Sub–Optimal Kalman Filter for Multimodeling Systems

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Abstract

In this paper, we study the optimal Kalman filtering problem for multiparameter singularly perturbed system (MSPS). The attention is focused on the design of the high–order approximate Kalman filters. It is shown that the resulting filters in fact remove ill–conditioning of the original full–order singularly perturbed Kalman filters. In addition the resulting filters can be used compared with the previously proposed result even if the Hamiltonian matrices for the fast subsystems have eigenvalues in common.

1 Introduction

Recently, filtering problems for the multiparameter singularly perturbed system (MSPS) have been investigated [3, 4, 10, 12]. Such problems arise in large scale dynamic systems. For example, the MSPS in practice is illustrated by the passenger car model [4]. A popular approach to deal with the filtering problem for the MSPS is the two–time–scale design method [1, 2]. However, it is well–known that an $O(\|\mu\|)$ (where $\mu = [\ \varepsilon_1 \quad \varepsilon_2\ ]$) accuracy is very often not sufficient [4].

In order to obtain the optimal solution of the filtering problems, we must solve the multiparameter algebraic Riccati equation (MARE), which are parameterized by small positive same order parameters ε_1 and ε_2. Various reliable approaches to the theory of the algebraic Riccati equation (ARE) have been well documented in many literatures (see e.g., [5, 6]). One of the approaches is the invariant subspace approach which is based on the Hamiltonian matrix [5]. However, such an approach is not adequate to the MSPS since for the computed solution there is no guarantee of symmetry when the ARE is ill–conditioned [5]. In order to avoid the numerical stiffness, the recursive algorithms for solving the MARE and the generalized multiparameter algebraic Lyapunov equation (GMALE) have been developed [10]. However, there exists the drawback that the recursive algorithm converges only to the approximation solution since the convergence of the recursive algorithm depend on the zero–order solutions. On the other hand, the exact slow–fast decomposition method for solving the MARE has been proposed in [4]. However, the resulting algorithm is restricted to the MSPS such that the Hamiltonian matrices for the fast subsystems have no eigenvalues in common (Assumption 5, [4]). Thus, we can not apply the technique proposed in [4] to the practical system, such as the Pareto optimal strategy of a multi–area power system [1]. Furthermore, so far, the loss of steady–state mean square error between the optimal filter and the resulting filter which is based on the exact decomposition technique has not been investigated.

In this paper, we study the optimal Kalman filtering problem for the MSPS. The results obtained are valid for steady state. We first investigate the uniqueness and boundedness of the solution to such MARE and establish its asymptotic structure. The proof of the existence of the solution to the MARE with asymptotic expansion is obtained by an implicit function theorem [2]. The main contribution of this paper is to propose the high–order approximate Kalman filters. Furthermore, we claim that the proposed filters can be constructed even if the Hamiltonian matrices for the fast subsystems have eigenvalues in common compared with the previous result [4]. Therefore, our proposed algorithm is extremely useful since the proposed algorithm apply to more realistic MSPS. As another important feature, it is shown that the high–order approximate Kalman filter achieves a performance which is $\dfrac{O(\|\mu\|^{2^{i+1}-1})}{2^i}$ close to the optimal mean square error. It is worth pointing out that the feature of the $\dfrac{O(\|\mu\|^{2^{i+1}-1})}{2^i}$ sub–optimality is established for the first time for the optimal filtering problem of the MSPS [4].

Notation: The superscript T denotes matrix transpose. $\det L$ denotes the determinant of square matrix L. I_n denotes the $n \times n$ identity matrix. $\| \cdot \|$ denotes its Euclidean norm for a matrix. $\mathbf{block-diag}$ denotes the block diagonal matrix. $\mathrm{vec}M$ denotes the column vector of the matrix M [7]. $\otimes$ denotes the Kronecker product. U_{lm} denotes a permutation matrix in the Kronecker matrix sense [7] such that $U_{lm}\mathrm{vec}M = \mathrm{vec}M^T,\ M \in \mathbf{R}^{l \times m}$.

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2 Optimal Kalman Filtering Problem

We consider the linear time–invariant MSPS

$$\dot{x}_0 = A_{00}x_0 + \sum_{j=1}^{2} A_{0j}x_j + \sum_{j=1}^{2} D_{0j}w_j, \quad (1a)$$

$$\varepsilon_i\dot{x}_i = A_{i0}x_0 + A_{ii}x_i + D_{ii}w_i,\ i = 1,\ 2, \quad (1b)$$

with the corresponding measurements

$$y_i = C_{i0}x_0 + C_{ii}x_i + v_i,\ i = 1,\ 2, \quad (2)$$

where $x_i \in \mathbf{R}^{n_i}$, $i = 0,\ 1,\ 2$ are state vectors, $y_i \in \mathbf{R}^{p_i}$, $i = 1,\ 2$ are system measurements, $w_i \in \mathbf{R}^{q_i}$, $i = 1,\ 2$ and $v_i \in \mathbf{R}^{r_i}$, $i = 1,\ 2$ are zero–mean stationary, Gaussian, mutually uncorrelated, white noise stochastic processes with intensities $W_i \geq 0$ and $V_i > 0$, respectively. All the matrices are constant matrices of appropriate dimensions.

