Algorithm for Solving Cross–Coupled Algebraic Riccati Equations Related to Mixed H_2/H_∞ Control Problems of Multimodeling Systems

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Abstract

In this paper, the mixed H_2/H_∞ control problems for the multiparameter singularly perturbed systems (MSPS) are discussed. In order to obtain the strategies, the algorithm for solving the cross–coupled algebraic Riccati equations (CARE) is proposed. Since a new algorithm is based on the Newton's method, the proposed method is computationally attractive and the implementation of the algorithm is easy. It is newly proven that the new algorithm guarantees the quadratic convergence. As a result, it is shown that the proposed algorithm succeed in improving the convergence rate dramatically compared with the previous results.

1 Introduction

The mixed H_2/H_∞ control problems have been studied by using the several approaches. In particular, a state feedback mixed H_2/H_∞ control problem has been formulated as a dynamic Nash game as in [1]. The resulting feedback controller is characterized by the solution to a pair of the cross–coupled algebraic Riccati equations (CARE). It is well known that the CARE plays an important role to Nash games [1]. In order to obtain the Nash equilibrium strategies, we must solve the CARE. Various reliable approaches to the theory of the CARE have been well–documented [2, 3, 7, 8]. These methods consist of the Riccati iterations [3, 7, 9] and the Lyapunov iterations [2, 8]. However, the convergence of the Riccati iterations were not proved exactly. Moreover, there exist no results for the convergence rate of the Lyapunov iterations. In fact, it is easy to verify that the convergence speed is very slow when we run the numerical example [8].

Multimodeling stability, control and filtering problems have been investigated extensively (see e.g., [4, 5]). The multimodeling problems arise in large–scale dynamic systems. Linear quadratic Nash games for the multiparameter singularly perturbed systems (MSPS) have been studied by using composite controller design [4]. When the parameters represent the small unknown perturbations whose values are not known exactly, the composite design is very useful. However, there exists drawback for the composite design. Firstly, the composite Nash equilibrium solution achieves only a performance which is $O(\|\mu\|)$ (where $\|\mu\|$ denotes the norm of the vector $[\varepsilon_1\ \varepsilon_2]$) close to the full–order performance. Secondly, since the closed–loop solution of the reduced Nash problem depends on the path along $\varepsilon_1/\varepsilon_2$ as $\|\mu\| \to 0$, we cannot expect that the closed–loop solution of the full problem tends to the closed–loop solution of the reduced problem [5]. Therefore, as long as the small perturbation parameters ε_j are known, much effort should be made towards finding the exact strategies which guarantees the Nash equilibrium without the ill–conditioning. From the above point of view, it is easily found that the mixed H_2/H_∞ control problems for the MSPS have also these disadvantages because the H_2/H_∞ control strategies are obtained by solving the CARE.

In this paper, the mixed H_2/H_∞ control problem for infinite horizon MSPS is considered. It is worth pointing out that although the H_∞ control problem for the MSPS has been investigated [10], the mixed H_2/H_∞ control problems has never been studied. Newton's method is applied to the parameterized CARE. The resulting algorithm consists of the generalized linear matrix equation (GLME). The quadratic convergence of the proposed algorithm is proved by using the Newton–Kantorovich theorem [11]. The sufficient conditions are provided such that the proposed algorithm converges to a positive semidefinite solution. It should be noted that the proof of the quadratic convergence property of the resulting algorithm by using the Newton–Kantorovich theorem has not been studied so far. Using the new algorithm, we will improve the convergence speed compared with the previous results [7, 8]. Finally, simulation results show that the proposed algorithm succeed in improving the convergence rate dramatically.

Notation: The notations used in this paper are fairly standard. The superscript T denotes matrix transpose. I_n denotes the $n \times n$ identity matrix. $\|\cdot\|$ denotes its Euclidean norm for a matrix. $\det M$ denotes the determinant of M. $\mathbf{Re}\lambda[M]$ denotes the real part of the eigenvalue of M. $\mathrm{vec}M$ denotes an ordered stack of the columns of M [12]. $\otimes$ denotes Kronecker product. U_{lm} denotes a permutation matrix in Kronecker matrix

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sense [12] such that $U_{lm}\mathrm{vec}M = \mathrm{vec}M^T$, $(M \in \mathbf{R}^{l\times m})$. **block − diag** denotes the block diagonal matrix.

2 Problem Formulation

Consider a linear time–invariant MSPS

$$\begin{aligned}
\dot{x}_0 &= \sum_{i=0}^{2} A_{0i}x_i + \sum_{i=1}^{2} D_{0i}w_i + \sum_{i=1}^{2} B_{0i}u_i, && \text{(1a)}\\
\varepsilon_1\dot{x}_1 &= A_{10}x_0 + A_{11}x_1 + D_{11}w_1 + B_{11}u_1, && \text{(1b)}\\
\varepsilon_2\dot{x}_2 &= A_{20}x_0 + A_{22}x_2 + D_{22}w_2 + B_{22}u_2, && \text{(1c)}\\
x_j(0) &= 0,\ j = 0,\ 1,\ 2,
\end{aligned}$$

with quadratic cost function

$$J(u,\ w) = \int_0^\infty z^T z dt, \quad z = Cx + Hu, \tag{2}$$

where

$$C = \begin{bmatrix} C_{10} & C_{11} & 0 \\ C_{20} & 0 & C_{22} \end{bmatrix},\ H = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix},$$

$$x = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix},\ w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},\ u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

and $x_j \in \mathbf{R}^{n_j}$, $j = 0,\ 1,\ 2$ are the state vector, $u_j \in \mathbf{R}^{m_j}$, $j = 1,\ 2$ are the control input, $w_j \in \mathbf{R}^{l_j}$, $j = 1,\ 2$ are the disturbance. Let us now assume that $C^T H = 0$, $H^T H = I_m$, $m_1 + m_2 = m$. All the matrices are constant matrices of appropriate dimensions. ε_1 and ε_2 are two small positive singular parameters of the same order of magnitude such that

$$0 < k_1 \le \alpha \equiv \frac{\varepsilon_1}{\varepsilon_2} \le k_2 < \infty. \tag{3}$$

Note that the fast state matrices $A_{jj}, j = 1,\ 2$ may be singular.

