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Numerical Solution of Output Feedback H_∞-Constrained LQG Control Problem

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***Abstract*— In this paper, the output feedback H_∞-constrained LQG control problem is investigated. The asymptotic structure along with the uniqueness and positive semidefiniteness of the solutions of the cross-coupled algebraic Riccati equations (CAREs) is newly established. The main purpose of this paper is to propose a new algorithm that is based on the Newton's method and the reduced-order algorithm for solving the CAREs. Particularly, it is noteworthy that the quadratic convergence under the appropriate initial condition and the reduction of the dimension for the matrix computation are both attained. As another important feature, when the disturbance attenuation level γ is small, the successive algorithm for solving CAREs is given for the first time. A numerical example is given to demonstrate the efficiency of the proposed algorithm.**

I. INTRODUCTION

Many modern control problems involve solving a set of cross-coupled algebraic Riccati equations (CAREs) (see for example [1], [2]). In [1], an output feedback H_∞-constrained LQG control problem has been formulated. In [2], the global existence of solution to a state feedback mixed H_2/H_∞ control problem has been studied using a dynamic Nash game approach. Although some algorithm for solving the different CAREs have been introduced in these literatures, there is no proof on the convergence of the related algorithm.

Up to now, various reliable approaches to the computation of the algebraic Riccati equation (ARE) have been well documented in many literatures (see e.g., [7]). One of the approaches is Newton's method [7]. In the past few decades, Newton's method has been applied to the CAREs (see e.g., [8]). However, if the initial conditions are not chosen adequately, the algorithm may not converge because the Newton's method guarantees the local convergence.

In recent years, Newton's method has been used for various control problems of the singularly perturbed systems (SPS) [3], [4]. It has been shown that Newton's method is very effective and reliable to solve the CAREs of the SPS. However, Newton's method for solving the CAREs related to the H_∞-constrained LQG control problem for the linear state space systems has not been investigated. The reason is that it is difficult to find an appropriate initial condition of the iterations for a state space system compared with the SPS.

In this paper, an output feedback H_∞-constrained LQG control problem is investigated from the viewpoint of numerical computation. The purpose of the paper is to analyze the asymptotic structure and the local uniqueness of the solution for such CAREs, and to develop the numerical algorithm to solve them. Since the proposed algorithm is based on Newton's method, it is quite different from the existing algorithm [1]. The quadratic convergence of the algorithm is proved under the sufficiently large disturbance attenuation level γ. The main idea of this paper is to utilize the theory of the SPS. That is, if the disturbance attenuation level γ is sufficiently large, the newly defined parameter $\varepsilon := \gamma^{-2}$ can be thought as a perturbation. As a result, the appropriate initial condition for the Newton's method can be chosen. Furthermore, in order to reduce the dimension of the matrix calculation, the reduced-order algorithm is combined with the Newton's method. As another important feature, when the disturbance attenuation level γ is small, the new successive algorithm for solving CAREs is given. Using such algorithm, the local uniqueness and the quadratic convergence are both attained for the small parameter γ. Finally, a numerical example is solved to show the validity of the proposed algorithm.

Notation: The notations used in this paper are fairly standard. The superscript T denotes matrix transpose. I_p denotes the $p \times p$ identity matrix. vec denotes the column vector of the matrix [6]. $\otimes$ denotes the Kronecker product. $E[\cdot]$ denotes the expectation.

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II. PROBLEM STATEMENT

Consider the following linear system

$$\dot{x}(t) = Ax(t) + D_1 w(t) + Bu(t), \tag{1a}$$
$$y(t) = Cx(t) + D_2 w(t), \tag{1b}$$

where, $x(t) \in \mathrm{R}^n$ is the state, $u(t) \in \mathrm{R}^{l_1}$ is the control input, $w(t) \in \mathrm{R}^{l_2}$ is the disturbance. All matrices above are of appropriate dimensions.

The H_∞-constrained LQG control problem addressed in this paper is as follows [1]:

Given the stabilizable and detectable plant (1), determine a dynamic compensator

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \tag{2a}$$
$$u(t) = C_c x_c(t), \tag{2b}$$

which satisfies the following design criteria:

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i. the following closed-loop system is asymptotically stable, i.e., $\tilde{A}$ is asymptotically stable.

