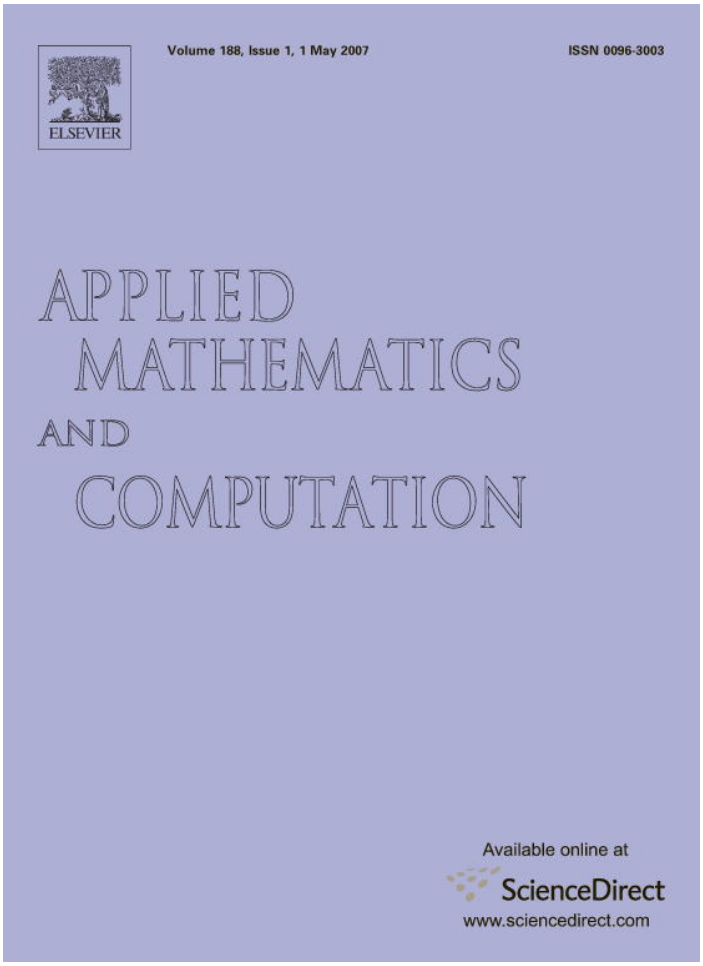

Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 188 (2007) 103–115
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

Newton's method for solving cross-coupled sign-indefinite algebraic Riccati equations for weakly coupled large-scale systems ☆

Hiroaki Mukaidani

Graduate School of Education, Hiroshima University, 1-1-1, Kagamiyama, Higashi-Hiroshima 739-8524, Japan

Abstract

In this paper, a new algorithm for solving cross-coupled sign-indefinite algebraic Riccati equations (CSAREs) for weakly coupled large-scale systems is proposed. It is shown that since the proposed algorithm is based on the Newton's method, the quadratic convergence is attained. Moreover, the local uniqueness of the convergence solutions for the CSAREs is investigated. Finally, in order to overcome the computation of large- and sparse-matrix related to the Newton's method, the fixed point algorithm and the alternating direction implicit (ADI) method are combined.

Keywords: Weakly coupled large-scale systems; Cross-coupled sign-indefinite algebraic Riccati equations (CSAREs); Newton's method; Newton–Kantorovich theorem; Fixed point algorithm; Fixed point theorem; Alternating direction implicit (ADI) method

1. Introduction

The robust equilibria in indefinite linear quadratic differential games under the disturbance input affecting the systems have been discussed in [1]. It is well known that in order to obtain the equilibrium strategy, the cross-coupled sign-indefinite algebraic Riccati equations (CSAREs) must be solved. In [2], the numerical algorithm that is based on the calculation of the eigenstructure for solving the soft-constrained Nash equilibria has been developed. However, the scalar case has only been considered. Moreover, the convergence rate is unclear. On the other hand, the Newton-type algorithm for solving the CSAREs seems to be reliable. However, it is well-known that if the initial conditions are not chosen adequately, the algorithm may not converge because the Newton's method guarantees the local convergence.

The control problems of weakly coupled large-scale systems have been studied by several researchers (see [1–10] and references therein). In particular, the Nash games for such systems have been investigated via the Lyapunov iterations [8,9]. However, the convergence speed is slow because the Lyapunov iterations have

☆ This work was supported in part by the Research Foundation for the Electrotechnology of Chubu (REFEC) and a Grant-in-Aid for Young Scientists Research (B)-18700013 from the Ministry of Education, Culture, Sports, Science and Technology of Japan.
E-mail address: mukaida@hiroshima-u.ac.jp

0096-3003/$ - see front matter
doi:10.1016/j.amc.2006.09.100

linear convergence. Moreover, the uniqueness of the convergence solutions for the CSAREs have not been discussed so far.

This paper investigates the numerical algorithm for solving the CSAREs of weakly coupled large-scale systems. The main contribution is to propose a new algorithm that is based on the Newton's method. After deriving the asymptotic structure of the CSAREs and taking into account such structure, the initial condition of the Newton's method is given. As a result, it is shown that the new algorithm has a quadratic convergence property even if the CSAREs has the sign-indefinite quadratic term [8,9]. Additionally, the existence and the local uniqueness of the solutions is proved via the Newton–Kantorovich theorem. As another important features, in order to overcome the computation of large- and sparse-matrix that arises in the Newton's method, the fixed point algorithm [7] and the alternating direction implicit (ADI) method [13,14] are combined. Finally, in order to demonstrate the efficiency of the algorithm, a computational example is included.

Notation: The notations used in this paper are fairly standard. The superscript T denotes the matrix transpose. I_n denotes the $n \times n$ identity matrix. **block diag** denotes the block diagonal matrix. $\|\cdot\|$ denotes its Euclidean norm for a matrix. $\otimes$ denotes the Kronecker product. δ_{ij} denotes the Kronecker delta. vec$M$ denotes the column vector of the matrix M. The space of $\boldsymbol{R}^k$-valued functions that are quadratically integrable on $(0,\infty)$ is denoted by $L_2^k(0,\infty)$.

2. Problem formulation

Consider the weakly coupled large-scale linear systems with N-players:

$$\begin{aligned}\dot{x}_i(t) &= A_{ii}x_i(t) + B_{ii}u_i(t) + \varepsilon \sum_{j=1,\ j\neq i}^{N} A_{ij}x_j(t) + \varepsilon \sum_{j=1,\ j\neq i}^{N} B_{ij}u_j(t) + E_{ii}w_i(t) + \varepsilon \sum_{j=1,\ j\neq i}^{N} E_{ij}w_j(t),\\ x_i(0) &= x_i^0, \quad i = 1,\ldots,N,\end{aligned} \tag{1}$$

where $x_i \in \boldsymbol{R}^{n_i}$, $i = 1,\ldots,N$ represent ith state vectors. $u_i \in \boldsymbol{R}^{m_i}$, $i = 1,\ldots,N$ represent ith control inputs. $w_i \in \boldsymbol{R}^{k_i}$, $i = 1,\ldots,N$ represent ith disturbance vectors. ε denotes a small positive weak coupling parameter which connect the other subsystems.

Let us introduce the partitioned matrices

$$A_\varepsilon := \begin{bmatrix} A_{11} & \varepsilon A_{12} & \cdots & \varepsilon A_{1N} \\ \varepsilon A_{21} & A_{22} & \cdots & \varepsilon A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon A_{N1} & \varepsilon A_{N2} & \cdots & A_{NN} \end{bmatrix},$$

$$B_{i\varepsilon} := \begin{bmatrix} \varepsilon^{1-\delta_{1i}} B_{1i} \\ \varepsilon^{1-\delta_{2i}} B_{2i} \\ \vdots \\ \varepsilon^{1-\delta_{Ni}} B_{Ni} \end{bmatrix},$$

$$E_\varepsilon := \begin{bmatrix} E_{11} & \varepsilon E_{12} & \cdots & \varepsilon E_{1N} \\ \varepsilon E_{21} & E_{22} & \cdots & \varepsilon E_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon E_{N1} & \varepsilon E_{N2} & \cdots & E_{NN} \end{bmatrix}.$$

