Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Guaranteed Cost Control of Multimodeling Systems

Hiroaki Mukaidani, Hidetoshi Oya and Hua Xu

Abstract— **The guaranteed cost control problem for multiparameter singularly perturbed systems (MSPS) with uncertainties is investigated. The main contribution of this paper is that an ε–independent controller is newly derived by solving the reduced–order ε–independent slow and fast algebraic Riccati equations (AREs) which is extremely smaller than the dimension of full–order ARE. It is shown that if these AREs have positive definite stabilizing solution then the uncertain closed–loop system with the new ε–independent controller is quadratically stable and has the cost bound. As another important feature, our new results can be applied to the standard and the nonstandard MSPS because the nonsingularity assumption of the fast state matrices are not needed compared with the existing result.**

I. INTRODUCTION

Multimodeling control problems have been studied extensively [1]–[11]. In order to obtain a controller, the multiparameter algebraic Riccati equation (MARE) must be solved. Although various reliable approaches for solving the MARE have been established (see e.g., [8]–[10]), the major limitation in these early approaches is that the small parameters are assumed to be known.

In recent years, the problem of the robust control of singularly perturbed systems (SPS) with parameter uncertainties has been widely studied in the literatures (see e.g., [12] and reference therein). In particular, it is well–known that the guaranteed cost control approach [13] which satisfies not only the robust stability, but also an adequate level of performance is very useful. This approach has the advantage of providing an upper bound on a given performance index. However, when the parameters ε_j are unknown, the guaranteed cost control [13] cannot be directly applied to the uncertain MSPS in the sense that the MARE cannot be solved numerically.

A popular approach to deal with the MSPS is the two–time–scale design method (see e.g., [1]–[6], [14]). When ε_j is very small or unknown, this previously used technique is very efficient. However, the nonsingularity assumption for the fast state matrices $A_{jj},\ j=1,\ 2$ are needed to obtain the slow subsystem. Furthermore, as long as the problem of stabilizing the uncertain SPS is considered, the assumption that the fast state uncertain matrix $A_{jj}+\Delta A_{jj}(t)$ is Hurwitz plays an important role in the study of this problem.

H. Mukaidani is with Graduate School of Education, Hiroshima University, 1–1–1 Kagamiyama, Higashi–Hiroshima, 739–8524 Japan. mukaida@hiroshima-u.ac.jp

H. Oya is with Department of Communications and Systems, University of Electro–Communications, 1–5–1 Chohugaoka, Chohu, Tokyo, 182–8585 Japan. hide-o@se.uec.ac.jp

H. Xu is with Graduate School of Business Sciences, The University of Tsukuba, 3–29–1, Otsuka, Bunkyo–ku, Tokyo, 112–0012 Japan. xuhua@gssm.otsuka.tsukuba.ac.jp

In this paper, the guaranteed cost control problem of the MSPS is newly investigated. Firstly, the unique and the bounded solution of the MARE with indefinite sign quadratic term and its asymptotic structure are established. Secondly, using the asymptotic structure, a new guaranteed cost controller which does not depend on the values of the small parameters ε_j is obtained. It is worth pointing out that the resulting controller is not based on the two–time–scale design method [1]–[6], [14]. Therefore, the nonsingularity assumptions for the fast state matrices A_{jj} and $A_{jj}+\Delta A_{jj}(t)$ are not needed. As another significant feature, the new method of the calculation for the guaranteed cost is proposed to obtain the ε–independent controller. In particular, since the proposed method is based on the reduced–order ARE with smaller dimension, the amount of computation required to get the ε–independent controller becomes extremely small in contrast with the case of solving the full–order MARE.

Notation: The notations used in this paper are fairly standard. The superscript T denotes matrix transpose. $\det L$ denotes the determinant of the square matrix L. I_p denotes the $p\times p$ identity matrix. block diag denotes the block diagonal matrix. $\text{vec}M$ denotes the column vector of the matrix M [15]. $\otimes$ denotes the Kronecker product. U_{pq} denotes the permutation matrix in the Kronecker matrix sense [15] such that $U_{pq}\text{vec}M=\text{vec}M^T,\ M\in\mathbf{R}^{p\times q}$. $E[\cdot]$ denotes the expectation.

II. PROBLEM STATEMENT

Consider the uncertain MSPS

$$\begin{aligned}\dot{x}_0(t) &= \Big[A_{00}+\sum_{i=1}^{2}D_{0i}F_{ii}(t)E_{i0}\Big]x_0(t)\\ &\quad+\sum_{i=1}^{2}\Big[A_{0i}+D_{0i}F_{ii}(t)E_{ii}\Big]x_i(t)\\ &\quad+\sum_{i=1}^{2}B_{0i}u_i(t), \qquad (1a)\\ \varepsilon_j\dot{x}_j(t) &= [A_{j0}+D_{jj}F_{jj}(t)E_{j0}]x_0(t)\\ &\quad+[A_{jj}+D_{jj}F_{jj}(t)E_{jj}]x_j(t)\\ &\quad+B_{jj}u_j(t), \qquad (1b)\\ F_{jj}^T(t)F_{jj}(t) &\le I_{s_j},\ j=1,\ 2, \qquad (1c)\end{aligned}$$

where $\varepsilon_j, j=1,\ 2$ are the small positive parameters, $x_j(t)\in\mathbf{R}^{n_j},\ j=0,\ 1,\ 2$ are the state vectors, $u_j(t)\in\mathbf{R}^{m_j},\ j=1,\ 2$ are the control input. Moreover, $F_{jj}(t)\in\mathbf{R}^{k_j\times s_j}$ are Lebesgue measurable matrix of the uncertain parameters satisfying (1c). All the matrices are the constant

0-7803-8335-4/04/$17.00 ©2004 AACC

matrices of appropriate dimensions. We assume that the ratio of the small positive parameter ε_j is bounded by some positive constants $\underline{k}$, $\bar{k}$ (see e.g., [1]–[3]),

$$0 < \underline{k} \le \alpha \equiv \frac{\varepsilon_2}{\varepsilon_1} \le \bar{k} < \infty. \tag{2}$$

It should be noted that the fast state matrices $A_{jj},\ j = 1,\ 2$ may be singular.

