039 \end{bmatrix}, \ \varepsilon A_{12} = \begin{bmatrix} 0.0024 & 0 & -0.087 & 0.002 \\ -0.185 & 0 & 1.11 & -0.011 \\ 0 & 0 & 0 & 0 \\ 0.222 & 0 & 8.17 & 0.004 \end{bmatrix}, \ \varepsilon A_{13} = \begin{bmatrix} 0.073 & 0 & -0.25 & 0.003 \\ -0.46 & 0 & 2.8 & -0.02 \\ 0 & 0 & 0 & 0 \\ 0.924 & 0 & 17.5 & 0.02 \end{bmatrix},$$

$$\varepsilon A_{21} = \begin{bmatrix} 0.021 & 0 & 0.121 & 0.003 \\ -1.1 & 0 & -1.62 & -0.015 \\ 0 & 0 & 0 & 0 \\ -2.43 & 0 & 1.37 & -0.034 \end{bmatrix}, \ A_{22} = \begin{bmatrix} -0.21 & 1 & -1.6 & -0.005 \\ -1.9 & -1.8 & 9.3 & -0.12 \\ 0 & 0 & 0 & 1 \\ -3.1 & 0 & -56 & 0.032 \end{bmatrix}, \ \varepsilon A_{23} = \begin{bmatrix} 0.06 & 0 & 0.46 & 0.002 \\ -1 & 0 & 1.49 & -0.04 \\ 0 & 0 & 0 & 0 \\ 0.12 & 0 & 29.8 & -0.028 \end{bmatrix},$$

$$\varepsilon A_{31} = \begin{bmatrix} -0.002 & 0 & 0.83 & 0 \\ -6.78 & 0 & -10.1 & 0.09 \\ 0 & 0 & 0 & 0 \\ -1.24 & 0 & 0.498 & -0.017 \end{bmatrix}, \ \varepsilon A_{32} = \begin{bmatrix} 0.011 & 0 & 0.22 & 0 \\ -2.1 & 0 & 1.7 & -0.123 \\ 0 & 0 & 0 & 0 \\ -0.07 & 0 & 6.38 & -0.011 \end{bmatrix}, \ A_{33} = \begin{bmatrix} -0.197 & 1 & -1.2 & -0.003 \\ -54.5 & -20 & 70.1 & -2.37 \\ 0 & 0 & 0 & 1 \\ -3.4 & 0 & -21.0 & -0.017 \end{bmatrix},$$

$$B_{11} = \begin{bmatrix} 0 \\ 36.1 \\ 0 \\ 0 \end{bmatrix}, \ B_{22} = \begin{bmatrix} 0 \\ 78.9 \\ 0 \\ 0 \end{bmatrix}, \ B_{33} = \begin{bmatrix} 0 \\ 1000 \\ 0 \\ 0 \end{bmatrix}, \ B_{ij} = 0, \ i \neq j.$$

Table 1.

k	$\|\mathcal{F}^{(k)}(0.5065)\|$	$\|\mathcal{F}^{(k)}(1.0e-01)\|$	$\|\mathcal{F}^{(k)}(1.0e-02)\|$	$\|\mathcal{F}^{(k)}(1.0e-03)\|$	$\|\mathcal{F}^{(k)}(1.0e-04)\|$
0	$2.6664e+01$	$5.2643e+00$	$5.2643e-01$	$5.2643e-02$	$5.2643e-03$
1	$8.3969e+00$	$3.4809e-01$	$3.3150e-03$	$3.3014e-05$	$3.3003e-07$
2	$4.1901e-01$	$1.2350e-03$	$1.3022e-07$	$4.1497e-11$	$3.7051e-11$
3	$1.2351e-03$	$5.0034e-09$	$2.1602e-11$		
4	$5.2579e-09$	$5.0034e-09$			
5	$5.8274e-11$	$6.6434e-11$			

Table 2. Number of iterations

ε	Newton's method	Fixed point itrations
0.5065	5	13
$1.0e-01$	5	7
$1.0e-02$	3	4
$1.0e-03$	2	3
$1.0e-04$	2	2

From this example point of view, it is worth pointing out that even if number of the subsystems is more than four, the required workspace for calculating the strategies is the same as the dimension of the subsystems. That is, even if the large–scale systems (1) are composed of N four–dimensional subsystems, the required workspace is four.

VII. CONCLUSION

In this paper, Nash games for the large–scale systems which are connected by the weak small coupling parameter have been studied. The new algorithm which combine Newton's method and the fixed point iterations for solving the large–scale CARE has been proposed. It should be noted that the proposed design method is quite different from the existing method such as the recursive approach [10] and the fixed point algorithm [6]. As a result, we have succeeded in improving the convergence rate dramatically because the proposed algorithm has the quadratic convergence. As another important feature, it has been newly shown that the resulting strategies achieve the high–order approximation of the optimal cost. Moreover, when the weak coupling parameter ε is unknown, it has been also shown that the proposed approximate strategies are equivalent to the classical linear quadratic approximate controllers. It is worth pointing out that the proposed approximate Nash equilibrium satisfies the ordinary linear quadratic approximate controller.

Finally, even if the large–scale systems are composed of N subsystems, the required workspace is the same as each subsystem dimension. Thus, the proposed design method can reduce the workspace dramatically.

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