By using above relations, the system (1) can be changed as

$$\dot{x}(t) = A_\varepsilon x(t) + \sum_{i=1}^{N} B_{i\varepsilon}u_i(t) + E_\varepsilon w(t), \tag{2}$$

where

$$x(t) := \left[x_1(t)^{\mathrm{T}}, \ldots, x_N(t)^{\mathrm{T}}\right]^{\mathrm{T}} \in \boldsymbol{R}^{\bar{n}}, \quad \bar{n} := \sum_{i=1}^{N} n_i,$$

$$w(t) := \left[w_1(t)^{\mathrm{T}}, \ldots, w_N(t)^{\mathrm{T}}\right]^{\mathrm{T}} \in \boldsymbol{R}^{\bar{k}}, \quad \bar{k} := \sum_{i=1}^{N} k_i.$$

The cost performance for each strategy subset is defined by

$$J_i(u_1, \ldots, u_N,\ w,\ x(0)) = \int_0^{\infty} \left[x^{\mathrm{T}}(t) Q_{i\varepsilon} x(t) + u_i^{\mathrm{T}}(t) R_{ii} u_i(t) + \mu \sum_{j=1,\ j\neq i}^{N} u_j^{\mathrm{T}}(t) R_{ij} u_j(t) - w^{\mathrm{T}}(t) V_{i\mu} w(t)\right] \mathrm{d}t, \tag{3}$$

where

$$Q_{i\varepsilon} = \begin{bmatrix} \varepsilon^{1-\delta_{i1}} Q_{i1} & \varepsilon Q_{i12} & \cdots & \varepsilon Q_{i1N} \\ \varepsilon Q_{i12}^{\mathrm{T}} & \varepsilon^{1-\delta_{i2}} Q_{i2} & \cdots & \varepsilon Q_{i2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon Q_{i1N}^{\mathrm{T}} & \varepsilon Q_{i2N}^{\mathrm{T}} & \cdots & \varepsilon^{1-\delta_{iN}} Q_{iN} \end{bmatrix} \in \boldsymbol{R}^{\bar{n}\times\bar{n}},$$

$$R_{ii} = R_{ii}^{\mathrm{T}} > 0 \in \boldsymbol{R}^{m_i \times m_i}, \quad R_{ij} = R_{ij}^{\mathrm{T}} \geqslant 0 \in \boldsymbol{R}^{m_j \times m_j},$$

$$V_{i\mu} = \textbf{block diag} \quad (\mu^{-(1-\delta_{i1})} V_{i1}, \ldots, \mu^{-(1-\delta_{iN})} V_{iN}) \geqslant 0 \in \boldsymbol{R}^{\bar{k}\times\bar{k}}, \quad i, j = 1, \ldots, N.$$

The state weight matrices $Q_{i\varepsilon}$ is symmetric and assumed to be sign-indefinite [1]. Furthermore, it should be noted that μ denotes a small positive parameter which is the same order for the parameter ε. That is, the following assumption is made.

Assumption 1. The ratio of the small positive parameters ε and μ is bounded by some positive constants $\tilde{k}$.

$$0 < \tilde{k} := \frac{\mu}{\varepsilon} < \infty. \tag{4}$$

For the matrices A_ε, $B_{i\varepsilon}$, $i = 1, \ldots, N$, the set $\mathscr{F}_N$ is defined by

$$\mathscr{F}_N := \left\{ (F_{1\varepsilon}, \ldots, F_{N\varepsilon}) \middle| A_\varepsilon + \sum_{j=1}^{N} B_{j\varepsilon} F_{j\varepsilon} \quad \text{is stable.} \right\}.$$

The soft-constrained Nash equilibrium strategy pair $(F_{1\varepsilon}^*, \ldots, F_{N\varepsilon}^*)$ is defined as satisfying the following conditions [1]:

$$\bar{J}_i(F_{1\varepsilon}^*, \ldots, F_{N\varepsilon}^*, x(0)) \leqslant \bar{J}_i(F_{1\varepsilon}^*, \ldots, F_{i-1\varepsilon}^*, F_{i\varepsilon}, F_{i+1\varepsilon}^*, \ldots, F_{N\varepsilon}^*, x(0)), \quad i = 1, \ldots, N, \tag{5}$$

where

$$\begin{aligned} \bar{J}_i(F_{1\varepsilon}, \ldots, F_{N\varepsilon}, x(0)) &:= \sup_{w \in L_2^{\bar{k}}(0,\infty)} J_i(F_{1\varepsilon}, \ldots, F_{N\varepsilon}, w, x(0)), J_i(F_{1\varepsilon}, \ldots, F_{N\varepsilon}, w, x(0)) \\ &= \int_0^{\infty} \left[x^{\mathrm{T}}(t) \left[Q_{i\varepsilon} + F_{i\varepsilon}^{\mathrm{T}} R_{ii} F_{i\varepsilon} + \mu \sum_{j=1,\ j\neq i}^{N} F_{j\varepsilon}^{\mathrm{T}} R_{ij} F_{j\varepsilon}\right] x(t) - w^{\mathrm{T}}(t) V_{i\mu} w(t)\right] \mathrm{d}t, \end{aligned}$$

for all $x(0)$ and for all $(F_{1\varepsilon}, \ldots, F_{N\varepsilon})$ that satisfy

$$\left(F_{1\varepsilon}^*,\ \ldots,\ F_{i-1\varepsilon}^*,\ F_{i\varepsilon},\ F_{i+1\varepsilon}^*,\ \ldots,\ F_{N\varepsilon}^*\right) \in \mathscr{F}_N.$$

It should be noted that the following assumption guarantees the existence of the admissible strategy.

Assumption 2. Each player uses the linear feedback strategy $u_i(t) = K_{i\varepsilon} x(t)$, $i = 1, \ldots, N$ such that the closed-loop system is asymptotically stable for sufficiently small parameters ε and μ.

Obviously, this assumption is made in order to obtain a stable system. Using the fact studied by [1], the soft-constrained feedback Nash equilibrium is given below.

Lemma 1. *Assume that there exist N real symmetric matrices $P_{i\varepsilon}$ and $W_{i\varepsilon}$, such that*

$$\begin{aligned}\mathscr{G}_i(P_{1\varepsilon},\ldots,P_{N\varepsilon}) := {} & P_{i\varepsilon}\left(A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon}P_{j\varepsilon}\right) + \left(A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon}P_{j\varepsilon}\right)^{\mathrm{T}} P_{i\varepsilon} + P_{i\varepsilon}S_{i\varepsilon}P_{i\varepsilon} + \mu \sum_{j=1,\ j\neq i}^{N} P_{j\varepsilon}S_{ij\varepsilon}P_{j\varepsilon} \\ & + P_{i\varepsilon}M_{i\mu}P_{i\varepsilon} + Q_{i\varepsilon} = 0, \end{aligned} \tag{6}$$

where

$$S_{i\varepsilon} := B_{i\varepsilon}R_{ii}^{-1}B_{i\varepsilon}^{\mathrm{T}}, \quad S_{ij\varepsilon} := B_{j\varepsilon}R_{jj}^{-1}R_{ij}R_{jj}^{-1}B_{j\varepsilon}^{\mathrm{T}}, \quad M_{i\mu} := E_\varepsilon V_{i\mu}^{-1}E_\varepsilon^{\mathrm{T}}.$$

$A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon}P_{j\varepsilon} + M_{i\mu}P_{i\varepsilon}$ is stable for $i = 1,\ldots,N$, $A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon}P_{j\varepsilon}$ is stable,

$$W_{i\varepsilon}\left(A_\varepsilon - \sum_{j=1,\ j\neq i}^{N} S_{j\varepsilon}P_{j\varepsilon}\right) + \left(A_\varepsilon - \sum_{j=1,\ j\neq i}^{N} S_{j\varepsilon}P_{j\varepsilon}\right)^{\mathrm{T}} W_{i\varepsilon} - W_{i\varepsilon}S_{i\varepsilon}W_{i\varepsilon} + \mu \sum_{j=1,\ j\neq i}^{N} P_{j\varepsilon}S_{ij\varepsilon}P_{j\varepsilon} + Q_{i\varepsilon} \geqslant 0. \tag{7}$$

Define the N-tuple $(F_{1\varepsilon}^*,\ldots,F_{N\varepsilon}^*)$ by

$$u_i^*(t) := F_{i\varepsilon}^* x(t) = -R_{ii}^{-1}B_{i\varepsilon}^{\mathrm{T}}P_{i\varepsilon}x(t), \quad i = 1,\ldots,N. \tag{8}$$

Then, $(F_{1\varepsilon}^*,\ldots,F_{N\varepsilon}^*) \in \mathscr{F}_N$ and this N-tuple is a soft-constrained Nash equilibrium. Furthermore, $\overline{J}_i(F_{1\varepsilon}^*,\ldots,F_{N\varepsilon}^*, x(0)) = x(0)^T P_{i\varepsilon}x(0)$.

