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A Computationally Efficient Numerical Algorithm for Solving Cross-Coupled Algebraic Riccati Equation and Its Application to Multimodeling Systems

Hiroaki Mukaidani

Abstract— **In this paper, a new algorithm for solving cross–coupled algebraic Riccati equation (CARE) is proposed. Since the new algorithm is based on the fixed point algorithm, the solutions can be obtained independently as the solution of the algebraic Lyapunov equation. As a result, the convergence and the positive semidefiniteness of the obtained solutions are guaranteed. In order to show the validity of the proposed algorithm, the linear quadratic infinite horizon Nash game for general multiparameter singularly perturbed systems (GMSPS) is applied. The local uniqueness and the asymptotic structure of the solutions to the cross–coupled multiparameter algebraic Riccati equation (CMARE) is newly established by means of implicit function theorem. Moreover, a new formulation related to the reduced–order CARE is derived. Utilizing the new formulation and the asymptotic structure of the solutions to CMARE, the approximate Nash strategy is constructed.**

I. INTRODUCTION

The linear quadratic Nash games and their applications have been studied widely in many literatures (see e.g., [1]). It is well–known that in order to obtain a Nash equilibrium strategy, the cross–coupled algebraic Riccati equations (CARE) must be solved. In [2], the Newton–type algorithm for solving the CARE has been applied. In [9], an algorithm that is called the Lyapunov iterations for solving the CARE has been derived. However, these researches have concentrated on determining feedback gain matrices for the 2–players Nash games. It should be noted that it is extremely hard to find the solution of the N–coupled CARE (see e.g., [3] and reference therein) because the required computational workspace is needed to N times of the dimension of the full–systems.

The control problems for the multiparameter singularly perturbed systems (MSPS) have been investigated extensively (see e.g., [5], [6], [7] and reference therein). Recent advance in the numerical algorithms for the singularly perturbed systems (SPS) and MSPS have allowed us to revisit the study on Nash games [8], [10], [11], [12]. The numerical computation approach is very powerful and reliable. It would be used to find feasible solutions and to raise the accuracy of the approximate Nash strategy. However, only 2-players Nash games have been studied. In order to avoid this constraint, it is very important to investigate the extension to N–players Nash games. Recently, the fixed point algorithm for solving the N–coupled CARE that is related to the large–scale weakly coupled systems has been considered [13]. However, since the proof of the convergence is based on the special structure of the weakly coupled systems, there is no convergence proof for the N–coupled CARE of the general systems. Moreover, there are no results for the uniqueness that is related to the solution of the CARE.

In this paper, a new algorithm for solving the N–coupled CARE of the general systems is proposed. Since the new algorithm is based on the fixed point algorithm, the required workspace for computing the solution is the same as each subsystem dimension. Moreover, the convergence and the positive semidefiniteness of the obtained solutions are guaranteed. It should be noted that these properties can be proved by using the successive approximation technique. It is worth pointing out that the proof of the convergence for the general system is newly given. In order to show the validity of the proposed algorithm, the linear quadratic infinite horizon Nash game for the general multiparameter singularly perturbed systems (GMSPS) is applied. After proving the boundedness and asymptotic structure of the solutions of the cross–coupled multiparameter algebraic Riccati equation (CMARE), the reduced–order solutions that is independent of the small parameter ε_i are computed via the new algorithm. Moreover, a new formulation related to the reduced–order CARE is introduced. Particularly, it is newly shown the local uniqueness for the solutions of the CMARE in a neighborhood of the reduced–order CARE. It should be noted that the local uniqueness is proved for the first time. Finally, utilizing the iterative solutions of the reduced–order CMARE, the approximate Nash strategy is constructed.

Notation: The notations used in this paper are fairly standard. **block diag** denotes the block diagonal matrix. $\otimes$ denotes the Kronecker product. $I_n \in \mathbf{R}^{n\times n}$ denotes the identity matrix. $O_{p\times q} \in \mathbf{R}^{p\times q}$ denotes the zero matrix.

II. PRELIMINARY RESULT

Consider a linear time–invariant system

$$\dot{z}(t) = Az(t) + \sum_{i=1}^{N} B_i u_i(t),\ z(0) = z^0, \tag{1}$$

and a quadratic cost function

$$J_i(u_1,\ \dots,u_N) = \frac{1}{2}\int_0^{\infty} [z^T Q_i z + u_i^T R_i u_i]dt, \tag{2}$$
$$Q_i = Q_i^T \geq 0,\ R_i^T = R_i > 0,$$

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H. Mukaidani is with Graduate School of Education, Hiroshima University, 1–1–1 Kagamiyama, Higashi–Hiroshima, 739–8524 Japan. mukaida@hiroshima-u.ac.jp

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where $z(t) \in \mathrm{R}^n$ is a state vector, $u_i(t) \in \mathrm{R}^{m_i}$ are the control inputs. All matrices above are of appropriate dimensions.