$\varepsilon_1, \varepsilon_2$ are the small positive singular parameters of the same order of magnitude [1] such that

$$0 < \underline{k} \leq \alpha := \frac{\varepsilon_1}{\varepsilon_2} \leq \bar{k} < \infty. \quad (3)$$

That is, we assume that the ratio of ε_1 and ε_2 is bounded by some positive constants. In this paper we design the high–order approximate Kalman filter to estimate system states x_i. The optimal Kalman filter of (1) and (2) is given by [4]

$$\dot{\xi}_0 = A_{00}\xi_0 + \sum_{j=1}^{2} A_{0j}\xi_j + \sum_{j=1}^{2} K_{0j}\nu_j, \quad (4a)$$

$$\varepsilon_i\dot{\xi}_i = A_{i0}\xi_0 + A_{ii}\xi_i + \sum_{j=1}^{2} K_{ij}\nu_j, \quad (4b)$$

$$\nu_i = y_i - C_{i0}\xi_0 - C_{ii}\xi_i,\ i = 1,\ 2, \quad (4c)$$

where the filter gains K_{ij} are obtained from

$$\begin{aligned} K_e &= X_eC^TV^{-1} = \Phi_e^{-1}K^{\mathrm{opt}} = \Phi_e^{-1}XC^TV^{-1} \\ &= \Phi_e^{-1}\begin{bmatrix} K_{01} & K_{02} \\ K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}, \quad (5) \\ \Phi_e &:= \begin{bmatrix} I_{n_0} & 0 \\ 0 & \Pi_e \end{bmatrix}, \\ \Pi_e &:= \mathbf{block-diag}\left(\varepsilon_1 I_{n_1} \quad \varepsilon_2 I_{n_2} \right). \end{aligned}$$

The matrix X_e is the positive semidefinite stabilizing solution of the following MARE

$$A_eX_e + X_eA_e^T - X_eSX_e + U_e = 0, \quad (6)$$

where

$$A_e := \begin{bmatrix} A_{00} & A_{0f} \\ \Pi_e^{-1}A_{f0} & \Pi_e^{-1}A_f \end{bmatrix},\ A_{0f} := \begin{bmatrix} A_{01} & A_{02} \end{bmatrix},$$

$$A_{f0} := \begin{bmatrix} A_{10}^T & A_{20}^T \end{bmatrix}^T,$$

$$A_f := \mathbf{block-diag}\left(A_{11} \quad A_{22} \right),$$

$$C := \begin{bmatrix} C_0 & C_f \end{bmatrix},\ C_0 := \begin{bmatrix} C_{10}^T & C_{20}^T \end{bmatrix}^T,$$

$$C_f := \mathbf{block-diag}\left(C_{11} \quad C_{22} \right),$$

$$D_e := \begin{bmatrix} D_0 \\ \Pi_e^{-1}D_f \end{bmatrix},\ D_0 := \begin{bmatrix} D_{01} & D_{02} \end{bmatrix},$$

$$D_f := \mathbf{block-diag}\left(D_{11} \quad D_{22} \right),$$

$$W := \mathbf{block-diag}\left(W_1 \quad W_2 \right),$$

$$V := \mathbf{block-diag}\left(V_1 \quad V_2 \right),$$

$$S := C^TV^{-1}C = \begin{bmatrix} S_{00} & S_{0f} \\ S_{0f}^T & S_f \end{bmatrix},$$

$$S_{00} := \sum_{j=1}^{2} C_{j0}^TV_j^{-1}C_{j0},$$

$$\begin{aligned} S_{0f} &:= \begin{bmatrix} S_{01} & S_{02} \end{bmatrix} \\ &= \begin{bmatrix} C_{10}^TV_1^{-1}C_{11} & C_{20}^TV_2^{-1}C_{22} \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} S_f &:= \mathrm{block-diag}\left(S_{11} \quad S_{22} \right) \\ &= \mathbf{block-diag}\left(C_{11}^TV_1^{-1}C_{11} \quad C_{22}^TV_2^{-1}C_{22} \right), \end{aligned}$$

$$U_e := D_eWD_e^T = \begin{bmatrix} U_{00} & U_{0f}\Pi_e^{-1} \\ \Pi_e^{-1}U_{0f}^T & \Pi_e^{-1}U_f\Pi_e^{-1} \end{bmatrix},$$

$$U_{00} := \sum_{j=1}^{2} D_{0j}W_jD_{0j}^T,$$

$$U_{0f} := \begin{bmatrix} U_{01} & U_{02} \end{bmatrix} = \begin{bmatrix} D_{01}W_1D_{11}^T & D_{02}W_2D_{22}^T \end{bmatrix},$$

$$\begin{aligned} U_f &:= \mathrm{block-diag}\left(U_{11} \quad U_{22} \right) \\ &= \mathbf{block-diag}\left(D_{11}W_1D_{11}^T \quad D_{22}W_2D_{22}^T \right). \end{aligned}$$

Since the matrices A_e and D_e contain the term of ε_i^{-1}–order, a solution X_e of the MARE (6), if it exists, must contain terms of order ε_i. Taking this fact into consideration, we look for a solution X_e to the MARE (6) with the structure

$$\begin{aligned} X_e &:= \begin{bmatrix} X_{00} & X_{0f} \\ X_{0f}^T & \Pi_e^{-1}X_f \end{bmatrix}, \\ X_{00} &= X_{00}^T,\ \Pi_e^{-1}X_f = X_f^T\Pi_e^{-1}, \\ X_{0f} &:= \begin{bmatrix} X_{01}^T \\ X_{02}^T \end{bmatrix}^T,\ X_f := \begin{bmatrix} X_{11} & \sqrt{\alpha}X_{12} \\ \frac{1}{\sqrt{\alpha}}X_{12}^T & X_{22} \end{bmatrix}. \end{aligned}$$

In the following analysis, we need some assumptions.