It should be noted that if $Q_{i\varepsilon} \geqslant 0$ and $S_{ij\varepsilon} \geqslant 0$ for all $i = 1,\ldots,N$, the matrix inequality (7) is trivially satisfied with $W_{i\varepsilon} = 0$ [1]. Then, only the CSAREs (6) should be solved.

In the following analysis, the basic assumption is needed.

Assumption 3. The triples $(A_{ii}, B_{ii}, \sqrt{Q_{ii}})$, $i = 1,\ldots,N$ are stabilizable and detectable.

3. Asymptotic structure of the CSAREs

Firstly, in order to obtain the strategy, the asymptotic structure of the CSAREs (6) is established. Since A_ε, $S_{i\varepsilon}$, $S_{ij\varepsilon}$ and $M_{i\mu}$ include the term of the small parameters ε and μ, the solution $P_{i\varepsilon}$ of the CSAREs (6), if it exists, must contain these parameters. Moreover, it should be noted that two parameters ε and μ are the same magnitude such that Assumption 1 holds. Taking these facts into account, the solution $P_{i\varepsilon}$ of the CSAREs (6) with the following structure is considered [4,8,9].

$$P_{i\varepsilon} := \begin{bmatrix} \varepsilon^{1-\delta_{i1}}P_{i1} & \varepsilon P_{i12} & \cdots & \varepsilon P_{i1N} \\ \varepsilon P_{i12}^{\mathrm{T}} & \varepsilon^{1-\delta_{i2}}P_{i2} & \cdots & \varepsilon P_{i2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon P_{i1N}^{\mathrm{T}} & \varepsilon P_{i2N}^{\mathrm{T}} & \cdots & \varepsilon^{1-\delta_{iN}}P_{iN} \end{bmatrix} \in \boldsymbol{R}^{\bar{n}\times\bar{n}}.$$

Substituting the matrices A_ε, $S_{i\varepsilon}$, $S_{ij\varepsilon}$, $M_{i\mu}$, $Q_{i\varepsilon}$ and $P_{i\varepsilon}$ into the CSAREs (6), letting $\varepsilon = 0$ and $\mu = 0$, and partitioning the CSAREs (6), the following reduced-order algebraic Riccati equations (AREs) are obtained, where $\overline{P}_{ii}$, $i = 1,\ldots,N$ be the 0-order solutions of the CSAREs (6) as $\varepsilon = \mu = 0$.

$$\overline{P}_{ii}A_{ii} + A_{ii}^{\mathrm{T}}\overline{P}_{ii} - \overline{P}_{ii}(S_{ii} - M_{ii})\overline{P}_{ii} + Q_{ii} = 0, \tag{9}$$

where $S_{ii} := B_{ii}R_{ii}^{-1}B_{ii}^{\mathrm{T}}$ and $M_{ii} := E_{ii}V_{ii}^{-1}E_{ii}^{\mathrm{T}}$.

In order to guarantee the existence of a positive semidefinite stabilizing solution of the ARE (9), the following condition is assumed.

Assumption 4. The ARE (9) has a positive semidefinite stabilizing solution such that $A_{ii} - S_{ii}\overline{P}_{ii}$ is stable.

The asymptotic expansion of the CSAREs (6) at $\varepsilon = \mu = 0$ is described by the following lemma.

Lemma 2. *Under Assumptions 1–4, there exist the small constants σ^* and ρ^* such that for all $\varepsilon \in (0,\sigma^*)$ and $\mu \in (0,\rho^*)$, the CSAREs (6) admits a unique positive semidefinite solution $P_{i\varepsilon}^*$ that can be written as*

$$P_{i\varepsilon} := P_{i\varepsilon}^* = \bar{P}_i + \mathrm{O}(\varepsilon) = \textbf{block diag}(\,0 \quad \cdots \quad \overline{P}_{ii} \quad \cdots \quad 0\,) + \mathrm{O}(\varepsilon). \tag{10}$$

Proof. The proof can be derived by using the implicit function theorem [7] for the CSAREs (6). Using the implicit function theorem, it can be shown that there exists a neighbourhood of $\varepsilon = \mu = 0$ and a unique function $P_{i\varepsilon} := \bar{P}_i + \mathrm{O}(\varepsilon)$. It should be noted that under Assumption 4, since the solution of the reduced-order ARE (9) is unique (see, e.g., Theorem 13.5 of [11]), $\bar{P}_i$ is a unique solution. Therefore, the CSAREs (6) has a unique positive semidefinite solution $P_{i\varepsilon}^*$ under the sufficiently small parameters ε and μ. □

4. Newton's method for solving CSAREs

In order to obtain the solution of CSAREs (6), the following useful algorithm is given. Consider the following iterative algorithm.

$$\begin{aligned}
&P_{i\varepsilon}^{(k+1)}\left(A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon}P_{j\varepsilon}^{(k)} + M_{i\mu}P_{i\varepsilon}^{(k)}\right) + \left(A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon}P_{j\varepsilon}^{(k)} + M_{i\mu}P_{i\varepsilon}^{(k)}\right)^{\mathrm{T}} P_{i\varepsilon}^{(k+1)} \\
&\quad - \sum_{j=1,\ j\neq i}^{N} P_{j\varepsilon}^{(k+1)} S_{j\varepsilon}P_{i\varepsilon}^{(k)} - \sum_{j=1,\ j\neq i}^{N} P_{i\varepsilon}^{(k)} S_{j\varepsilon}P_{j\varepsilon}^{(k+1)} + \mu \sum_{j=1,\ j\neq i}^{N} P_{j\varepsilon}^{(k+1)} S_{ij\varepsilon}P_{j\varepsilon}^{(k)} + \mu \sum_{j=1,\ j\neq i}^{N} P_{j\varepsilon}^{(k)} S_{ij\varepsilon}P_{j\varepsilon}^{(k+1)} \\
&\quad + \sum_{j=1,\ j\neq i}^{N} P_{j\varepsilon}^{(k)} S_{j\varepsilon}P_{i\varepsilon}^{(k)} + \sum_{j=1,\ j\neq i}^{N} P_{i\varepsilon}^{(k)} S_{j\varepsilon}P_{j\varepsilon}^{(k)} + P_{i\varepsilon}^{(k)} S_{i\varepsilon}P_{i\varepsilon}^{(k)} - \mu \sum_{j=1,\ j\neq i}^{N} P_{j\varepsilon}^{(k)} S_{ij\varepsilon}P_{j\varepsilon}^{(k)} \\
&\quad - P_{i\varepsilon}^{(k)} M_{i\mu}P_{i\varepsilon}^{(k)} + Q_{i\varepsilon} = 0, \quad k = 0,1,\ldots,
\end{aligned} \tag{11a}$$

$$P_{i\varepsilon}^{(k)} := \begin{bmatrix} \varepsilon^{1-\delta_{i1}}P_{i1}^{(k)} & \varepsilon P_{i12}^{(k)} & \cdots & \varepsilon P_{i1N}^{(k)} \\ \varepsilon P_{i12}^{(k)T} & \varepsilon^{1-\delta_{i2}}P_{i2}^{(k)} & \cdots & \varepsilon P_{i2N}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon P_{i1N}^{(k)T} & \varepsilon P_{i2N}^{(k)T} & \cdots & \varepsilon^{1-\delta_{iN}}P_{iN}^{(k)} \end{bmatrix} \tag{11b}$$

with the initial conditions

$$P_{i\varepsilon}^{(0)} = \overline{P}_i = \textbf{block diag}(\,0 \quad \cdots \quad \overline{P}_{ii} \quad \cdots \quad 0\,). \tag{12}$$

The algorithm (11a) can be constructed by setting $P_{i\varepsilon}^{(k+1)} = P_{i\varepsilon}^{(k)} + \Delta P_{i\varepsilon}^{(k)}$ and neglecting $\mathrm{O}(\Delta^2)$ term. The following theorem indicates that the algorithm (11a) is Newton's method.