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{D}w(t), \quad (3a)$$
$$z(t) = \tilde{E}_\infty \tilde{x}(t), \quad (3b)$$

where $z(t) \in \mathrm{R}^{k_2}$ is the controlled output and

$$\tilde{x}(t) := \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}, \ \tilde{A} := \begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix},$$
$$\tilde{D} := \begin{bmatrix} D_1 \\ B_c D_2 \end{bmatrix}, \ \tilde{E}_\infty := \begin{bmatrix} E_{1\infty} & 0 \end{bmatrix}.$$

ii. the closed-loop transfer function $H(s) := \tilde{E}_\infty(sI_n - \tilde{A})^{-1}\tilde{D}$ from $w(t)$ to $z(t) := E_{1\infty}x(t)$ satisfies the constraint

$$\|\tilde{E}_\infty(sI_n - \tilde{A})^{-1}\tilde{D}\|_\infty \le \gamma, \quad (4)$$

where $\gamma > 0$ is a given disturbance attenuation level; and

iii. the performance functional

$$J(A_c,\ B_c,\ C_c) := \lim_{t\to\infty} \frac{1}{t} E\left(\int_0^t [x^T(s)R_1 x(s) + u^T(s)R_2 u(s)]ds\right) \quad (5)$$

is minimized.

It should be noted that there is no direct feedthrough term because the particular case is only considered. Such case has been investigated in [1]. However, it would be easy to generalize.

Without loss of generality, the following basic assumptions are made [1].

Assumption 1: $D_1 D_2^T = 0$ is assumed, which effectively implies that plant disturbance and sensor noise are uncorrelated.

Assumption 2: $(A,\ B,\ C)$ is assumed to be stabilizable and detectable.

The following lemma is already known [1].

Lemma 1: If $(A_c,\ B_c,\ C_c)$ solves the auxiliary minimization problem then there exist Q, P and $\hat{Q}$ such that

$$A_c := A - Q\bar{\Sigma} - \Sigma P + \gamma^{-2} Q R_{1\infty}, \quad (6a)$$
$$B_c := QC^T V_2^{-1}, \quad (6b)$$
$$C_c := -R_2^{-1}B^T P, \quad (6c)$$

and such that Q, P and $\hat{Q}$ satisfy

$$L_1(Q,\ P,\ \hat{Q}) := AQ + QA^T + V_1 + \gamma^{-2}QR_{1\infty}Q - Q\bar{\Sigma}Q = 0, \quad (7a)$$
$$L_2(Q,\ P,\ \hat{Q}) := (A + \gamma^{-2}[Q+\hat{Q}]R_{1\infty})^T P + P(A + \gamma^{-2}[Q+\hat{Q}]R_{1\infty}) + R_1 - P\Sigma P = 0, \quad (7b)$$
$$L_3(Q,\ P,\ \hat{Q}) := (A - \Sigma P + \gamma^{-2}QR_{1\infty})\hat{Q} + \hat{Q}(A - \Sigma P + \gamma^{-2}QR_{1\infty})^T + \gamma^{-2}\hat{Q}R_{1\infty}\hat{Q} + Q\bar{\Sigma}Q = 0, \quad (7c)$$

where

$$V_1 = D_1 D_1^T,\ V_2 = D_2 D_2^T,\ R_1 = E_1^T E_1,\ R_2 = E_2^T E_2,$$
$$R_{1\infty} = E_{1\infty}^T E_{1\infty},\ \Sigma = BR_2^{-1}B^T,\ \bar{\Sigma} = C^T V_2^{-1} C.$$

It should be noted that since the equation (7a) is the ordinary ARE and the ARE (7a) is decoupled from the equations (7b) and (7c), it can be solved independently by using the Schur method [9]. Thus, it is enough to consider the CARE (7b) and (7c).

III. PRELIMINARY

Let us consider the following equations that are defined as the parameter $\varepsilon := \gamma^{-2}$.

$$M_1(\varepsilon,\ P,\ \hat{Q}) := (A + \varepsilon[Q+\hat{Q}]R_{1\infty})^T P + P(A + \varepsilon[Q+\hat{Q}]R_{1\infty}) + R_1 - P\Sigma P = 0, \quad (8a)$$
$$M_2(\varepsilon,\ P,\ \hat{Q}) := (A - \Sigma P + \varepsilon QR_{1\infty})\hat{Q} + \hat{Q}(A - \Sigma P + \varepsilon QR_{1\infty})^T + \varepsilon\hat{Q}R_{1\infty}\hat{Q} + Q\bar{\Sigma}Q = 0, \quad (8b)$$

Setting $\varepsilon = 0$ for the previous equations (8), the following equations hold.

$$M_1(0,\ P^{[0]},\ \hat{Q}^{[0]}) := A^T P^{[0]} + P^{[0]}A + R_1 - P^{[0]}\Sigma P^{[0]} = 0, \quad (9a)$$
$$M_2(0,\ P^{[0]},\ \hat{Q}^{[0]}) := (A - \Sigma P^{[0]})\hat{Q}^{[0]} + \hat{Q}^{[0]}(A - \Sigma P^{[0]})^T + Q\bar{\Sigma}Q = 0, \quad (9b)$$

where $P^{[0]}$ and $\hat{Q}^{[0]}$ are zeroth-order solutions of the equations (8).

Using (9), the asymptotic structure of the solutions $P = P(\varepsilon)$ and $\hat{Q} = \hat{Q}(\varepsilon)$ of the CAREs (8) as $M_k(\varepsilon,\ P,\ \hat{Q}) = 0$ is established.