Assumption 1 *The triples* $(A_{ii}^T,\ C_{ii}^T,\ D_{ii}^T)$, $i = 1,\ 2$ *are stabilizable and detectable.*

Assumption 2

$$\mathrm{rank}\begin{bmatrix} sI_{n_0} - A_{00}^T & -A_{f0}^T & C_0^T \\ -A_{0f}^T & -A_f^T & C_f^T \end{bmatrix} = \bar{n}, \quad (7a)$$

$$\mathrm{rank}\begin{bmatrix} sI_{n_0} - A_{00} & -A_{0f} & D_0 \\ -A_{f0} & -A_f & D_f \end{bmatrix} = \bar{n}, \quad (7b)$$

with $\forall s \in \mathbf{C}$, $\mathrm{Re}[s] \geq 0$ *and* $\bar{n} := \sum_{j=0}^{2} n_j$.

Assumption 3 *The Hamiltonian matrices* $T_{ii},\ i=1,\ 2$ *are nonsingular, where*

$$T_{ii} := \begin{bmatrix} A_{ii}^T & -S_{ii} \\ -U_{ii} & -A_{ii} \end{bmatrix}.$$

First, we investigate the asymptotic structure of the MARE (6). In order to avoid the ill–conditioned caused by the large parameter ε_i^{-1} which is included in the MARE (6), we introduce the following useful lemma [10, 12].

Lemma 1 *The MARE* (6) *is equivalent to the following generalized multiparameter algebraic Riccati equation (GMARE)* (8)

$$\mathcal{F}(X) = AX^T + XA^T - XSX^T + U = 0, \qquad (8)$$

where $A = \Phi_e A_e,\ U = \Phi_e U_e \Phi_e$ *and* $X = \Phi_e X_e$.

The GMARE (8) can be partitioned into

$$\begin{aligned} f_1 &= A_{00}X_{00} + X_{00}A_{00}^T + A_{0f}X_{0f}^T + X_{0f}A_{0f}^T \\ &\quad -X_{00}S_{00}X_{00} - X_{0f}S_fX_{0f}^T \\ &\quad -X_{00}S_{0f}X_{0f}^T - X_{0f}S_{0f}^TX_{00} + U_{00} = 0, \qquad (9a) \\ f_2 &= A_{0f}X_f^T + A_{00}X_{0f}\Pi_e + X_{00}A_{f0}^T + X_{0f}A_f^T \\ &\quad -X_{00}S_{00}X_{0f}\Pi_e - X_{0f}S_{0f}^TX_{0f}\Pi_e \\ &\quad -X_{00}S_{0f}X_f^T - X_{0f}S_fX_f^T + U_{0f} = 0, \qquad (9b) \\ f_3 &= A_fX_f^T + X_fA_f^T + A_{f0}X_{0f}\Pi_e + \Pi_eX_{0f}^TA_{f0}^T \\ &\quad -X_fS_fX_f^T - \Pi_eX_{0f}^TS_{0f}X_f^T - X_fS_{0f}^TX_{0f}\Pi_e \\ &\quad -\Pi_eX_{0f}^TS_{00}X_{0f}\Pi_e + U_f = 0. \qquad (9c) \end{aligned}$$

It is assumed that the limit of α exists as ε_1 and ε_2 tend to zero (see e.g., [1, 2]), that is

$$\bar{\alpha} = \lim_{\substack{\varepsilon_1 \to +0 \\ \varepsilon_2 \to +0}} \alpha. \qquad (10)$$

Let $\bar{X}_{00}$, $\bar{X}_{f0}$ and $\bar{X}_f$ be the limiting solutions of the above equation (9) as $\varepsilon_1 \to +0,\ \varepsilon_2 \to +0$, then we obtain the following equations

$$\begin{aligned} &A_{00}\bar{X}_{00} + \bar{X}_{00}A_{00}^T + A_{0f}\bar{X}_{0f}^T + \bar{X}_{0f}A_{0f}^T \\ &\quad -\bar{X}_{00}S_{00}\bar{X}_{00} - \bar{X}_{0f}S_f\bar{X}_{0f}^T \\ &\quad -\bar{X}_{00}S_{0f}\bar{X}_{0f}^T - \bar{X}_{0f}S_{0f}^T\bar{X}_{00} + U_{00} = 0, \qquad (11a) \\ &A_{0f}\bar{X}_f^T + \bar{X}_{00}A_{f0}^T + \bar{X}_{0f}A_f \\ &\quad -\bar{X}_{00}S_{0f}\bar{X}_f^T - \bar{X}_{0f}S_f\bar{X}_f^T + U_{0f} = 0, \qquad (11b) \\ &A_f\bar{X}_f^T + \bar{X}_fA_f^T + A_f^T\bar{X}_f - \bar{X}_fS_f\bar{X}_f^T + U_f = 0, \qquad (11c) \end{aligned}$$

