Available online at www.sciencedirect.com

SCIENCE DIRECT®

automatica

Automatica 39 (2003) 2157–2167

www.elsevier.com/locate/automatica

Brief Paper

New results for near-optimal control of linear multiparameter singularly perturbed systems☆

Hiroaki Mukaidani[a,*], Hua Xu[b], Koichi Mizukami[c]

[a] Graduate School of Education, Hiroshima University, Kagamiyama Higashi-Hiroshima, Hiroshima 739-8524, Japan
[b] Graduate School of Business Sciences, The University of Tsukuba, Otsuka, Bunkyou-ku, Tokyo 112-0012, Japan
[c] Faculty of Engineering, Hiroshima Kokusai Gakuin University, Nakano, Aki-ku, Hiroshima 739-0321, Japan

Received 6 November 2001; received in revised form 30 June 2003; accepted 18 July 2003

Abstract

In this paper, we consider the linear quadratic optimal control problem for multiparameter singularly perturbed systems in which N lower-level fast subsystems are interconnected through a higher-level slow subsystem. Different from the existing methods, a new method is developed to design a near-optimal controller which does not depend on the unknown small parameters. It is shown that the resulting controller in fact achieves an $O(\|\mu\|^2)$ approximation to the optimal cost of the original optimal control problem.

Keywords: Multiparameter singularly perturbed systems (MSPS); Multiparameter algebraic Riccati equation (MARE); Near-optimal control; ε-independent controller

1. Introduction

The deterministic and stochastic multimodeling stability, control, filtering and dynamic games have been investigated extensively by several researchers (see e.g., Khalil, 1979, 1980, 1981; Khalil & Kokotović, 1978, 1979a, b; Özgüner, 1979; Salman, Lee, & Boustany, 1990; Coumarbatch & Gajić, 2000a, b; Gajić, 1988; Gajić & Khalil, 1986; Wang, Paul, & Wu, 1994). In order to obtain the optimal solution to the multimodeling problems, we must solve the multiparameter algebraic Riccati equation (MARE), which is parameterized by the small positive same order parameters $\varepsilon_j, j = 1, \dots, N$. Various reliable approaches for solving the MARE have been well documented in literatures (see e.g., Coumarbatch & Gajić, 2000a, b; Mukaidani, Xu, & Mizukami, 2002). However, a limitation of these approaches is that the small parameters are assumed to be known. Thus, it is not applicable to a large class of problems where the parameters represent small unknown perturbations whose values are not known exactly. On the other hand, although it is well known that a popular approach to deal with the multiparameter singularly perturbed systems is the two-time-scale design method (see e.g., Kokotović, Khalil, & O'Reilly, 1986; Wang et al., 1994), the existing controller only achieves $O(\|\mu\|)$ (where $\|\mu\|$ denotes the norm of the vector $[\varepsilon_1 \ \cdots \ \varepsilon_N]$) approximation of the optimal cost.

In this paper, we study the linear quadratic optimal control problem for nonstandard multiparameter singularly perturbed systems (MSPS). The considered MSPS is more general compared with Mukaidani and Mizukami (2001) and is based on the specific structure of the lower-level multi-fast subsystems and a higher-level slow subsystem (Özgüner, 1979). We first investigate the unique and bounded solution of the MARE and establish its asymptotic structure. Using the asymptotic structure, a new near-optimal controller which does not depend on the values of the small parameters is obtained. It is newly shown that the resulting controller achieves $O(\|\mu\|^2)$ approximation of the optimal cost. As another important feature, we prove that the new near-optimal controller is equivalent to the existing one in

☆ This paper was partially presented at IFAC workshop on singular solutions and perterbations, Bucharest, October 2001. This paper was recommended for publication in revised form by Associate Editor Thor I. Fossen under the direction of Editor Hassan Khalil.

* Corresponding author. Tel.: +81-824-24-7155; fax: +81-824-24-7155.

E-mail addresses: mukaida@hiroshima-u.ac.jp (H. Mukaidani), xuhua@gssm.otsuka.tsukuba.ac.jp (H. Xu), mizukami@cs.hkg.ac.jp (K. Mizukami).

0005-1098/$ - see front matter © 2003 Published by Elsevier Ltd.
doi:10.1016/S0005-1098(03)00248-6

the case of the standard and the nonstandard MSPS. We claim that the proposed controller includes the composite near-optimal controller (Khalil & Kokotović, 1979a) as a special case since the proposed controller can be constructed even if the fast state matrices are singular. Moreover, we also claim that the near-optimal controller via the descriptor variable approach (Wang et al., 1994) is equivalent to the proposed controller under certain conditions. Therefore, we emphasize that the composite controller obtained by decomposing the full systems and the approximation controller obtained by eliminating ε_j of the full controller are identical.

Notation: The superscript T denotes matrix transpose. $\det L$ denotes the determinant of square matrix L. I_n denotes the $n \times n$ identity matrix. $\|\cdot\|$ denotes its Euclidean norm for a matrix. block-diag denotes the block diagonal matrix. $\mathrm{vec}\, M$ denotes the column vector of the matrix M (Magnus & Neudecker, 1999). $\otimes$ denotes the Kronecker product. U_{lm} denotes a permutation matrix in the Kronecker matrix sense (Magnus & Neudecker, 1999) such that $U_{lm}\,\mathrm{vec}\, M = \mathrm{vec}\, M^{\mathrm{T}}$, $M \in \mathbf{R}^{l\times m}$. $E[\,\cdot\,]$ denotes the expection operator.

2. Multiparameter singularly perturbed systems

We consider a specific structure of N-lower-level multi-fast subsystems interconnected through the dynamics of a higher-level slow subsystem (See e.g., Özgüner, 1979).

$$\dot{x}_0(t) = A_{00}x_0(t) + \sum_{j=1}^{N} A_{0j}x_j(t) + \sum_{j=1}^{N} B_{0j}u_j(t),\ x_0(0) = x_0^0, \tag{1a}$$

$$\varepsilon_j\dot{x}_j(t) = A_{j0}x_0(t) + A_{jj}x_j(t) + B_{jj}u_j(t),$$
$$x_j(0) = x_j^0,\ j = 1,2,\ldots,N, \tag{1b}$$

$$y_0(t) = C_{00}x_0(t), \tag{1c}$$

$$y_j(t) = C_{j0}x_0(t) + C_{jj}x_j(t),\ j = 1,2,\ldots,N, \tag{1d}$$

where $x_j \in \mathbf{R}^{n_j}$, $j = 0,1,\ldots,N$ are the state vectors, $u_j \in \mathbf{R}^{m_j}$, $j = 1,2,\ldots,N$ are the control inputs, $y_j \in \mathbf{R}^{l_j}$, $j = 0,1,\ldots,N$ are the outputs. We assume that the ratios of the small positive parameter $\varepsilon_j > 0$, $j = 1,2,\ldots,N$ are bounded by some positive constants $\underline{k}_{ij}$, $\bar{k}_{ij}$ (see e.g., Khalil, 1979, 1980, 1981; Khalil & Kokotović, 1978, 1979a, b),

$$0 < \underline{k}_{ij} \leqslant \alpha_{ij} \equiv \frac{\varepsilon_j}{\varepsilon_i} \leqslant \bar{k}_{ij} < \infty. \tag{2}$$

Note that one of the fast state matrices A_{jj}, $j = 1,2,\ldots,N$ may be singular. The performance criterion is given by

$$J = \frac{1}{2}\int_0^{\infty}\left(y^{\mathrm{T}}(t)y(t) + \sum_{j=1}^{N} u_j^{\mathrm{T}}(t)R_ju_j(t)\right)\mathrm{d}t = \frac{1}{2}\int_0^{\infty}\left(x^{\mathrm{T}}(t)Qx(t) + \sum_{j=1}^{N} u_j^{\mathrm{T}}(t)R_ju_j(t)\right)\mathrm{d}t, \tag{3}$$

where

$$y(t)^{\mathrm{T}} := [y_0(t)^{\mathrm{T}}\ \ y_1(t)^{\mathrm{T}}\ \ \cdots\ \ y_N(t)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbf{R}^{\bar{l}},$$

$$\bar{l} := \sum_{j=0}^{N} l_j,$$

$$x(t)^{\mathrm{T}} := [x_0(t)^{\mathrm{T}}\ \ x_1(t)^{\mathrm{T}}\ \ \cdots\ \ x_N(t)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbf{R}^{\bar{n}},$$

$$\bar{n} := \sum_{j=0}^{N} n_j,$$

$$Q := \begin{bmatrix} Q_{00} & Q_{0f} \\ Q_{0f}^{\mathrm{T}} & Q_f \end{bmatrix},\ Q_{00} := \sum_{j=0}^{N} C_{j0}^{\mathrm{T}}C_{j0},$$

$$Q_{0f} := [Q_{01}\ \ \cdots\ \ Q_{0N}] = [C_{10}^{\mathrm{T}}C_{11}\ \ \cdots\ \ C_{N0}^{\mathrm{T}}C_{NN}],$$

$$Q_f := \text{block-diag}(Q_{11}\ \ \cdots\ \ Q_{NN}) = \text{block-diag}(C_{11}^{\mathrm{T}}C_{11}\ \ \cdots\ \ C_{NN}^{\mathrm{T}}C_{NN}).$$