Theorem 1: Then there exists small $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0,\ \bar{\varepsilon})$, the CAREs (8) admits the unique solutions P and $\hat{Q}$ in the neighborhood of $\varepsilon = 0$, which can be written as

$$P(\varepsilon) = P^{[0]} + O(\varepsilon), \quad (10a)$$
$$\hat{Q}(\varepsilon) = \hat{Q}^{[0]} + O(\varepsilon). \quad (10b)$$

Proof: It can be done by applying the implicit function theorem to the CAREs (8). To do so, it is enough to show that the corresponding Jacobian is nonsingular at $\varepsilon = 0$. Taking the partial derivative of the function $M_k(\varepsilon,\ P,\ \hat{Q})$, $k = 1,\ 2$ with respect to P, $\hat{Q}$ and setting $\varepsilon = 0$ result in (11).

$$\hat{\mathcal{J}}(0,\ P^{[0]},\ \hat{Q}^{[0]}) = \begin{bmatrix} \Psi_1 & 0 \\ \Psi_3 & \Psi_2 \end{bmatrix}, \quad (11)$$

where

$$\hat{\mathcal{J}}(\varepsilon,\ P,\ \hat{Q}) := \begin{bmatrix} \dfrac{\partial \mathrm{vec} M_1}{\partial (\mathrm{vec} P)^T} & \dfrac{\partial \mathrm{vec} M_1}{\partial (\mathrm{vec} \hat{Q})^T} \\ \dfrac{\partial \mathrm{vec} M_2}{\partial (\mathrm{vec} P)^T} & \dfrac{\partial \mathrm{vec} M_2}{\partial (\mathrm{vec} \hat{Q})^T} \end{bmatrix}, \quad (12)$$

$$\Psi_1 := (A - \Sigma P^{[0]})^T \otimes I_n + I_n \otimes (A - \Sigma P^{[0]})^T,$$
$$\Psi_2 := (A - \Sigma P^{[0]}) \otimes I_n + I_n \otimes (A - \Sigma P^{[0]}),$$
$$\Psi_3 := -\hat{Q}^{[0]} \otimes \Sigma - \Sigma \otimes \hat{Q}^{[0]}.$$

Obviously, $A - \Sigma P^{[0]}$ is nonsingular because the ARE (9a) has the positive semidefinite stabilizing solution under Assumption 2. Thus, $\hat{\mathcal{J}}(\varepsilon,\ P,\ \hat{Q})$ is nonsingular at $\varepsilon = 0$. The conclusion of Theorem 1 is obtained directly by using the implicit function theorem. ■

IV. NEWTON'S METHOD

In order to obtain the solutions of the CAREs (8), the following new algorithm is given.

$$\begin{aligned}
&(A - \Sigma P^{(i)} + \gamma^{-2}[Q + \hat{Q}^{(i)}]R_{1\infty})^T P^{(i+1)} \\
&\quad P^{(i+1)}(A - \Sigma P^{(i)} + \gamma^{-2}[Q + \hat{Q}^{(i)}]R_{1\infty}) \\
&\quad +\gamma^{-2} R_{1\infty} \hat{Q}^{(i+1)} P^{(i)} + \gamma^{-2} P^{(i)} \hat{Q}^{(i+1)} R_{1\infty} \\
&\quad -\gamma^{-2} R_{1\infty} \hat{Q}^{(i)} P^{(i)} - \gamma^{-2} P^{(i)} \hat{Q}^{(i)} R_{1\infty} \\
&\quad +P^{(i)} \Sigma P^{(i)} + R_1 = 0, \qquad (13a)\\
&(A - \Sigma P^{(i)} + \gamma^{-2}[Q + \hat{Q}^{(i)}]R_{1\infty})\hat{Q}^{(i+1)} \\
&\quad +\hat{Q}^{(i+1)}(A - \Sigma P^{(i)} + \gamma^{-2}[Q + \hat{Q}^{(i)}]R_{1\infty})^T \\
&\quad -\Sigma P^{(i+1)} \hat{Q}^{(i)} - \hat{Q}^{(i)} P^{(i+1)} \Sigma \\
&\quad +\Sigma P^{(i)} \hat{Q}^{(i)} + \hat{Q}^{(i)} P^{(i)} \Sigma + Q \bar{\Sigma} Q \\
&\quad -\gamma^{-2} \hat{Q}^{(i)} R_{1\infty} \hat{Q}^{(i)} = 0, \qquad (13b)
\end{aligned}$$

where $P^{(0)}$ and $\hat{Q}^{(0)}$ satisfy the AREs (9) as $P^{(0)} = P^{[0]}$ and $\hat{Q}^{(0)} = \hat{Q}^{[0]}$, respectively.