where

$$\begin{aligned} \bar{X}_f &:= \begin{bmatrix} \bar{X}_{11} & \sqrt{\bar{\alpha}}\bar{X}_{12} \\ \dfrac{1}{\sqrt{\bar{\alpha}}}\bar{X}_{12}^T & \bar{X}_{22} \end{bmatrix}, \qquad (12) \\ \bar{X}_{ii} &= \bar{X}_{ii}^T,\ i = 0,\ 1,\ 2. \end{aligned}$$

Note that the ARE (11c) is asymmetric. However, it can be seen that the ARE (11c) admits at least a symmetric positive semidefinite stabilizing solution as follows [12].

Lemma 2 *Under Assumption 1, the ARE* (11c) *admits a unique symmetric positive semidefinite stabilizing solution* $\bar{X}_f$ *which can be written as*

$$\bar{X}_f^* := \mathbf{block-diag}\begin{pmatrix} \bar{X}_{11}^* & \bar{X}_{22}^* \end{pmatrix}, \qquad (13)$$

where $\bar{X}_{ii}^*$ *is a unique symmetric positive semidefinite stabilizing solution of the following AREs respectively*

$$A_{ii}\bar{X}_{ii}^* + \bar{X}_{ii}^*A_{ii}^T - \bar{X}_{ii}^*S_{ii}\bar{X}_{ii}^* + U_{ii} = 0,\ i = 1,\ 2. \qquad (14)$$

Substituting the solution of (11c) into (11b) and substituting $\bar{X}_{0f}^*$ into (11a) and making some lengthy calculations (the detail is omitted for brevity), we obtain the following 0–order equations (15)

$$\mathcal{A}\bar{X}_{00}^* + \bar{X}_{00}^*\mathcal{A}^T - \bar{X}_{00}^*\mathcal{S}\bar{X}_{00}^* + \mathcal{U} = 0, \qquad (15a)$$

$$\bar{X}_{0f}^* = \begin{bmatrix} -\bar{X}_{00} & I_{n_0} \end{bmatrix} H_2H_4^{-1} \begin{bmatrix} I_{\bar{n}} \\ \bar{X}_f \end{bmatrix} \qquad (15b)$$

$$A_f\bar{X}_f^* + \bar{X}_f^*A_f^T - \bar{X}_f^*S_f\bar{X}_f^* + U_f = 0, \qquad (15c)$$

where

$$H_1 := \begin{bmatrix} A_{00}^T & -S_{00} \\ -U_{00} & -A_{00} \end{bmatrix},\ H_2 := \begin{bmatrix} A_{f0}^T & -S_{0f} \\ -U_{0f} & -A_{0f} \end{bmatrix},$$

$$H_3 := \begin{bmatrix} A_{0f}^T & -S_{0f}^T \\ -U_{0f}^T & -A_{f0} \end{bmatrix},\ H_4 := \begin{bmatrix} A_f^T & -S_f \\ -U_f & -A_f \end{bmatrix},$$

$$H_0 := \begin{bmatrix} \mathcal{A}^T & -\mathcal{S} \\ -\mathcal{U} & -\mathcal{A} \end{bmatrix} = H_1 - H_2H_4^{-1}H_3.$$

Note that Assumption 1 ensures that the matrix $A_f - \bar{X}_f^*S_f$ is nonsingular because the matrices $A_{ii} - \bar{X}_{ii}^*S_{ii},\ i = 1,\ 2$ are nonsingular. Moreover, Assumption 3 ensures that H_4 are also nonsingular because $\Omega^TH_4\Omega = \mathbf{block-diag}\begin{pmatrix} T_{11} & T_{22} \end{pmatrix}$, where

$$\Omega = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_2} & 0 \\ 0 & I_{n_1} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix}. \qquad (16)$$

In the following we established the relation between the GMARE (8) and the 0–order equations (15). Before doing that, we give the results for the ARE (15a) [12].

Lemma 3 *Under Assumptions 1–3, there exist a matrix* $\mathcal{C} \in \mathbf{R}^{\bar{p}\times n_0},\ \bar{p} := p_1 + p_2$ *and a matrix* $\mathcal{D} \in \mathbf{R}^{n_0\times\bar{q}},\ \bar{q} := q_1 + q_2$ *such that* $\mathcal{S} = \mathcal{C}^TV^{-1}\mathcal{C},\ \mathcal{U} = \mathcal{D}W\mathcal{D}^T$. *Moreover, the triple* $(\mathcal{A}^T,\ \mathcal{C}^T,\ \mathcal{D}^T)$ *is stabilizable and detectable.*

Since the triple $(\mathcal{A}^T,\ \mathcal{C}^T,\ \mathcal{D}^T)$ is stabilizable and detectable, the ARE (15a) admits a unique stabilizing positive semidefinite symmetric solution, denoted by $\bar{X}_{00}^*$, and $\mathcal{A} - \bar{X}_{00}^*\mathcal{S}$ is stable.