The decision makers are required to select the closed loop strategy u_i^*, if they exist, such that

$$\begin{aligned} & J_i(u_1^*, \ \dots, \ u_N^*) \\ \leq \ & J_i(u_1^*, \ \dots, \ u_{i-1}^*, \ u_i, \ u_{i+1}^*, \ \dots, \ u_N^*), \\ & i = 1, \ \dots, N. \end{aligned} \tag{3}$$

The set $(u_1^*, \ \dots, u_N^*)$ are called Nash optimal strategy. Nash inequality shows that u_i^* regulates the state to zero with minimum input and output energy. The following lemma is already known [1].

Lemma 1: There exists an admissible strategy such that the inequality (3) holds iff the following cross–coupled algebraic Riccati equation (CARE) has solution $X_i \geq 0$.

$$\begin{aligned} & X_i \left(A - \sum_{j=1}^{N} S_j X_j \right) + \left(A - \sum_{j=1}^{N} S_j X_j \right)^T X_i \\ & \quad + X_i S_i X_i + Q_i = 0, \end{aligned} \tag{4}$$

where $S_i = B_i R_i^{-1} B_i^T$.

Then the closed–loop linear Nash equilibrium solutions to the problem are given by

$$u_i^*(t) = -R_i^{-1} B_i^T X_i z(t). \tag{5}$$

In order to obtain a Nash equilibrium strategy (5), the CARE (4) need to be solved. Although the algorithm that is based on the fixed point algorithm has been investigated [9], only 2–players Nash games have been studied. Thus, a new algorithm and its convergence proof for solving the N–coupled CARE of the general systems are newly investigated.

It is shown that the algorithm converges to the positive semidefinite stabilizing solutions of (4) under the following control–oriented assumption.

Assumption 1: The triples $(A, \ B_i, \ \sqrt{Q_i}), \ i = 1, \ \dots, N$ are stabilizable and detectable.

Now, let us consider the following fixed point algorithm for solving the CARE (4).

$$\begin{aligned} & X_i^{(n+1)} \left(A - \sum_{j=1}^{N} S_j X_j^{(n)} \right) + \left(A - \sum_{j=1}^{N} S_j X_j^{(n)} \right)^T X_i^{(n+1)} \\ & \quad + X_i^{(n)} S_i X_i^{(n)} + Q_i = 0, \end{aligned} \tag{6}$$

where $X_i^{(0)}$ is the solutions of the following algebraic Riccati equations (AREs).

$$\begin{aligned} & X_1^{(0)} A + A^T X_1^{(0)} - X_1^{(0)} S_1 X_1^{(0)} + Q_1 = 0, \\ & X_2^{(0)} (A - S_1 X_1^{(0)}) + (A - S_1 X_1^{(0)})^T X_2^{(0)} \\ & \quad - X_2^{(0)} S_2 X_2^{(0)} + Q_2 = 0, \\ & \dots \\ & X_N^{(0)} \left(A - \sum_{j=1}^{N-1} S_j X_j^{(0)} \right) + \left(A - \sum_{j=1}^{N-1} S_j X_j^{(0)} \right)^T X_N^{(0)} \\ & \quad - X_N^{(0)} S_N X_N^{(0)} + Q_N = 0. \end{aligned}$$

Theorem 1: Under Assumption 1, the positive semidefinite solutions of the CARE (4) exist. It is obtained by performing the fixed point algorithm (6).

Proof: We give the proof by using the successive approximation technique [15]. Firstly, we take any stabilizable linear strategy $u_i^{(0)}(t, \ z) = -R_i^{-1} B_i^T X_i^{(0)} z(t)$. Then, the following minimization problems need to be considered.

$$\dot{z}(t) = \left(A - \sum_{j=1, \ j \neq i}^{N} S_j X_j^{(0)} \right) z(t) + B_i u_i(t), \tag{7a}$$

$$V_i(z, \ t) = \min_{u_i} \int_t^{\infty} [z^T Q_i z + u_i^T R_i u_i] d\tau. \tag{7b}$$

Corresponding Hamiltonians to the Nash differential games for each control agent are given below.