Let us introduce the partitioned matrices

$$\begin{aligned}
&\Pi_e := \textbf{block diag}\left(\varepsilon_1 I_{n_1} \quad \varepsilon_2 I_{n_2} \right),\\
&A_e := \begin{bmatrix} A_{00} & A_{0f} \\ \Pi_e^{-1}A_{f0} & \Pi_e^{-1}A_f \end{bmatrix},\\
&A_{0f} := \begin{bmatrix} A_{01} & A_{02} \end{bmatrix},\ A_{f0} := \begin{bmatrix} A_{10}^T & A_{20}^T \end{bmatrix}^T,\\
&A_f := \textbf{block diag}\left(A_{11} \quad A_{22} \right),\\
&B_e := \begin{bmatrix} B_0 \\ \Pi_e^{-1}B_f \end{bmatrix},\ B_0 := \begin{bmatrix} B_{01} & B_{02} \end{bmatrix},\\
&B_f := \textbf{block diag}\left(B_{11} \quad B_{22} \right),\\
&D_e := \begin{bmatrix} D_0 \\ \Pi_e^{-1}D_f \end{bmatrix},\ D_0 := \begin{bmatrix} D_{01} & D_{02} \end{bmatrix},\\
&D_f := \textbf{block diag}\left(D_{11} \quad D_{22} \right),\\
&F(t) := \textbf{block diag}\left(F_{11}(t) \quad F_{22}(t) \right),\\
&E := \begin{bmatrix} E_0 & E_f \end{bmatrix},\ E_0 := \begin{bmatrix} E_{10}^T & E_{20}^T \end{bmatrix}^T,\\
&E_f := \textbf{block diag}\left(E_{11} \quad E_{22} \right).
\end{aligned}$$

By using above relations, the MSPS (1) can be changed as

$$\dot{x}(t) \ = \ [A_e + D_e F(t) E] x(t) + B_e u(t), \tag{3}$$

where

$$\begin{aligned}
x(t) &:= \begin{bmatrix} x_0^T(t) & x_1^T(t) & x_2^T(t) \end{bmatrix}^T \in \mathbf{R}^n,\\
u(t) &:= \begin{bmatrix} u_1^T(t) & u_2^T(t) \end{bmatrix}^T \in \mathbf{R}^m,\\
n &:= n_0 + n_1 + n_2,\ m \ := \ m_1 + m_2.
\end{aligned}$$

For technical simplification, without loss of generality we shall make the following basic assumptions.

Assumption 1: There exists a constant parameter $\sigma^* > 0$ such that the pair $(A_e,\ B_e)$ is stabilizable for $\|\varepsilon\| := \sqrt{\varepsilon_1^2 + \varepsilon_2^2} \in (0,\ \sigma^*]$, $\varepsilon := \begin{bmatrix} \varepsilon_1 & \varepsilon_2 \end{bmatrix}$.

Assumption 2: The pairs $(A_{jj},\ B_{jj}),\ j = 1,\ 2$ are stabilizable.

Associated with the system (1) is the cost function

$$\mathcal{J} = \int_0^\infty [x^T(t) Q x(t) + u^T(t) R u(t)] dt, \tag{4}$$

where Q and R are given positive definite symmetric matrices.

Definition 1: A control law $u(t) = Kx(t)$ is said to be a quadratic guaranteed cost control with the associated cost matrix $X_e > 0$ for the MSPS (1) and the cost function (4) if the closed–loop system is quadratically stable and the closed–loop value of the cost function (4) satisfies the bound $\mathcal{J} \le \mathcal{J}^*$ for all admissible uncertainties, that is,

$$\frac{d}{dt} x^T(t) X_e x(t) + x^T(t)[Q + K^T R K] x(t) \le 0, \tag{5}$$

where $\mathcal{J}^*$ is the guaranteed cost.

The objective of this paper is to design an ε–independent guaranteed cost control law $u(t) = Kx(t)$ for the uncertain MSPS (1).

III. PRELIMINARY

Before constructing the guaranteed cost controller, let us establish the stability condition for the following autonomous uncertain MSPS (6) without the control input.

$$\begin{aligned}
\dot{x}_0(t) \ &= \ \left[\bar{A}_{00} + \sum_{i=1}^{2} D_{0i} F_{ii}(t) E_{i0} \right] x_0(t)\\
&\quad + \sum_{i=1}^{2} \left[\bar{A}_{0i} + D_{0i} F_{ii}(t) E_{ii} \right] x_i(t), \qquad (6a)\\
\varepsilon_j \dot{x}_j(t) \ &= \ [\bar{A}_{j0} + D_{jj} F_{jj}(t) E_{j0}] x_0(t)\\
&\quad + [\bar{A}_{jj} + D_{jj} F_{jj}(t) E_{jj}] x_j(t), \qquad (6b)\\
j \ &= \ 1,\ 2.
\end{aligned}$$

It is easy to verify that the uncertain MSPS (6) is equivalent to the following uncertain MSPS.

$$\dot{x}(t) = [\bar{A}_e + D_e F(t) E] x(t). \tag{7}$$

Associated with the uncertain MSPS (7) is the cost function

$$J = \int_0^\infty x^T(t) \bar{Q} x(t) dt, \tag{8}$$

where $\bar{Q}$ is given positive definite symmetric matrices.

Definition 2: The matrix $P_e > 0$ is said to be the quadratic cost matrix for the uncertain MSPS (7) if the following inequality holds

$$\frac{d}{dt} x^T(t) P_e x(t) + x^T(t) \bar{Q} x(t) \le 0. \tag{9}$$

The following result is already known in [13].