The new algorithm (13) can be constructed by setting $P^{(i+1)} = P^{(i)} + \Delta P^{(i)}$ and $\hat{Q}^{(i+1)} = \hat{Q}^{(i)} + \Delta \hat{Q}^{(i)}$, and neglecting $O(\Delta^2)$ term.

Theorem 2: Suppose that there exist solutions to the CAREs (7). It can be obtained by performing the algorithm (13) which is equal to Newton's method.

Proof: Taking the vec-operator transformation on both sides of (7) results in

$$\begin{bmatrix} \mathrm{vec} L_2(Q,\ P^{(i)},\ \hat{Q}^{(i)}) \\ \mathrm{vec} L_3(Q,\ P^{(i)},\ \hat{Q}^{(i)}) \end{bmatrix} = \Phi^{(i)} \begin{bmatrix} \mathrm{vec} P^{(i)} \\ \mathrm{vec} \hat{Q}^{(i)} \end{bmatrix} + \begin{bmatrix} \mathrm{vec} H^{(i)} \\ \mathrm{vec} J^{(i)} \end{bmatrix}, \quad (14)$$

where

$$\Phi^{(i)} = \begin{bmatrix} I_n \otimes E^{(i)T} + E^{(i)T} \otimes I_n & \Lambda^{(i)} \\ -\hat{Q}^{(i)} \otimes \Sigma - \Sigma \otimes \hat{Q}^{(i)} & I_n \otimes E^{(i)} + E^{(i)} \otimes I_n \end{bmatrix},$$

$$E^{(i)} = A - \Sigma P^{(i)} + \gamma^{-2}[Q + \hat{Q}^{(i)}]R_{1\infty},$$
$$\Lambda^{(i)} = P^{(i)} \otimes (\gamma^{-2} R_{1\infty}) + (\gamma^{-2} R_{1\infty}) \otimes P^{(i)},$$
$$H^{(i)} = -\gamma^{-2} R_{1\infty} \hat{Q}^{(i)} P^{(i)} - \gamma^{-2} P^{(i)} \hat{Q}^{(i)} R_{1\infty} + P^{(i)} \Sigma P^{(i)} + R_1,$$
$$J^{(i)} = \Sigma P^{(i)} \hat{Q}^{(i)} + \hat{Q}^{(i)} P^{(i)} \Sigma + Q \bar{\Sigma} Q - \gamma^{-2} \hat{Q}^{(i)} R_{1\infty} \hat{Q}^{(i)}.$$

Moreover, taking the vec-operator transformation on both sides of (13) results in

$$\Phi^{(i)} \begin{bmatrix} \mathrm{vec} P^{(i+1)} \\ \mathrm{vec} \hat{Q}^{(i+1)} \end{bmatrix} + \begin{bmatrix} \mathrm{vec} H^{(i)} \\ \mathrm{vec} J^{(i)} \end{bmatrix} = 0. \quad (15)$$

Subtracting (14) from (15) and using (12), it is easy to verify that

$$\begin{bmatrix} \mathrm{vec} P^{(i+1)} \\ \mathrm{vec} \hat{Q}^{(i+1)} \end{bmatrix} = \begin{bmatrix} \mathrm{vec} P^{(i)} \\ \mathrm{vec} \hat{Q}^{(i)} \end{bmatrix} - [\hat{\mathcal{J}}(\varepsilon, P^{(i)},\ \hat{Q}^{(i)})]^{-1} \times \begin{bmatrix} \mathrm{vec} L_2(Q,\ P^{(i)},\ \hat{Q}^{(i)}) \\ \mathrm{vec} L_3(Q,\ P^{(i)},\ \hat{Q}^{(i)}) \end{bmatrix}. \quad (16)$$

This is the desired result. ■

Newton's method is well-known and is widely used to find a solution of algebraic nonlinear equations. Its local convergence properties are well understood [5]. Particularly, it is highly expected that the proposed algorithm can converge to the adequate solutions because the initial conditions are close to the exact solutions with the structure of (10) under the sufficiently small parameter $\varepsilon = \gamma^{-2}$. The following theorem indicates the local quadratic convergence and the uniqueness for the convergence solutions.

Theorem 3: Assume that the conditions of Theorem 1 hold. Then, there exists a small σ^* such that for all $\varepsilon \in (0,\ \sigma^*)$, Newton's method (13) converges to the exact solution of P^* and $\hat{Q}^*$ with the rate of the quadratic convergence. Moreover, the convergence solutions P^* and $\hat{Q}^*$ are unique solution of the CAREs (8) in the neighborhood of the initial conditions $P^{(0)} = P^{[0]}$, $\hat{Q}^{(0)} = \hat{Q}^{[0]}$, respectively. That is, the following relations are satisfied.