Let us introduce the partitioned matrices

$$A_e = \Pi_e^{-1}A,\ D_e = \Pi_e^{-1}D,\ B_e = \Pi_e^{-1}B,$$
$$U_e = \Pi_e^{-1}U\Pi_e^{-1},\ S_e = \Pi_e^{-1}S\Pi_e^{-1},$$
$$D = \begin{bmatrix} D_1 & D_2 \end{bmatrix},\ B = \begin{bmatrix} B_1 & B_2 \end{bmatrix},$$
$$D_1 = \begin{bmatrix} D_{01} \\ D_{11} \\ 0 \end{bmatrix},\ D_2 = \begin{bmatrix} D_{02} \\ 0 \\ D_{22} \end{bmatrix},\ B_1 = \begin{bmatrix} B_{01} \\ B_{11} \\ 0 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} B_{02} \\ 0 \\ B_{22} \end{bmatrix},\ \Pi_e = \begin{bmatrix} I_{n_0} & 0 & 0 \\ 0 & \varepsilon_1 I_{n_1} & 0 \\ 0 & 0 & \varepsilon_2 I_{n_2} \end{bmatrix},$$
$$A = \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & 0 \\ A_{20} & 0 & A_{22} \end{bmatrix},$$
$$U = DD^T = \begin{bmatrix} U_{00} & U_{01} & U_{02} \\ U_{01}^T & U_{11} & 0 \\ U_{02}^T & 0 & U_{22} \end{bmatrix},$$
$$S = BB^T = \begin{bmatrix} S_{00} & S_{01} & S_{02} \\ S_{01}^T & S_{11} & 0 \\ S_{02}^T & 0 & S_{22} \end{bmatrix},$$
$$Q = C^T C = \begin{bmatrix} Q_{00} & Q_{01} & Q_{02} \\ Q_{01}^T & Q_{11} & 0 \\ Q_{02}^T & 0 & Q_{22} \end{bmatrix}.$$

We now consider the mixed H_2/H_∞ control problem for the MSPS (1) under the following basic assumptions [1].

Assumption 1 *There exists an* $\|\mu\|^* > 0$ *such that the triplets* $(A_e,\ D_e,\ C)$ *and* $(A_e,\ B_e,\ C)$ *are stabilizable and detectable for all* $\|\mu\| \in (0, \|\mu\|^*]$, *where* $\|\mu\| := \sqrt{\varepsilon_1\varepsilon_2}$.

Assumption 2 (i) *The triplets* $(A_{11},\ D_{11},\ C_{11})$ *and* $(A_{11},\ B_{11},\ C_{11})$ *are stabilizable and detectable.*
(ii) *The triplets* $(A_{22},\ D_{22},\ C_{22})$ *and* $(A_{22},\ B_{22},\ C_{22})$ *are stabilizable and detectable.*

These conditions are quite natural. The mixed H_2/H_∞ control problem is formulated as a two–player Nash game [1] associated with a prescribed disturbance attenuation level γ,

$$\begin{aligned}
J_1(u,\ w) &= \int_0^\infty \gamma^2 w^T w dt - J(u,\ w), && \text{(4a)}\\
J_2(u,\ w) &= J(u,\ w). && \text{(4b)}
\end{aligned}$$

The first is used to reflect an H_∞ criterion, while the second is used for the H_2 optimality requirement. The purpose is to find a linear feedback controller $u^* = Kx$ such that

$$\begin{aligned}
J_1(u^*,\ w^*) &\le J_1(u^*,\ w), && \text{(5a)}\\
J_2(u^*,\ w^*) &\le J_2(u,\ w^*), && \text{(5b)}
\end{aligned}$$

where w^* represents the worst–case disturbance. When $J_1(u^*, w^*) \ge 0$, we have

$$\sup_{w\in H_w} \frac{\sqrt{J(u^*, w)}}{\|w\|} \le \gamma, \tag{6}$$

the H_∞ criterion, where H_w denotes an appropriate Hilbert space. The second Nash inequality shows that u^* regulates the state to zero with minimum output energy when the disturbance is at its worst value w^*. The following lemma is already known [1].

Lemma 1 *Under Assumptions 1 and 2, there exists an admissible controller such that* (5) *iff the following full–order CARE*

$$\begin{aligned}
&A_e^T X_e + X_e A_e + Q + \gamma^2 X_e U_e X_e \\
&\quad - X_e S_e Y_e - Y_e S_e X_e + Y_e S_e Y_e = 0, && \text{(7a)}\\
&A_e^T Y_e + Y_e A_e + Q - Y_e S_e Y_e \\
&\quad + \gamma^2 Y_e U_e X_e + \gamma^2 X_e U_e Y_e = 0, && \text{(7b)}
\end{aligned}$$

have solutions $X_e \geq 0$ and $Y_e \geq 0$, where

$$X_e = \begin{bmatrix} X_{00} & \varepsilon_1 X_{10}^T & \varepsilon_2 X_{20}^T \\ \varepsilon_1 X_{10} & \varepsilon_1 X_{11} & \sqrt{\varepsilon_1\varepsilon_2} X_{21}^T \\ \varepsilon_2 X_{20} & \sqrt{\varepsilon_1\varepsilon_2} X_{21} & \varepsilon_2 X_{22} \end{bmatrix},$$

$$Y_e = \begin{bmatrix} Y_{00} & \varepsilon_1 Y_{10}^T & \varepsilon_2 Y_{20}^T \\ \varepsilon_1 Y_{10} & \varepsilon_1 Y_{11} & \sqrt{\varepsilon_1\varepsilon_2} Y_{21}^T \\ \varepsilon_2 Y_{20} & \sqrt{\varepsilon_1\varepsilon_2} Y_{21} & \varepsilon_2 Y_{22} \end{bmatrix}.$$

Then, the closed–loop strategies to the full–order problem are given by

$$w^* = \gamma^{-2} D_e^T X_e x, \tag{8a}$$

$$u^* = -B_e^T Y_e x. \tag{8b}$$

3 Asymptotic Structure

In order to obtain the solutions of the CARE (7), we introduce the following useful lemma.