The limiting behavior of X_e as the parameter $\|\mu\| = \sqrt{\varepsilon_1\varepsilon_2} \to +0$ is described by the following theorem.

Proceedings of the American Control Conference
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Theorem 1 *Under Assumptions 1–3, there exists a small σ^* such that for all $\|\mu\| \in (0,\ \sigma^*)$ the MARE* (6) *admits a symmetric positive semidefinite stabilizing solution X_e which can be written as*

$$X_e = \Phi_e^{-1} \begin{bmatrix} \bar{X}_{00}^* + O(\|\mu\|) & \bar{X}_{0f}^* + O(\|\mu\|) \\ \Pi_e\{\bar{X}_{0f}^* + O(\|\mu\|)\}^T & \bar{X}_f^* + O(\|\mu\|) \end{bmatrix}$$
$$= \begin{bmatrix} \bar{X}_{00}^* + O(\|\mu\|) & \bar{X}_{0f}^* + O(\|\mu\|) \\ \{\bar{X}_{0f}^* + O(\|\mu\|)\}^T & \Pi_e^{-1}\{\bar{X}_f^* + O(\|\mu\|)\} \end{bmatrix}. \quad (17)$$

Proof: We apply the implicit function theorem [2] to (9). To do so, it is enough to show that the corresponding Jacobian is nonsingular at $\|\mu\| = 0$. It can be shown, after some algebra, that the Jacobian of (9) in the limit as $\|\mu\| \to 0$ is given by

$$\begin{aligned} \mathbf{J} &= \nabla\mathcal{F} = \frac{\partial \text{vec}(f_1,\ f_2,\ f_3)}{\partial \text{vec}(X_{00},\ X_{0f},\ X_f)^T}\bigg|_\phi \\ &= \begin{bmatrix} \mathbf{J}_{00} & \mathbf{J}_{01} & 0 \\ \mathbf{J}_{10} & \mathbf{J}_{11} & \mathbf{J}_{12} \\ 0 & 0 & \mathbf{J}_{22} \end{bmatrix}, \quad (18) \\ \phi &= (\|\mu\| = 0,\ X_{00} = \bar{X}_{00}^*,\ X_{0f} = \bar{X}_{0f}^*,\ X_f = \bar{X}_f^*), \end{aligned}$$

where

$$\begin{aligned} &\mathbf{J}_{00} = \Gamma_1 \otimes I_{n_0} + I_{n_0} \otimes \Gamma_1, \\ &\mathbf{J}_{01} = \Gamma_3 \otimes I_{n_0} + (I_{n_0} \otimes \Gamma_3)U_{n_0\hat{n}}, \\ &\mathbf{J}_{10} = \Gamma_2 \otimes I_{n_0},\ \mathbf{J}_{11} = \Gamma_4 \otimes I_{n_0}, \\ &\mathbf{J}_{12} = (I_{\hat{n}} \otimes \Gamma_3)U_{\hat{n}\hat{n}},\ \mathbf{J}_{22} = \Gamma_4 \otimes I_{\hat{n}} + I_{\hat{n}} \otimes \Gamma_4, \\ &\Gamma_1 = A_{00} - \bar{X}_{00}^* S_{00} - \bar{X}_{0f}^* S_{0f}^T, \\ &\Gamma_3 = A_{0f} - \bar{X}_{00}^* S_{0f} - \bar{X}_{0f}^* S_f. \end{aligned}$$

The Jacobian (18) can be expressed as

$$\det\mathbf{J} = \det\mathbf{J}_{22} \cdot \det\mathbf{J}_{11} \cdot \det[\Gamma_0 \otimes I_{n_0} + I_{n_0} \otimes \Gamma_0], \quad (19)$$

where $\Gamma_0 := \Gamma_1 - \Gamma_2\Gamma_4^{-1}\Gamma_3$. Obviously, $\mathbf{J}_{ii},\ j = 1,\ 2$ are nonsingular because the matrix $\Gamma_4 = A_f - \bar{X}_f^* S_f$ is stable under Assumption 1. After some straightforward but tedious algebra, we see that $\mathcal{A} - \bar{X}_{00}^*\mathcal{S} = \Gamma_1 - \Gamma_2\Gamma_4^{-1}\Gamma_3 = \Gamma_0$. Therefore, the matrix Γ_0 is also stable if Assumption 2 holds. Thus, $\det\mathbf{J} \neq 0$, i.e., $\mathbf{J}$ is nonsingular at $\|\mu\| = 0$. The conclusion of Theorem 2 is obtained directly by using the implicit function theorem.