$$\begin{aligned} & H_i \left(t, \ z, \ u_1^{(0)}, \ \dots, u_i, \ \dots, u_N^{(0)}, \ p_i^{(0)} \right) \\ & = z(t)^T Q_i z(t) + u_i(t)^T R_i u_i(t) \\ & \quad + p_i^{(0)T}(t) \left[\left(A - \sum_{j=1, \ j \neq i}^{N} S_j X_j^{(0)} \right) z(t) + B_i u_i(t) \right], \end{aligned} \tag{8}$$

where

$$\begin{aligned} & \frac{\partial}{\partial z} V_i^{(0)}(z, \ t) = p_i^{(0)}(t), \ i = 1, \ 2, \ \dots, N, \\ & \dot{z}(t) = \left(A - \sum_{j=1}^{N} S_j X_j^{(0)} \right) z(t), \\ & V_i^{(0)}(z, \ t) = \int_t^{\infty} z^T \left(Q_i + X_i^{(0)} S_i X_i^{(0)} \right) z d\tau. \end{aligned}$$

The equilibrium controls must satisfy the following equation

$$\frac{\partial H_i}{\partial u_i} = 0 \ \Rightarrow \ u_i^{(1)}(t, \ z) = -\frac{1}{2} R_i^{-1} B_i^T p_i^{(0)}(t). \tag{9}$$

Note that $\dfrac{\partial}{\partial z} V_i^{(0)}(z, \ t)$ along the system trajectory can be calculated from (10).

$$\frac{\partial}{\partial z} V_i^{(0)}(z, \ t) \cdot \dot{z}(t) = \frac{d}{dt} V_i^{(0)}(z, \ t), \ i = 1, \ 2, \ \dots . \tag{10}$$

In fact, we obtain the following equations (11).

$$\begin{aligned} & \frac{\partial}{\partial z} V_1^{(0)}(z, \ t) \cdot \left(A - \sum_{j=1}^{N} S_j X_j^{(0)} \right) z(t) \\ & = \ -z(t)^T \left(Q_i + X_i^{(0)} S_i X_i^{(0)} \right) z(t). \end{aligned} \tag{11}$$

Assume that these simple partial differential equations (11) have solutions of the following form

$$V_i^{(0)}(z, \ t) = z^T(t) X_i^{(1)} z(t). \tag{12}$$

A partial differentiation to (12) gives

$$\frac{\partial}{\partial z} V_i^{(0)}(z, \ t) = 2 X_i^{(1)} z(t) = p_i^{(0)}(t). \tag{13}$$

Therefore, using (11) and (13), for any $z(t)$ we have

$$X_i^{(1)}\left(A-\sum_{j=1}^{N}S_jX_j^{(0)}\right)+\left(A-\sum_{j=1}^{N}S_jX_j^{(0)}\right)^TX_i^{(1)}$$
$$+X_i^{(0)T}S_iX_i^{(0)}+Q_i=0. \quad (14)$$

Thus, from (9) and (13), we get

$$u_i^{(1)}(t,\ z)=-R_i^{-1}B_iX_i^{(1)}z(t),\ X_i^{(1)}\geq 0. \quad (15)$$

Repeating the above steps, we get $u_i^{(2)}(t,\ z) = -R_i^{-1}B_iX_i^{(2)}z(t),\ X_i^{(2)} \geq 0$. Continuing the same procedure, we get the sequences of the solution matrices. Finally, by using the monotonicity result of the successive approximations and the minimization technique in the negative gradient direction [15], we get a monotone decreasing sequence

$$V_i^{(n+1)}(z,\ t) \quad \leq \quad V_i^{(n)}(z,\ t), \quad (16)$$

where $V_i^{(n)}(z,\ t)\geq 0$. Thus, these sequences (6) are convergent. Note that the sequences $p_i^{(n)}(t)$ and $u_i^{(n)}(t,\ z)$ are also convergent, since $\dfrac{\partial}{\partial z}V_i^{(n)}(z,\ t) = p_i^{(n)}(t),\ u_i^{(n)}(t,\ z) = -\dfrac{1}{2}R_i^{-1}B_i^Tp_i^{(n-1)}(t)$. Consequently, from the method of successive approximations [15], the convergence proof is completed.

Second, we prove that $X_i^{(n)}$ is positive semidefinite and $\mathcal{A}(n):=A-\sum_{j=1}^{N}S_jX_j^{(n)}$ is stable. The first stage is to prove that $\mathcal{A}(n)$ is stable. The proof is done by using mathematical induction. When $n=0$, $\mathcal{A}(0)$ is stable because $X_i^{(0)}$ is the stabilizing solution of the ARE. Next $n=k$, we assume that $\mathcal{A}(k)$ is stable. Substituting $n=k$ into (7) instead of $n=0$, the minimization problem (7) produce a stabilizing control given by