Let the optimal control for the regulator problem (1) and (3) be

$$u_{\mathrm{opt}}(t) = K_{\mathrm{opt}}x(t) = [u_{1\mathrm{opt}}(t)^{\mathrm{T}}\ \ \cdots\ \ u_{N\mathrm{opt}}(t)^{\mathrm{T}}]^{\mathrm{T}} = -R^{-1}B_e^{\mathrm{T}}P_ex(t), \tag{4}$$

where P_e satisfies the MARE

$$P_eA_e + A_e^{\mathrm{T}}P_e - P_eS_eP_e + Q = 0 \tag{5}$$

with

$$\Pi_e := \text{block-diag}(\varepsilon_1 I_{n_1}\ \ \cdots\ \ \varepsilon_N I_{n_N}),$$

$$A_e := \begin{bmatrix} A_{00} & A_{0f} \\ \Pi_e^{-1}A_{f0} & \Pi_e^{-1}A_f \end{bmatrix},$$

$$A_{0f} := [A_{01}\ \ \cdots\ \ A_{0N}],\ A_{f0} := [A_{10}^{\mathrm{T}}\ \ \cdots\ \ A_{N0}^{\mathrm{T}}]^{\mathrm{T}},$$

$$A_f := \text{block-diag}(A_{11}\ \ \cdots\ \ A_{NN}),$$

$$S_e := B_eR^{-1}B_e^{\mathrm{T}} = \begin{bmatrix} S_{00} & S_{0f}\Pi_e^{-1} \\ \Pi_e^{-1}S_{0f}^{\mathrm{T}} & \Pi_e^{-1}S_f\Pi_e^{-1} \end{bmatrix},$$

$$S_{00} := B_0 R^{-1} B_0^{\mathrm{T}} = \sum_{j=1}^{N} B_{0j} R_j^{-1} B_{0j}^{\mathrm{T}},$$

$$S_{0f} := B_0 R^{-1} B_f^{\mathrm{T}} = [S_{01} \ \cdots \ S_{0N}]$$

$$= \left[B_{01} R_1^{-1} B_{11}^{\mathrm{T}} \quad \cdots \quad B_{0N} R_N^{-1} B_{NN}^{\mathrm{T}} \right],$$

$$S_f := B_f R^{-1} B_f^{\mathrm{T}} = \text{block-diag}(S_{11} \ \cdots \ S_{NN})$$

$$= \text{block-diag}(B_{11} R_1^{-1} B_{11}^{\mathrm{T}} \ \cdots \ B_{NN} R_N^{-1} B_{NN}^{\mathrm{T}}),$$

$$B_e := \begin{bmatrix} B_0 \\ \Pi_e^{-1} B_f \end{bmatrix}, \ B_0 := [B_{01} \ \cdots \ B_{0N}],$$

$$B_f := \text{block-diag}(B_{11} \ \cdots \ B_{NN}),$$

$$R := \text{block-diag}(R_1 \ \cdots \ R_N).$$

Since the matrices A_e and B_e contain the term of ε_j^{-1}, a solution P_e of the MARE (5), if it exists, must contain terms of ε_j. Taking this fact into consideration, we look for a solution P_e of the MARE (5) with the structure

$$P_e := \begin{bmatrix} P_{00} & P_{f0}^{\mathrm{T}} \Pi_e \\ \Pi_e P_{f0} & \Pi_e P_f \end{bmatrix}, \ P_{00} = P_{00}^{\mathrm{T}},$$

$$P_{f0} := \begin{bmatrix} P_{10} \\ \vdots \\ P_{N0} \end{bmatrix},$$

$$P_f :=$$

$$\begin{bmatrix} P_{11} & \alpha_{12} P_{21}^{\mathrm{T}} & \alpha_{13} P_{31}^{\mathrm{T}} & \cdots & \alpha_{1N} P_{N1}^{\mathrm{T}} \\ P_{21} & P_{22} & \alpha_{23} P_{32}^{\mathrm{T}} & \cdots & \alpha_{2N} P_{N2}^{\mathrm{T}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{(N-1)1} & P_{(N-1)2} & P_{(N-1)3} & \cdots & \alpha_{(N-1)N} P_{N(N-1)}^{\mathrm{T}} \\ P_{N1} & P_{N2} & P_{N3} & \cdots & P_{NN} \end{bmatrix},$$

$$\Pi_e P_f = P_f^{\mathrm{T}} \Pi_e.$$

In the following analysis, we need some assumptions. Specifically, in order to guarantee the existence of the reduced-order algebraic Riccati equation (ARE) and its standard stabilizability and the detectability conditions when $\|\mu\| := \sqrt{\varepsilon_1^2 + \varepsilon_2^2 + \cdots + \varepsilon_N^2} \to +0$, Assumptions 2 and 3 are needed (Mukaidani, 2001). These assumptions play an important role in proving Lemma 6 which will be given later.

Assumption 1. The triples $(A_{jj},\ B_{jj},\ C_{jj})$, $j = 1, 2, \ldots, N$ are stabilizable and detectable.

Assumption 2.

$$\text{rank} \begin{bmatrix} sI_{n_0} - A_{00} & -A_{0f} & B_0 \\ -A_{f0} & -A_f & B_f \end{bmatrix} = \bar{n}, \tag{6a}$$

$$\text{rank} \begin{bmatrix} sI_{n_0} - A_{00}^{\mathrm{T}} & -A_{f0}^{\mathrm{T}} & C_0^{\mathrm{T}} \\ -A_{0f}^{\mathrm{T}} & -A_f^{\mathrm{T}} & C_f^{\mathrm{T}} \end{bmatrix} = \bar{n}, \tag{6b}$$

where

$$C_0 := \begin{bmatrix} C_{00} \\ C_{10} \\ \vdots \\ C_{N0} \end{bmatrix}, \quad C_f := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ C_{11} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & C_{NN} \end{bmatrix},$$

with $\forall s \in \mathbf{C}$, $\mathrm{Re}[s] \geqslant 0$.

Assumption 3. The Hamiltonian matrices

$$T_{jj} := \begin{bmatrix} A_{jj} & -S_{jj} \\ -Q_{jj} & -A_{jj}^{\mathrm{T}} \end{bmatrix}, \ j = 1, 2, \ldots, N$$

are nonsingular.

Before investigating the optimal control problem, we investigate the asymptotic structure of the MARE (5). Let us introduce the scaling matrices $\Phi_e := \text{block-diag}(I_{n_0} \ \Pi_e)$. In order to avoid the ill-conditioned caused by the large parameter ε_j^{-1} which is included in the MARE (5), we introduce the following useful lemma.

Lemma 4. *The MARE (5) is equivalent to the following generalized multiparameter algebraic Riccati equation (GMARE) (7)*

$$\mathscr{G}(P) = P^{\mathrm{T}} A + A^{\mathrm{T}} P - P^{\mathrm{T}} S P + Q = 0, \tag{7}$$

where

$$A := \begin{bmatrix} A_{00} & A_{0f} \\ A_{f0} & A_f \end{bmatrix}, \quad S := \begin{bmatrix} S_{00} & S_{0f} \\ S_{0f}^{\mathrm{T}} & S_f \end{bmatrix},$$

$$P := \begin{bmatrix} P_{00} & P_{f0}^{\mathrm{T}} \Pi_e \\ P_{f0} & P_f \end{bmatrix}.$$