The remainder of the proof is to show that X_e is the positive semidefinite stabilizing solution. Firstly, from the matrix (17), we get

$$\begin{aligned} z^T X_e z &= \begin{bmatrix} z_1^T & z_2^T \end{bmatrix} \begin{bmatrix} \bar{X}_{00}^* & \bar{X}_{0f}^* \\ \bar{X}_{0f}^{*T} & \Pi_e^{-1}\bar{X}_f^* \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ &= \Pi_e^{-1}\{z_2^T \bar{X}_f^* z_2 + O(\|\mu\|)\}, \end{aligned}$$

where $z^T = \begin{bmatrix} z_1^T & z_2^T \end{bmatrix} \in \mathbf{R}^{\bar{n}}$. Therefore, $X_e \geq 0$ for small $\|\mu\|$ because the matrix $\bar{X}_f^*$ is the positive semidefinite stabilizing solution of the ARE (15c). Secondly, using (17), we obtain

$$A_e - X_e S = \Phi_e^{-1}\left(\begin{bmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_3 & \Gamma_4 \end{bmatrix} + O(\|\mu\|)\right).$$

Taking into account the fact that Γ_4 and Γ_0 are stable matrix, if the parameter $\|\mu\|$ is very small, $A_e - X_e S$ is stable by applying the Theorem 1 in [1]. ■

3 High–order Approximate Kalman Filters for the Nonstandard MSPS

The required solution of the MARE (6) exists under Assumptions 1–3. Our attention is focused on the design of the high–order approximate Kalman filters. Such filters are obtained by performing the algorithm which is based on the Kleinman algorithm [6]. If $\|\mu\|$ is very small, it is obvious that the high–order approximate Kalman filter gain (5) can be obtained as

$$K_{\text{app}}^{(i)} = X^{(i)} C^T V^{-1}, \quad (20)$$

where

$$\begin{aligned} &(A - X^{(i)}S)X^{(i+1)T} + X^{(i+1)}(A - X^{(i)}S)^T \\ &\quad + X^{(i)}SX^{(i)T} + U = 0,\ \ i = 0,\ 1,\ 2,\ \cdots, \quad (21a) \\ &X^{(i)} = \begin{bmatrix} X_{00}^{(i)} & X_{0f}^{(i)} \\ \Pi_e X_{0f}^{(i)T} & X_f^{(i)} \end{bmatrix}, \quad (21b) \end{aligned}$$

with the initial condition obtained from

$$X^{(0)} = X^{\text{app}} = \begin{bmatrix} \bar{X}_{00}^* & \bar{X}_{0f}^* \\ \Pi_e\bar{X}_{0f}^{*T} & \bar{X}_f^* \end{bmatrix}. \quad (22)$$

The Kleinman algorithm (21) which is based on the generalized multiparameter algebraic Lyapunov equation (GMALE) has the feature given in the following theorem.

Theorem 2 *Under Assumptions 1–3, there exists a small $\bar{\sigma}$ such that for all $\|\mu\| \in (0,\ \bar{\sigma})$, $\bar{\sigma} \leq \sigma^*$ the iterative algorithm* (21) *converges to the exact solution of X_e with the rate of quadratic convergence. That is, the following conditions are satisfied.*

$$\|X^{(i)} - X\| \leq \frac{O(\|\mu\|^{2^i})}{2^i\beta\gamma} = \frac{O(\|\mu\|^{2^i})}{2^i},\ i = 0, 1, \cdots, (23)$$

where

$$\begin{aligned} &\gamma := 2\|S\| < \infty,\ \beta := \|[\nabla\mathcal{F}(X^{(0)})]^{-1}\|, \\ &\eta := \beta \cdot \|\mathcal{F}(X^{(0)})\|,\ \theta := \beta\eta\gamma,\ \nabla\mathcal{F}(X) = \frac{\partial \text{vec}\mathcal{F}(X)}{\partial(\text{vec}X)^T}. \end{aligned}$$

Proof: The proof follows directly by applying Newton–Kantorovich theorem [9, 11] for the GMARE (8). We now verify that function $\mathcal{F}(X)$ is differentiable on a certain convex set. Using the fact that

$$\nabla\mathcal{F}(X) = (A - SX) \otimes I_{\bar{n}} + [I_{\bar{n}} \otimes (A - SX)]U_{\bar{n}\bar{n}},$$

Proceedings of the American Control Conference
Denver, Colorado June 4-6, 2003

we have

$$\|\nabla\mathcal{F}(X_1) - \nabla\mathcal{F}(X_2)\| \le \gamma\|X_1 - X_2\|,$$

where $\gamma = 2\|S\|$. Moreover, using the fact that

$$\nabla\mathcal{F}(X^{(0)}) = \begin{bmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_3 & \Gamma_4 \end{bmatrix} \otimes I_{\bar{n}} + I_{\bar{n}} \otimes \begin{bmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_3 & \Gamma_4 \end{bmatrix} U_{\bar{n}\bar{n}},$$

it follows that $\nabla\mathcal{F}(X^{(0)})$ is nonsingular because Γ_4 and Γ_0 are stable under Assumptions 1 and 2 (see e.g., Theorem 1 [1]). Therefore, there exists β such that $\beta = \|[\nabla\mathcal{F}(X^{(0)})]^{-1}\|$. On the other hand, we verify that $\mathcal{F}(X^{(0)}) = O(\|\mu\|)$. Hence, there exists η such that $\|[\nabla\mathcal{F}(X^{(0)})]^{-1}\| \cdot \|\mathcal{F}(X^{(0)})\| = \eta = O(\|\mu\|)$. Thus, there exists θ such that $\theta = \beta\gamma\eta < 2^{-1}$ because of $\eta = O(\|\mu\|)$. Using Newton–Kantorovich theorem [9], the strict error estimate is given by (23). Therefore, the proof is completed. ■

When $\|\mu\|$ is sufficiently small, we know from Theorem 3 that the resulting filter gain (20) will be sufficiently close to the optimal Kalman filters gain (5).