$$u_i^{(k+1)}(t,\ z)=-R_i^{-1}B_i^TX_i^{(k+1)}z(t). \quad (17)$$

It is obvious from the method of successive approximations [15] that $\mathcal{A}(k+1)$ is stable since it is the stable matrix of the closed–loop system. Thus, $\mathcal{A}(n)$ is stable for all $n\in\mathbf{N}$. The next stage is to prove that $X_i^{(n)}$ is positive semidefinite matrix. This proof is also done by using mathematical induction. When $n=0$, it is obvious that $X_i^{(n)}$ is positive semidefinite matrix because $X_i^{(0)}$ is the positive semidefinite solution of the ARE. Next $n=k$, we assume that $X_i^{(k)}$ is positive semidefinite matrix. Using the Lyapunov theory [14] and fact that $\mathcal{A}(n)$ is stable, $X_i^{(k+1)}$ is positive semidefinite matrix. Thus, $X_i^{(n)}$ is positive semidefinite matrix for all $n\in\mathbf{N}$. Consequently, the proof of Theorem 1 is completed. ■

It should be noted that the convergence rate of the fixed point algorithm is unclear to date.

III. PROBLEM FORMULATION

In order to show the validity of the proposed algorithm (6), the linear quadratic infinite horizon Nash game for the MSPS is applied. Consider a linear time–invariant GMSPS

$$\dot{x}_0 \ = \ \sum_{i=0}^{N}A_{0i}x_i+\sum_{i=1}^{N}B_{0i}u_i,\ x_0(0)=x_0^0, \quad (18a)$$
$$\varepsilon_i\dot{x}_i \ = \ A_{i0}x_0+A_{ii}x_i+B_{ii}u_i,\ x_i(0)=x_i^0, \quad (18b)$$

with the quadratic cost functions

$$J_i(u_1,\ \dots,u_N)=\frac{1}{2}\int_0^\infty[y_i^Ty_i+u_i^TR_{ii}u_i]dt, \quad (19a)$$
$$y_i=C_{i0}x_0+C_{ii}x_i=C_ix, \quad (19b)$$
$$x=\left[\begin{array}{cccc} x_0^T & x_1^T & \cdots & x_N^T\end{array}\right]^T, \quad (19c)$$
$$R_{ii}>0,\ i=1,\ \dots,N,$$

where $x_i\in\mathbf{R}^{n_i},\ i=0,\ 1,\ \dots,N$ are the state vectors, $u_i\in\mathbf{R}^{m_i},\ i=1,\ \dots,N$ are the control inputs, $y_i\in\mathbf{R}^{l_i},\ i=0,\ 1,\ \dots,N$ are the outputs. It is supposed that the ratios of the small positive parameter $\varepsilon_i>0, i=1,\ \dots,N$ are bounded by some positive constants $\underline{k}_{ij},\ \bar{k}_{ij}$ (see e.g., [5], [6]),

$$0<\underline{k}_{ij}\leq\alpha_{ij}\equiv\frac{\varepsilon_j}{\varepsilon_i}\leq\bar{k}_{ij}<\infty. \quad (20)$$

Furthermore, it is also assumed that the limit of α_{ij} exists as ε_i and ε_j tend to zero, that is

$$\bar{\alpha}_{ij}=\lim_{\substack{\varepsilon_j\to+0\\ \varepsilon_i\to+0}}\alpha_{ij}. \quad (21)$$

Let us introduce the partitioned matrices

$$A:=\left[\begin{array}{cc} A_{00} & A_{0f}\\ A_{f0} & A_f\end{array}\right],$$
$$A_{0f}:=\left[\begin{array}{ccc} A_{01} & \cdots & A_{0N}\end{array}\right],\ A_{f0}:=\left[\begin{array}{ccc} A_{10}^T & \cdots & A_{N0}^T\end{array}\right]^T,$$
$$A_f:=\textbf{block diag}\left(\begin{array}{ccc} A_{11} & \cdots & A_{NN}\end{array}\right),$$
$$B_1:=\left[\begin{array}{c} B_{10}\\ B_{11}\\ 0\\ \vdots\\ 0\end{array}\right],\ B_i:=\left[\begin{array}{c} B_{i0}\\ \vdots\\ B_{ii}\\ \vdots\\ 0\end{array}\right],\ B_N:=\left[\begin{array}{c} B_{0N}\\ 0\\ 0\\ \vdots\\ B_{NN}\end{array}\right],$$
$$S_i:=B_iR_{ii}^{-1}B_i^T$$
$$=\left[\begin{array}{cccccccc} S_{i00} & 0 & \cdots & 0 & S_{i0i} & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ S_{i0i}^T & 0 & \cdots & 0 & S_{iii} & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\end{array}\right],$$

$$Q_i := C_i C_i^T = \begin{bmatrix} Q_{i00} & 0 & \cdots & 0 & Q_{i0i} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ Q_{i0i}^T & 0 & \cdots & 0 & Q_{iii} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Without loss of generality, the following basic assumptions (see e.g., [7], [9]) are made.