Lemma 2 *The CARE* (7) *is equivalent to the following GCMARE* (9)*, respectively.*

$$A^T X + X^T A + Q + \gamma^{-2} X^T U X \\ -X^T S Y - Y^T S X + Y^T S Y = 0, \tag{9a}$$

$$A^T Y + Y^T A + Q - Y^T S Y \\ +\gamma^{-2} Y^T U X + \gamma^{-2} X^T U Y = 0, \tag{9b}$$

where

$$X_e = \Pi_e X = X^T \Pi_e,\ X_{ii} = X_{ii}^T,\ i = 0,\ 1,\ 2,$$

$$X = \begin{bmatrix} X_{00} & \varepsilon_1 X_{10}^T & \varepsilon_2 X_{20}^T \\ X_{10} & X_{11} & \sqrt{\alpha}^{-1} X_{21}^T \\ X_{20} & \sqrt{\alpha} X_{21} & X_{22} \end{bmatrix},$$

$$Y_e = \Pi_e Y = Y^T \Pi_e,\ Y_{ii} = Y_{ii}^T,\ i = 0,\ 1,\ 2,$$

$$Y = \begin{bmatrix} Y_{00} & \varepsilon_1 Y_{10}^T & \varepsilon_2 Y_{20}^T \\ Y_{10} & Y_{11} & \sqrt{\alpha}^{-1} Y_{21}^T \\ Y_{20} & \sqrt{\alpha} Y_{21} & Y_{22} \end{bmatrix}.$$

Proof: The proof is identical to the proof of Lemma 3 in [7]. ∎

After substituting X and Y into the GCMARE (9), we obtain the following equations as $\varepsilon_j \to +0,\ j = 1,\ 2$, where $\bar{X}_{lm},\ \bar{Y}_{lm},\ lm = 00,\ 10,\ 20,\ 11,\ 21,\ 22$ are the 0–order solutions of the GCMARE (9).

$$A^T \bar{X} + \bar{X}^T A + Q + \gamma^{-2} \bar{X}^T U \bar{X} \\ -\bar{X}^T S \bar{Y} - \bar{Y}^T S \bar{X} + \bar{Y}^T S \bar{Y} = 0, \tag{10a}$$

$$A^T \bar{Y} + \bar{Y}^T A + Q - \bar{Y}^T S \bar{Y} \\ +\gamma^{-2} \bar{Y}^T U \bar{X} + \gamma^{-2} \bar{X}^T U \bar{Y} = 0, \tag{10b}$$

where

$$\bar{X} = \begin{bmatrix} \bar{X}_{00} & 0 & 0 \\ \bar{X}_{10} & \bar{X}_{11} & 0 \\ \bar{X}_{20} & 0 & \bar{X}_{22} \end{bmatrix},\ \bar{Y} = \begin{bmatrix} \bar{Y}_{00} & 0 & 0 \\ \bar{Y}_{10} & \bar{Y}_{11} & 0 \\ \bar{Y}_{20} & 0 & \bar{Y}_{22} \end{bmatrix}.$$

The following theorem will establish the relation between the solutions X and Y and the solutions $\bar{X}$ and $\bar{Y}$ for the reduced–order equations (10).

Theorem 1 *Assume that*

$$\det \nabla \mathcal{F}(\bar{\mathcal{P}}) \neq 0, \tag{11}$$

where $\bar{X}_{21} = 0$, $\bar{Y}_{21} = 0$ and

$$\begin{aligned} \nabla \mathcal{F}(\mathcal{P}) &= \frac{\partial \mathrm{vec} \mathcal{F}(\mathcal{P})}{\partial (\mathrm{vec} \mathcal{P})^T} \\ &= [(\tilde{A} - \tilde{S}\mathcal{P} - \mathcal{J}\tilde{S}\mathcal{P}\mathcal{J})^T \otimes I_N] U_{2N2N} \\ &\quad + I_N \otimes (\tilde{A} - \tilde{S}\mathcal{P} - \mathcal{J}\tilde{S}\mathcal{P}\mathcal{J})^T \\ &\quad - [(\tilde{S}\mathcal{J}\mathcal{P} - \tilde{G}\mathcal{P}\mathcal{J})^T \otimes \mathcal{J}] U_{2N2N} \\ &\quad - \mathcal{J} \otimes (\tilde{S}\mathcal{J}\mathcal{P} - \tilde{G}\mathcal{P}\mathcal{J})^T, \end{aligned} \tag{12}$$

$$\mathcal{F}(\mathcal{P}) := \tilde{A}^T \mathcal{P} + \mathcal{P}^T \tilde{A} + \tilde{Q} - \mathcal{P}^T \tilde{S} \mathcal{P} \\ -\mathcal{J}\mathcal{P}^T \tilde{S} \mathcal{J}\mathcal{P} - \mathcal{P}^T \mathcal{J} \tilde{S} \mathcal{P} \mathcal{J} + \mathcal{J}\mathcal{P}^T \tilde{G} \mathcal{P} \mathcal{J},$$

$$\bar{\mathcal{P}} := \begin{bmatrix} \bar{X} & 0 \\ 0 & \bar{Y} \end{bmatrix},\ \mathcal{P} = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix},\ \tilde{A} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix},$$

$$\tilde{Q} = \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix},\ \tilde{S} = \begin{bmatrix} -\gamma^{-2} U & 0 \\ 0 & S \end{bmatrix},\ \tilde{G} = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix},$$

$$\mathcal{J} = \begin{bmatrix} 0 & I_N \\ I_N & 0 \end{bmatrix},\ N = n_0 + n_1 + n_2.$$