Proof. Firstly, by direct calculation we verify that $P_e = \Phi_e P$. Secondly, it is easy to verify that $A = \Phi_e A_e$, $S = \Phi_e S_e \Phi_e$. Hence,

$$A^{\mathrm{T}} P = A_e^{\mathrm{T}} \Phi_e \Phi_e^{-1} P_e = A_e^{\mathrm{T}} P_e, \ P^{\mathrm{T}} S P$$

$$= P_e \Phi_e^{-1} \Phi_e S_e \Phi_e \Phi_e^{-1} P_e = P_e S_e P_e.$$

By using the similar calculation, we can immediately rewrite (5) as (7). □

The GMARE (7) can be partitioned into

$$f_1 = P_{00}^{\mathrm{T}} A_{00} + A_{00}^{\mathrm{T}} P_{00} + P_{f0}^{\mathrm{T}} A_{f0} + A_{f0}^{\mathrm{T}} P_{f0} - P_{00}^{\mathrm{T}} S_{00} P_{00} - P_{f0}^{\mathrm{T}} S_f P_{f0} - P_{00}^{\mathrm{T}} S_{0f} P_{f0} - P_{f0}^{\mathrm{T}} S_{0f}^{\mathrm{T}} P_{00} + Q_{00} = 0, \tag{8a}$$

$$f_2 = A_{00}^{\mathrm{T}} P_{f0}^{\mathrm{T}} \Pi_e + A_{f0}^{\mathrm{T}} P_f + P_{00}^{\mathrm{T}} A_{0f} + P_{f0}^{\mathrm{T}} A_f - P_{00}^{\mathrm{T}} S_{00} P_{f0}^{\mathrm{T}} \Pi_e - P_{f0}^{\mathrm{T}} S_{0f}^{\mathrm{T}} P_{f0}^{\mathrm{T}} \Pi_e - P_{00}^{\mathrm{T}} S_{0f} P_f - P_{f0}^{\mathrm{T}} S_f P_f + Q_{0f} = 0, \tag{8b}$$

$$f_3 = P_f^{\mathrm{T}} A_f + A_f^{\mathrm{T}} P_f + \Pi_e P_{f0} A_{0f} + A_{0f}^{\mathrm{T}} P_{f0}^{\mathrm{T}} \Pi_e - P_f^{\mathrm{T}} S_f P_f - P_f^{\mathrm{T}} S_{0f}^{\mathrm{T}} P_{f0}^{\mathrm{T}} \Pi_e - \Pi_e P_{f0} S_{0f} P_f - \Pi_e P_{f0} S_{00} P_{f0}^{\mathrm{T}} \Pi_e + Q_f = 0. \tag{8c}$$

It is assumed that the limit of α_{ij} exists as ε_i and ε_j tend to zero (see e.g., Khalil, 1979, 1980, 1981; Khalil & Kokotović, 1978, 1979a, b), that is

$$\bar{\alpha}_{ij} = \lim_{\substack{\varepsilon_j \to +0 \\ \varepsilon_i \to +0}} \alpha_{ij}. \tag{9}$$

Let $\bar{P}_{00}$, $\bar{P}_{f0}$ and $\bar{P}_f$ be the limiting solutions of the above equation (8) as $\varepsilon_j \to +0,\ j = 1, \ldots, N$, then we obtain the following equations:

$$\bar{P}_{00}^{\mathrm{T}} A_{00} + A_{00}^{\mathrm{T}} \bar{P}_{00} + \bar{P}_{f0}^{\mathrm{T}} A_{f0} + A_{f0}^{\mathrm{T}} \bar{P}_{f0} - \bar{P}_{00}^{\mathrm{T}} S_{00} \bar{P}_{00} - \bar{P}_{f0}^{\mathrm{T}} S_f \bar{P}_{f0} - \bar{P}_{00}^{\mathrm{T}} S_{0f} \bar{P}_{f0} - \bar{P}_{f0}^{\mathrm{T}} S_{0f}^{\mathrm{T}} \bar{P}_{00} + Q_{00} = 0, \tag{10a}$$

$$A_{f0}^{\mathrm{T}} \bar{P}_f + \bar{P}_{00}^{\mathrm{T}} A_{0f} + \bar{P}_{f0}^{\mathrm{T}} A_f - \bar{P}_{00}^{\mathrm{T}} S_{0f} \bar{P}_f - \bar{P}_{f0}^{\mathrm{T}} S_f \bar{P}_f + Q_{0f} = 0, \tag{10b}$$

$$\bar{P}_f^{\mathrm{T}} A_f + A_f^{\mathrm{T}} \bar{P}_f - \bar{P}_f^{\mathrm{T}} S_f \bar{P}_f + Q_f = 0, \tag{10c}$$

where

$$\bar{P}_f := \begin{bmatrix} \bar{P}_{11} & \bar{\alpha}_{12} \bar{P}_{21}^{\mathrm{T}} & \bar{\alpha}_{13} \bar{P}_{31}^{\mathrm{T}} & \cdots & \bar{\alpha}_{1N} \bar{P}_{N1}^{\mathrm{T}} \\ \bar{P}_{21} & \bar{P}_{22} & \bar{\alpha}_{23} \bar{P}_{32}^{\mathrm{T}} & \cdots & \bar{\alpha}_{2N} \bar{P}_{N2}^{\mathrm{T}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{P}_{(N-1)1} & \bar{P}_{(N-1)2} & \bar{P}_{(N-1)3} & \cdots & \bar{\alpha}_{(N-1)N} \bar{P}_{N(N-1)}^{\mathrm{T}} \\ \bar{P}_{N1} & \bar{P}_{N2} & \bar{P}_{N3} & \cdots & \bar{P}_{NN} \end{bmatrix},$$

$$\bar{P}_{jj} = \bar{P}_{jj}^{\mathrm{T}},\ j = 0, 1, 2, \ldots, N. \tag{11}$$

Note that the ARE (10c) admits an asymmetric solution. However, it can be seen that the ARE (10c) admits at least a symmetric positive semidefinite stabilizing solution as follows.

Theorem 5. *Under Assumption* 1, *the ARE* (10c) *admits a unique symmetric positive semidefinite stabilizing solution* $\bar{P}_f$ *which can be written as*

$$\bar{P}_f^* := \text{block-diag}(\bar{P}_{11}^* \ \cdots \ \bar{P}_{NN}^*), \tag{12}$$

where $\bar{P}_{jj}^*$ *is a unique symmetric positive semidefinite stabilizing solution for the following AREs, respectively,*

$$\bar{P}_{jj}^* A_{jj} + A_{jj}^{\mathrm{T}} \bar{P}_{jj}^* - \bar{P}_{jj}^* S_{jj} \bar{P}_{jj}^* + Q_{jj} = 0,\ j = 1, 2, \ldots, N.$$

Proof. Substituting (12) into the ARE (10c) as $\bar{P}_f^* \to \bar{P}_f$, it is easy to verify that $\bar{P}_f^* A_f + A_f^{\mathrm{T}} \bar{P}_f^* - \bar{P}_f^* S_f \bar{P}_f^* + Q_f = 0$. Furthermore, it can be seen that $\bar{P}_f^* = \bar{P}_f^{*\mathrm{T}} \geqslant 0$ and the following matrix $A_f - S_f \bar{P}_f^*$ is stable because $\bar{P}_{jj}^*$ is a unique symmetric positive semidefinite stabilizing solution under Assumption 1.

$$A_f - S_f \bar{P}_f^* = \text{block-diag}(A_{11} - S_{11} \bar{P}_{11}^* \ \cdots \ A_{NN} - S_{NN} \bar{P}_{NN}^*).$$

Consequently, there exists a unique solution of the ARE (10c) and its solution is (12) itself. $\square$

Assumption 1 ensures that $A_{jj} - S_{jj} \bar{P}_{jj}^*,\ j = 1, 2, \ldots, N$ are nonsingular. Substituting the solution of (10c) into (10b) and substituting $\bar{P}_{f0}^*$ into (10a) and making some lengthy calculations (the detail is omitted for brevity), we obtain the following 0-order equations (13)

$$\bar{P}_{00}^* \mathscr{A} + \mathscr{A}^{\mathrm{T}} \bar{P}_{00}^* - \bar{P}_{00}^* \mathscr{S} \bar{P}_{00}^* + \mathscr{Q} = 0, \tag{13a}$$

$$\bar{P}_{f0}^* = -N_2^{\mathrm{T}} + N_1^{\mathrm{T}} \bar{P}_{00}^*, \Leftrightarrow \bar{P}_{j0}^{*\mathrm{T}} = -[\bar{P}_{00}^* D_{0j} + (A_{j0}^{\mathrm{T}} \bar{P}_{jj}^* + Q_{0j})] D_{jj}^{-1} = \begin{bmatrix} \bar{P}_{jj}^* & -I_{n_j} \end{bmatrix} T_{jj}^{-1} T_{j0} \begin{bmatrix} I_{n_0} \\ \bar{P}_{00}^* \end{bmatrix}, \tag{13b}$$

$$\bar{P}_f^* A_f + A_f^{\mathrm{T}} \bar{P}_f^* - \bar{P}_f^* S_f \bar{P}_f^* + Q_f = 0, \Leftrightarrow \bar{P}_{jj}^* A_{jj} + A_{jj}^{\mathrm{T}} \bar{P}_{jj}^* - \bar{P}_{jj}^* S_{jj} \bar{P}_{jj}^* + Q_{jj} = 0, \tag{13c}$$

where

$$\mathscr{A} := A_{00} + N_1 A_{f0} + S_{0f} N_2^{\mathrm{T}} + N_1 S_f N_2^{\mathrm{T}},$$

$$\mathscr{S} := S_{00} + N_1 S_{0f}^{\mathrm{T}} + S_{0f} N_1^{\mathrm{T}} + N_1 S_f N_1^{\mathrm{T}},$$

$$\mathscr{Q} := Q_{00} - N_2 A_{f0} - A_{f0}^{\mathrm{T}} N_2^{\mathrm{T}} - N_2 S_f N_2^{\mathrm{T}},$$