Assumption 2:
1) The triples $(A,\ B_i,\ C_i)$, $i = 1,\ \ldots,N$ are stabilizable and detectable.
2) The triples $(A_{ii},\ B_{ii},\ C_{ii})$, $i = 1,\ \ldots,N$ are stabilizable and detectable.

The decision makers are required to select the closed loop strategy u_i^*, if they exist, such that (3) holds. The following lemma is already known [12], [13].

Lemma 2: There exists an admissible strategy such that the inequality (3) holds iff the generalized cross–coupled multiparameter algebraic Riccati equations (GCMAREs)

$$P_i^T \left(A - \sum_{j=1}^{N} S_j P_j \right) + \left(A - \sum_{j=1}^{N} S_j P_j \right)^T P_i + P_i^T S_i P_i + Q_i = 0, \quad (22)$$

have solutions $P_{ie} := \Phi_e P_i \geq 0$, where

$$\Phi_e := \textbf{block diag}\left(I_{n_0} \quad \varepsilon_1 I_{n_1} \quad \cdots \quad \varepsilon_N I_{n_N} \right),$$
$$\Pi_e := \textbf{block diag}\left(\varepsilon_1 I_{n_1} \quad \cdots \quad \varepsilon_N I_{n_N} \right),$$
$$P_i := \begin{bmatrix} P_{i00} & P_{if0}^T \Pi_e \\ P_{if0} & P_{if} \end{bmatrix},\ P_{i00} = P_{i00}^T,$$
$$P_{if0} := \begin{bmatrix} P_{i10} \\ \vdots \\ P_{iN0} \end{bmatrix},\ \Pi_e P_{if} = P_{if}^T \Pi_e,$$
$$P_{if} := \begin{bmatrix} P_{i11} & \alpha_{i12} P_{i21}^T & \alpha_{i13} P_{i31}^T & \cdots & \alpha_{i1N} P_{iN1}^T \\ P_{i21} & P_{i22} & \alpha_{i23} P_{i32}^T & \cdots & \alpha_{i2N} P_{iN2}^T \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{i(N-1)1} & P_{i(N-1)2} & P_{i(N-1)3} & \cdots & \alpha_{i(N-1)N} P_{iN(N-1)}^T \\ P_{iN1} & P_{iN2} & P_{iN3} & \cdots & P_{iNN} \end{bmatrix}.$$

Then the closed–loop linear Nash equilibrium solutions to the full–order problem are given by

$$u_i^*(t) = -R_{ii}^{-1} B_i^T P_i x(t). \quad (23)$$

It should be noted that it is impossible to solve the GCMARE (22) if the small perturbed parameter ε_i are unknown. In fact, it is well–known that the small perturbed parameter ε_i are often not known exactly [5], [6]. Thus, our purpose is to find the approximate Nash strategies.

IV. ASYMPTOTIC STRUCTURE

Nash equilibrium strategies for the GMSPS will be studied under the following basic assumption.

Assumption 3: The Hamiltonian matrices T_{iii}, $i = 1,\ \ldots\ N$ are nonsingular, where

$$T_{iii} := \begin{bmatrix} A_{ii} & -S_{iii} \\ -Q_{iii} & -A_{ii}^T \end{bmatrix}. \quad (24)$$

Under Assumptions 2 and 3, using the similar technique in [12], the following zeroth–order equations of the GCMARE (22) are given as $\|\mu\| := \sqrt{\varepsilon_1^2 + \cdots + \varepsilon_N^2} \to +0$ where $\mu := [\varepsilon_1\ \cdots\ \varepsilon_N]^T$. Moreover, the new formulation related to the reduced–order CARE is also given.

$$\bar{P}_{i00} \left(A_s - \sum_{j=1}^{N} S_{s_j} \bar{P}_{j00} \right) + \left(A_s - \sum_{j=1}^{N} S_{s_j} \bar{P}_{j00} \right)^T \bar{P}_{i00} + \bar{P}_{i00}^T S_{s_i} \bar{P}_{i00} + Q_{s_i} = 0, \quad (25a)$$

$$A_{ii}^T \bar{P}_{iii} + \bar{P}_{iii} A_{ii} - \bar{P}_{iii} S_{iii} \bar{P}_{iii} + Q_{iii} = 0, \quad (25b)$$

$$\bar{P}_{ikl} = 0,\ k > l,\ \bar{P}_{ijj} = 0,\ i \neq j \quad (25c)$$

$$\begin{bmatrix} \bar{P}_{110} & \bar{P}_{210} & \cdots & \bar{P}_{N10} \end{bmatrix} = \begin{bmatrix} \bar{P}_{111} \\ -I_{n_1} \end{bmatrix}^T T_{111}^{-1} T_{101} \begin{bmatrix} I_{n_0} & 0 & \cdots & 0 \\ \bar{P}_{100} & \bar{P}_{200} & \cdots & \bar{P}_{N00} \end{bmatrix}, \quad (25d)$$