Under Assumptions 1 and 2, the GCMARE (9) *admits the solutions X and Y such that these matrices possess a power series expansion at $\|\mu\| = 0$. That is,*

$$X = \bar{X} + O(\|\mu\|), \tag{13a}$$

$$Y = \bar{Y} + O(\|\mu\|). \tag{13b}$$

Proof: We apply the implicit function theorem [6] to the GCMARE (9). To do so, it is enough to show that the corresponding Jacobian is nonsingular at $\|\mu\| = 0$. It can be shown, after some algebra, that the Jacobian of (9) in the limit as $\|\mu\| \to +0$ is given by

$$J_{\bar{\mathcal{P}}} = \lim_{\|\mu\| \to +0} \frac{\partial \mathrm{vec} \mathcal{F}(\mathcal{P})}{\partial (\mathrm{vec} \mathcal{P})^T} = \nabla \mathcal{F}(\bar{\mathcal{P}}). \tag{14}$$

Therefore, using the assumption (11), $J_{\bar{\mathcal{P}}}$ is nonsingular at $\|\mu\| = 0$. The conclusion of Theorem 1 is obtained directly by using the implicit function theorem. ∎

4 Newton's method

In order to improve the convergence rate of the Lyapunov iterations [2], we propose the following new algorithm which is based on the Newton's method [11].

$$\Phi^{(n)T} \mathcal{P}^{(n+1)} + \mathcal{P}^{(n+1)T} \Phi^{(n)} \\ -\Theta^{(n)T} \mathcal{P}^{(n+1)} \mathcal{J} - \mathcal{J} \mathcal{P}^{(n+1)T} \Theta^{(n)} + \Xi^{(n)} = 0, \tag{15}$$

$$n = 0,\ 1,\ \cdots,$$

where

$$\Phi^{(n)} := \tilde{A} - \tilde{S}\mathcal{P}^{(n)} - \mathcal{J}\tilde{S}\mathcal{P}^{(n)}\mathcal{J} = \begin{bmatrix} \Phi_1^{(n)} & 0 \\ 0 & \Phi_2^{(n)} \end{bmatrix},$$

$$\Theta^{(n)} := \tilde{S}\mathcal{J}\mathcal{P}^{(n)} - \tilde{G}\mathcal{P}^{(n)}\mathcal{J} = \begin{bmatrix} 0 & \Theta_1^{(n)} \\ \Theta_2^{(n)} & 0 \end{bmatrix},$$

$$\Xi^{(n)} := \tilde{Q} + \mathcal{P}^{(n)T}\tilde{S}\mathcal{P}^{(n)} + \mathcal{J}\mathcal{P}^{(n)T}\tilde{S}\mathcal{J}\mathcal{P}^{(n)} + \mathcal{P}^{(n)T}\mathcal{J}\tilde{S}\mathcal{P}^{(n)}\mathcal{J} - \mathcal{J}\mathcal{P}^{(n)T}\tilde{G}\mathcal{P}^{(n)}\mathcal{J} = \begin{bmatrix} \Xi_1^{(n)} & 0 \\ 0 & \Xi_2^{(n)} \end{bmatrix},$$

$$\Phi_i^{(n)} := \begin{bmatrix} \Phi_{00i}^{(n)} & \Phi_{01i}^{(n)} & \Phi_{02i}^{(n)} \\ \Phi_{10i}^{(n)} & \Phi_{11i}^{(n)} & \mu\Phi_{12i}^{(n)} \\ \Phi_{20i}^{(n)} & \mu\Phi_{21i}^{(n)} & \Phi_{22i}^{(n)} \end{bmatrix},$$

$$\Theta_i^{(n)} := \begin{bmatrix} \Theta_{00i}^{(n)} & \Theta_{01i}^{(n)} & \Theta_{02i}^{(n)} \\ \Theta_{10i}^{(n)} & \Theta_{11i}^{(n)} & \mu\Theta_{12i}^{(n)} \\ \Theta_{20i}^{(n)} & \mu\Theta_{21i}^{(n)} & \Theta_{22i}^{(n)} \end{bmatrix},$$

$$\Xi_i^{(n)} := \begin{bmatrix} \Xi_{00i}^{(n)} & \Xi_{01i}^{(n)} & \Xi_{02i}^{(n)} \\ \Xi_{01i}^{(n)T} & \Xi_{11i}^{(n)} & \mu\Xi_{21i}^{(n)T} \\ \Xi_{02i}^{(n)T} & \mu\Xi_{21i}^{(n)} & \Xi_{22i}^{(n)} \end{bmatrix}, \ i = 1,\ 2,$$

$$\mathcal{P}^{(n)} = \begin{bmatrix} X^{(n)} & 0 \\ 0 & Y^{(n)} \end{bmatrix},$$

$$X^{(n)} = \begin{bmatrix} X_{00}^{(n)} & \varepsilon_1 X_{10}^{(n)T} & \varepsilon_2 X_{20}^{(n)T} \\ X_{10}^{(n)} & X_{11}^{(n)} & \sqrt{\alpha}^{-1} X_{21}^{(n)T} \\ X_{20}^{(n)} & \sqrt{\alpha} X_{21}^{(n)} & X_{22}^{(n)} \end{bmatrix},$$