Theorem 1. *Suppose that there exist a solution to the CSAREs (6). It can be obtained by performing the algorithm (11) which is equal to the Newton's method.*

Proof. Taking the vec-operator transformation on both sides of (11a) and $\mathscr{G}_i = \mathscr{G}_i(P_{1\varepsilon}^{(k)},\ldots,P_{N\varepsilon}^{(k)}) = 0$ and subtracting these equations, it is easy to verify the following equation:

$$\begin{bmatrix} \mathrm{vec}P_{1\varepsilon}^{(k+1)} \\ \vdots \\ \mathrm{vec}P_{N\varepsilon}^{(k+1)} \end{bmatrix} = \begin{bmatrix} \mathrm{vec}P_{1\varepsilon}^{(k)} \\ \vdots \\ \mathrm{vec}P_{N\varepsilon}^{(k)} \end{bmatrix} - \left[\nabla\mathscr{G}\left(P_{1\varepsilon}^{(k)},\ldots,P_{N\varepsilon}^{(k)}\right)\right]^{-1} \times \begin{bmatrix} \mathrm{vec}\mathscr{G}_1\left(P_{1\varepsilon}^{(k)},\ldots,P_{N\varepsilon}^{(k)}\right) \\ \vdots \\ \mathrm{vec}\mathscr{G}_N\left(P_{1\varepsilon}^{(k)},\ldots,P_{N\varepsilon}^{(k)}\right) \end{bmatrix}, \tag{13}$$

where

$$\nabla\mathscr{G}(P_{1\varepsilon},\ \ldots,\ P_{N\varepsilon}) = \frac{\partial([\mathrm{vec}\mathscr{G}_1]^{\mathrm{T}},\ldots,[\mathrm{vec}\mathscr{G}_N]^{\mathrm{T}})^{\mathrm{T}}}{\partial([\mathrm{vec}P_{1\varepsilon}]^{\mathrm{T}},\ldots,[\mathrm{vec}P_{N\varepsilon}]^{\mathrm{T}})}, \quad \mathscr{G} := \mathscr{G}(\mathscr{G}_1,\ldots,\mathscr{G}_N).$$

This is the desired result. □

The following theorem indicates that the proposed algorithm (11) which is based on the Newton's method attains the quadratic convergence.

Theorem 2. *Under Assumptions 1–4, there exist the small constants $\bar{\sigma}$ and $\bar{\rho}$ such that for all $\varepsilon \in (0,\bar{\sigma}), \bar{\sigma} \leqslant \sigma^*$ and $\mu \in (0,\bar{\rho}), \bar{\rho} \leqslant \rho^*$, the iterative algorithm (11) converges to the exact solution of $P^*_{i\varepsilon}$ with the rate of the quadratic convergence, where $P^{(k)}_{i\varepsilon}$ is positive semidefinite matrix and $A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon}P^{(k)}_{j\varepsilon} + M_{i\mu}P^{(k)}_{i\varepsilon}$ is stable. Moreover, the convergence solutions attain a local unique solution $P^*_{i\varepsilon}$ of the CSAREs (6) in the neighborhood of the initial condition $P^{(0)}_{i\varepsilon} = \overline{P}_i$. That is, the following conditions are satisfied.*

$$\|P^{(k)}_{i\varepsilon} - P^*_{i\varepsilon}\| = \mathrm{O}(\varepsilon^{2^k}), \tag{14a}$$

$$\mathrm{Re}\lambda\left[A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon}P^{(k)}_{j\varepsilon} + M_{i\mu}P^{(k)}_{i\varepsilon}\right] < 0,\ \ k = 0,1,\ldots. \tag{14b}$$

In order to prove the theorem, the following fact must be needed.

4.1. Newton–Kantorovich theorem [12]

Assume that $F{:}\boldsymbol{R}^n \to \boldsymbol{R}^n$ is differentiable on a convex set D. Suppose that the inverse of map F exists and moreover it is differentiable on set D and that $\|F'(\boldsymbol{x}) - F'(\boldsymbol{y})\| \leqslant \gamma\|\boldsymbol{x} - \boldsymbol{y}\|$ for all $\boldsymbol{x},\boldsymbol{y} \in D$. Suppose that there is an $\boldsymbol{x}^0 \in D$ such that $\|F'(\boldsymbol{x}^0)^{-1}\| \leqslant \beta$, $\|F'(\boldsymbol{x}^0)^{-1}F(\boldsymbol{x}^0)\| \leqslant \eta$ and $\theta := \beta\gamma\eta < 1/2$. Assume that $S := \{\boldsymbol{x}: \|\boldsymbol{x} - \boldsymbol{x}^0\| \leqslant t^*\} \subset D$, $t^* = \frac{1-\sqrt{1-2\theta}}{\beta\gamma}$. Then Newton iterations $\boldsymbol{x}^{k+1} = \boldsymbol{x}^k - F'(\boldsymbol{x}^k)^{-1}F(\boldsymbol{x}^k)$, $k = 0,1,\ldots$, are well defined and converge to a solution $\boldsymbol{x}^*$ of $F(\boldsymbol{x}) = 0$ in S. Moreover, the solution $\boldsymbol{x}^*$ is unique in $\tilde{S} \cap D$, where $\tilde{S} := \{\boldsymbol{x}: \|\boldsymbol{x} - \boldsymbol{x}^0\| \leqslant \tilde{t}\} \subset D$, $\tilde{t} = \frac{1+\sqrt{1-2\theta}}{\beta\gamma}$ and error estimate is given by $\|\boldsymbol{x}^* - \boldsymbol{x}^k\| \leqslant \frac{(2\theta)^{2^k}}{2^k\beta\gamma} = 2^{1-k}(2\theta)^{2^k-1}\eta$, $k = 0,1,\ldots$.

Proof. The proof is given directly by applying the Newton–Kantorovich theorem [12] for the CSAREs (6). It is immediately obtained from the CSAREs (6) that there exists a positive scalar γ such that for any $P^a_{i\varepsilon}$ and $P^b_{i\varepsilon}$

$$\left\|\nabla\mathscr{G}\left(P^a_{1\varepsilon},\ldots,P^a_{N\varepsilon}\right) - \nabla\mathscr{G}\left(P^b_{1\varepsilon},\ldots,P^b_{N\varepsilon}\right)\right\| \leqslant \gamma\left\|\left(\left[\mathrm{vec}P^a_{1\varepsilon}\right]^{\mathrm{T}},\ldots,\left[\mathrm{vec}P^a_{N\varepsilon}\right]^{\mathrm{T}}\right) - \left(\left[\mathrm{vec}P^b_{1\varepsilon}\right]^{\mathrm{T}},\ldots,\left[\mathrm{vec}P^b_{N\varepsilon}\right]^{\mathrm{T}}\right)\right\|. \tag{15}$$

Moreover, it is easy to verify that

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{11}|_{\varepsilon=0} & \cdots & \mathbf{J}_{1N}|_{\varepsilon=0} \\ \vdots & \ddots & \vdots \\ \mathbf{J}_{N1}|_{\varepsilon=0} & \cdots & \mathbf{J}_{NN}|_{\varepsilon=0} \end{bmatrix} = \begin{bmatrix} \mathscr{D}_A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathscr{D}_A \end{bmatrix}, \tag{16}$$

where

$$\mathbf{J}_{ij} = \frac{\partial\mathrm{vec}\mathscr{G}_i}{\partial[\mathrm{vec}P_{j\varepsilon}]^{\mathrm{T}}}, \mathscr{D}_A = \mathbf{block\ diag}(\,\mathbf{D}_{11} \quad \cdots \quad \mathbf{D}_{NN}\,),$$

$$\mathbf{D}_{ii} := D_{ii}^{\mathrm{T}} \otimes I_{n_i} + I_{n_i} \otimes D_{ii}^{\mathrm{T}}, D_{ii} := A_{ii} - (S_{ii} - M_{ii})\overline{P}_{ii}.$$

Thus, since $\mathbf{J}$ is nonsingular under Assumption 4, for small ε and μ,

$$\nabla\mathscr{G}\left(P^{(0)}_{1\varepsilon},\ldots,P^{(0)}_{N\varepsilon}\right) = \nabla\mathscr{G}\left(\overline{P}_1,\ldots,\overline{P}_N\right) = \mathbf{J} + \mathrm{O}(\varepsilon)$$

is also nonsingular. Therefore, there exists β such that $\beta = \|[\nabla\mathscr{G}(\overline{P}_1,\ldots,\overline{P}_N)]^{-1}\|$. On the other hand, since $\|\mathscr{G}(\overline{P}_1,\ldots,\overline{P}_N)\| = \mathrm{O}(\varepsilon)$, there exists η such that $\eta = \|[\nabla\mathscr{G}(\overline{P}_1,\ldots,\overline{P}_N)]^{-1}\| \cdot \|\mathscr{G}(\overline{P}_1,\ldots,\overline{P}_N)\| = \mathrm{O}(\varepsilon)$. Thus, there exists θ such that $\theta = \beta\eta\gamma < 2^{-1}$ because $\eta = \mathrm{O}(\varepsilon)$. Finally, the Newton–Kantorovich theorem results in the desired results (14).