Theorem 3 *Under Assumptions 1–3, the use of the high–order approximate Kalman filter gain* (20) *results in*

$$\text{Trace } W_e = \text{Trace } X_e + \frac{O(\|\mu\|^{2^{i+1}-1})}{2^i},\ i = 0, 1, \cdots \tag{24}$$

where Trace X_e *is the optimal steady–state mean square error, while* Trace W_e *is the high–order sub–optimal steady–state mean square error and* W_e *is a positive semidefinite solution of the following multiparameter algebraic Lyapunov equation (MALE)*

$$\begin{aligned}(A_e - X_e^{(i)}S)W_e + W_e(A_e - X_e^{(i)}S)^T \\ +X_e^{(i)}SX_e^{(i)} + U_e = 0,\end{aligned} \tag{25}$$

with $X_e^{(i)} := \Phi_e^{-1}X^{(i)}$.

Proof: Subtracting (6) from (25) we find that $V_e = W_e - X_e$ satisfies the following MALE

$$\begin{aligned}(A_e - X_e^{(i)}S)V_e + V_e(A_e - X_e^{(i)}S)^T \\ +(X_e - X_e^{(i)})S(X_e - X_e^{(i)}) = 0.\end{aligned} \tag{26}$$

Similarly, subtracting (6) from (21a) we also get the MALE

$$\begin{aligned}(A_e - X_e^{(i)}S)(X_e^{(i+1)} - X_e) \\ +(X_e^{(i+1)} - X_e)(A_e - X_e^{(i)}S)^T \\ +(X_e - X_e^{(i)})S(X_e - X_e^{(i)}) = 0.\end{aligned} \tag{27}$$

Therefore, it is easy to verify that $V_e = X_e^{(i+1)} - X_e$ [8] because $A_e - X_e^{(i)}S$ is stable from Theorem 1 in [1]. Consequently we obtain that

$$\begin{aligned}\|V_e\| &= \|W_e - X_e\| = \|X_e^{(i+1)} - X_e\| \\ &\le \|\Phi_e^{-1}\| \cdot \|X^{(i+1)} - X\| \le \|\mu\|^{-1}\|X^{(i+1)} - X\| \\ &= \frac{O(\|\mu\|^{2^{i+1}-1})}{2^i}.\end{aligned} \tag{28}$$

Hence

$$V_e = W_e - X_e = \frac{O(\|\mu\|^{2^{i+1}-1})}{2^i}, \tag{29}$$

which implies (24). ■

4 Numerical Example

In order to demonstrate the efficiency of our proposed algorithm, we have run a numerical example. The system matrix is given as a modification of [4].

$$A_{00} = \begin{bmatrix} 0 & 0 & 1 & 0.8755 \\ 0 & 0 & 1 & -1.79 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_{01} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\ A_{02} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_{10} = A_{20} = \mathbf{0} \in \mathbf{R}^{2\times 4},$$

$$A_{11} = A_{22} = \begin{bmatrix} 0 & 6.0435 \\ -6.0435 & 0 \end{bmatrix},$$

$$D_{01} = D_{02} = \mathbf{0} \in \mathbf{R}^4,\ D_{11} = D_{22} = \begin{bmatrix} -0.1 & 0 \end{bmatrix}^T,$$

$$C_{10} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},\ C_{20} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix},$$

$$C_{11} = C_{22} = I_2,\ V_1 = V_2 = I_2,\ W_1 = W_2 = 1.$$

The small parameters are chosen as $\varepsilon_1 = \varepsilon_2 = 0.1$. It should be noted that we cannot apply the technique proposed in [4] to the MSPS since the Hamiltonian matrices $T_{jj},\ j = 1,\ 2$ have eigenvalues in common. We give a solution of the MARE (6) in Table 1. In order to verify the exactitude of the solution, we calculate the remainder per iteration by substituting $X_e^{(i)}$ into the MARE (6). In Table 2 we present results for the error $\|\mathcal{F}(X_e^{(i)})\|$ per iterations. We find that the solution of the MARE (6) converge to the exact solution with accuracy of $\|\mathcal{F}(X_e^{(i)})\| < 10^{-10}$ after 3 iterative iterations. It can be seen that the initial guess (22) for the algorithm (21a) is quite good and the proposed algorithm is the quadratic convergence.