$$Y^{(n)} = \begin{bmatrix} Y_{00}^{(n)} & \varepsilon_1 Y_{10}^{(n)T} & \varepsilon_2 Y_{20}^{(n)T} \\ Y_{10}^{(n)} & Y_{11}^{(n)} & \sqrt{\alpha}^{-1} Y_{21}^{(n)T} \\ Y_{20}^{(n)} & \sqrt{\alpha} Y_{21}^{(n)} & Y_{22}^{(n)} \end{bmatrix},$$

and the initial condition $\mathcal{P}^{(0)}$ has the following form

$$\mathcal{P}^{(0)} = \begin{bmatrix} X^{(0)} & 0 \\ 0 & Y^{(0)} \end{bmatrix},$$

$$X^{(0)} = \begin{bmatrix} \bar{X}_{00} & \varepsilon_1 \bar{X}_{10}^T & \varepsilon_2 \bar{X}_{20}^T \\ \bar{X}_{10} & \bar{X}_{11} & 0 \\ \bar{X}_{20} & 0 & \bar{X}_{22} \end{bmatrix},$$

$$Y^{(0)} = \begin{bmatrix} \bar{Y}_{00} & \varepsilon_1 \bar{Y}_{10}^T & \varepsilon_2 \bar{Y}_{20}^T \\ \bar{Y}_{10} & \bar{Y}_{11} & 0 \\ \bar{Y}_{20} & 0 & \bar{Y}_{22} \end{bmatrix}.$$

The main result of this section is as follows.

Theorem 2 *Let us assume that the condition* (11) *holds. Under Assumptions 1 and 2, the new iterative algorithm* (15) *converges to the exact solution $\mathcal{P}^*$ of the GCMARE* (9) *with the rate of the quadratic convergence. Moreover, the unique bounded solution $\mathcal{P}^*$ of the GCMARE* (9) *is in the neighborhood of the matrix $\mathcal{P}^{(0)}$. That is, the following condition is satisfied.*

$$\|\mathcal{P}^{(n)} - \mathcal{P}^*\| \le O(\|\mu\|^{2^n}),\ n = 0,\ 1,\ \cdots, \quad (16)$$

where

$$\mathcal{P} = \mathcal{P}^* = \begin{bmatrix} X^* & 0 \\ 0 & Y^* \end{bmatrix},\ \mathcal{L} := 6\|\tilde{S}\| + 2\|\tilde{G}\|,$$

$$\beta := \|[\nabla\mathcal{F}(\mathcal{P}^{(0)})]^{-1}\|,\ \eta := \beta \cdot \|\mathcal{F}(\mathcal{P}^{(0)})\|,\ \theta := \beta\eta\mathcal{L}.$$

Proof: The proof is given directly by applying the Newton–Kantorovich theorem [11] for the GCMARE (9). Taking the partial derivative of the function $\mathcal{F}(\mathcal{P})$ with respect to $\mathcal{P}$ yields (12). It is obvious that $\nabla\mathcal{F}(\mathcal{P})$ is continuous at for all $\mathcal{P}$. Thus, it is immediately obtained from the equation (12) that

$$\|\nabla\mathcal{F}(\mathcal{P}_1) - \nabla\mathcal{F}(\mathcal{P}_2)\| \le \mathcal{L}\|\mathcal{P}_1 - \mathcal{P}_2\|. \quad (17)$$

Moreover, using the fact that

$$\nabla\mathcal{F}(\mathcal{P}^{(0)}) = \nabla\mathcal{F}(\bar{\mathcal{P}}) + O(\|\mu\|), \quad (18)$$

it follows that $\nabla\mathcal{F}(\mathcal{P}^{(0)})$ is nonsingular under the condition (11) for sufficiently small $\|\mu\|$. Therefore, there exists β such that $\beta = \|[\nabla\mathcal{F}(\mathcal{P}^{(0)})]^{-1}\|$. On the other hand, since $\mathcal{F}(\mathcal{P}^{(0)}) = O(\|\mu\|)$, there exists η such that $\eta = \|[\nabla\mathcal{F}(\mathcal{P}^{(0)})]^{-1}\| \cdot \|\mathcal{F}(\mathcal{P}^{(0)})\| = O(\|\mu\|)$. Thus, there exists θ such that $\theta = \beta\eta\mathcal{L} < 2^{-1}$ because of $\eta = O(\|\mu\|)$. Now, let us define

$$t^* \equiv \frac{1}{\beta\mathcal{L}}[1 - \sqrt{1 - 2\theta}]. \quad (19)$$

Using the Newton–Kantorovich theorem, we can show that $\mathcal{P}^*$ is the unique solution in the subset $\mathcal{S} \equiv \{\ \mathcal{P} : \|\mathcal{P}^{(0)} - \mathcal{P}\| \le t^*\ \}$. Moreover, using the Newton–Kantorovich theorem, the error estimate is given by

$$\|\mathcal{P}^{(n)} - \mathcal{P}^*\| \le \frac{(2\theta)^{2^n}}{2^n\beta\mathcal{L}},\ n = 1,\ \cdots. \quad (20)$$

Substituting $2\theta = O(\|\mu\|)$ into (20), we have (16). ∎

Remark 1 *It is well–known that the solution of the GCMARE* (9) *is not unique and several non–negative solutions exist. In this paper, it is very important to note that if the initial conditions $\Pi_e X^{(0)}$ and $\Pi_e Y^{(0)}$ are the positive semidefinite solutions, the new algorithm* (15) *converge to the required positive semidefinite solution in the same way as the Lyapunov iterations.*

5 High–Oreder Approximate H_2/H_∞ Control

In this section, the high–order approximate H_2/H_∞ control is given. Such control is obtained by using the iterative solution (15).

$$\bar{w}^{(n)} = \gamma^{-2}D^T X^{(n)}x,\ n = 0,\ 1,\ \cdots, \quad (21a)$$
$$\bar{u}^{(n)} = -B^T Y^{(n)}x,\ n = 0,\ 1,\ \cdots. \quad (21b)$$

Corollary 1 *Let us assume that the condition* (11) *holds and* $\mathbf{Re}\lambda[E^{-1}(A + \gamma^{-2}UX^{(0)} - SY^{(0)})] < 0$. *Under Assumptions 1 and 2, the following result holds.*

$$J_i(\bar{u}^{(n)},\ \bar{w}^{(n)}) = J_i(u^*,\ w^*) + O(\|\mu\|^{2^n}),\ i = 1,\ 2, \quad (22)$$

where $J_i(u^*,\ w^*)$, $i = 1,\ 2$ are the equilibrium values satisfying (5).