Second, the local uniqueness of the solution is discussed. Now, let us define $t^* \equiv \frac{1}{\gamma\beta}\left[1 - \sqrt{1-2\theta}\right]$. Clearly, $S \equiv \{P_{i\varepsilon} : \|P_{i\varepsilon} - P^{(0)}_{i\varepsilon}\| \leqslant t^*\}$ is in the convex set D. In the sequel, since $\|P_{i\varepsilon} - P^{(0)}_{i\varepsilon}\| = \mathrm{O}(\varepsilon)$ holds for a small ε, the local uniqueness of $P^*_{i\varepsilon}$ is guaranteed in the neighbourhood of $\varepsilon = \mu = 0$ for a subset S by applying the Newton–Kantorovich theorem. □

5. A numerical algorithm for solving the large-scale Lyapunov equations (CLALEs)

When the cross-coupled large-scale algebraic Lyapunov equations (11a) is solved, the existence of the cross-coupled term

$$-\sum_{j=1,\ j\neq i}^{N} P_{j\varepsilon}^{(k+1)} S_{j\varepsilon} P_{i\varepsilon}^{(k)} - \sum_{j=1,\ j\neq i}^{N} P_{i\varepsilon}^{(k)} S_{j\varepsilon} P_{j\varepsilon}^{(k+1)} + \mu \sum_{j=1,\ j\neq i}^{N} P_{j\varepsilon}^{(k+1)} S_{ij\varepsilon} P_{j\varepsilon}^{(k)} + \mu \sum_{j=1,\ j\neq i}^{N} P_{j\varepsilon}^{(k)} S_{ij\varepsilon} P_{j\varepsilon}^{(k+1)}$$

in CLALEs (11a) makes it difficult to solve this equation directly due to the large-dimension as $N \times \bar{n}$ larger than the dimensions $\bar{n} := \sum_{i=1}^{N} n_i$. Thus, in order to avoid the cross-coupled term, a new decoupling algorithm that is based on the fixed point algorithm [7] is established. Taking into account the fact that $S_{j\varepsilon} P_{i\varepsilon}^{(k)} = \mathrm{O}(\varepsilon),\ i \neq j$, let us consider CLALEs (17) in its general form.

$$X_{i\varepsilon} \Lambda_{i\varepsilon} + \Lambda_{i\varepsilon}^{\mathrm{T}} X_{i\varepsilon} + \varepsilon \sum_{j=1,\ j\neq i}^{N} \left(X_{j\varepsilon} \Phi_{j\varepsilon} + \Phi_{j\varepsilon}^{T} X_{j\varepsilon} \right) + U_{i\varepsilon} = 0, \quad i = 1, \ldots, N, \tag{17}$$

where

$$X_{i\varepsilon} := \begin{bmatrix} \varepsilon^{1-\delta_{i1}} X_{i1} & \varepsilon X_{i12} & \cdots & \varepsilon X_{i1N} \\ \varepsilon X_{i12}^{\mathrm{T}} & \varepsilon^{1-\delta_{i2}} X_{i2} & \cdots & \varepsilon X_{i2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon X_{i1N}^{\mathrm{T}} & \varepsilon X_{i2N}^{\mathrm{T}} & \cdots & \varepsilon^{1-\delta_{iN}} X_{iN} \end{bmatrix},$$

$$\Lambda_{i\varepsilon} := \begin{bmatrix} \Lambda_{i1} & \varepsilon \Lambda_{i12} & \cdots & \varepsilon \Lambda_{i1N} \\ \varepsilon \Lambda_{i21} & \Lambda_{i2} & \cdots & \varepsilon \Lambda_{i2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon \Lambda_{iN1} & \varepsilon \Lambda_{iN2} & \cdots & \Lambda_{iN} \end{bmatrix},$$

$$\Phi_{i\varepsilon} := \begin{bmatrix} \Phi_{i1} & \varepsilon \Phi_{i12} & \cdots & \varepsilon \Phi_{i1N} \\ \varepsilon \Phi_{i21} & \Phi_{i2} & \cdots & \varepsilon \Phi_{i2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon \Phi_{iN1} & \varepsilon \Phi_{iN2} & \cdots & \Phi_{iN} \end{bmatrix},$$

$$U_{i\varepsilon} := \begin{bmatrix} \varepsilon^{1-\delta_{i1}} U_{i1} & \varepsilon U_{i12} & \cdots & \varepsilon U_{i1N} \\ \varepsilon U_{i12}^{\mathrm{T}} & \varepsilon^{1-\delta_{i2}} U_{i2} & \cdots & \varepsilon U_{i2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon U_{i1N}^{\mathrm{T}} & \varepsilon U_{i2N}^{\mathrm{T}} & \cdots & \varepsilon^{1-\delta_{iN}} U_{iN} \end{bmatrix}.$$

It should be noted that

$$P_{i\varepsilon}^{(k+1)} \Rightarrow X_{i\varepsilon}, P_{j\varepsilon}^{(k+1)} \Rightarrow X_{j\varepsilon}, A_{\varepsilon} - \sum_{j=1}^{N} S_{j\varepsilon} P_{j\varepsilon}^{(k)} + M_{i\mu} P_{i\varepsilon}^{(k)} \Rightarrow \Lambda_{i\varepsilon}, -S_{j\varepsilon} P_{i\varepsilon}^{(k)} + \mu S_{ij\varepsilon} P_{j\varepsilon}^{(k)} \Rightarrow \varepsilon \Phi_{j\varepsilon},$$

$$\sum_{j=1,\ j\neq i}^{N} P_{i\varepsilon}^{(k)} S_{j\varepsilon} P_{j\varepsilon}^{(k)} + \sum_{j=1,\ j\neq i}^{N} P_{j\varepsilon}^{(k)} S_{j\varepsilon} P_{i\varepsilon}^{(k)} + P_{i\varepsilon}^{(k)} S_{i\varepsilon} P_{i\varepsilon}^{(k)} - \mu \sum_{j=1,\ j\neq i}^{N} P_{j\varepsilon}^{(k)} S_{ij\varepsilon} P_{j\varepsilon}^{(k)} + Q_{i\varepsilon} \Rightarrow U_{i\varepsilon},$$

where $\Rightarrow$ represents the replacement. Without loss of generality, the following condition is assumed for CLALEs (17).

Assumption 5. $\Lambda_{i1}, \ldots, \Lambda_{iN}$, $i = 1, \ldots, N$ are stable.

The algorithm (18) for solving CLALEs (17) is given as follows:

$$X_{i\varepsilon}^{(n+1)}\Lambda_{i\varepsilon}+\Lambda_{i\varepsilon}^{\mathrm{T}}X_{i\varepsilon}^{(n+1)}+\varepsilon\sum_{j=1,\ j\neq i}^{N}(X_{j\varepsilon}^{(n)}\Phi_{j\varepsilon}+\Phi_{j\varepsilon}^{\mathrm{T}}X_{j\varepsilon}^{(n)})+U_{i\varepsilon}=0,\quad i=1,\ldots,N,\quad n=0,1,\ldots, \tag{18}$$

where

$$X_{i\varepsilon}^{(0)}=0,\quad i=1,\ldots,N.$$

It should be noted that the numerical algorithm (18) can be carried out independently for each solution. The following theorem indicates the convergence of algorithm (18).