$$N_1^{\mathrm{T}} := -\bar{A}_f^{-\mathrm{T}}\bar{A}_{0f}^{\mathrm{T}} = [-D_{01}D_{11}^{-1} \ \cdots \ -D_{0N}D_{NN}^{-1}]^{\mathrm{T}}$$
$$= [N_{11} \ \cdots \ N_{1N}]^{\mathrm{T}},$$

$$N_2^{\mathrm{T}} := \bar{A}_f^{-\mathrm{T}}\bar{Q}_{0f}^{\mathrm{T}} = [\bar{Q}_{01}D_{11} \ \cdots \ \bar{Q}_{0N}D_{NN}]^{\mathrm{T}}$$
$$= [N_{21} \ \cdots \ N_{2N}]^{\mathrm{T}},$$

$$\bar{A}_{0f} := A_{0f} - S_{0f}\bar{P}_f^* = [D_{01} \ \cdots \ D_{0N}],$$
$$\bar{A}_f := A_f - S_f\bar{P}_f^* = \text{block-diag}(D_{11} \ \cdots \ D_{NN}),$$
$$\bar{Q}_{0f} := Q_{0f} + A_{f0}^{\mathrm{T}}\bar{P}_f^* = [\bar{Q}_{01} \ \cdots \ \bar{Q}_{0N}],$$
$$D_{0j} := A_{0j} - S_{0j}\bar{P}_{jj}^*, \ D_{jj} := A_{jj} - S_{jj}\bar{P}_{jj}^*,$$
$$\bar{Q}_{0j} := Q_{0j} + A_{j0}^{\mathrm{T}}\bar{P}_{jj}^*, \ j = 1, 2, \ldots, N.$$

In the following we established the relation between the GMARE (7) and the 0-order equations (13). Before doing that, we give the results for the AREs (13).

Lemma 6. *Under Assumptions 1–3, the following results hold*:

(i) *The matrices $\mathscr{A}$, $\mathscr{S}$ and $\mathscr{Q}$ do not depend on $\bar{P}_{jj}^*$, $j = 1, 2, \ldots, N$. That is, following formulations are satisfied*:

$$\begin{bmatrix} \mathscr{A} & -\mathscr{S} \\ -\mathscr{Q} & -\mathscr{A}^{\mathrm{T}} \end{bmatrix} = T_{00} - \sum_{j=1}^{N} T_{0j}T_{jj}^{-1}T_{j0}, \tag{14}$$

where

$$T_{00} := \begin{bmatrix} A_{00} & -S_{00} \\ -Q_{00} & -A_{00}^{\mathrm{T}} \end{bmatrix}, \ T_{0j} := \begin{bmatrix} A_{0j} & -S_{0j} \\ -Q_{0j} & -A_{j0}^{\mathrm{T}} \end{bmatrix},$$
$$T_{j0} := \begin{bmatrix} A_{j0} & -S_{0j}^{\mathrm{T}} \\ -Q_{0j}^{\mathrm{T}} & -A_{0j}^{\mathrm{T}} \end{bmatrix}, \ j = 1, 2, \ldots, N.$$

(ii) *There exist a matrix $\mathscr{B} := [B_{01} + N_{11}B_{11} \ \cdots \ B_{0N} + N_{1N}B_{NN}] \in \mathbf{R}^{n_0 \times \bar{m}}$, $\bar{m} := \sum_{j=1}^{N} m_j$ and a matrix $\mathscr{C}$ with the same dimension as C_0 such that $\mathscr{S} = \mathscr{B}R^{-1}\mathscr{B}^{\mathrm{T}}$, $\mathscr{Q} = \mathscr{C}^{\mathrm{T}}\mathscr{C}$. Moreover, the triple $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ is stabilizable and detectable.*

Proof. Since the proof can be performed by using the dual argument in Mukaidani (2003), it is omitted. □

Taking into account the fact that $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ is stabilizable and detectable, the ARE (13a) admits a unique stabilizing positive semidefinite symmetric solution, denoted by $\bar{P}_{00}^*$, and $\mathscr{A} - \mathscr{S}\bar{P}_{00}^*$ is stable.

The limiting behavior of P_e as the parameter $\|\mu\| \to +0$ is described by the following theorem.

Theorem 7. *Under Assumptions* 1–3, *there exists a small σ^* such that for all $\|\mu\| \in (0, \sigma^*)$ the MARE* (5) *admits a symmetric positive semidefinite stabilizing solution P_e which can be written as*

$$P_e = \Phi_e \begin{bmatrix} \bar{P}_{00}^* + O(\|\mu\|) & [\bar{P}_{f0}^* + O(\|\mu\|)]^{\mathrm{T}}\Pi_e \\ \bar{P}_{f0}^* + O(\|\mu\|) & \bar{P}_f^* + O(\|\mu\|) \end{bmatrix}$$
$$= \begin{bmatrix} \bar{P}_{00}^* + O(\|\mu\|) & [\bar{P}_{f0}^* + O(\|\mu\|)]^{\mathrm{T}}\Pi_e \\ \Pi_e[\bar{P}_{f0}^* + O(\|\mu\|)] & \Pi_e[\bar{P}_f^* + O(\|\mu\|)] \end{bmatrix}. \tag{15}$$

In order to prove Theorem 7, we need the following useful lemma (Mukaidani & Mizukami, 2000).

Lemma 8. *Let us consider the linear time-invariant MSPS*

$$\dot{z}_1(t) = [F_{11} + O(\|\mu\|)]z_1(t) + [F_{12} + O(\|\mu\|)]z_2(t),$$
$$z_1(0) = z_1^0, \tag{16a}$$

$$\Pi_e\dot{z}_2(t) = [F_{21} + O(\|\mu\|)]z_1(t) + [F_{22} + O(\|\mu\|)]z_2(t),$$
$$z_2(0) = z_2^0, \tag{16b}$$

where $\|\mu\|$ is a small positive parameter, $z_j \in \mathbf{R}^{l_j}$, $j = 1, 2$ are the state vectors. All matrices above are of appropriate dimensions.

If F_{22} and $F_0 = F_{11} - F_{12}F_{22}^{-1}F_{21}$ are stable, then there exists a small perturbation parameter $||\bar{\mu}|| > 0$ such that for all $\|\mu\| \in (0, ||\bar{\mu}||]$ system (16) *is asymptotically stable.*

Proof. Since the proof can be performed by using the similar technique in Mukaidani and Mizukami (2000), it is omitted. □

Using the Lemma 8, let us prove Theorem 7.

Proof. We apply the implicit function theorem (Gajić, 1988) to (8). To do so, it is enough to show that the corresponding Jacobian is nonsingular at $\|\mu\| = 0$. It can be shown, after some algebra, that the Jacobian of (8) in the limit as $\|\mu\| \to 0$ is given by

$$\mathbf{J} = \nabla\mathbf{F} = \left.\frac{\partial \mathrm{vec}(f_1, \ f_2, \ f_3)}{\partial \mathrm{vec}(P_{00}, \ P_{f0}, \ P_f)^{\mathrm{T}}}\right|_{\|\mu\|=0, \ P_{00}=\bar{P}_{00}^*, \ P_{f0}=\bar{P}_{f0}^*, \ P_f=\bar{P}_f^*}$$
$$= \begin{bmatrix} \mathbf{J}_{00} & \mathbf{J}_{01} & 0 \\ \mathbf{J}_{10} & \mathbf{J}_{11} & \mathbf{J}_{12} \\ 0 & 0 & \mathbf{J}_{22} \end{bmatrix}, \tag{17}$$

where

$$\mathbf{J}_{00} = I_{n_0} \otimes \bar{A}_{00}^{\mathrm{T}} + \bar{A}_{00}^{\mathrm{T}} \otimes I_{n_0},$$

$$\mathbf{J}_{01} = (I_{n_0} \otimes \bar{A}_{f0}^{\mathrm{T}}) U_{n_0 \hat{n}} + \bar{A}_{f0}^{\mathrm{T}} \otimes I_{n_0},$$

$$\mathbf{J}_{10} = \bar{A}_{0f}^{\mathrm{T}} \otimes I_{n_0} = (\bar{A}_{0f}^{\mathrm{T}} \otimes I_{n_0}) U_{n_0 n_0},$$

$$\mathbf{J}_{11} = \bar{A}_{f}^{\mathrm{T}} \otimes I_{n_0}, \ \mathbf{J}_{22} = I_{\hat{n}} \otimes \bar{A}_{f}^{\mathrm{T}} + \bar{A}_{f}^{\mathrm{T}} \otimes I_{\hat{n}},$$

$$\bar{A}_{00} = A_{00} - S_{00}\bar{P}_{00}^{*} - S_{0f}\bar{P}_{f0}^{*},$$

$$\bar{A}_{f0} = A_{f0} - S_{0f}^{\mathrm{T}}\bar{P}_{00}^{*} - S_{f}\bar{P}_{f0}^{*},$$