Proof: Since it is done by using the similar technique proposed in [4], it is omitted. ∎

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6 Example

In order to demonstrate the efficiency of our proposed algorithm, we have run a numerical example. The system matrix is given below [10]. It should be noted that the system matrix is given as a modification of practical systems (see e.g., Appendix A [4]) which is originated in the power plant.

$$A_{00} = \begin{bmatrix} 0 & 0 & 4.5 & 0 & 1 \\ 0 & 0 & 0 & 4.5 & -1 \\ 0 & 0 & -0.05 & 0 & -0.1 \\ 0 & 0 & 0 & -0.05 & 0.1 \\ 0 & 0 & 32.7 & -32.7 & 0 \end{bmatrix},$$

$$A_{01} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\ A_{02} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.4 & 0 & 0 \end{bmatrix},$$

$$A_{20} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.4 & 0 \end{bmatrix},$$

$$A_{11} = A_{22} = \begin{bmatrix} -0.05 & 0.05 \\ 0 & -0.1 \end{bmatrix},$$

$$D_{01} = D_{02} = B_{01} = B_{02} = \mathbf{0} \in \mathbf{R}^5,$$

$$D_{11} = D_{22} = \begin{bmatrix} 0 \\ 0.05 \end{bmatrix},\ B_{11} = B_{22} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix},$$

$$C = \begin{bmatrix} \mathbf{block-diag}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \end{pmatrix} \\ 0 \end{bmatrix} \in \mathbf{R}^{11\times 9},$$

$$H = \begin{bmatrix} 0 \\ \mathbf{block-diag}\begin{pmatrix} 1 & 1 \end{pmatrix} \end{bmatrix} \in \mathbf{R}^{11\times 2}.$$

The H_∞ control problem has been considered in [10]. In this paper, in order to guarantee the optimality for the cost function, the H_2/H_∞ control is applied to the MSPS (1). The numerical results are obtained for small parameter $\varepsilon_1 = \varepsilon_2 = 10^{-3}$. It is found that there exists the solution of the H_∞ control problem for all $\gamma \in \{\gamma \mid 0.501673 < \gamma\}$ via the MATLAB. Now, we choose as $\gamma = 1.0$ to solve the GCMARE (9). We give the initial condition (10) and solutions of the GCMARE (9) as the convergence solution $\mathcal{P}^{(5)}$, respectively.

We find that the solution of the GCMARE (9) converges to the exact solution with accuracy of $\|\mathcal{F}(\mathcal{P}^{(n)})\| < e-10$ after 5 iterative iterations. In order to verify the exactitude of the solution, we calculate the remainder per iteration by substituting $\mathcal{P}^{(n)}$ into the GCMARE (9) in Table 1, where we present results for the error $\|\mathcal{F}(\mathcal{P}^{(n)})\|$. It can be seen that the initial guess (10) for the algorithm (15) is quite good and the proposed algorithm has the quadratic convergence property. Table 2 shows the results of iterations for both the Lyapunov iterations and the proposed algorithm. On the other hand, in Table 3, we give the results of the CPU time when we have run the Lyapunov iterations versus Newton's method. The CPU time represents the average based on the computations of 10 runs. It is easy to verify that the convergence speed of the Lyapunov iterations is slow and needs more CPU times compared with the Newton's method. Therefore, the simulation results have been shown that the proposed algorithm succeed in improving the convergence rate dramatically and it does not need to much CPU time compared with the Lyapunov iterations.

Table 1. Errors per Iteration

i	$\|\mathcal{F}(\mathcal{P}^{(n)})\|$
0	$9.1844e-01$
1	$5.3443e-02$
2	$4.5728e-04$
3	$1.7249e-10$
4	$5.6420e-12$

Table 2. Number of iterations

ε	Lyapunov iterations	Newton's method
10^{-2}	29	6
10^{-3}	29	5
10^{-4}	29	4
10^{-5}	29	3
10^{-6}	29	3
10^{-7}	29	3
10^{-8}	29	2

Table 3. CPU time [s]

ε	Lyapunov iterations	Newton's method
10^{-2}	3.8047	3.2750
10^{-3}	3.7610	3.4187
10^{-4}	3.2421	2.8891
10^{-5}	3.6235	3.2343
10^{-6}	4.1844	3.8234
10^{-7}	3.5968	3.2079
10^{-8}	3.0469	2.6375

7 Conclusion

The mixed H_2/H_∞ control problem for infinite horizon MSPS have been studied. The new algorithm to solve the GCMARE which is based on the Newton's method has been newly proposed. Consequently, the resulting algorithm achieves the quadratic convergence. Moreover, it is very important to note that the resulting algorithm is quite different from the existing method [7]. Comparing with Lyapunov iterations [2], even if the singular perturbation parameter is extremely small, we have succeeded in improving the convergence rate dramatically.