$$\|P^{(i)} - P^*\| \le O(\varepsilon^{2^i}),\ i = 0,\ 1,\ \dots, \quad (17a)$$
$$\|\hat{Q}^{(i)} - \hat{Q}^*\| \le O(\varepsilon^{2^i}),\ i = 0,\ 1,\ \dots. \quad (17b)$$

Proof: The proof of this theorem can be done by using Newton-Kantorovich theorem [5]. It is immediately obtained from the equation (12) that there exists the positive scalar constant $\mathcal{L}(\varepsilon)$ such that for any P^a, $\hat{Q}^a$, P^b and $\hat{Q}^b$,

$$\begin{aligned} &\|\hat{\mathcal{J}}(\varepsilon,\ P^a,\ \hat{Q}^a) - \hat{\mathcal{J}}(\varepsilon,\ P^b,\ \hat{Q}^b)\| \\ &\le \mathcal{L}(\varepsilon)\|(P^a,\ \hat{Q}^a) - (P^b,\ \hat{Q}^b)\|, \end{aligned} \quad (18)$$

where $\mathcal{L}(\varepsilon) := 6\varepsilon\|R_{1\infty}\| + 6\|\Sigma\|$.

Moreover, using (10), we get

$$\hat{\mathcal{J}}(\varepsilon,\ P^{(0)},\ \hat{Q}^{(0)}) = \hat{\mathcal{J}}(0,\ P^{[0]},\ \hat{Q}^{[0]}) + O(\varepsilon). \quad (19)$$

Hence, it follows that $\hat{\mathcal{J}}(\varepsilon,\ P^{(0)},\ \hat{Q}^{(0)})$ is nonsingular under $\det \hat{\mathcal{J}}(0,\ P^{[0]},\ \hat{Q}^{[0]}) \neq 0$ for sufficiently small ε. Therefore, there exists β such that $\beta = \|[\hat{\mathcal{J}}(\varepsilon,\ P^{(0)},\ \hat{Q}^{(0)})]^{-1}\|$. On the other hand, since $L_k(Q,\ P^{(0)},\ \hat{Q}^{(0)}) = O(\varepsilon)$, there exists η such that $\eta = \|[\hat{\mathcal{J}}(\varepsilon,\ P^{(0)},\ \hat{Q}^{(0)})]^{-1}\| \cdot$

$\|L_k(Q,\ P^{(0)},\ \hat{Q}^{(0)})\| = O(\varepsilon)$. Thus, there exists θ such that $\theta = \beta\eta\mathcal{L}(\varepsilon) < 2^{-1}$ because $\eta = O(\varepsilon)$. Finally, using the Newton-Kantorovich theorem, we can show that P^* and $\hat{Q}^*$ are the unique solution in the subset. Moreover, the error estimate is given by (17). ■

It may be noted that the Newton's method (13) is well defined. That is, the Lyapunov equations in (13) are solvable in each step due to the following reason. For the sufficiently large parameter γ, the following equation holds.

$$\Phi^{(i)} \to \Psi^{(i)} := \begin{bmatrix} \Psi_1^{(i)} & 0 \\ \Psi_3^{(i)} & \Psi_2^{(i)} \end{bmatrix}, \tag{20}$$

where

$$\begin{aligned} &\Psi_1^{(i)} := (A - \Sigma P^{(i)})^T \otimes I_n + I_n \otimes (A - \Sigma P^{(i)})^T, \\ &\Psi_2^{(i)} := (A - \Sigma P^{(i)}) \otimes I_n + I_n \otimes (A - \Sigma P^{(i)}), \\ &\Psi_3^{(i)} := -\hat{Q}^{(i)} \otimes \Sigma - \Sigma \otimes \hat{Q}^{(i)}, \\ &A - \Sigma P^{(i)} = A - \Sigma P^{[0]} + O(\varepsilon). \end{aligned}$$

The matrix $A - \Sigma P^{[0]}$ is stable because the ARE (9a) has the positive semidefinite stabilizing solutions. Thus, if the parameter $\varepsilon = \gamma^{-2}$ is small, $A - \Sigma P^{(i)}$ is also stable. Finally, the Newton's method (13) is well defined for each step.

The algorithm is now summarized.

Step 1. Solve (7a) for Q. Calculate $P^{[0]}$ and $\hat{Q}^{[0]}$ by using the initial conditions (9).

Step 2. Compute the solutions $P^{(i+1)}$ and $\hat{Q}^{(i+1)}$ by using the following linear equation.

$$\begin{bmatrix} \mathrm{vec}P^{(i+1)} \\ \mathrm{vec}\hat{Q}^{(i+1)} \end{bmatrix} = -[\Phi^{(i)}]^{-1} \begin{bmatrix} \mathrm{vec}H^{(i)} \\ \mathrm{vec}J^{(i)} \end{bmatrix}. \tag{21}$$

Step 3. If $i \geq 1$ check for

$$\mathcal{E}(\varepsilon) := \sum_{k=1}^{3} \|L_k(Q,\ P^{(i)},\ \hat{Q}^{(i)})\| < \phi \tag{22}$$

for a given convergence criterion $\phi > 0$.