$$\bar{A}_{0} = \bar{A}_{00} - \bar{A}_{0f}\bar{A}_{f}^{-1}\bar{A}_{f0}.$$

The Jacobian (17) can be expressed as $\det \mathbf{J} = \det \mathbf{J}_{22} \cdot \det \mathbf{J}_{11} \cdot \det[I_{n_0} \otimes \bar{A}_0^{\mathrm{T}} + \bar{A}_0^{\mathrm{T}} \otimes I_{n_0}]$, where $\bar{A}_0 \equiv \bar{A}_{00} - \bar{A}_{0f}\bar{A}_f^{-1}\bar{A}_{f0}$. Obviously, $\mathbf{J}_{jj}$, $j = 1, 2$ are nonsingular because the matrix $\bar{A}_f = A_f - S_f\bar{P}_f^{*}$ is stable under Assumption 1. After some straightforward but tedious algebra, we see that $\mathscr{A} - \mathscr{S}\bar{P}_{00}^{*} = \bar{A}_{00} - \bar{A}_{0f}\bar{A}_f^{-1}\bar{A}_{f0} = \bar{A}_0$. Therefore, the matrix $\bar{A}_0$ is also stable if Assumption 2 holds. Thus, $\det \mathbf{J} \neq 0$, i.e., $\mathbf{J}$ is nonsingular at $\|\mu\| = 0$. The conclusion of Theorem 7 is obtained directly by using the implicit function theorem.

The remainder of the proof is to show that P_e is the positive semidefinite stabilizing solution. Firstly, from (15), we obtain

$$A_e - S_e P_e = \Phi_e^{-1}\left(\begin{bmatrix} \bar{A}_{00} & \bar{A}_{0f} \\ \bar{A}_{f0} & \bar{A}_f \end{bmatrix} + O(\|\mu\|)\right).$$

The matrices $\bar{A}_f$ and $\bar{A}_0$ are stable since Assumptions 1 and 2 hold. Therefore, if parameter $\|\mu\|$ is very small, $A_e - S_e P_e$ is stable by applying the Lemma 8. Finally, the property $P_e \geqslant 0$ follows now since the stabilizing solution of (5) is always positive semidefinite. See more detail in Mukaidani (2003). □

3. Near-optimal control for the nonstandard MSPS

The required solution of the MARE (5) exists under Assumptions 1–3. Our attention is focused on the specific linear state feedback controller which does not depend on the values of the small parameters. Such a linear state feedback controller is obtained by eliminating $O(\|\mu\|)$ item of the linear state feedback controller (4). If $\|\mu\|$ is very small, it is obvious that the linear state feedback controller (4) can be approximated as

$$\begin{aligned} u_{\mathrm{app}}(t) &= [u_{1\mathrm{app}}(t)^{\mathrm{T}} \ \cdots \ u_{N\mathrm{app}}(t)^{\mathrm{T}}]^{\mathrm{T}} = -R^{-1}B^{\mathrm{T}}P_{\mathrm{app}}x(t) \\ &= -R^{-1}B^{\mathrm{T}}\begin{bmatrix} \bar{P}_{00}^{*} & 0 \\ \bar{P}_{f0}^{*} & \bar{P}_{f}^{*} \end{bmatrix} x(t), \end{aligned} \tag{18}$$

where $B = \Phi_e B_e$.

Even if our controller designing process is quite different from the composite controller designing process (Khalil & Kokotović, 1979a; Wang et al., 1994; Xu, Mukaidani, & Mizukami, 1997). We can show that the resulting controller (18) is the same as the existing one. Firstly, we show that controller (18) is equivalent to the near-optimal controller proposed in Wang et al. (1994) under certain conditions.

According to Wang et al. (1994), the near-optimal control is given by

$$u_{\mathrm{des}}(t) = -R^{-1}B^{\mathrm{T}}Yx(t), \tag{19}$$

where

$$Y^{\mathrm{T}}A + A^{\mathrm{T}}Y - Y^{\mathrm{T}}SY + Q = 0, \ Y = \begin{bmatrix} Y_0 & 0 \\ Y_{f0} & Y_f \end{bmatrix},$$
$$Y_0 = Y_0^{\mathrm{T}}. \tag{20}$$

Note that the matrix Y in (20) is not unique. However, if Y_f is chosen as $\bar{P}_f^{*}$, the following result holds.

Lemma 9. *Under Assumptions* 1–3, *if Y_f is chosen as $\bar{P}_f^{*}$, there exists a unique stabilizing solution Y which satisfy the GMARE* (20). *Such a solution is given by $Y = P_{\mathrm{app}}$. Furthermore, controller* (19) *is equivalent to the near-optimal controller* (18).

Proof. Under Assumptions 1 and 3, there exists a unique stabilizing solution P_f^{*} of the ARE (13c). Then it is shown that the GMARE (20) are equivalent to the AREs (13) under $Y_f = P_f^{*}$. Thus, there exists the solution $Y \equiv P_{\mathrm{app}}$. Moreover, if Y_f is chosen as $\bar{P}_f^{*}$, the matrix Y is unique. Under Assumption 2, there exists a unique stabilizing solution $\bar{P}_{00}^{*}$ of the ARE (13a). Taking into consideration the fact that $\bar{A}_f = A_f - S_f Y_f = A_f - S_f\bar{P}_f^{*}$ and $\bar{A}_0 = \mathscr{A} - \mathscr{S}Y_{00} = \mathscr{A} - \mathscr{S}\bar{P}_{00}^{*}$ are stable, the matrix $A - SY = A - SP_{\mathrm{app}}$ is also stable. Therefore, the matrix Y is the stabilizing solution. For the rest of the proof of Lemma 9, it is easy to verify that controller (19) is equivalent to controller (18) because of $Y = P_{\mathrm{app}}$. □

Using the result of Lemma 9, we will give an important interpretation. Based on the optimal controller (4), we can change the form as $u_{\mathrm{app}}(t) = \lim_{\|\mu\| \to +0} u_{\mathrm{opt}}(t) = u_{\mathrm{des}}(t)$ when $Y_f = \bar{P}_f^{*}$. Thus, we claim that the composite controller obtained by decomposing the full systems and the approximation controller obtained by eliminating ε_j of the full controller are identical. Moreover, the similar results for the standard MSPS will be shown in the next section for a special case.

Secondly, we establish stability properties of the closed-loop system. Substituting $u_{\mathrm{app}}(t)$ into MSPS (1a) and (1b), we have

$$\dot{x} = \Phi_e^{-1}\left(\begin{bmatrix} A_{00} & A_{0f} \\ A_{f0} & A_f \end{bmatrix} - BR^{-1}B^{\mathrm{T}}\begin{bmatrix} \bar{P}_{00}^{*} & 0 \\ \bar{P}_{f0}^{*} & \bar{P}_f^{*} \end{bmatrix}\right)$$

$$x = \Phi_e^{-1}\begin{bmatrix} \bar{A}_{00} & \bar{A}_{0f} \\ \bar{A}_{f0} & \bar{A}_f \end{bmatrix} x = (A_e - S_e P_{\mathrm{app}e})x. \tag{21}$$

Lemma 10. *Under Assumptions* 1–3, *there exists a small* $\hat{\sigma}$ *such that for all* $\|\mu\| \in (0,\ \hat{\sigma})$, $\hat{\sigma} \leqslant \sigma^*$ *the closed-system* (21) *is asymptotically stable.*

Proof. The matrices $\bar{A}_f$ and $\bar{A}_0$ are stable since Assumptions 1 and 2 hold. Therefore, by direct applying Theorem 1 in Khalil and Kokotović (1979a, b), we have obtained the required result. □

When $\|\mu\|$ is sufficiently small, we know from Theorem 7 that the resulting controller (18) will be close to the optimal controller (4). In an optimization problem it is of interest to check whether the resulting value of the cost function will be near to its optimal value. The optimal value J_{opt} is obtained with controller (4) which optimizes the cost for the actual system (1).

Theorem 11. *Under Assumptions* 1–3, *the use of the approximation controller* (18) *results in* J_{app} *satisfying*

$$J_{\mathrm{app}} = J_{\mathrm{opt}} + O(\|\mu\|^2), \tag{22}$$

where $J_{\mathrm{opt}} = \frac{1}{2}x(0)^{\mathrm{T}}P_e x(0)$.

Before proving this theorem, we introduce the following useful lemma (Mukaidani et al., 2001; Mukaidani, 2003).