Theorem 3. *Under Assumption 5, there exists the small constant $\tilde{\sigma}$ such that for all $\varepsilon\in(0,\tilde{\sigma})$, the fixed point algorithm (18) converges to an exact solution $X_{i\varepsilon}$ with a linear convergence.*

Proof. The CLALEs (17) can be changed as follows.

$$\boldsymbol{\Lambda}(\varepsilon)\begin{bmatrix}\mathrm{vec}X_{1\varepsilon}^{(n+1)}\\ \mathrm{vec}X_{2\varepsilon}^{(n+1)}\\ \vdots\\ \mathrm{vec}X_{N\varepsilon}^{(n+1)}\end{bmatrix}=-\varepsilon\begin{bmatrix}0&\boldsymbol{\Phi}_{2\varepsilon}&\cdots&\boldsymbol{\Phi}_{N\varepsilon}\\ \boldsymbol{\Phi}_{1\varepsilon}&0&\cdots&\boldsymbol{\Phi}_{N\varepsilon}\\ \vdots&\vdots&\ddots&\vdots\\ \boldsymbol{\Phi}_{1\varepsilon}&\boldsymbol{\Phi}_{2\varepsilon}&\cdots&0\end{bmatrix}\begin{bmatrix}\mathrm{vec}X_{1\varepsilon}^{(n)}\\ \mathrm{vec}X_{2\varepsilon}^{(n)}\\ \vdots\\ \mathrm{vec}X_{N\varepsilon}^{(n)}\end{bmatrix}-\begin{bmatrix}\mathrm{vec}U_{1\varepsilon}\\ \mathrm{vec}U_{2\varepsilon}\\ \vdots\\ \mathrm{vec}U_{N\varepsilon}\end{bmatrix}, \tag{19}$$

where

$$\boldsymbol{\Lambda}(\varepsilon):=\textbf{block diag}[\,\boldsymbol{\Lambda}_{1\varepsilon}\quad\boldsymbol{\Lambda}_{2\varepsilon}\quad\cdots\quad\boldsymbol{\Lambda}_{N\varepsilon}\,],\quad\boldsymbol{\Lambda}_{i\varepsilon}:=\Lambda_{i\varepsilon}^{\mathrm{T}}\otimes I_{\bar{n}}+I_{\bar{n}}\otimes\Lambda_{i\varepsilon}^{\mathrm{T}},$$

and

$$\boldsymbol{\Phi}_{i\varepsilon}:=\Phi_{i\varepsilon}^{\mathrm{T}}\otimes I_{\bar{n}}+I_{\bar{n}}\otimes\Phi_{i\varepsilon}^{\mathrm{T}}.$$

Since Assumption 5 holds, for sufficient small ε there exists $[\boldsymbol{\Lambda}(\varepsilon)]^{-1}$ because

$$\lim_{\varepsilon\to+0}\boldsymbol{\Lambda}(\varepsilon)=\textbf{block diag}[\,\boldsymbol{\Lambda}_1\quad\boldsymbol{\Lambda}_2\quad\cdots\quad\boldsymbol{\Lambda}_N\,], \tag{20}$$

where $\boldsymbol{\Lambda}_i:=\boldsymbol{\Lambda}_{i\varepsilon}|_{\varepsilon=0}$.

Therefore, it is easy to verify that there exists the small constant $\tilde{\sigma}$ such that for all $\varepsilon\in(0,\tilde{\sigma})$:

$$\varepsilon\|[\boldsymbol{\Lambda}(\varepsilon)]^{-1}\|\left\|\begin{bmatrix}0&\boldsymbol{\Phi}_{2\varepsilon}&\cdots&\boldsymbol{\Phi}_{N\varepsilon}\\ \boldsymbol{\Phi}_{1\varepsilon}&0&\cdots&\boldsymbol{\Phi}_{N\varepsilon}\\ \vdots&\vdots&\ddots&\vdots\\ \boldsymbol{\Phi}_{1\varepsilon}&\boldsymbol{\Phi}_{2\varepsilon}&\cdots&0\end{bmatrix}\right\|<1. \tag{21}$$

Finally, using the fixed point theorem, it can be shown that the algorithm (18) attains the linear convergence. □

When each algebraic Lyapunov equation (ALE) (18) is solved, the dimension of the workspace as $\bar{n}:=\sum_{i=1}^{N}n_i$ larger than the dimensions n_i is needed. Thus, in order to reduce the dimension of the workspace, a new algorithm for solving the ALE (18) which is based on the alternating direction implicit (ADI) method [13,14] is established. Let us consider the following ALE (22), in a general form of the ALE (18).

$$\mathscr{X}_\varepsilon\Psi_\varepsilon+\Psi_\varepsilon^{T}\mathscr{X}_\varepsilon+\mathscr{U}_\varepsilon=0. \tag{22}$$

In particular, the following special matrices $\mathscr{X}_\varepsilon$, Ψ_ε and $\mathscr{U}_\varepsilon$ which are related to the CLALEs (17) are considered because the other case $i=2,\ldots,N$ can be changed into the similar form by using the similarity transformation $\mathscr{T}_i$, where

$$\mathscr{X}_\varepsilon:=\mathscr{T}_i^{-1}X_{i\varepsilon}^{(n+1)}\mathscr{T}_i,\quad\Psi_\varepsilon:=\mathscr{T}_i^{-1}\Lambda_{i\varepsilon}\mathscr{T}_i,$$

$$\mathscr{U}_\varepsilon := \mathscr{T}_i^{-1}\left[\varepsilon \sum_{j=1,\ j\neq i}^{N}\left(X_{j\varepsilon}^{(n)}\Phi_{j\varepsilon} + \Phi_{j\varepsilon}^{T}X_{j\varepsilon}^{(n)}\right) + U_{i\varepsilon}\right]\mathscr{T}_i,$$

$$\mathscr{T}_i := \begin{bmatrix} 0 & \cdots & I_{n_i} & \cdots & 0 \\ \vdots & \textbf{block diag}(1\ldots1) & \vdots & \ddots & \vdots \\ I_{n_i} & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \textbf{block diag}(1\ldots1) & \vdots \\ 0 & \cdots & 0 & \cdots & I_{n_N} \end{bmatrix},$$

$$\mathscr{X}_\varepsilon := \begin{bmatrix} \mathscr{X}_{11} & \varepsilon\mathscr{X}_{12} & \cdots & \varepsilon\mathscr{X}_{1N} \\ \varepsilon\mathscr{X}_{12}^{\mathrm{T}} & \varepsilon\mathscr{X}_{22} & \cdots & \varepsilon\mathscr{X}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon\mathscr{X}_{1N}^{\mathrm{T}} & \varepsilon\mathscr{X}_{2N}^{\mathrm{T}} & \cdots & \varepsilon\mathscr{X}_{NN} \end{bmatrix} := \begin{bmatrix} \mathscr{X}_{11} & \varepsilon\mathscr{X}_{1f} \\ \varepsilon\mathscr{X}_{1f}^{\mathrm{T}} & \varepsilon\mathscr{X}_{f} \end{bmatrix},$$

$$\Psi_\varepsilon := \begin{bmatrix} \Psi_{11} & \varepsilon\Psi_{12} & \cdots & \varepsilon\Psi_{1N} \\ \varepsilon\Psi_{21} & \Psi_{22} & \cdots & \varepsilon\Psi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon\Psi_{N1} & \varepsilon\Psi_{N2} & \cdots & \Psi_{NN} \end{bmatrix} := \begin{bmatrix} \Psi_{11} & \varepsilon\Psi_{1f} \\ \varepsilon\Psi_{f1} & \Psi_{f} \end{bmatrix},$$

$$\mathscr{U}_\varepsilon := \begin{bmatrix} \mathscr{U}_{11} & \varepsilon\mathscr{U}_{12} & \cdots & \varepsilon\mathscr{U}_{1N} \\ \varepsilon\mathscr{U}_{12}^{\mathrm{T}} & \varepsilon\mathscr{U}_{22} & \cdots & \varepsilon\mathscr{U}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon\mathscr{U}_{1N}^{\mathrm{T}} & \varepsilon\mathscr{U}_{2N}^{\mathrm{T}} & \cdots & \varepsilon\mathscr{U}_{NN} \end{bmatrix} := \begin{bmatrix} \mathscr{U}_{11} & \varepsilon\mathscr{U}_{1f} \\ \varepsilon\mathscr{U}_{1f}^{\mathrm{T}} & \varepsilon\mathscr{U}_{f} \end{bmatrix}.$$

In order to guarantee the existence of the solution and the convergence of the algorithm, another assumption is needed.

Assumption 6. $\Psi_{11},\ldots,\Psi_{NN}$ are stable.