Lemma 1: Suppose there exist a symmetric positive definite matrix $P_e > 0$ and a positive scalar parameter μ such that for all uncertain matrices (1c), the following multiparameter algebraic Riccati equation (MARE) is satisfied.

$$P_e \bar{A}_e + \bar{A}_e^T P_e + \mu P_e D_e D_e^T P_e + \mu^{-1} E^T E + \bar{Q} = 0, \tag{10}$$

where

$$\begin{aligned}
&\bar{A}_e := \begin{bmatrix} \bar{A}_{00} & \bar{A}_{0f} \\ \Pi_e^{-1}\bar{A}_{f0} & \Pi_e^{-1}\bar{A}_f \end{bmatrix},\\
&\bar{A}_{0f} := \begin{bmatrix} \bar{A}_{01} & \bar{A}_{02} \end{bmatrix},\ \bar{A}_{f0} := \begin{bmatrix} \bar{A}_{10}^T & \bar{A}_{20}^T \end{bmatrix}^T,\\
&\bar{A}_f := \textbf{block diag}\left(\bar{A}_{11} \quad \bar{A}_{22} \right),\\
&P_e := \begin{bmatrix} P_{00} & P_{f0}^T \Pi_e \\ \Pi_e P_{f0} & \Pi_e P_f \end{bmatrix},\\
&P_{00} = P_{00}^T,\ \Pi_e P_f = P_f^T \Pi_e,\\
&P_{f0} := \begin{bmatrix} P_{10} \\ P_{20} \end{bmatrix},\ P_f := \begin{bmatrix} P_{11} & \alpha P_{21}^T \\ P_{21} & P_{22} \end{bmatrix}.
\end{aligned}$$

Then the autonomous uncertain MSPS (7) is quadratically stable and the corresponding value of the cost function (8) satisfies the following inequality (11).

$$J \le x^T(0) P_e x(0). \tag{11}$$

In order to investigate the solvability condition of the MARE (10), let us introduce the following useful lemma [10].

Lemma 2: The MARE (10) is equivalent to the following generalized multiparameter algebraic Riccati equation (GMARE) (12)

$$P^T\bar{A} + \bar{A}^TP + \mu P^TDD^TP + \mu^{-1}E^TE + \bar{Q} = 0, \quad (12a)$$
$$P_e = \Phi_e^TP = P^T\Phi_e, \quad (12b)$$

where

$$\Phi_e := \begin{bmatrix} I_{n_0} & 0 \\ 0 & \Pi_e \end{bmatrix},\ \bar{A} := \begin{bmatrix} \bar{A}_{00} & \bar{A}_{0f} \\ \bar{A}_{f0} & \bar{A}_f \end{bmatrix},$$
$$D := \begin{bmatrix} D_0 \\ D_f \end{bmatrix},\ P := \begin{bmatrix} P_{00} & P_{0f} \\ P_{f0} & P_f \end{bmatrix},$$
$$P_{0f} = P_{f0}^T\Pi_e := \begin{bmatrix} \varepsilon_1P_{10}^T & \varepsilon_2P_{20}^T \end{bmatrix}.$$

In order to simplify the notation of the GMARE (12a), we introduce the following matrices.

$$S := \mu DD^T = \begin{bmatrix} S_{00} & S_{0f} \\ S_{0f}^T & S_f \end{bmatrix},$$
$$S_{00} := \mu D_0D_0^T = \sum_{i=1}^{2}\mu D_{0i}D_{0i}^T,$$
$$S_{0f} := \mu D_0D_f^T = \begin{bmatrix} S_{01} & S_{02} \end{bmatrix},$$
$$S_f := \mu D_fD_f^T = \text{block diag}\begin{pmatrix} S_{11} & S_{22} \end{pmatrix},$$
$$S_{0j} := \mu D_{0j}D_{jj}^T,\ S_{jj} := \mu D_{jj}D_{jj}^T,$$
$$U := \mu^{-1}E^TE + \bar{Q} = \begin{bmatrix} U_{00} & U_{0f} \\ U_{0f}^T & U_f \end{bmatrix},$$
$$U_{00} := \sum_{i=0}^{2}\mu^{-1}E_{i0}^TE_{i0} + \bar{Q}_{00},\ U_{0f} := \begin{bmatrix} U_{01} & U_{02} \end{bmatrix},$$
$$U_f := \text{block diag}\begin{pmatrix} U_{11} & U_{22} \end{pmatrix},$$
$$U_{0j} := \mu^{-1}E_{j0}^TE_{jj} + \bar{Q}_{0j},\ U_{jj} := \mu^{-1}E_{jj}^TE_{jj} + \bar{Q}_{jj}.$$
$$\bar{Q} := \begin{bmatrix} \bar{Q}_{00} & \bar{Q}_{0f} \\ \bar{Q}_{0f}^T & \bar{Q}_f \end{bmatrix},\ \bar{Q}_{0f} := \begin{bmatrix} \bar{Q}_{01} & \bar{Q}_{02} \end{bmatrix},$$
$$\bar{Q}_f := \text{block diag}\begin{pmatrix} \bar{Q}_{11} & \bar{Q}_{22} \end{pmatrix}.$$

The GMARE (12a) can be partitioned into

$$\begin{aligned} f_1 &= P_{00}^T\bar{A}_{00} + \bar{A}_{00}^TP_{00} + P_{f0}^T\bar{A}_{f0} + \bar{A}_{f0}^TP_{f0} \\ &\quad + P_{00}^TS_{00}P_{00} + P_{f0}^TS_fP_{f0} \\ &\quad + P_{00}^TS_{0f}P_{f0} + P_{f0}^TS_{0f}^TP_{00} + U_{00} = 0, \quad (13a) \\ f_2 &= \bar{A}_{00}^TP_{f0}^T\Pi_e + \bar{A}_{f0}^TP_f + P_{00}^T\bar{A}_{0f} + P_{f0}^T\bar{A}_f \\ &\quad + P_{00}^TS_{00}P_{f0}^T\Pi_e + P_{f0}^TS_{0f}^TP_{f0}^T\Pi_e \\ &\quad + P_{00}^TS_{0f}P_f + P_{f0}^TS_fP_f + U_{0f} = 0, \quad (13b) \\ f_3 &= P_f^T\bar{A}_f + \bar{A}_f^TP_f + \Pi_eP_{f0}\bar{A}_{0f} + \bar{A}_{0f}^TP_{f0}^T\Pi_e \\ &\quad + P_f^TS_fP_f + P_f^TS_{0f}^TP_{f0}^T\Pi_e + \Pi_eP_{f0}S_{0f}P_f \\ &\quad + \Pi_eP_{f0}S_{00}P_{f0}^T\Pi_e + U_f = 0. \quad (13c) \end{aligned}$$