$$\begin{bmatrix} \bar{P}_{120} & \bar{P}_{220} & \cdots & \bar{P}_{N20} \end{bmatrix} = \begin{bmatrix} \bar{P}_{222} \\ -I_{n_2} \end{bmatrix}^T T_{222}^{-1} T_{202} \begin{bmatrix} 0 & I_{n_0} & \cdots & 0 \\ \bar{P}_{100} & \bar{P}_{200} & \cdots & \bar{P}_{N00} \end{bmatrix}, \quad (25e)$$

$$\vdots$$

$$\begin{bmatrix} \bar{P}_{1N0} & \bar{P}_{2N0} & \cdots & \bar{P}_{NN0} \end{bmatrix} = \begin{bmatrix} \bar{P}_{NNN} \\ -I_{n_N} \end{bmatrix}^T T_{NNN}^{-1} T_{N0N} \begin{bmatrix} 0 & 0 & \cdots & I_{n_0} \\ \bar{P}_{100} & \bar{P}_{200} & \cdots & \bar{P}_{N00} \end{bmatrix}, \quad (25f)$$

where

$$\begin{bmatrix} A_s & \star \\ \star & -A_s^T \end{bmatrix} = \begin{bmatrix} A_{00} & \star \\ \star & -A_{00}^T \end{bmatrix} - \sum_{i=1}^{N} T_{i0i} T_{iii}^{-1} T_{ii0},$$

$$\begin{bmatrix} \star & -S_{s_i} \\ -Q_{s_i} & \star \end{bmatrix} = T_{i00} - T_{0ii} T_{iii}^{-1} T_{i0i},$$

$$T_{i00} = \begin{bmatrix} A_{00} & -S_{i00} \\ -Q_{i00} & -A_{00}^T \end{bmatrix},\ T_{i0i} = \begin{bmatrix} A_{0i} & -S_{i0i} \\ -Q_{i0i} & -A_{i0}^T \end{bmatrix},$$

$$T_{ii0} = \begin{bmatrix} A_{i0} & -S_{i0i}^T \\ -Q_{i0i}^T & -A_{0i}^T \end{bmatrix},\ i = 1,\ \ldots,N.$$

The following theorem shows the relation between the solutions P_i and the zeroth–order solutions $\bar{P}_{ikl}$ $i = 1,\ \ldots,N,\ k \geq l,\ 0 \leq k,\ l \leq N$.

Theorem 2: Suppose that the condition (26) of the top of this page holds. Under Assumptions 2 and 3, there is a neighborhood $\mathcal{V}(0)$ of $\|\mu\| = 0$ such that for all $\|\mu\| \in \mathcal{V}(0)$ there exists a solution $P_i = P_i(\varepsilon_1,\ \ldots,\varepsilon_N)$. These solutions are unique in a neighborhood of $\bar{P}_i = P_i(0,\ \ldots,0)$. Then, the GCMARE (22) possess the power series expansion at

$$\det \begin{bmatrix} \hat{A}_s^T \otimes I_{n_0} + I_{n_0} \otimes \hat{A}_s^T & -(S_{s_2}\bar{P}_{100}) \otimes I_{n_0} - I_{n_0} \otimes (S_{s_2}\bar{P}_{100}) & \cdots & -(S_{s_N}\bar{P}_{100}) \otimes I_{n_0} - I_{n_0} \otimes (S_{s_N}\bar{P}_{100}) \\ -(S_{s_1}\bar{P}_{200}) \otimes I_{n_0} - I_{n_0} \otimes (S_{s_1}\bar{P}_{200}) & \hat{A}_s^T \otimes I_{n_0} + I_{n_0} \otimes \hat{A}_s^T & \cdots & -(S_{s_N}\bar{P}_{200}) \otimes I_{n_0} - I_{n_0} \otimes (S_{s_N}\bar{P}_{200}) \\ \vdots & \vdots & \vdots & \vdots \\ -(S_{s_1}\bar{P}_{N00}) \otimes I_{n_0} - I_{n_0} \otimes (S_{s_1}\bar{P}_{N00}) & -(S_{s_2}\bar{P}_{N00}) \otimes I_{n_0} - I_{n_0} \otimes (S_{s_2}\bar{P}_{N00}) & \cdots & \hat{A}_s^T \otimes I_{n_0} + I_{n_0} \otimes \hat{A}_s^T \end{bmatrix} \neq 0, \quad (26)$$

where $\hat{A}_s := A_s - \sum_{j=1}^{N} S_{s_j}\bar{P}_{j00}$ and $\hat{A}_s$ are stable matrix.