Lemma 12. *Consider the iterative algorithm which is based on the Kleinman algorithm*

$$P^{(i+1)\mathrm{T}}(A - SP^{(i)}) + (A - SP^{(i)})^{\mathrm{T}}P^{(i+1)} + P^{(i)\mathrm{T}}SP^{(i)} + Q = 0,$$

$$P^{(0)} = P_{\mathrm{app}},\ i = 0, 1, 2, 3, \ldots \tag{23}$$

with

$$P^{(i)} = \begin{bmatrix} P_{00}^{(i)} & P_{f0}^{(i)\mathrm{T}}\Pi_e \\ P_{f0}^{(i)} & P_f^{(i)} \end{bmatrix}. \tag{24}$$

Under Assumptions 1–3, *there exists a small* $\bar{\sigma}$ *such that for all* $\|\mu\| \in (0,\ \bar{\sigma})$, $\bar{\sigma} \leqslant \sigma^*$ *the iterative algorithm* (23) *converges to the exact solution of* $P_e = \Phi_e P = P^{\mathrm{T}}\Phi_e$ *with the rate of quadratic convergence, where* $P_e^{(i)} = \Phi_e P^{(i)} = P^{(i)\mathrm{T}}\Phi_e$ *is the positive semidefinite solution. That is, the following condition is satisfied*:

$$\|P^{(i)} - P\| = O(\|\mu\|^{2^i}),\ i = 0, 1, 2, \ldots. \tag{25}$$

Now, let us prove Theorem 11.

Proof. When u_{app} is used, the value of the performance index is

$$J_{\mathrm{app}} = \tfrac{1}{2}x(0)^{\mathrm{T}}W_e x(0), \tag{26}$$

where W_e is a positive semidefinite solution of the multiparameter algebraic Lyapunov equation (MALE)

$$W_e(A_e - S_e P_{\mathrm{app}e}) + (A_e - S_e P_{\mathrm{app}e})^{\mathrm{T}}W_e + P_{\mathrm{app}e}S_e P_{\mathrm{app}e} + Q = 0, \tag{27}$$

where $P_{\mathrm{app}e} = \Phi_e P_{\mathrm{app}}$. Subtracting (5) from (27) we find that $V_e = W_e - P_e$ satisfies the following MALE:

$$V_e(A_e - S_e P_{\mathrm{app}e}) + (A_e - S_e P_{\mathrm{app}e})^{\mathrm{T}}V_e + (P_e - P_{\mathrm{app}e})S_e(P_e - P_{\mathrm{app}e}) = 0. \tag{28}$$

Similarly, subtracting (5) from (23) we also get the MALE

$$(P_e^{(i+1)} - P_e)(A_e - S_e P_e^{(i)}) + (A_e - S_e P_e^{(i)})^{\mathrm{T}}(P_e^{(i+1)} - P_e) + (P_e - P_e^{(i)})S_e(P_e - P_e^{(i)}) = 0, \tag{29}$$

where $P_e^{(i)} = \Phi_e P^{(i)}$. When $i = 0$, taking $P_e^{(0)} = P_{\mathrm{app}e}$ into account we have

$$(P_e^{(1)} - P_e)(A_e - S_e P_{\mathrm{app}e}) + (A_e - S_e P_{\mathrm{app}e})^{\mathrm{T}}(P_e^{(1)} - P_e) + (P_e - P_{\mathrm{app}e})S_e(P_e - P_{\mathrm{app}e}) = 0.$$

Therefore, it is easy to verify that $V_e = P_e^{(1)} - P_e$ because $A_e - S_e P_{\mathrm{app}e}$ is stable from Lemma 10. Using Lemma 12 we obtain that $\|V_e\| = \|W_e - P_e\| \leqslant \|P^{(1)} - P\| = O(\|\mu\|^2)$. Hence $V_e = W_e - P_e = O(\|\mu\|^2)$, which implies (22). □

Consequently, the resulting controller (18) achieves $O(\|\mu\|^2)$ approximation of the optimal cost compared with the existing controller (Khalil & Kokotović, 1979a; Wang et al., 1994) in case where the fast subsystems have the special form.

So far, for the multiparameter optimal control problem, it is merely shown that all the near-optimal controls, without the knowledge of the small parameter vector μ, achieve an $O(\|\mu\|)$ approximation of the optimal performance value (Khalil & Kokotović, 1979a; Wang et al., 1994). In the rest of this section, we give the reason why the resulting controller (18) achieves a better performance.

We assume that there exists a strong interconnection among the fast state variables. That is, the matrices A_f, S_f and Q_f are the nonblock-diagonal matrix. Furthermore, instead of Assumption 1 we assume that the triples $(A_f,\ \sqrt{S}_f,\ \sqrt{Q}_f)$ are stabilizable and detectable. By following the similar steps in the proof of Theorem 7, it is also shown that the solution P_e of the MARE (5) can be written as

$$P_e = \Phi_e P = \begin{bmatrix} \bar{P}_{00} + O(\|\mu\|) & [\bar{P}_{f0} + O(\|\mu\|)]^{\mathrm{T}} \Pi_e \\ \Pi_e[\bar{P}_{f0} + O(\|\mu\|)] & \Pi_e[\bar{P}_f + O(\|\mu\|)] \end{bmatrix}, \quad (30)$$

where the matrices $\bar{P}_{00}$, $\bar{P}_{f0}$ and $\bar{P}_f$ satisfy equation (10).

It is very important to note that if we find the matrix which satisfy the GARE (10c), the following controller will achieve the $O(\|\mu\|^2)$ approximation of the optimal cost because of $\|\bar{K}_{\mathrm{app}} - K_{\mathrm{opt}}\| \leqslant \|R^{-1} \begin{bmatrix} B_0^{\mathrm{T}} & B_f^{\mathrm{T}} \end{bmatrix}\| \cdot \|\bar{P} - P\| = O(\|\mu\|)$, where

$$\bar{u}_{\mathrm{app}}(t) = \bar{K}_{\mathrm{app}} x(t) = -R^{-1} \begin{bmatrix} B_0^{\mathrm{T}} & B_f^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \bar{P}_{00} & 0 \\ \bar{P}_{f0} & \bar{P}_f \end{bmatrix} x(t). \quad (31)$$

However, we cannot solve the asymmetric GARE (10c) without the knowledge of the small perturbation parameters ε_j (Khalil & Kokotović, 1979a). Moreover, although the descriptor variable approach is applicable, it is also hard to solve the GMARE (20) which has a form (11) from various solutions. On the other hand, when the fast subsystems have the special form (1b), it is possible to obtain the solution of the GARE (10c) without the information of the small parameters. Therefore, the resulting controller (18) achieves $O(\|\mu\|^2)$ approximation. By similar reason, we can obtain the near-optimal controller which admit the $O(\|\mu\|^2)$ approximation via the descriptor variable approach for a special case of the fast subsystems.

4. Near-optimal control for the standard MSPS

In this section, we will show that the near-optimal controller (18) is equivalent to the existing composite optimal controller (Özgüner, 1979) for the standard MSPS. We assume that all of the fast state matrices $A_{jj},\ j = 1, 2, \ldots, N$ of (1b) are nonsingular. According to Özgüner (1979), the near-optimal closed-loop control is given by

$$\begin{aligned} u_{j\mathrm{com}}(t) = &-[(I_{m_j} - R_j^{-1} B_{jj}^{\mathrm{T}} X_{jj} A_{jj}^{-1} B_{jj}) \tilde{R}_j^{-1} (\tilde{D}_j^{\mathrm{T}} \tilde{C}_{j0} + \tilde{B}_{0j}^{\mathrm{T}} X_{00}) \\ &+ R_j^{-1} B_{jj}^{\mathrm{T}} X_{jj} A_{jj}^{-1} A_{j0}] x_0(t) \\ &- R_j^{-1} B_{jj}^{\mathrm{T}} X_{jj} x_j(t),\ j = 1, 2, \ldots, N, \end{aligned} \quad (32)$$

where $\tilde{B}_{0j} = B_{0j} - A_{0j} A_{jj}^{-1} B_{jj}$, $\tilde{C}_{j0} = C_{j0} - C_{jj} A_{jj}^{-1} A_{j0}$, $\tilde{R}_j = R_j + \tilde{D}_j^{\mathrm{T}} \tilde{D}_j$, $\tilde{D}_j = -C_{jj} A_{jj}^{-1} B_{jj}$.