It is assumed that the limit of α exists as ε_1 and ε_2 tend to zero (see e.g., [1]–[3]), that is

$$\bar{\alpha} := \lim_{\substack{\varepsilon_1\to+0 \\ \varepsilon_2\to+0}}\alpha = \lim_{\substack{\varepsilon_1\to+0 \\ \varepsilon_2\to+0}}\frac{\varepsilon_2}{\varepsilon_1}. \quad (14)$$

Let $\bar{P}_{00}$, $\bar{P}_{f0}$ and $\bar{P}_f$ be the limiting solutions of the above equation (13) as $\varepsilon_j \to +0,\ j = 1,\ 2$, then we obtain the following equations

$$\begin{aligned} &\bar{P}_{00}^T\bar{A}_{00} + \bar{A}_{00}^T\bar{P}_{00} + \bar{P}_{f0}^T\bar{A}_{f0} + \bar{A}_{f0}^T\bar{P}_{f0} \\ &\quad + \bar{P}_{00}^TS_{00}\bar{P}_{00} + \bar{P}_{f0}^TS_f\bar{P}_{f0} \\ &\quad + \bar{P}_{00}^TS_{0f}\bar{P}_{f0} + \bar{P}_{f0}^TS_{0f}^T\bar{P}_{00} + U_{00} = 0, \quad (15a) \\ &\bar{A}_{f0}^T\bar{P}_f + \bar{P}_{00}^T\bar{A}_{0f} + \bar{P}_{f0}^T\bar{A}_f \\ &\quad + \bar{P}_{00}^TS_{0f}\bar{P}_f + \bar{P}_{f0}^TS_f\bar{P}_f + U_{0f} = 0, \quad (15b) \\ &\bar{P}_f^T\bar{A}_f + \bar{A}_f^T\bar{P}_f + \bar{P}_f^TS_f\bar{P}_f + U_f = 0, \quad (15c) \end{aligned}$$

where

$$\bar{P}_f := \begin{bmatrix} \bar{P}_{11} & \bar{\alpha}\bar{P}_{21}^T \\ \bar{P}_{21} & \bar{P}_{22} \end{bmatrix},\ \bar{P}_{jj} = \bar{P}_{jj}^T,\ j = 0,\ 1,\ 2. \quad (16)$$

Note that the ARE (15c) is asymmetric. However, it will be shown that the ARE (15c) admits at least a symmetric positive definite stabilizing solution under the appropriate conditions. In the following analysis, we need a weaker assumption than the nonsingularity assumption of the fast state matrices.

Assumption 3: The Hamiltonian matrices $T_{jj},\ j = 1,\ 2$ are nonsingular, where

$$T_{jj} := \begin{bmatrix} \bar{A}_{jj} & S_{jj} \\ -U_{jj} & -\bar{A}_{jj}^T \end{bmatrix}.$$

We will establish the relation between the GMARE (12a) and the zeroth–order equations (15). Firstly, the following sets are newly defined.

$\Gamma_{f_j} := \{\mu > 0|$ The ARE $\bar{P}_{jj}\bar{A}_{jj} + \bar{A}_{jj}^T\bar{P}_{jj} + \bar{P}_{jj}S_{jj}\bar{P}_{jj} + U_{jj} = 0$ has a positive definite stabilizing solution.$\}$, $j = 1,\ 2$.

$\mu_{f_j} := \sup\{\mu|\mu \in \Gamma_{f_j}\}$.

Theorem 1: Under Assumption 3, if we select a parameter $0 < \mu < \mu_f := \min\{\mu_{f_1},\ \mu_{f_2}\}$, then the ARE (15c) admits a unique symmetric positive definite stabilizing solution $\bar{P}_f$ which can be written as

$$\bar{P}_f^* := \text{block diag}\begin{pmatrix} \bar{P}_{11}^* & \bar{P}_{22}^* \end{pmatrix}, \quad (17)$$

where $\bar{P}_{jj}^*$ is a unique symmetric positive definite stabilizing solution for the following ARE, respectively

$$\bar{P}_{jj}^*\bar{A}_{jj} + \bar{A}_{jj}^T\bar{P}_{jj}^* + \bar{P}_{jj}^*S_{jj}\bar{P}_{jj}^* + U_{jj} = 0,\ j = 1,\ 2.$$

Proof: Substituting the matrix (17) into the ARE (15c) as $\bar{P}_f^* \to \bar{P}_f$, it is easy to verify that $\bar{P}_f^*\bar{A}_f + \bar{A}_f^T\bar{P}_f^* + \bar{P}_f^*S_f\bar{P}_f^* + U_f = 0$. Furthermore, it can be seen that $\bar{P}_f^* = \bar{P}_f^{*T} > 0$ and the following matrix $\bar{A}_f + S_f\bar{P}_f^*$ is stable because $\bar{P}_{jj}^*$ is the unique symmetric positive definite stabilizing solution.

$$\begin{aligned} &\bar{A}_f + S_f\bar{P}_f^* \\ = &\ \text{block diag}\begin{pmatrix} \bar{A}_{11} + S_{11}\bar{P}_{11}^* & \bar{A}_{22} + S_{22}\bar{P}_{22}^* \end{pmatrix}. \end{aligned}$$

Consequently, there exists the unique solution of the ARE (15c) and its solution is (17) itself. ∎