It should be noted that the matrix computation of our algorithm needs two times dimension of the full–order CMARE compared with the Lyapunov iterations. Thus, it seems to be formidable for the proposed algorithm. This drawback must be avoided by all means because the MSPS includes the numerous fast subsystems. This problem will be addressed in future investigations.

$$X^{(0)} = \begin{bmatrix} \bar{X}_{00} & \varepsilon_1 \bar{X}_{10}^T & \varepsilon_2 \bar{X}_{20}^T \\ \bar{X}_{10} & \bar{X}_{11} & 0 \\ \bar{X}_{20} & 0 & \bar{X}_{22} \end{bmatrix}, \ \bar{X}_{00} = \begin{bmatrix} 3.7133e+00 & 3.6038e-01 & 3.2010e+01 & 6.4624e-14 & 2.4552e-01 \\ 3.6038e-01 & 3.7133e+00 & 2.0421e-13 & 3.2010e+01 & -2.4552e-01 \\ 3.2010e+01 & 1.3319e-13 & 6.5090e+02 & -2.0457e+02 & 4.8400e+00 \\ -2.8560e-13 & 3.2010e+01 & -2.0457e+02 & 6.5090e+02 & -4.8400e+00 \\ 2.4552e-01 & -2.4552e-01 & 4.8400e+00 & -4.8400e+00 & 1.7478e+00 \end{bmatrix},$$

$$\bar{X}_{10} = \begin{bmatrix} 5.2916e+01 & -9.0222e-14 & 1.0465e+03 & -3.3809e+02 & 7.9970e+00 \\ 1.4558e+01 & 8.3316e-14 & 2.5005e+02 & -9.2913e+01 & 2.1958e+00 \end{bmatrix},$$

$$\bar{X}_{20} = \begin{bmatrix} 9.1116e-14 & 5.2916e+01 & -3.3809e+02 & 1.0465e+03 & -7.9970e+00 \\ -1.2361e-13 & 1.4558e+01 & -9.2913e+01 & 2.5005e+02 & -2.1958e+00 \end{bmatrix},$$

$$\bar{X}_{11} = \begin{bmatrix} 3.6935e+01 & 7.9373e+00 \\ 7.9373e+00 & 1.6947e+01 \end{bmatrix}, \ \bar{X}_{22} = \begin{bmatrix} 3.6935e+01 & 7.9373e+00 \\ 7.9373e+00 & 1.6947e+01 \end{bmatrix},$$

$$Y^{(0)} = \begin{bmatrix} \bar{Y}_{00} & \varepsilon_1 \bar{Y}_{10}^T & \varepsilon_2 \bar{Y}_{20}^T \\ \bar{Y}_{10} & \bar{Y}_{11} & 0 \\ \bar{Y}_{20} & 0 & \bar{Y}_{22} \end{bmatrix}, \ \bar{Y}_{00} = \begin{bmatrix} 3.8150e+00 & 3.8110e-01 & 3.2877e+01 & 2.1161e-13 & 2.4431e-01 \\ 3.8110e-01 & 3.8150e+00 & 1.6920e-13 & 3.2877e+01 & -2.4431e-01 \\ 3.2877e+01 & 1.8630e-13 & 6.6051e+02 & -2.0697e+02 & 4.8362e+00 \\ -1.7764e-15 & 3.2877e+01 & -2.0697e+02 & 6.6051e+02 & -4.8362e+00 \\ 2.4431e-01 & -2.4431e-01 & 4.8362e+00 & -4.8362e+00 & 1.7547e+00 \end{bmatrix},$$

$$\bar{Y}_{10} = \begin{bmatrix} 5.5739e+01 & 1.4397e-13 & 1.0832e+03 & -3.5113e+02 & 8.2091e+00 \\ 1.7378e+01 & 8.6035e-14 & 2.9194e+02 & -1.0976e+02 & 2.5717e+00 \end{bmatrix},$$

$$\bar{Y}_{20} = \begin{bmatrix} 2.2926e-13 & 5.5739e+01 & -3.5113e+02 & 1.0832e+03 & -8.2091e+00 \\ -3.6731e-14 & 1.7378e+01 & -1.0976e+02 & 2.9194e+02 & -2.5717e+00 \end{bmatrix},$$

$$\bar{Y}_{11} = \begin{bmatrix} 3.7496e+01 & 9.0012e+00 \\ 9.0012e+00 & 1.9007e+01 \end{bmatrix}, \ \bar{Y}_{22} = \begin{bmatrix} 3.7496e+01 & 9.0012e+00 \\ 9.0012e+00 & 1.9007e+01 \end{bmatrix},$$

$$X^{(5)} = \begin{bmatrix} X_{00}^{(5)} & \varepsilon_1 X_{10}^{(5)T} & \varepsilon_2 X_{20}^{(5)T} \\ X_{10}^{(5)} & X_{11}^{(5)} & \sqrt{\alpha}^{-1} X_{21}^{(5)T} \\ X_{20}^{(5)} & \sqrt{\alpha} X_{21}^{(5)} & X_{22}^{(5)} \end{bmatrix}, \ X_{00}^{(5)} = \begin{bmatrix} 3.7318e+00 & 3.5834e-01 & 3.2414e+01 & -1.3395e-01 & 2.4760e-01 \\ 3.5834e-01 & 3.7318e+00 & -1.3395e-01 & 3.2414e+01 & -2.4760e-01 \\ 3.2414e+01 & -1.3395e-01 & 6.6025e+02 & -2.1004e+02 & 4.9057e+00 \\ -1.3395e-01 & 3.2414e+01 & -2.1004e+02 & 6.6025e+02 & -4.9057e+00 \\ 2.4760e-01 & -2.4760e-01 & 4.9057e+00 & -4.9057e+00 & 1.7472e+00 \end{bmatrix},$$

$$X_{10}^{(5)} = \begin{bmatrix} 5.3086e+01 & -5.6524e-02 & 1.0556e+03 & -3.4322e+02 & 6.6619e+00 \\ 1.4558e+01 & -3.5609e-14 & 2.5196e+02 & -9.3934e+01 & 1.7101e+00 \end{bmatrix},$$

$$X_{20}^{(5)} = \begin{bmatrix} -5.6524e-02 & 5.3086e+01 & -3.4322e+02 & 1.0556e+03 & -6.6619e+00 \\ -1.0252e-14 & 1.4558e+01 & -9.3934e+01 & 2.5196e+02 & -1.7101e+00 \end{bmatrix},$$