In the above, X_{00} is the unique stabilizing positive semidefinite symmetric solution of the following ARE:

$$\begin{aligned} &X_{00}(A_r - B_r R_r^{-1} D_r^{\mathrm{T}} C_r) + (A_r - B_r R_r^{-1} D_r^{\mathrm{T}} C_r)^{\mathrm{T}} X_{00} \\ &- X_{00} B_r R_r^{-1} B_r^{\mathrm{T}} X_{00} + C_r^{\mathrm{T}} (I_{\bar{l}} - D_r R_r^{-1} D_r^{\mathrm{T}}) C_r = 0, \end{aligned} \quad (33)$$

where

$$R_r = R + D_r^{\mathrm{T}} D_r,\ A_r = A_{00} - \sum_{j=1}^{N} A_{0j} A_{jj}^{-1} A_{j0},$$

$$\begin{aligned} B_r &= B_0 - A_{0f} A_f^{-1} B_f \\ &= [B_{01} - A_{01} A_{11}^{-1} B_{11} \ \cdots \ B_{0N} - A_{0N} A_{NN}^{-1} B_{NN}], \end{aligned}$$

$$\begin{aligned} C_r &= C_0 - C_f A_f^{-1} A_{f0} \\ &= [C_{00}^{\mathrm{T}} (C_{10} - C_{11} A_{11}^{-1} A_{10})^{\mathrm{T}} \ \cdots \ (C_{N0} - C_{NN} A_{NN}^{-1} A_{N0})^{\mathrm{T}}]^{\mathrm{T}}, \end{aligned}$$

$$D_r = -C_f A_f^{-1} B_f = -\begin{bmatrix} 0 & \cdots & 0 \\ C_{11} A_{11}^{-1} B_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & C_{NN} A_{NN}^{-1} B_{NN} \end{bmatrix}.$$

$X_{jj},\ j = 1, 2, \ldots, N$ are the unique stabilizing positive semidefinite solution of the following AREs:

$$X_{jj} A_{jj} + A_{jj}^{\mathrm{T}} X_{jj} - X_{jj} S_{jj} X_{jj} + Q_{jj} = 0. \quad (34)$$

It is well known from Kokotović et al. (1986) that controller (32) is identical with the following controller:

$$\begin{aligned} u_{j\mathrm{com}}(t) = &-R_j^{-1} B_{j0}^{\mathrm{T}} X_{00} x_0(t) - R_j^{-1} B_{jj}^{\mathrm{T}} X_{j0} x_0(t) \\ &- R_j^{-1} B_{jj}^{\mathrm{T}} X_{jj} x_j(t), \end{aligned} \quad (35)$$

where $X_{j0},\ j = 1, 2, \ldots, N$ are

$$\begin{aligned} X_{j0}^{\mathrm{T}} = &[X_{00}(S_{0j} X_{jj} - A_{0j}) \\ &- (A_{j0}^{\mathrm{T}} X_{jj} + Q_{0j})](A_{jj} - S_{jj} X_{jj})^{-1}. \end{aligned} \quad (36)$$

Furthermore, the composite optimal controller $u_{\mathrm{com}}(t) = [u_{1\mathrm{com}}(t)^{\mathrm{T}} \ \cdots \ u_{N\mathrm{com}}(t)^{\mathrm{T}}]^{\mathrm{T}}$ can be rewritten as the following composite controller:

$$u_{\mathrm{com}}(t) = -R^{-1} B^{\mathrm{T}} \begin{bmatrix} X_{00} & 0 & 0 & \cdots & 0 \\ X_{10} & X_{11} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{N0} & 0 & 0 & \cdots & X_{NN} \end{bmatrix} x(t). \quad (37)$$

The following theorem gives a relation between the proposed controller (18) and the composite optimal controller (37).

Theorem 13. *Under Assumptions* 1–3, *the following identities*

$$X_{jj} = \bar{P}_{jj}^*,\ X_{j0} = \bar{P}_{j0}^*,\ X_{00} = \bar{P}_{00}^*,\ j = 1, 2, \ldots, N \quad (38)$$

hold. Hence, the resulting near-optimal controller (18) *is the same as the composite optimal controller* (37).

Proof. For the proof, see Appendix A. □

From Theorem 13, we claim that the new near-optimal controller includes the existing composite optimal controller (37) as a special case because our controller can be constructed even if one of the fast state matrices A_{jj} is singular and the small positive parameters have different values.

5. Numerical example

In order to demonstrate the efficiency of our proposed controller, we have run a numerical example.

Consider the following optimal control problem:

$$\dot{x}_0(t)=\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x_0(t)+\begin{bmatrix} 0 \\ 2 \end{bmatrix} x_1(t)+\begin{bmatrix} 0 \\ 3 \end{bmatrix} x_2(t) + \sum_{j=1}^{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_j(t), \quad x_0(0)=x_0^0, \tag{39a}$$

$$\varepsilon_1\dot{x}_1(t)=\begin{bmatrix} 1 & 0.2 \end{bmatrix} x_0(t)+u_1(t),\ x_1(0)=x_1^0, \tag{39b}$$

$$\varepsilon_2\dot{x}_2(t)=\begin{bmatrix} 1 & 0.3 \end{bmatrix} x_0(t)+u_2(t),\ x_2(0)=x_2^0 \tag{39c}$$

with a performance index

$$J=\frac{1}{2}\int_0^{\infty}\left(x_0^{\mathrm{T}}(t)x_0(t)+2\sum_{j=1}^{2}\{x_j^{\mathrm{T}}(t)x_j(t)+u_j^{\mathrm{T}}(t)u_j(t)\}\right)\mathrm{d}t. \tag{40}$$

Note that the above system (39) is the nonstandard MSPS because of $A_{11}=A_{22}=0$.

Referring the design procedure, the near-optimal control is

$$u_{\mathrm{app}}(t) = \begin{bmatrix} -1.5295 & -5.2582\times 10^{-1} & -1.0000 & 0 \\ -1.7943 & -7.8872\times 10^{-1} & 0 & -1.0000 \end{bmatrix} x(t). \tag{41}$$

Now, letting $\varepsilon_1=\varepsilon_2=0.1$, the optimal feedback control is

$$u_{\mathrm{opt}}(t) = \begin{bmatrix} -1.4430 & -5.4031\times 10^{-1} & -1.0644 & -1.0940\times 10^{-1} \\ -1.6360 & -7.5160\times 10^{-1} & -8.8270\times 10^{-2} & -1.1514 \end{bmatrix} x(t). \tag{42}$$

We evaluate the costs using the near-optimal controller (41). We assume that the initial conditions are zero mean independent random vector with covariance matrix $E[x(0)x(0)^{\mathrm{T}}]=I_4$. The average values of the performance index are $E[J_{\mathrm{app}}]=1.7297$, $E[J_{\mathrm{opt}}]=1.6964$. Hence, the loss of performance J_{app} is less than 1.9630% compared with J_{opt}. The values of the cost functional for various ε_1, ε_2 are given in Table 1, where

Table 1

ε_1	ε_2	$E[J_{\mathrm{app}}]$	$E[J_{\mathrm{opt}}]$	ϕ
10^{-1}	10^{-1}	1.7297	1.6964	3.3291
10^{-1}	5×10^{-2}	1.6241	1.6061	3.6085
10^{-2}	10^{-2}	1.4098	1.4094	4.6778
10^{-2}	5×10^{-3}	1.3992	1.3990	4.6297
10^{-3}	10^{-3}	1.3779	1.3779	4.8666
10^{-3}	5×10^{-4}	1.3768	1.3768	4.7574
10^{-4}	10^{-4}	1.3747	1.3747	4.8862

$$\phi=\frac{E[J_{\mathrm{app}}]-E[J_{\mathrm{opt}}]}{\|\mu\|^2}=\frac{E[J_{\mathrm{app}}]-E[J_{\mathrm{opt}}]}{\varepsilon_1\varepsilon_2}.$$

It is easy to verify that $J_{\mathrm{app}}=J_{\mathrm{opt}}+O(\|\mu\|^2)$ because of $\phi<\infty$.

6. Conclusion

In this paper, we have studied the optimal control problem associated with the MSPS. The main contribution of this paper is to propose the new designing method of the ε-independent controller. Note that our designing method is quite different from the existing methods such as the two-time-scale design method and the descriptor variable approach. Furthermore, we have proven that the resulting controller achieves $O(\|\mu\|^2)$ approximation of the optimal cost compared with the existing result for a special case of the fast subsystems. Finally, we have proved that the composite controller obtained by decomposing the full systems and the approximation controller obtained by eliminating ε_j of the full controller are identical under the certain condition.

Appendix A. Proof of Theorem 13

Proof. First, comparing (34) with (13c) $X_{jj}=\bar{P}_{jj}^*$, $j=1,2,\ldots,N$ yields directly. Second, comparing (36) with (13b) and noting that $X_{jj}=\bar{P}_{jj}^*$, we have the conclusion that $X_{j0}=\bar{P}_{j0}^*$, $j=1,2,\ldots,N$ if $X_{00}=\bar{P}_{00}^*$. Therefore, the remainder of the proof is to show that $X_{00}=\bar{P}_{00}^*$. In order to do that, we only need to show that the ARE (33) and (13a) are the same equations. Before showing these relations, let us define the following matrices (Kokotović et al., 1986, pp. 115)

$$H=I_{\hat{n}}+\text{block-diag}\,(R_1^{-1}B_{11}^{\mathrm{T}}\bar{P}_{11}^*D_{11}^{-1}B_{11} \cdots R_N^{-1}B_{NN}^{\mathrm{T}}\bar{P}_{NN}^*D_{NN}^{-1}B_{NN}). \tag{A.1}$$

Then,

$$H^{-1} = I_{\hat{n}} - \text{block-diag}\,(R_1^{-1}B_{11}^{\mathrm{T}}\bar{P}_{11}^{*}A_{11}^{-1}B_{11} \quad \cdots \quad R_N^{-1}B_{NN}^{\mathrm{T}}\bar{P}_{NN}^{*}A_{NN}^{-1}B_{NN}). \quad (\text{A.2})$$