In this situation, we obtain the following zeroth–order equations (18) [10]

$$\bar{P}_{00}^*\mathcal{A} + \mathcal{A}^T\bar{P}_{00}^* + \bar{P}_{00}^*\mathcal{S}\bar{P}_{00}^* + \mathcal{U} = 0, \tag{18a}$$

$$\bar{P}_{j0}^* = \begin{bmatrix} \bar{P}_{jj}^* & -I_{n_j} \end{bmatrix} T_{jj}^{-1}T_{j0}\begin{bmatrix} I_{n_0} \\ \bar{P}_{00}^* \end{bmatrix}, \tag{18b}$$

$$\bar{P}_{jj}^*\bar{A}_{jj} + \bar{A}_{jj}^T\bar{P}_{jj}^* + \bar{P}_{jj}^*S_{jj}\bar{P}_{jj}^* + U_{jj} = 0, \tag{18c}$$

where $j = 1,\ 2$,

$$\mathcal{T} := \begin{bmatrix} \mathcal{A} & \mathcal{S} \\ -\mathcal{U} & -\mathcal{A}^T \end{bmatrix} = T_{00} - \sum_{j=1}^{2} T_{0j}T_{jj}^{-1}T_{j0},$$

$$T_{00} = \begin{bmatrix} \bar{A}_{00} & S_{00} \\ -U_{00} & -\bar{A}_{00}^T \end{bmatrix},\ T_{0j} = \begin{bmatrix} \bar{A}_{0j} & S_{0j} \\ -U_{0j} & -\bar{A}_{j0}^T \end{bmatrix},$$

$$T_{j0} = \begin{bmatrix} \bar{A}_{j0} & S_{0j}^T \\ -U_{0j}^T & -\bar{A}_{0j}^T \end{bmatrix},\ j = 1,\ 2.$$

Secondly, let us define the following set.

$\Gamma_s := \{0 < \mu |$ The ARE (18a) has a positive definite stabilizing solution.$\}$

$\mu_s := \sup\{\mu | \mu \in \Gamma_s\}$.

As a result, for every $0 < \mu < \bar{\mu} = \min\{\mu_s,\ \mu_f\}$, the AREs (18a) and (18c) have the positive definite stabilizing solutions. Hence, the limiting behavior of P_e as the parameter $\|\varepsilon\| = \sqrt{\varepsilon_1^2 + \varepsilon_2^2} \to +0$ is described by the following theorem.

Theorem 2: Under Assumption 3, suppose there exists a positive scalar $\bar{\mu} := \min\{\mu_s,\ \mu_f\}$ such that for all $0 < \mu < \bar{\mu}$, the AREs (18a) and (18c) have the positive definite stabilizing solutions. Then there exists a small constant $\bar{\sigma}(\le \sigma^*)$ such that for all $\|\varepsilon\| \in (0,\ \bar{\sigma})$ and any $\mu(< \bar{\mu})$, the MARE (10) admits the symmetric positive definite stabilizing solution P_e which can be written as

$$P_e = \begin{bmatrix} \bar{P}_{00}^* + O(\|\varepsilon\|) & [\bar{P}_{f0}^* + O(\|\varepsilon\|)]^T\Pi_e \\ \Pi_e[\bar{P}_{f0}^* + O(\|\varepsilon\|)] & \Pi_e[\bar{P}_f^* + O(\|\varepsilon\|)] \end{bmatrix}. \tag{19}$$

In this case, the matrix $P_e > 0$ is called the guaranteed cost matrix for the uncertain MSPS (7).

Proof: We apply the implicit function theorem [5] to (13). To do so, it is enough to show that the corresponding Jacobian is nonsingular at $\|\varepsilon\| = 0$. It can be shown, after some algebra, that the Jacobian matrix of (13) in the limit as $\|\varepsilon\| \to 0$ is given by

$$\mathbf{J} = \nabla\mathbf{F} = \begin{bmatrix} \mathbf{J}_{00} & \mathbf{J}_{01} & 0 \\ \mathbf{J}_{10} & \mathbf{J}_{11} & \mathbf{J}_{12} \\ 0 & 0 & \mathbf{J}_{22} \end{bmatrix}, \tag{20}$$

where

$$\begin{aligned}
&\mathbf{J}_{00} = I_{n_0} \otimes \hat{A}_{00}^T + \hat{A}_{00}^T \otimes I_{n_0}, \\
&\mathbf{J}_{01} = (I_{n_0} \otimes \hat{A}_{f0}^T)U_{n_0\hat{n}} + \hat{A}_{f0}^T \otimes I_{n_0}, \\
&\mathbf{J}_{10} = \hat{A}_{0f}^T \otimes I_{n_0} = (\hat{A}_{0f}^T \otimes I_{n_0})U_{n_0n_0}, \\
&\mathbf{J}_{11} = \hat{A}_f^T \otimes I_{n_0},\ \mathbf{J}_{22} = I_{\hat{n}} \otimes \hat{A}_f^T + \hat{A}_f^T \otimes I_{\hat{n}}, \\
&\hat{A}_{00} = \bar{A}_{00} + S_{00}\bar{P}_{00}^* + S_{0f}\bar{P}_{f0}^*,\ \hat{A}_{0f} = \bar{A}_{0f} + S_{0f}\bar{P}_f^*, \\
&\hat{A}_{f0} = \bar{A}_{f0} + S_{0f}^T\bar{P}_{00}^* + S_f\bar{P}_{f0}^*,\ \hat{A}_f = \bar{A}_f + S_f\bar{P}_f^*, \\
&\hat{A}_f = \text{block diag}\begin{pmatrix} H_{11} & H_{22} \end{pmatrix}, \\
&H_{jj} := \bar{A}_{jj} + S_{jj}\bar{P}_{jj}^*,\ \hat{n} := n_1 + n_2.
\end{aligned}$$

The Jacobian (20) can be expressed as

$$\det\mathbf{J} = \det\mathbf{J}_{22} \cdot \det\mathbf{J}_{11} \cdot \det[I_{n_0} \otimes \hat{A}_0^T + \hat{A}_0^T \otimes I_{n_0}], \tag{21}$$

where $\hat{A}_0 \equiv \hat{A}_{00} - \hat{A}_{0f}\hat{A}_f^{-1}\hat{A}_{f0}$. Obviously, $\mathbf{J}_{jj},\ j = 1,\ 2$ are nonsingular because the matrix $\hat{A}_f = \bar{A}_f + S_f\bar{P}_f^*$ is stable from Theorem 1. After some straightforward but tedious algebra, we see that $\mathcal{A} + \mathcal{S}\bar{P}_{00}^* = \hat{A}_{00} - \hat{A}_{0f}\hat{A}_f^{-1}\hat{A}_{f0} = \hat{A}_0$. Therefore, the matrix $\hat{A}_0$ is also stable because the assumption that the ARE (18a) has the positive definite stabilizing solution is satisfied. Thus, $\det\mathbf{J} \ne 0$, i.e., $\mathbf{J}$ is nonsingular at $\|\varepsilon\| = 0$. The asymptotic structure of P_e is obtained directly by using the implicit function theorem.