Table 2.

i	$\|\mathcal{F}(X_e^{(i)})\|$
0	$5.412527504675319 \times 10^{-3}$
1	$2.171750232633503 \times 10^{-6}$
2	$1.983629567766865 \times 10^{-10}$
3	$6.129528501153141 \times 10^{-14}$

5 Conclusions

The optimal Kalman filtering problem for MSPS has been investigated. The new design method of the

Proceedings of the American Control Conference
Denver, Colorado June 4-6, 2003

Table 1.

$$X_e = \begin{bmatrix} X_{00} & X_{0f} \\ X_{0f}^T & \Pi_e^{-1} X_f \end{bmatrix} = \begin{bmatrix} X_{00} & X_{01} & X_{02} \\ X_{01}^T & \varepsilon_1^{-1} X_{11} & \frac{1}{\sqrt{\varepsilon_1 \varepsilon_2}} X_{12} \\ X_{02}^T & \frac{1}{\sqrt{\varepsilon_1 \varepsilon_2}} X_{12}^T & \varepsilon_2^{-1} X_{22} \end{bmatrix}$$

$$X_{00} = \begin{bmatrix} 5.4694 \times 10^{-1} & -8.6872 \times 10^{1} & 8.4483 \times 10^{-2} & 5.3570 \times 10^{-1} \\ -8.6872 \times 10^{-1} & 5.4383 & 1.2040 & -2.3840 \\ 8.4483 \times 10^{-2} & 1.2040 & 4.6080 \times 10^{-1} & -4.2233 \times 10^{-1} \\ 5.3570 \times 10^{-1} & -2.3840 & -4.2233 \times 10^{-1} & 1.1077 \end{bmatrix}$$

$$X_{01} = \begin{bmatrix} -8.0319 \times 10^{-3} & 6.0658 \times 10^{-3} \\ -3.5836 \times 10^{-2} & -1.1246 \times 10^{-2} \\ -9.2557 \times 10^{-3} & 5.4330 \times 10^{-4} \\ 1.4901 \times 10^{-2} & 6.6275 \times 10^{-3} \end{bmatrix}, \ X_{02} = \begin{bmatrix} -7.5605 \times 10^{-3} & 6.2316 \times 10^{-3} \\ -1.5508 \times 10^{-2} & -1.1129 \times 10^{-2} \\ -6.4282 \times 10^{-3} & 6.2053 \times 10^{-4} \\ -1.4165 \times 10^{-3} & 6.4898 \times 10^{-3} \end{bmatrix}$$

$$X_{11} = \begin{bmatrix} 7.1318 \times 10^{-2} & -4.1516 \times 10^{-4} \\ -4.1516 \times 10^{-4} & 7.1291 \times 10^{-2} \end{bmatrix}, \ X_{22} = \begin{bmatrix} 7.0728 \times 10^{-2} & -4.2174 \times 10^{-4} \\ -4.2174 \times 10^{-4} & 7.0709 \times 10^{-2} \end{bmatrix}$$

$$X_{12} = \begin{bmatrix} 3.1628 \times 10^{-4} & -5.0918 \times 10^{-6} \\ -4.4501 \times 10^{-6} & 3.0306 \times 10^{-4} \end{bmatrix}$$

high–order sub–optimal Kalman filters has been proposed. As a result, solving the high–dimension and ill–conditioned MARE has been replaced by solving the low–order and well–conditioned ALE. Furthermore, the proposed filters can be implemented even if the fast state matrices are singular and the Hamiltonian matrices for the fast subsystems have eigenvalues in common compared with the existing results [4].

References

[1] H. K. Khalil and P. V. Kokotovic, "Control strategies for decision makers using different models of the same system," *IEEE Trans. Automat. Contr.*, vol.23, pp.289–298, 1978.

[2] Z. Gajic, "The existence of a unique and bounded solution of the algebraic Riccati equation of multimodel estimation and control problems," *Syst. Control Lett.*, vol.10, pp.185–190, 1988.

[3] Z. Gajic and H. K. Khalil, "Multimodel strategies under randam disturbances and imperfect partial observations," *Automatica*, vol.22, pp.121–125, 1986.

[4] C. Coumarbatch and Z. Gajic, "Parallel optimal Kalman filtering for stochastic systems in multimodeling form," *Transactions on ASME, Journal of Dynamic Systems, Measurement, and Control*, vol.122, pp.542–550, 2000.

[5] A. J. Laub, "A Schur method for solving algebraic Riccati equations," *IEEE Trans. Automat. Contr.*, vol.24, pp.913–921, 1979.

[6] D. L. Kleinman, "On the iterative technique for Riccati equation computations," *IEEE Trans. Automat. Contr.*, vol.13, pp.114–115, 1968.

[7] J. R. Magnus and H. Neudecker, Matrix Differential Calculus with Applications in Statistics and Econometrics, John Wiley and Sons, New York, 1999.

[8] K. Zhou, Essentials of Robust Control, Prentice–Hall, New Jersey, 1998.

[9] T. Yamamoto, "A method for finding sharp error bounds for Newton's method under the Kantorvich assumptions," *Numerische Mathematik*, vol.49, pp.203–220, 1986.

[10] H. Mukaidani, T. Shimomura and H. Xu, "Recursive approach of H_∞ optimal filtering for multiparameter singularly perturbed systems," *15th IFAC World Congress*, CD–Rom, Barcelona, 2002.

[11] H. Mukaidani, H. Xu and K. Mizukami, "Near–optimal control of linear multiparameter singularly perturbed systems," *IEEE Trans. Automat. Contr.*, vol. 47, pp.2051-2057, Dec. 2002.

[12] H. Mukaidani, "Near–optimal Kalman filters for multiparameter singularly perturbed linear systems," *IEEE Transactions Circuits and Systems I*, no. 50, 2003 (to appear).