$\|\mu\| = 0$. That is, the following form is satisfied.

$$P_i = \bar{P}_i + O(\|\mu\|) = \begin{bmatrix} \bar{P}_{i00} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \bar{P}_{i10} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{P}_{ii0} & 0 & \cdots & 0 & \bar{P}_{iii} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{P}_{iN0} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} + O(\|\mu\|). \quad (27)$$

Proof: First, zeroth-order solutions for the asymptotic structure of GCMARE (22) are established. Under Assumption 3, the following equality holds.

$$\begin{bmatrix} A_{ii} & -S_{iii} \\ -Q_{ii} & -A_{iii}^T \end{bmatrix} = \begin{bmatrix} I_{n_i} & 0 \\ \bar{P}_{iii} & I_{n_i} \end{bmatrix} \begin{bmatrix} \hat{A}_{ii} & -S_{ii} \\ 0 & -\hat{A}_{iii}^T \end{bmatrix} \begin{bmatrix} I_{n_i} & 0 \\ -\bar{P}_{iii} & I_{n_i} \end{bmatrix},$$

where $\hat{A}_{iii} := A_{ii} - S_{iii}\bar{P}_{iii}$. Since T_{iii} is nonsingular, $\hat{A}_{iii}$ is also nonsingular. This means that T_{iii}^{-1} can be expressed explicitly in terms of $\hat{A}_{iii}^{-1}$. Therefore, using the above result, the formulations (25) are obtained. These transformations can be done by the lengthy, but direct algebraic manipulations, which are omitted here.

For the local uniqueness of the solutions $P_i = P_i(\varepsilon_1, \ldots, \varepsilon_N)$, it is enough to verify that the corresponding Jacobian is nonsingular at $\|\mu\| = 0$. Formally calculating the derivative of the GCMARE (22) and after some tedious algebra, the left-hand side of (26) is obtained. Setting $\|\mu\| = 0$ and using (25), the condition (26) is obtained. Finally, the implicit function theorem implies that there is a unique solutions map $P_i = P_i(\varepsilon_1, \ldots, \varepsilon_N)$ and a neighborhood $\mathcal{V}(0)$ of $\|\mu\| = 0$ because the condition (26) is equivalent to the corresponding Jacobian at $\|\mu\| = 0$. ■

It is well–known that the GCMARE (22) could have several positive definite solutions and even some indefinite solutions [4]. However, since the implicit function theorem admits the unique solution at the neighborhood of $\|\mu\| = 0$, the uniqueness of these solutions with the form (11) are guaranteed if the optimal solutions exist.

Although the proposed linear equations (25a) seem to be formidable because these equations are coupled with other solutions $\bar{P}_{i00}$, these equations can be solved easily by applying the proposed algorithm (6).

Let us consider the following algorithm (28) that is based on the fixed point algorithm (6) for solving the linear equations (25a).

$$\bar{P}_{i00}^{(n+1)}\left(A_s - \sum_{j=1}^{N} S_{s_j}\bar{P}_{j00}^{(n)}\right) + \left(A_s - \sum_{j=1}^{N} S_{s_j}\bar{P}_{j00}^{(n)}\right)^T \bar{P}_{i00}^{(n+1)} + \bar{P}_{i00}^{(n)T} S_{s_i}\bar{P}_{i00}^{(n)} + Q_{s_i} = 0, \quad (28)$$

where

$$A_s^T\bar{P}_{100}^{(0)} + \bar{P}_{100}^{(0)}A_s - \bar{P}_{100}^{(0)}S_{s_1}\bar{P}_{100}^{(0)} + Q_{s_1} = 0,$$
$$(A_s - S_{s_1}\bar{P}_{100}^{(0)})^T\bar{P}_{200}^{(0)} + \bar{P}_{200}^{(0)}(A_s - S_{s_1}\bar{P}_{100}^{(0)}) - \bar{P}_{200}^{(0)}S_{s_2}\bar{P}_{200}^{(0)} + Q_{s_2} = 0,$$
$$\cdots$$
$$\bar{P}_{N00}^{(0)}\left(A_s - \sum_{j=1}^{N-1} S_{s_j}\bar{P}_{j00}^{(n)}\right) + \left(A_s - \sum_{j=1}^{N-1} S_{s_j}\bar{P}_{j00}^{(n)}\right)^T \bar{P}_{N00}^{(n+1)} + \bar{P}_{N00}^{(n)}S_{s_N}\bar{P}_{N00}^{(n)} + Q_{s_N} = 0.$$

Lemma 3: Suppose that there exists the unique positive semidefinite stabilizing solution of the CARE (25a). It is obtained by performing fixed point algorithm (28). That is, the algorithm (28) converges to the exact solution of the CARE (25a).