The remainder of the proof is to show that P_e is the positive semidefinite stabilizing solution. Since this proof can be done by using Schur complement similarly as in [11], it is omitted. ∎

IV. GUARANTEED COST CONTROL FOR THE MSPS

In this section, we consider not the MARE but the GMARE because these equations are equivalent under the mathematical argument in Lemma 2. The following useful result is known [13].

Lemma 3: Under Assumption 1, suppose there exists a matrix X satisfying $X_e = X_e^T := \Phi_e X > 0$ and the positive scalar parameter ν such that for all uncertain matrices (1c), the following GMARE satisfies.

$$\begin{aligned} &X^TA + A^TX + X^T(\nu DD^T - BR^{-1}B^T)X \\ &+\nu^{-1}E^TE + Q = 0, \end{aligned} \tag{22}$$

where

$$A := \begin{bmatrix} A_{00} & A_{0f} \\ A_{f0} & A_f \end{bmatrix},\ B := \begin{bmatrix} B_0 \\ B_f \end{bmatrix},$$

$$R := \text{block diag}\begin{pmatrix} R_{11} & R_{22} \end{pmatrix} > 0,$$

$$Q := \begin{bmatrix} Q_{00} & Q_{0f} \\ Q_{0f}^T & Q_f \end{bmatrix} > 0,\ Q_{0f} := \begin{bmatrix} Q_{01} & Q_{02} \end{bmatrix},$$

$$Q_f := \text{block diag}\begin{pmatrix} Q_{11} & Q_{22} \end{pmatrix},$$

$$X := \begin{bmatrix} X_{00} & X_{f0}^T\Pi_e \\ X_{f0} & X_f \end{bmatrix},\ X_{00} = X_{00}^T > 0,$$

$$\Pi_eX_f = X_f^T\Pi_e,\ X_{f0} := \begin{bmatrix} X_{10} \\ X_{20} \end{bmatrix},$$

$$X_f := \begin{bmatrix} X_{11} & \alpha X_{21}^T \\ X_{21} & X_{22} \end{bmatrix}.$$

Then the closed–loop uncertain MSPS with the linear state feedback control law (23) is the guaranteed cost control

$$u(t) = Kx(t) = -R^{-1}B^T Xx(t). \tag{23}$$

Moreover, the corresponding value of the cost function (4) satisfies the following inequality (24).

$$J \leq x^T(0)\Phi_e Xx(0). \tag{24}$$

Applying the similar analysis used in Theorem 2, the existence condition of the GMARE (22) is studied. Firstly, we introduce the following matrices.

$$\begin{aligned}
&V := \nu DD^T - BR^{-1}B^T = \begin{bmatrix} V_{00} & V_{0f} \\ V_{0f}^T & V_f \end{bmatrix}, \\
&V_{00} := \nu D_0 D_0^T - B_0 R^{-1} B_0^T \\
&\quad = \sum_{i=1}^{2} (\nu D_{0i} D_{0i}^T - B_{0i} R_{ii}^{-1} B_{0i}^T), \\
&V_{0f} := \nu D_0 D_f^T - B_0 R^{-1} B_f^T = \begin{bmatrix} V_{01} & V_{02} \end{bmatrix}, \\
&V_f := \nu D_f D_f^T - B_f R^{-1} B_f^T = \text{block diag}\begin{pmatrix} V_{11} & V_{22} \end{pmatrix}, \\
&V_{0j} := \nu D_{0j} D_{jj}^T - B_{0j} R_{jj}^{-1} B_{jj}^T, \\
&V_{jj} := \nu D_{jj} D_{jj}^T - B_{jj} R_{jj}^{-1} B_{jj}^T, \\
&W := \nu^{-1}E^T E + Q = \begin{bmatrix} W_{00} & W_{0f} \\ W_{0f}^T & W_f \end{bmatrix}, \\
&W_{0f} := \begin{bmatrix} W_{01} & W_{02} \end{bmatrix}, \ W_f := \text{block diag}\begin{pmatrix} W_{11} & W_{22} \end{pmatrix}, \\
&W_{0j} := \nu^{-1}E_{j0}^T E_{jj} + Q_{0j}, \ W_{jj} := \nu^{-1}E_{jj}^T E_{jj} + Q_{jj}.
\end{aligned}$$

In order to guarantee the existence of the solution of the GMARE (22), the nonsingularity assumptions of the Hamiltonian matrices are needed.

Assumption 4: The Hamiltonian matrices $Z_{jj}, \ j = 1, \ 2$ are nonsingular, where

$$Z_{jj} := \begin{bmatrix} A_{jj} & V_{jj} \\ -W_{jj} & -A_{jj}^T \end{bmatrix}.$$

Secondly, let us define the reduced–order AREs (25).

$$\bar{X}_{00}^*\Xi + \Xi^T \bar{X}_{00}^* + \bar{X}_{00}^*\mathcal{V}\bar{X}_{00}^* + \mathcal{W} = 0, \tag{25a}$$

$$\bar{X}_{j0}^* = \begin{bmatrix} \bar{X}_{jj}^* & -I_{n_j} \end{bmatrix} Z_{jj}^{-1} Z_{j0} \begin{bmatrix} I_{n_0} \\ \bar{X}_{00}^* \end{bmatrix}, \tag{25b}$$

$$\bar{X}_{jj}^* A_{jj} + A_{jj}^T \bar{X}_{jj}^* + \bar{X}_{jj}^* V_{jj} \bar{X}_{jj}^* + W_{jj} = 0, \tag{25c}$$

where

$$\begin{aligned}
&\begin{bmatrix} \Xi & \mathcal{V} \\ -\mathcal{W} & \Xi^T \end{bmatrix} = Z_{00} - Z_{01}Z_{11}^{-1}Z_{10} - Z_{02}Z_{22}^{-1}Z_{20}, \\
&Z_{00} = \begin{bmatrix} A_{00} & V_{00} \\ -W_{00} & -A_{00}^T \end{bmatrix}, \ Z_{0j} = \begin{bmatrix} A_{0j} & V_{0j} \\ -W_{0j} & -A_{j0}^T \end{bmatrix}, \\
&Z_{j0} = \begin{bmatrix} A_{j0} & V_{0j}^T \\ -W_{0j}^T & -A_{0j}^T \end{bmatrix}, \ j = 1, \ 2.
\end{aligned}$$

Now, let us define the design parameters.