Step 4. If convergence is not achieved in Step 3, increment $i \to i+1$ and go to Step 2. Otherwise, stop Newton iterations (13) and compute the controller (2).

V. REDUCED-ORDER COMPUTATION OF THE NEWTON'S METHOD

Ones need to solve the linear equation (21) with quite large dimension $2n^2 \times 2n^2$. Thus, in order to avoid this drawback, a computation method to solve these linear equations is established.

Let us consider the following differential equations.

$$\begin{aligned} \dot{\mathbf{P}} &= E^{(i)T}\mathbf{P} + \mathbf{P}E^{(i)} \\ &\quad + \gamma^{-2}R_{1\infty}\hat{\mathbf{Q}}P^{(i)} + \gamma^{-2}P^{(i)}\hat{\mathbf{Q}}R_{1\infty} + H^{(i)}, \end{aligned} \tag{23a}$$

$$\dot{\hat{\mathbf{Q}}} = -\Sigma\mathbf{P}\hat{Q}^{(i)} - \hat{Q}^{(i)}\mathbf{P}\Sigma + E^{(i)}\hat{\mathbf{Q}} + \hat{\mathbf{Q}}E^{(i)T} + J^{(i)}, \tag{23b}$$

where

$$\mathbf{P} := \mathbf{P}(t),\ \hat{\mathbf{Q}} := \hat{\mathbf{Q}}(t),\ \mathbf{P}(0) = I_n,\ \hat{\mathbf{Q}}(0) = I_n.$$

It is important to note that the stability of the differential equations (23) is guaranteed because the matrix (20) is stable for sufficiently large γ. Thus, the solutions of (23) tend to some finite values as $t \to \infty$. Finally, the required solutions of the linear algebraic equations (21) can be obtained as the convergence solutions. It should be noted that a fourth order Runge-Kutta method is used to integrate the differential equations (23).

In this case, since the required workspace for the matrix calculus is $2n \times 2n$, the proposed computation method is very attractive in the sense that it is easy to implement. For example, in the next numerical example, when the dimension $n = 2$ the proposed algorithm requires 4×4 dimensions, while the algorithm (21) requires 8×8 dimensions for the matrix calculation. It is concluded that such example results in a 50% reduction of the workspace compared with the existing result [1].

VI. SUCCESSIVE ALGORITHM FOR SOLVING CAREs WITH SMALL PARAMETER

If the disturbance attenuation level γ is not sufficiently large, the initial condition (9) will be not adequate. Hence, in order to attain the quadratic convergence of the proposed algorithm (13) for any small parameter γ, the successive algorithm is newly proposed.

The idea is given below. It is assumed that the step of the successive algorithm is k. Let us define $\varepsilon^{[k]} := \left(\gamma^{[k]}\right)^{-2}$, and suppose that

$$\gamma^{[0]} > \gamma^{[1]} > \cdots > \gamma^{[k]} \Leftrightarrow \varepsilon^{[0]} < \varepsilon^{[1]} < \cdots < \varepsilon^{[k]}.$$

Then, if the following inequality (24) holds, the next approximate solutions $P^{[k+1]}$, $\hat{Q}^{[k+1]}$, $k = 1,\ 2,\ \ldots$ can be computed successively by choosing the new initial condition $P^{(0)} = P^{[k]}$, $\hat{Q}^{(0)} = \hat{Q}^{[k]}$. This idea is to exploit the fact that for large γ (small ε) the problem is approximated by LQG which provides reliable starting solution.

$$\theta^{[k]}(\varepsilon^{[k]}) := \theta^{[k]} = \beta^{[k]}\eta^{[k]}\mathcal{L}^{[k]} < 2^{-1}, \tag{24}$$

where

$$\begin{aligned} &\mathcal{L}^{[k]}(\varepsilon^{[k]}) := \mathcal{L}^{[k]} = 6\varepsilon^{[k]}\|R_{1\infty}\| + 6\|\Sigma\|, \\ &\beta^{[k]}(\varepsilon^{[k]}) := \beta^{[k]} = \|[\hat{\mathcal{J}}(\varepsilon^{[k]},\ P^{[k]},\ \hat{Q}^{[k]})]^{-1}\|, \\ &\eta^{[k]}(\varepsilon^{[k]}) := \eta^{[k]} = \beta^{[k]} \cdot \|\mathcal{M}(\varepsilon^{[k]},\ P^{[k]},\ \hat{Q}^{[k]})\|, \\ &\mathcal{M}(\varepsilon^{[k]},\ P^{[k]},\ \hat{Q}^{[k]}) := \begin{bmatrix} M_1(\varepsilon^{[k]},\ P^{[k]},\ \hat{Q}^{[k]}) \\ M_2(\varepsilon^{[k]},\ P^{[k]},\ \hat{Q}^{[k]}) \end{bmatrix}, \\ &k = 0,\ 1,\ \ldots . \end{aligned}$$

The successive algorithm for relatively small parameter γ is given as follows.