As a result, the ALE (22) can be changed as follows by partitioning.

$$\mathscr{X}_{11}\Psi_{11} + \Psi_{11}^{\mathrm{T}}\mathscr{X}_{11} + \varepsilon^2\left(\mathscr{X}_{1f}\Psi_{f1} + \Psi_{f1}^{\mathrm{T}}\mathscr{X}_{1f}^{\mathrm{T}}\right) + \mathscr{U}_{11} = 0, \tag{23a}$$

$$\mathscr{X}_{f}\Psi_{f} + \Psi_{f}^{\mathrm{T}}\mathscr{X}_{f} + \varepsilon\left(\mathscr{X}_{1f}^{\mathrm{T}}\Psi_{1f} + \Psi_{1f}^{\mathrm{T}}\mathscr{X}_{1f}\right) + \mathscr{U}_{f} = 0, \tag{23b}$$

$$\mathscr{X}_{11}\Psi_{1f} + \mathscr{X}_{1f}\Psi_{f} + \Psi_{11}^{\mathrm{T}}\mathscr{X}_{1f} + \varepsilon\Psi_{f1}^{\mathrm{T}}\mathscr{X}_{f} + \mathscr{U}_{1f} = 0. \tag{23c}$$

Firstly, using the implicit function, the asymptotic structure of the ALE (23) is established.

Lemma 3. *Under Assumption 6, the ALE (23) has unique solutions $\mathscr{X}_{11}$, $\mathscr{X}_{1f}$ and $\mathscr{X}_{f}$ such that these matrices possess a power series expansion at $\varepsilon = 0$. That is,*

$$\mathscr{X}_{11} := \sum_{m=0}^{\infty}\varepsilon^m\mathscr{X}_{11}^{(m)}, \quad \mathscr{X}_{1f} := \sum_{m=0}^{\infty}\varepsilon^m\mathscr{X}_{1f}^{(m)}, \quad \mathscr{X}_{f} := \sum_{m=0}^{\infty}\varepsilon^m\mathscr{X}_{f}^{(m)}. \tag{24}$$

Proof. It can be done by applying the implicit function theorem to the partitioned ALE (23). To do so, it is enough to show that the corresponding Jacobian is nonsingular at $\varepsilon = 0$. Since the detailed proof is the same as the proof that is given by [8], it is omitted. □

Secondly, the methodology for solving the ALE (23) for the matrix solutions $\mathscr{X}_{11}$, $\mathscr{X}_{1f}$ and $\mathscr{X}_{f}$ is given. Substituting the matrices $\mathscr{X}_{11}$, $\mathscr{X}_{1f}$ and $\mathscr{X}_{f}$ into the ALEs (23) and equating successively coefficients of equal powers of ε, the following linear equations are obtained.

$$\mathcal{X}_{11}^{(m+2)}\Psi_{11}+\Psi_{11}^{\mathrm{T}}\mathcal{X}_{11}^{(m+2)}+\mathcal{X}_{1f}^{(m)}\Psi_{f1}+\Psi_{f1}^{\mathrm{T}}\mathcal{X}_{1f}^{(m)\mathrm{T}}=0, \tag{25a}$$

$$\mathcal{X}_{f}^{(m+2)}\Psi_{f}+\Psi_{f}^{\mathrm{T}}\mathcal{X}_{f}^{(m+2)}+\mathcal{X}_{1f}^{(m+1)\mathrm{T}}\Psi_{1f}+\Psi_{1f}^{\mathrm{T}}\mathcal{X}_{1f}^{(m+1)}=0, \tag{25b}$$

$$\mathcal{X}_{1f}^{(m+2)}\Psi_{f}+\Psi_{11}^{\mathrm{T}}\mathcal{X}_{1f}^{(m+2)}+\mathcal{X}_{11}^{(m+2)}\Psi_{1f}+\Psi_{f1}^{\mathrm{T}}\mathcal{X}_{f}^{(m+1)}=0, \tag{25c}$$

where

$$\mathcal{X}_{11}^{(0)}\Psi_{11}+\Psi_{11}^{\mathrm{T}}\mathcal{X}_{11}^{(0)}+\mathcal{U}_{11}=0,\quad \mathcal{X}_{f}^{(0)}\Psi_{f}+\Psi_{f}^{\mathrm{T}}\mathcal{X}_{f}^{(0)}+\mathcal{U}_{f}=0,$$

$$\mathcal{X}_{11}^{(0)}\Psi_{1f}+\mathcal{X}_{1f}^{(0)}\Psi_{f}+\Psi_{11}^{\mathrm{T}}\mathcal{X}_{1f}^{(0)}+\mathcal{U}_{1f}=0,$$

$$\mathcal{X}_{11}^{(1)}=0,\quad \mathcal{X}_{f}^{(1)}\Psi_{f}+\Psi_{f}^{\mathrm{T}}\mathcal{X}_{f}^{(1)}+\mathcal{X}_{1f}^{(0)\mathrm{T}}\Psi_{1f}+\Psi_{1f}^{\mathrm{T}}\mathcal{X}_{1f}^{(0)}=0,$$

$$\mathcal{X}_{1f}^{(1)}\Psi_{f}+\Psi_{11}^{\mathrm{T}}\mathcal{X}_{1f}^{(1)}+\Psi_{f1}^{\mathrm{T}}\mathcal{X}_{f}^{(0)}=0.$$

It should be noted that the successive approximations (25) are independent of the small parameter ε. Moreover, the approach used in this paper is quite different because the proposed successive approximations (25) are based on not the existing algorithm [9] but Maclaurin series expansions. Thus, the desired solutions with any approximation are obtained by solving the linear equations directly.

Let us consider the following Sylvester's equations (26), in a general form of the ALEs (25b) and (25c)

$$AY+YB=C, \tag{26}$$

where the matrices $A\in\boldsymbol{R}^{p\times p}$, $B\in\boldsymbol{R}^{q\times q}$ and $C\in\boldsymbol{R}^{p\times q}$ are given, and the solution matrix $Y\in\boldsymbol{R}^{p\times q}$ is to be determined.

The ADI iterative method [13,14] for the solution of (26) proceeds by strictly alternating between the solution of the two equations

$$(A-\delta_{l+1}I_p)Y_{2l+1}=Y_{2l}(-B-\delta_{l+1}I_q)+C, \tag{27a}$$

$$Y_{2l+2}(-B-\tau_{l+1}I_q)=(A-\tau_{l+1}I_p)Y_{2l+1}-C, \tag{27b}$$

for $l=0,1,2,\ldots$ Here Y_0 is a given initial approximate solution, and the δ_l and τ_l are real or complex parameters chosen so that the computed approximate solutions Y_l converge rapidly to the solution $\hat{Y}$ of the Sylvester equation (26) as l increases.

If the matrices A and B are dense, then the direct solution method by Golub et al. [15] can be used. This method determines the real Schur factorization of A and brings B into Hessenberg form by orthogonal similarity transformation [14]. However, when the matrices $A(=\Psi_f^{\mathrm{T}})$ and $B(=\Psi_f)$ that appear in (25b) are large and sparse, iterative solution technique has to be employed. In fact, it is clear that

$$\Psi_f=\begin{bmatrix}\Psi_{22}&\cdots&\varepsilon\Psi_{2N}\\ \vdots&\ddots&\vdots\\ \varepsilon\Psi_{N2}&\cdots&\Psi_{NN}\end{bmatrix}\rightarrow\begin{bmatrix}\Psi_{22}&\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&\Psi_{NN}\end{bmatrix},\quad(\varepsilon\rightarrow+0)$$

are large and sparse for sufficiently small ε. Thus, since the ADI iterative method is an attractive technique in this case, such method will be used.