$\nu_f := \min\{\nu_{f_1}, \ \nu_{f_2}\}$, where $\nu_{f_j} := \sup\{\nu | \nu \in \Lambda_{f_j}\}$ and $\Lambda_{f_j} := \{\nu > 0|$ The AREs (25c) have a positive definite stabilizing solution, respectively.$\}$

$\nu_s := \sup\{\nu | \nu \in \Lambda_s\}$, where $\Lambda_s := \{0 < \nu|$ The ARE (25a) has a positive definite stabilizing solution.$\}$

As a result, for every $0 < \nu < \bar{\nu} = \min\{\nu_s, \ \nu_f\}$, the AREs (25a) and (25c) have the positive definite stabilizing solutions. Hence, there exists a small $\hat{\sigma}(\leq \sigma^*)$ such that for all $\|\varepsilon\| \in (0, \ \hat{\sigma})$ and for any $\nu(< \bar{\nu})$, the GMARE (22) has a solution X which can be written as

$$X = \begin{bmatrix} \bar{X}_{00}^* + O(\|\varepsilon\|) & [\bar{X}_{f0}^* + O(\|\varepsilon\|)]^T \Pi_e \\ \bar{X}_{f0}^* + O(\|\varepsilon\|) & \bar{X}_f^* + O(\|\varepsilon\|) \end{bmatrix}. \tag{26}$$

We now give a new design approach to the construction of the guaranteed cost controller. The new ε–independent guaranteed cost controller can be obtained by solving reduced–order slow and fast AREs (25). The ε–independent guaranteed cost controller are obtained by neglecting the term of $O(\|\varepsilon\|)$ of the guaranteed cost controller (23). The reason why the proposed approximation is adopted is that these parameters are unknown and $O(\|\varepsilon\|)$ term is sufficiently small. If $\|\varepsilon\|$ is very small, it is obvious that the guaranteed cost controller (23) can be changed as

$$\begin{aligned}
u(t) &\approx u_{\text{app}}(t) = K_{\text{app}}x(t) \\
&= -R^{-1}\begin{bmatrix} B_0^T & B_f^T \end{bmatrix} \begin{bmatrix} \bar{X}_{00}^* & 0 & 0 \\ \bar{X}_{10}^* & \bar{X}_{11}^* & 0 \\ \bar{X}_{20}^* & 0 & \bar{X}_{22}^* \end{bmatrix} x(t).
\end{aligned} \tag{27}$$

Since the proposed ε–independent controller (27) is very close to the exact one (23) because of $\|K - K_{\text{app}}\| = O(\|\varepsilon\|)$, it is expected that the proposed controller (27) works well as the guaranteed cost control. The main result of this section is as follows.

Theorem 3: Under Assumptions 1, 2 and 4, if we select a parameter $0 < \nu < \bar{\nu} = \min\{\nu_s, \ \nu_f\}$, then there exists a small $\tilde{\sigma} > 0$ such that for all $\|\varepsilon\| \in (0, \ \tilde{\sigma})$, the uncertain closed–loop MSPS is quadratically stable and the cost (4) has the upper bound via the ε–independent controller (27). That is, the approximate controller (27) is the guaranteed cost controller.

Proof: It is enough to show that the GMARE (12a) has the solution P as $A + BK_{\text{app}} \to \bar{A}$, $K_{\text{app}}^T RK_{\text{app}} + Q \to \bar{Q}$. Substituting $A + BK_{\text{app}}$ and $K_{\text{app}}^T RK_{\text{app}} + Q$ into $\bar{A}$ and $\bar{Q}$ respectively, it follows:

$$\begin{aligned}
&P^T(A + BK_{\text{app}}) + (A + BK_{\text{app}})^T P + \lambda P^T DD^T P \\
&\quad + \lambda^{-1}E^T E + K_{\text{app}}^T RK_{\text{app}} + Q = 0.
\end{aligned} \tag{28}$$

The proof of the existence of P is obtained by the implicit function theorem [5]. Using the similar manner used in the proof of Theorem 2, it is easy to prove that

$$P = \begin{bmatrix} \bar{X}_{00}^* + O(\|\varepsilon\|) & [\bar{X}_{f0}^* + O(\|\varepsilon\|)]^T \Pi_e \\ \bar{X}_{f0}^* + O(\|\varepsilon\|) & \bar{X}_f^* + O(\|\varepsilon\|) \end{bmatrix} \tag{29}$$

under $\lambda = \nu$.

Since $\bar{X}_{00}$ and $\bar{X}_f$ are positive definite solution of the AREs (25a) and (25c) respectively, we find that the solution

$P_e = \Phi_e P$ is the positive definite solution. Thus, the proof of Theorem 3 is completed. ■

According to the existing results [13], we need to solve the full–order GMARE (22) to calculate the bound of the cost $x(0)^T P_e x(0) = x(0)^T \Phi_e P x(0)$ for every $0 < \nu < \bar{\nu}$. However, to solve the GMARE (22), information of each parameter ε_j is needed. Therefore, the parameter independent calculation method of the cost bound is newly established.