Step 1. Solve (7a) for Q. When $k = 0$, solve the initial conditions (9).

Step 2. For sufficiently large $\gamma^{[0]}$, compute $\varepsilon^{[0]} = \left(\gamma^{[0]}\right)^{-2}$. Moreover, compute $\beta^{[0]}$ and $\eta^{[0]}$.

Step 3. If the following inequality holds, solve (15) for $P^{[1]}$, $\hat{Q}^{[1]}$ by using the Newton's method (13) with (23) under $P^{(0)} = P^{[0]}$, $\hat{Q}^{(0)} = \hat{Q}^{[0]}$.

$$\theta^{[0]} = \beta^{[0]}\eta^{[0]}\mathcal{L}^{[0]} < 2^{-1}. \tag{25}$$

Table 3.

k	$\gamma^{[k]}$	$\theta^{[k]} < 2^{-1}$
$k = 0 \sim 446$	$\gamma^{[k]} = 50.0 - 0.1 \times k$	ALL O.K.
$k = 446 \sim 648$	$\gamma^{[k]} = 5.4 - 0.01 \times (k - 446)$	ALL O.K.
$k = 648 \sim 678$	$\gamma^{[k]} = 3.38 - 0.001 \times (k - 648)$	ALL O.K.

If the inequality (25) does not hold, increase $\varepsilon^{[0]}$ and go to Step 2.

Step 4. Choose $\varepsilon^{[1]}$ such that $\varepsilon^{[0]} < \varepsilon^{[1]}$. Let $k = 1$ and check for the inequality (24). If this inequality holds, solve (13) with (23) for $P^{[2]}$, $\hat{Q}^{[2]}$ by using the Newton's method under $P^{(0)} = P^{[1]}$, $\hat{Q}^{(0)} = \hat{Q}^{[1]}$.

Step 5. Choose $\varepsilon^{[k]}$ such that $\varepsilon^{[0]} < \varepsilon^{[1]} < \cdots < \varepsilon^{[k]}$. If this inequality (24) holds under the fact that there exists $[\hat{\mathcal{J}}(\varepsilon^{[k]},\ P^{[k]},\ \hat{Q}^{[k]})]^{-1}$, increment $k \to k+1$ and solve (13) with (23) for $P^{[k+1]}$, $\hat{Q}^{[k+1]}$ by using the Newton's method under $P^{(0)} = P^{[k]}$, $\hat{Q}^{(0)} = \hat{Q}^{[k]}$.

Step 6. Repeat Step 5 until the desired disturbance attenuation level γ is attained. If the desired γ is attained, stop. Otherwise declare that there is no controller for $\varepsilon^{[k]} = \left(\gamma^{[k]}\right)^{-2}$.

The main result for above algorithm is stated as follows.

Theorem 4: Assume that the conditions of Theorem 1 hold and the CAREs (8) have solution. Suppose that for $\varepsilon^{[k]} = \left(\gamma^{[k]}\right)^{-2}$, $P^{[k]}$, $\hat{Q}^{[k]}$, the Jacobian $\hat{\mathcal{J}}(\varepsilon^{[k]},\ P^{[k]},\ \hat{Q}^{[k]})$ is nonsingular at $\varepsilon = \varepsilon^{[k]}$. Then, there exists a small $\bar{\sigma}^*$ such that for all $\varepsilon \in (0,\ \bar{\sigma}^*)$, Newton's method (15) converges to the exact solution of P^* and $\hat{Q}^*$ with the rate of the quadratic convergence. Moreover, for each step k, the convergence solutions P^* and $\hat{Q}^*$ are unique solution of the CAREs (10) in the neighborhood of the initial conditions $P^{(0)} = P^{[k]}$, $\hat{Q}^{(0)} = \hat{Q}^{[k]}$, respectively. That is, the following relations are satisfied.

$$\|P^{(i)} - P^*\| \le 2^{1-i}(2\theta^{[k]})^{2^i-1}\eta^{[k]}, \tag{26a}$$

$$\|\hat{Q}^{(i)} - \hat{Q}^*\| \le 2^{1-i}(2\theta^{[k]})^{2^i-1}\eta^{[k]}, \tag{26b}$$

where $0 < 2\theta^{[k]} < 1$.