$$X_{11}^{(5)} = \begin{bmatrix} 3.8681e+01 & 8.4009e+00 \\ 8.4009e+00 & 1.7076e+01 \end{bmatrix}, \ X_{22}^{(5)} = \begin{bmatrix} 3.8681e+01 & 8.4009e+00 \\ 8.4009e+00 & 1.7076e+01 \end{bmatrix}, \ X_{21}^{(5)} = \begin{bmatrix} -5.6223e-01 & -1.5430e-01 \\ -1.5430e-01 & -4.2769e-02 \end{bmatrix},$$

$$Y^{(5)} = \begin{bmatrix} Y_{00}^{(5)} & \varepsilon_1 Y_{10}^{(5)T} & \varepsilon_2 Y_{20}^{(5)T} \\ Y_{10}^{(5)} & Y_{11}^{(5)} & \sqrt{\alpha}^{-1} Y_{21}^{(5)T} \\ Y_{20}^{(5)} & \sqrt{\alpha} Y_{21}^{(5)} & Y_{22}^{(5)} \end{bmatrix}, \ Y_{00}^{(5)} = \begin{bmatrix} 3.8344e+00 & 3.7844e-01 & 3.3299e+01 & -1.4166e-01 & 2.4625e-01 \\ 3.7844e-01 & 3.8344e+00 & -1.4166e-01 & 3.3299e+01 & -2.4625e-01 \\ 3.3299e+01 & -1.4166e-01 & 6.7015e+02 & -2.1260e+02 & 4.8999e+00 \\ -1.4166e-01 & 3.3299e+01 & -2.1260e+02 & 6.7015e+02 & -4.8999e+00 \\ 2.4625e-01 & -2.4625e-01 & 4.8999e+00 & -4.8999e+00 & 1.7536e+00 \end{bmatrix},$$

$$Y_{10}^{(5)} = \begin{bmatrix} 5.5945e+01 & -6.8681e-02 & 1.0931e+03 & -3.5668e+02 & 6.8048e+00 \\ 1.7378e+01 & -2.8948e-14 & 2.9421e+02 & -1.1096e+02 & 1.9823e+00 \end{bmatrix},$$

$$Y_{20}^{(5)} = \begin{bmatrix} -6.8681e-02 & 5.5945e+01 & -3.5668e+02 & 1.0931e+03 & -6.8048e+00 \\ 1.3661e-14 & 1.7378e+01 & -1.1096e+02 & 2.9421e+02 & -1.9823e+00 \end{bmatrix},$$

$$Y_{11}^{(5)} = \begin{bmatrix} 3.9351e+01 & 9.5556e+00 \\ 9.5556e+00 & 1.9175e+01 \end{bmatrix}, \ Y_{22}^{(5)} = \begin{bmatrix} 3.9351e+01 & 9.5556e+00 \\ 9.5556e+00 & 1.9175e+01 \end{bmatrix}, \ Y_{21}^{(5)} = \begin{bmatrix} -5.9900e-01 & -1.8561e-01 \\ -1.8561e-01 & -5.6336e-02 \end{bmatrix}.$$

References

[1] D. J. N. Limebeer, B. D. O. Anderson and B. Hendel, "A Nash game approach to mixed H_2/H_∞ control," *IEEE Trans. A.C.*, Vol.39, pp.69–82, 1994.

[2] T. Li and Z. Gajić, "Lyapunov iterations for solving coupled algebraic Lyapunov equations of Nash differential games and algebraic Riccati equations of Zero–sum games," *New Trends in Dynamic Games and Applications*, Birkhauser, pp.333–351, 1994.

[3] G. Freiling, G. Jank and H. Abou–Kandil, "On global existence of solutions to coupled matrix Riccati equations in closed–loop Nash games," *IEEE Trans. A.C.*, Vol.41, pp.264–269, 1996.

[4] H. K. Khalil, "Multimodel design of a Nash strategy," *J. Opt. Theory and Appl.*, Vol31, pp.553–564, 1980.

[5] H. K. Khalil and P. V. Kokotovic, "Control of linear systems with multiparameter singular perturbations," *Automatica*, Vol.15, pp.197–207, 1979.

[6] Z. Gajić, D. Petkovski and X. Shen, *Singularly Perturbed and Weakly Coupled Linear System-a Recursive Approach, Lecture Notes in Control and Information Sciences*, Vol.140, Springer–Verlag, 1990.

[7] H. Mukaidani, H. Xu and K. Mizukami, "A new algorithm for solving cross–coupled algebraic Riccati equations of singularly perturbed Nash games," in *Proc. 39th IEEE CDC*, pp.3648–3653, 2000.

[8] H. Mukaidani, H. Xu and K. Mizukami, "Recursive algorithm for mixed H_2/H_∞ control problem of singularly perturbed systems," *Int. J. Systems Sciences*, Vol.31, pp.1299–1312, 2000.

[9] H. Mukaidani, H. Xu and K. Mizukami, "A new algorithm for solving cross–coupled algebraic Riccati equations of singularly perturbed systems for mixed H_2/H_∞ control problem," in *Proc. 9th Int. Symp. Dynamic Games and Applications*, pp.365–374, 2000.

[10] H. Mukaidani, T. Shimomura and K. Mizukami, Asymptotic expansions and new numerical algorithm of the algebraic Riccati equation for multiparameter singularly perturbed systems, *J. Math. Analysis and Applications*, vol.267, no.1, pp.209–234, 2002.

[11] T. Yamamoto, "A method for finding sharp error bounds for Newton's method under the Kantorvich assumptions," *Numerische Mathematik*, vol.49, pp.203–220, 1986.

[12] J. R. Magnus and H. Neudecker, *Matrix differential calculus with applications in statistics and econometrics*, John Wiley and Sons, 1999.