If $\|\varepsilon\|$ is very small, then the guaranteed cost given by (24) can be changed as follows

$$x(0)^T P_e x(0) = x_0(0) \bar{X}_{00} x_0(0) + O(\|\varepsilon\|). \tag{30}$$

Thus, in order to calculate the bound of the cost, our new idea is to use only the solution $\bar{X}_{00}$ of the reduced–order ARE (25a). That is, we can neglect the $O(\|\varepsilon\|)$ term of the cost (30) if $\|\varepsilon\|$ is sufficiently small. Therefore, the amount of the computation required to get the ε–independent controller becomes extremely small compared with the case of solving the full–order GMARE (22) because the approximate cost can be computed by the small dimension which are the same as the slow subsystems.

Remark 1: It can be noted that the bound obtained in Theorem 3 depends on the initial condition $x(0)$. To remove this dependence on $x(0)$, we assume that $x(0)$ is a zero mean random variable satisfying $E[x(0)x^T(0)] = I_n$. In this case, it is interesting to point out that the guaranteed cost becomes

$$x(0)^T P_e x(0) = \text{Trace } P_e = \text{Trace } \bar{X}_{00} + O(\|\varepsilon\|). \tag{31}$$

Finally, we give an algorithm for the guaranteed cost control problem of the uncertain MSPS.

Step 1. Search the minimum parameter $\nu_f = \min\{\nu_{f_1},\ \nu_{f_2}\}$ such that the reduced–order AREs (25c) have positive definite stabilizing solution $\bar{X}_{jj}$ by using the bisection method.

Step 2. Secondly, search the minimum parameter $\nu_s (\leq \nu_f)$ such that the reduced–order ARE (25a) has positive definite stabilizing solution $\bar{X}_{00}$ by using the bisection method.

Step 3. Choose any parameter ν such that $0 < \nu < \bar{\nu} = \min\{\nu_s,\ \nu_f\}$ and calculate Ξ, $\mathcal{V}$ and $\mathcal{W}$ via the matrices Z_{00}, Z_{0j}, Z_{j0} and $Z_{jj},\ j = 1,\ 2$.

Step 4. Compute the positive definite stabilizing solution $\bar{X}_{00}$ and calculate the approximate guaranteed cost

$$f(\nu) = \text{Trace } \bar{X}_{00}, \tag{32}$$

where we have neglected the $O(\|\varepsilon\|)$ term.

Step 5. Find a $\nu = \hat{\nu}$ that minimizes the approximate cost $f(\nu)$ for all $0 < \nu < \bar{\nu}$.

Step 6. Using the obtained $\nu = \hat{\nu}$, design the ε–independent controller (27).

V. CONCLUSION

The guaranteed cost control problem for the uncertain MSPS has been studied. By solving the reduced–order slow and fast AREs, the new ε–independent controller can be obtained. The new technique has the following advantages. 1. The proposed method does not need the information for the small parameters. 2. The required work space is the same as the reduced–order slow and fast subsystems. 3. Our new results apply to the standard and the non–standard MSPS without the nonsingularity assumption of the fast subsystems although the fast subsystems include the uncertainty. Therefore, we have succeeded in applying the new design approach to more practical MSPS.

REFERENCES

[1] H. K. Khalil and P. V. Kokotović, “Control strategies for decision makers using different models of the same system,” *IEEE Trans. Automatic Control*, vol. 23, pp. 289–298, Apr. 1978.

[2] H. K. Khalil and P. V. Kokotović, “Control of linear systems with multiparameter singular perturbations,” *Automatica*, vol. 15, no. 2, pp. 197–207, 1979.

[3] H. K. Khalil, “Stabilization of multiparameter singularly perturbed systems,” *IEEE Transactions on Automatic Control*, vol. 24, pp. 790–791, May 1979.

[4] Z. Gajic and H. K. Khalil, “Multimodel strategies under random disturbances and imperfect partial observations,” *Automatica*, vol. 22, no. 1, pp. 121–125, 1986.

[5] Z. Gajic, “The existence of a unique and bounded solution of the algebraic Riccati equation of multimodel estimation and control problems,” *Syst. Control Lett.*, vol. 10, pp. 185–190, 1988.

[6] Y-Y. Wang, P. M. Frank and N. E. Wu, “Near–optimal control of nonstandard singularly perturbed systems,” *Automatica*, vol. 30, no. 2, pp. 277–292, 1994.

[7] C. Coumarbatch and Z. Gajic, “Exact decomposition of the algebraic Riccati equation of deterministic multimodeling optimal control problems,” *IEEE Trans. Automatic Control*, vol. 45, pp. 790–794, Apr. 2000.

[8] C. Coumarbatch and Z. Gajic, “Parallel optimal Kalman filtering for stochastic systems in multimodeling form,” *Trans. ASME, J. Dynamic Systems, Measurement, and Control*, vol. 122, pp. 542–550, Sept. 2000.

[9] H. Mukaidani, H. Xu and K. Mizukami, Near–optimal control of linear multiparameter singularly perturbed systems, *IEEE Trans. Automatic Control*, vol. 47, pp.2051-2057, Dec. 2002.

[10] H. Mukaidani, T. Shimomura and K. Mizukami, Asymptotic expansions and new numerical algorithm of the algebraic Riccati equation for multiparameter singularly perturbed systems, *J. Math. Analysis and Applications*, vol.267, pp.209–234, 2002.

[11] H. Mukaidani, H. Xu and K. Mizukami, New results for near–optimal control of linear multiparameter singularly perturbed systems, *Automatica*, vol.39, no.12, pp.2157–2167, 2003.

[12] H. Mukaidani and K. Mizukami, “The guaranteed cost control problem of uncertain singularly perturbed systems,” *J. Math. Analysis and Applications*, vol. 251, pp. 716–735, 2000.

[13] I. R. Petersen and D. C. McFarlane, “Optimal guaranteed cost control and filtering for uncertain linear systems,” *IEEE Trans. Automat. Contr.*, vol.39, pp. 1971–1977, Sept. 1994.

[14] P. V. Kokotović, H. K. Khalil and J. O’Reilly, *Singular Perturbation Methods in Control: Analysis and Design*, New York: Academic Press, 1986.

[15] J. R. Magnus and H. Neudecker, *Matrix Differential Calculus with Applications in Statistics and Econometrics*, New York: John Wiley and Sons, 1999.