Proof: Since the proof of Theorem 4 can be done by using the Newton-Kantorovich, it is omitted. ■

VII. NUMERICAL EXAMPLE

In order to demonstrate the efficiency of the proposed algorithm, a simple numerical example is given. The system matrices are given below.

$$A = \begin{bmatrix} -0.1610 & 1 \\ -6.0040 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0.9955 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$E_1 = E_{1\infty} = \begin{bmatrix} 0.55 & 1.32 \\ 0 & 0 \end{bmatrix},\ E_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0.0713 & 0 \\ 1.0002 & 0 \end{bmatrix},\ D_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

In order to verify the accuracy of the solution, the remainder per iteration is computed by substituting $P^{(i)}$ and $\hat{Q}^{(i)}$ into the CAREs (7). Table 1 shows the error per iterations. In case of $\gamma = 5$, it should be noted that the algorithm (13) converges to the exact solution with accuracy of $\mathcal{E}(\varepsilon) < 1.0e-10$ after four iterations. Hence, it can be seen from Table 1 that the algorithm (13) attains the quadratic convergence.

The required iterations of the proposed algorithm (13) versus the existing algorithm [1] are presented in Table 2. It can be seen from Table 2 that the proposed algorithm have relatively small number of iterations than the existing algorithm [1]. Hence, the resulting algorithm of this paper is very reliable.

Table 1.

k	$\mathcal{E}(\varepsilon)$
0	$4.9642e-01$
1	$2.9217e-02$
2	$1.4495e-04$
3	$2.7991e-09$
4	$1.1071e-14$

Table 2. Number of iterations

γ	Newton's Method	Existing Method [1]
3.35	4	7
5	4	6
10	3	4
15	3	4
50	2	3

Second, in order to verify the validity of the proposed successive algorithm in the previous section, the iterations are carried out. Since large parameter $\gamma^{[0]} = 50.0$ at $k = 0$, $\theta^{[0]} = 0.110758 < 0.5$ holds, the quadratic convergence can be verified without the simulation. Thus, for sufficiently large γ, it can be concluded that the uniqueness of the solution is guaranteed if the inequality (25) holds. The simulation results via the successive algorithm are given for $\gamma = 3.35 \sim 50.0$. It is observed that since for all k, the inequality (24) ($\theta^{[k]} < 2^{-1}$) satisfy, the Newton's method (13) has the quadratic convergence. Moreover, the uniqueness of the convergence solutions are guaranteed at the neighbourhood of each $\varepsilon = \varepsilon^{[k]} = \left(\gamma^{[k]}\right)^{-2}$. That is, if the disturbance attenuation level γ is started from $\gamma^{[0]} = 50.0$, the initial conditions $X^{[0]}$ and $Y^{[0]}$ of (9) satisfy the inequality (24). For the next step, if $\gamma^{[k]}$ is chosen such that $\gamma^{[0]} > \gamma^{[1]} > \cdots > \gamma^{[k]}$, for all k, the inequality (24) also holds. Therefore, when the solutions $X^{(i)}$ and $Y^{(i)}$ are solved by using the Newton's method (13), the quadratic convergence is attained. In fact, for all k, this useful phenomenon has been observed. Moreover, the local uniqueness would be attained at the neighbourhood of each $\varepsilon = \varepsilon^{[k]}$. It should be noted that we succeed in obtaining the required solution by repeating the successive algorithm recursively until the desired disturbance attenuation level $\gamma = 3.35$. In this case, it may be also noted that the error estimations (26) satisfy.

Finally, for $\gamma = 3.35$, the exact solutions Q, P and $\hat{Q}$ are calculated by using the algorithms (13) and (23).

$$Q = \begin{bmatrix} 0.28226819916847 & 0.08274100319703 \\ 0.08274100319703 & 1.66010046329206 \end{bmatrix},$$
$$P = \begin{bmatrix} 7.68772857895541 & -0.17152234337753 \\ -0.17152234337754 & 1.50657592238958 \end{bmatrix},$$
$$\hat{Q} = \begin{bmatrix} 0.04023096332073 & -0.03287931839280 \\ -0.03287931839280 & 0.15710718229110 \end{bmatrix}.$$

It is worth pointing out that the solutions can be solved with fast convergence. Moreover, it is noteworthy that the convergence proof and the convergence rate has been given compared with the existing result for the first time [1].

VIII. CONCLUSION

The numerical algorithm for solving the output feedback H_∞-constrained LQG control problem has been investigated. In order to solve the resulting CAREs, the new algorithm that is based on the Newton's method has been derived. It has been shown that the quadratic convergence is guaranteed under the obtained appropriate initial condition. It is noteworthy that the convergence proof of the algorithm has been given for the first time compared with the existing result [1]. Moreover, the local uniqueness of the solutions has been proved for any parameter γ. As another important contribution, the successive algorithm for solving CAREs was established for small parameter γ. As a result, even if the parameter γ is relatively small, the quadratic convergence and the uniqueness of the iterative solutions are both guaranteed. Furthermore, in order to reduce the dimension of the matrix calculation, the reduced-order algorithm has been combined with the Newton's method. Thus, the computation for the algebraic manipulation can be carried out as the same dimension of the matrix of each CAREs. Finally, the numerical example has shown the excellent results.

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