Proof: Since this result is obtained directly by applying the result of Theorem 1, it is omitted. ■

It can be seen that these equations (28) can be solved rather easily due to other solutions independent structure.

V. APPROXIMATE NASH STRATEGY

Using the result (27), the N–order approximate Nash strategy is given.

$$\bar{u}_i(t) := -R_{ii}^{-1}B_i^T\bar{P}_i x(t), \ i = 1, \ldots, N, \quad (29)$$

Theorem 3: Under Assumptions 2 and 3, the use of the approximate Nash strategy (29) results in $J_i(\bar{u}_1, \ldots, \bar{u}_N)$ satisfying

$$J_i(\bar{u}_1, \ldots, \bar{u}_N) = J_i(u_1^*, \ldots, u_N^*) + O(\|\mu\|), \quad (30)$$
$$i = 1, \ldots, N,$$

where $J_i(u_1^*, \ldots, u_N^*)$ are the optimal equilibrium values of the cost functions (19a).

Proof: When $\bar{u}_i(t)$ is used, the equilibrium value of the cost performances are

$$J_i(\bar{u}_1, \ldots, \bar{u}_N) = x^T(0)X_{ie}x(0), \quad (31)$$

where X_{ie} is the positive semidefinite solution of the following multiparameter algebraic Lyapunov equation (MALE)

$$X_{ie}\left(A_e-\sum_{j=1}^{N}S_{je}\bar{P}_{je}\right)+\left(A_e-\sum_{j=1}^{N}S_{je}\bar{P}_{je}\right)^T X_{ie} +Q_i+\bar{P}_{ie}S_{ie}\bar{P}_{ie}=0, \quad (32)$$

with $A_e=\Phi_e^{-1}A$, $S_{ie}=\Phi_e^{-1}S_i\Phi_e^{-1}$ and $P_{ie}=\Phi_eP_{ie}$.

Subtracting (22) from (32), it is easy to verify that $V_{ie}=X_{ie}-P_{ie}$, $P_{ie}:=\Phi_eP_i$ satisfies the following MALE.

$$\begin{aligned}&V_{ie}\left(A_e-\sum_{j=1}^{N}S_{je}\bar{P}_{je}\right)+\left(A_e-\sum_{j=1}^{N}S_{je}\bar{P}_{je}\right)^T V_{ie}\\&+(\bar{P}_{ie}-P_{ie})S_{ie}(\bar{P}_{ie}-P_{ie})\\&+\sum_{j=1,\ j\neq i}^{N}P_{ie}S_{je}(\bar{P}_{je}-P_{je})\\&+\sum_{j=1,\ j\neq i}^{N}(\bar{P}_{je}-P_{je})S_{je}P_{ie}=0.\end{aligned} \quad (33)$$

Using the relation $\bar{P}_{ie}-P_{ie}=O(\|\mu\|)$, the following MALE holds.

$$V_{ie}\left(A_e-\sum_{j=1}^{N}S_{je}\bar{P}_{je}\right)+\left(A_e-\sum_{j=1}^{N}S_{je}\bar{P}_{je}\right)^T V_{ie} =O(\|\mu\|). \quad (34)$$

Thus, it is easy to verify that $V_{ie}=O(\|\mu\|)$ because $A_e-\sum_{j=1}^{N}S_{je}\bar{P}_{je}$ is stable by using the standard Lyapunov theorem [14] for sufficiently small $\|\mu\|$. Consequently, the equality (30) holds. ■

Although ε_i is unknown, it is possible to design the approximate Nash strategy which achieves the $O(\|\mu\|)$ approximation for the equilibrium value of the cost functional.

VI. CONCLUSION

In this paper, the new algorithm for solving CARE has been proposed. Since the new algorithm is based on the fixed point algorithm, it is possible to compute the solution with each subsystem dimension. Furthermore, the convergence and the positive semidefiniteness of the solutions have been newly proved. After establishing the new numerical algorithm, the linear quadratic infinite horizon Nash game for the GMSPS has been investigated. As a result, we have succeeded in obtaining the approximate Nash strategy. As another important feature, the new formulation related to the reduced-order CARE has been extended to N–players Nash games. Furthermore, it has also been proved that the local uniqueness for the solutions of the CMARE in a neighborhood of the reduced–order CARE is attained. It would be worth pointing out that the local uniqueness of the original solutions is proved for the first time.

Regarding the reduced order CARE, it is desirable to discuss in more details the computational amount compared with the full order solution. This feature will be investigated in the near future.

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