6. Computational example

In order to demonstrate the efficiency of the proposed algorithm, a computational example is given. The system matrices are given as follows:

$$A_{11}=\begin{bmatrix}0&1&-0.266&-0.009\\ -2.75&-2.78&-1.36&-0.037\\ 0&0&0&1\\ -4.95&0&-55.5&-0.039\end{bmatrix},$$

$$\varepsilon A_{12} = \begin{bmatrix} 0.0024 & 0 & -0.087 & 0.002 \\ -0.185 & 0 & 1.11 & -0.011 \\ 0 & 0 & 0 & 0 \\ 0.222 & 0 & 8.17 & 0.004 \end{bmatrix},$$

$$\varepsilon A_{13} = \begin{bmatrix} 0.073 & 0 & -0.25 & 0.003 \\ -0.46 & 0 & 2.8 & -0.02 \\ 0 & 0 & 0 & 0 \\ 0.924 & 0 & 17.5 & 0.02 \end{bmatrix},$$

$$\varepsilon A_{21} = \begin{bmatrix} 0.021 & 0 & 0.121 & 0.003 \\ -1.1 & 0 & -1.62 & -0.015 \\ 0 & 0 & 0 & 0 \\ -2.43 & 0 & 1.37 & -0.034 \end{bmatrix},$$

$$A_{22} = \begin{bmatrix} -0.21 & 1 & -1.6 & -0.005 \\ -1.9 & -1.8 & 9.3 & -0.12 \\ 0 & 0 & 0 & 1 \\ -3.1 & 0 & -56 & 0.032 \end{bmatrix},$$

$$\varepsilon A_{23} = \begin{bmatrix} 0.06 & 0 & 0.46 & 0.002 \\ -1 & 0 & 1.49 & -0.04 \\ 0 & 0 & 0 & 0 \\ 0.12 & 0 & 29.8 & -0.028 \end{bmatrix},$$

$$\varepsilon A_{31} = \begin{bmatrix} -0.002 & 0 & 0.83 & 0 \\ -6.78 & 0 & -10.1 & 0.09 \\ 0 & 0 & 0 & 0 \\ -1.24 & 0 & 0.498 & -0.017 \end{bmatrix},$$

$$\varepsilon A_{32} = \begin{bmatrix} 0.011 & 0 & 0.22 & 0 \\ -2.1 & 0 & 1.7 & -0.123 \\ 0 & 0 & 0 & 0 \\ -0.07 & 0 & 6.38 & -0.011 \end{bmatrix},$$

$$A_{33} = \begin{bmatrix} -0.197 & 1 & -1.2 & -0.003 \\ -54.5 & -20 & 70.1 & -2.37 \\ 0 & 0 & 0 & 1 \\ -3.4 & 0 & -21.0 & -0.017 \end{bmatrix},$$

$$B_{11} = \begin{bmatrix} 0 \\ 36.1 \\ 0 \\ 0 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 0 \\ 78.9 \\ 0 \\ 0 \end{bmatrix}, \quad B_{33} = \begin{bmatrix} 0 \\ 1000 \\ 0 \\ 0 \end{bmatrix}, \quad B_{ij} = 0, \quad i \neq j,$$

$$E_{11} = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0.1 \end{bmatrix},$$

Table 1
Error per iterations

k	$\|\mathcal{G}^{(k)}(1.0e-01)\|$	$\|\mathcal{G}^{(k)}(1.0e-02)\|$	$\|\mathcal{G}^{(k)}(1.0e-03)\|$	$\|\mathcal{G}^{(k)}(1.0e-04)\|$
0	3.5262	3.5262e−01	3.5262e−02	3.5262e−03
1	6.1345e−01	4.8088e−03	4.8160e−05	4.8188e−07
2	3.0517e−02	2.2293e−06	2.2402e−10	2.5366e−12
3	1.8473e−05	3.3376e−11	1.0869e−12	
4	2.1005e−11			

$$E_{33} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}, \quad E_{ij} = 0, \quad i \neq j,$$

$$V_{ii} = \mathbf{diag}(1 \quad 2 \quad 2 \quad 1),$$

$$V_1 = \mathbf{block\ diag}\left(V_{ii} \quad \mu^{-1}I_4 \quad \mu^{-1}I_4\right),$$

$$V_2 = \mathbf{block\ diag}\left(\mu^{-1}I_4 \quad V_{ii} \quad \mu^{-1}I_4\right),$$

$$V_3 = \mathbf{block\ diag}\left(\mu^{-1}I_4 \quad \mu^{-1}I_4 \quad V_{ii}\right),$$

$$Q_1 = \mathbf{block\ diag}(0.5I_4 \quad \mathrm{O}_{8\times 8}),$$

$$Q_2 = \mathbf{block\ diag}(\mathrm{O}_{4\times 4} \quad 0.5I_4 \quad \mathrm{O}_{4\times 4}),$$

$$Q_3 = \mathbf{block\ diag}(\mathrm{O}_{8\times 8} \quad 0.5I_4),$$

$$R_{11} = R_{22} = R_{33} = 1, \quad R_{12} = R_{13} = 0.2,$$

$$R_{23} = R_{21} = 0.3, \quad R_{31} = R_{32} = 0.1.$$

The small parameters are chosen as $\varepsilon = 0.01$ and $\mu = 0.005$. It should be noted that the algorithm (11a) converges to the exact solution with accuracy of $\|\mathcal{G}^{(k)}(\varepsilon)\| < 1.0e{-}10$ after three iterations, where

$$\left\|\mathcal{G}^{(k)}(\varepsilon)\right\| := \sum_{i=1}^{3} \left\|\mathcal{G}_i\left(P_{1\varepsilon}^{(k)},\ P_{2\varepsilon}^{(k)},\ P_{3\varepsilon}^{(k)}\right)\right\|. \tag{28}$$

In order to verify the exactitude of the solution, the remainder per iteration by substituting $P_{i\varepsilon}^{(k)}$ into the CSAREs (6) is computed. In Table 1, the results of the error $\|\mathcal{G}^{(k)}(\varepsilon)\|$ per iterations are given for several values ε and $\mu = 0.5\varepsilon$. As a result, it can be seen that the algorithm (11a) has the quadratic convergence.

7. Conclusions

In this paper, a new algorithm for solving the CSAREs for weakly coupled large-scale systems has been proposed. Comparing with the existing result [8,9], the considered equation has the sign-indefinite quadratic term. It is noteworthy that although the proposed design method is based on the Newton's method, the convergence rate has been newly proved as a quadratic convergence. Moreover, the local uniqueness of the convergence solutions for the CSAREs have been proved for the first time by using the Newton–Kantorovich theorem. As another important feature, in order to overcome the computation of large- and sparse-matrix, the fixed point algorithm and the ADI method have been combined. As a result, both fast convergence and a reduced-order calculation are attained. Finally, the computational example has shown the excellent results.

References

[1] W.A.V.D. Broek, J.C. Engwerda, J.M. Schumacher, Robust equilibria in indefinite linear-quadratic differential games, J. Optim. Theory Appl. 119 (3) (2003) 565–595.
[2] J.C. Engwerda, A numerical algorithm to find soft-constrained Nash equilibria in scalar LQ-games, Int. J. Control 79 (6) (2006) 592–603.

[3] J.D. Delacour, M. Darwish, J. Fantin, Control strategies for large-scale power systems, Int. J. Control 27 (5) (1978) 753–767.
[4] X. Shen, Q. Xia, M. Rao, V. Gourishankar, Optimal control for large-scale systems: a recursive approach, Int. J. Syst. Sci. 25 (12) (1994) 2235–2244.
[5] Z. Gajić, I. Borno, General transformation for block diagonalization of weakly coupled linear systems composed of N-subsystems, IEEE Trans. Circ. Syst. I 47 (6) (2000) 909–912.
[6] B. Petrovic, Z. Gajić, The recursive solution of linear quadratic Nash games for weakly interconnected systems, J. Optim. Theory Appl. 56 (3) (1988) 463–477.
[7] Z. Gajić, D. Petkovski, X. Shen, Singularly Perturbed and Weakly Coupled Linear System – A Recursive Approach, Lecture Notes in Control and Information Sciences, vol. 140, Springer-Verlag, Berlin, 1990.
[8] H. Mukaidani, Numerical Computation of Nash Strategy for Large-Scale Systems, in Proc. of 2004 American Control Conference, Boston, June 2004. pp. 5641–5646.
[9] H. Mukaidani, Optimal numerical strategy for Nash games of weakly coupled large-scale systems, Dyn. Continuous, Discrete Impulsive Syst., Ser. B: Appl. Algorithms 13 (2) (2006) 249–268.
[10] H. Mukaidani, Numerical computation for H_2 state feedback control of large-scale systems, Dyn. Continuous, Discrete Impulsive Syst., Ser. B: Appl. Algorithms 12 (2) (2005) 281–296.
[11] K. Zhou, J.C. Doyle, K. Glover, Robust and Optimal Control, Prentice Hall, New Jersey, 1996.
[12] T. Yamamoto, A method for finding sharp error bounds for Newton's method under the Kantorovich assumptions, Numer. Math. 49 (2–3) (1986) 203–220.
[13] T. Penzl, A cyclic low rank smith method for large sparse Lyapunov equations with applications in model reduction and optimal control, SIAM J. Sci. Comput. 21 (4) (1999) 1401–1418.
[14] N. Levenberg, L. Reichel, A generalized ADI iterative method, Numer. Math. 66 (1) (1993) 215–233.
[15] G. Golub, S. Nash, C. Van Loan, A Hessenberg–Schur method for the problem $AX + XB = C$, IEEE Trans. Autom. Control 24 (6) (1979) 909–913.