Thus, using (A.2) and the ARE (13c) we have

$$H^{-\mathrm{T}}RH^{-1} = R + D_r^{\mathrm{T}}D_r = R_r > 0 \Leftrightarrow HR^{-1}H^{\mathrm{T}} = R_r^{-1}. \quad (\text{A.3})$$

Let us further introduce six useful identities.

$$A_{jj}^{-1} + A_{jj}^{-1}S_{jj}\bar{P}_{jj}^{*}D_{jj}^{-1} = D_{jj}^{-1}, \quad (\text{A.4a})$$

$$A_{jj}^{-1} + D_{jj}^{-1}S_{jj}\bar{P}_{jj}^{*}A_{jj}^{-1} = D_{jj}^{-1}, \quad (\text{A.4b})$$

$$I_{n_j} + S_{jj}\bar{P}_{jj}^{*}D_{jj}^{-1} = A_{jj}D_{jj}^{-1}, \quad (\text{A.4c})$$

$$I_{n_j} + \bar{P}_{jj}^{*}S_{jj}D_{jj}^{-\mathrm{T}} = A_{jj}^{\mathrm{T}}D_{jj}^{-\mathrm{T}}, \quad (\text{A.4d})$$

$$Q_{0j}^{\mathrm{T}} - Q_{jj}A_{jj}^{-1}A_{j0} = \bar{Q}_{0j}^{\mathrm{T}} + D_{jj}^{\mathrm{T}}\bar{P}_{jj}^{*}A_{jj}^{-1}A_{j0}, \quad (\text{A.4e})$$

$$-D_{0j} + N_{1j}S_{jj}\bar{P}_{jj}^{*} = N_{1j}A_{jj},\ j = 1, 2, \ldots, N. \quad (\text{A.4f})$$

Then, we obtain

$$\begin{aligned} -B_rR_r^{-1}D_r^{\mathrm{T}} &= [S_{01} - A_{01}A_{11}^{-1}S_{11} \;\; S_{02} - A_{02}A_{22}^{-1}S_{22} \;\; \cdots \\ &\quad \times S_{0N} - A_{0N}A_{NN}^{-1}S_{NN}]\text{block-diag}(D_{11}^{-\mathrm{T}}C_{11}^{\mathrm{T}} \\ &\quad + \bar{P}_{11}^{*}D_{11}^{-1}S_{11}D_{11}^{-\mathrm{T}}C_{11}^{\mathrm{T}} \;\; \cdots \;\; D_{NN}^{-\mathrm{T}}C_{NN}^{\mathrm{T}} \\ &\quad + \bar{P}_{NN}^{*}D_{NN}^{-1}S_{NN}D_{NN}^{-\mathrm{T}}C_{NN}^{\mathrm{T}}). \end{aligned} \quad (\text{A.5})$$

Hence, we have (A.4e), (A.4f) and (A.5)

$$\begin{aligned} A_r - B_rR_r^{-1}D_r^{\mathrm{T}}C_r &= A_{00} - \sum_{j=1}^{N} A_{0j}A_{jj}^{-1}A_{j0} \\ &\quad + \sum_{j=1}^{N}(S_{0j} + N_{1j}S_{jj})D_{jj}^{-\mathrm{T}}(\bar{Q}_{0j}^{\mathrm{T}} + D_{jj}^{\mathrm{T}}\bar{P}_{jj}^{*}A_{jj}^{-1}A_{j0}) \\ &= A_{00} + N_1A_{f0} + S_{0f}N_2^{\mathrm{T}} + N_1S_fN_2^{\mathrm{T}} = \mathscr{A}. \end{aligned} \quad (\text{A.6})$$

Now, considering (A.4a), we have

$$B_rH = B_0 + [N_{11}B_{11} \;\; \cdots \;\; N_{1N}B_{NN}] = \mathscr{B}. \quad (\text{A.7})$$

Hence, using fact that (A.7), we have

$$B_rR_r^{-1}B_r^{\mathrm{T}} = B_rHR^{-1}H^{\mathrm{T}}B_r^{\mathrm{T}} = \mathscr{B}R^{-1}\mathscr{B}^{\mathrm{T}} = \mathscr{S}. \quad (\text{A.8})$$

Finally, using the identities of (A.8) it is straightforward but tedious to verify that

$$D_rR_r^{-1}D_r^{\mathrm{T}} = \text{block-diag}\,(C_{11}D_{11}^{-1}S_{11}D_{11}^{-\mathrm{T}}C_{11}^{\mathrm{T}} \quad \cdots \quad C_{NN}D_{NN}^{-1}S_{NN}D_{NN}^{-\mathrm{T}}C_{NN}^{\mathrm{T}}). \quad (\text{A.9})$$

Moreover, using (13c), we get

$$C_r^{\mathrm{T}}C_r = Q_{00} - \sum_{j=1}^{N}(\bar{Q}_{0j}A_{jj}^{-1}A_{j0} + A_{j0}^{\mathrm{T}}A_{jj}^{-\mathrm{T}}\bar{Q}_{0j}^{\mathrm{T}}) + \sum_{j=1}^{N}A_{j0}^{\mathrm{T}}A_{jj}^{-\mathrm{T}}\bar{P}_{jj}^{*}S_{jj}\bar{P}_{jj}^{*}A_{jj}^{-1}A_{j0}. \quad (\text{A.10})$$

Since $-Q_{jj} = A_{jj}^{\mathrm{T}}\bar{P}_{jj}^{*} + \bar{P}_{jj}^{*}A_{jj} - \bar{P}_{jj}^{*}S_{jj}\bar{P}_{jj}^{*}$, $j = 1, 2, \ldots, N$, it follows that

$$\begin{aligned} C_r^{\mathrm{T}}C_r - C_r^{\mathrm{T}}D_rR_r^{-1}D_r^{\mathrm{T}}C_r &= Q_{00} - \sum_{j=1}^{N}(\bar{Q}_{0j}A_{jj}^{-1}A_{j0} + A_{j0}^{\mathrm{T}}A_{jj}^{-\mathrm{T}}\bar{Q}_{0j}^{\mathrm{T}} \\ &\quad + \bar{Q}_{0j}D_{jj}^{-1}A_{j0} + A_{j0}^{\mathrm{T}}D_{jj}^{-\mathrm{T}}\bar{Q}_{0j}^{\mathrm{T}} - \bar{Q}_{0j}A_{jj}^{-1}A_{j0} \\ &\quad - A_{j0}^{\mathrm{T}}A_{jj}^{-\mathrm{T}}\bar{Q}_{0j}^{\mathrm{T}} + \bar{Q}_{0j}D_{jj}^{-1}S_{jj}D_{jj}^{-\mathrm{T}}\bar{Q}_{0j}^{\mathrm{T}}) = \mathscr{Q}. \end{aligned} \quad (\text{A.11})$$

In consequence, we have $X_{00} = \bar{P}_{00}^{*}$, hence, $X_{j0} = \bar{P}_{j0}^{*}$, $j = 1, 2, \ldots, N$. The proof of Theorem 13 is completed. □

References

Coumarbatch, C., & Gajić, Z. (2000a). Exact decomposition of the algebraic Riccati equation of deterministic multimodeling optimal control problems. *IEEE Transactions on Automatic Control*, *45*, 790–794.

Coumarbatch, C., & Gajić, Z. (2000b). Parallel optimal Kalman filtering for stochastic systems in multimodeling form. *Transactions on ASME, Journal of Dynamic Systems, Measurement, and Control*, *122*, 542–550.

Gajić, Z. (1988). The existence of a unique and bounded solution of the algebraic Riccati equation of multimodel estimation and control problems. *Systems & Control Letters*, *10*, 185–190.

Gajić, Z., & Khalil, H. K. (1986). Multimodel strategies under random disturbances and imperfect partial observations. *Automatica*, *22*, 121–125.

Khalil, H. K. (1979). Stabilization of multiparameter singularly perturbed systems. *IEEE Transactions on Automatic Control*, *24*, 790–791.

Khalil, H. K. (1980). Multimodel design of a Nash strategy. *Journal of Optimization Theory and Applications*, *31*, 553–564.

Khalil, H. K. (1981). Asymptotic stability of nonlinear Multiparameter singularly perturbed systems. *Automatica*, *17*, 797–804.

Khalil, H. K., & Kokotović, P. V. (1978). Control strategies for decision makers using different models of the same system. *IEEE Transactions on Automatic Control*, *23*, 289–298.

Khalil, H. K., & Kokotović, P. V. (1979a). Control of linear systems with multiparameter singular perturbations. *Automatica*, *15*, 197–207.

Khalil, H. K., & Kokotović, P. V. (1979b). *D*-Stability and multi-parameter singular perturbation. *SIAM Journal on Control and Optimization*, *15*, 197–207.

Kokotović, P. V., Khalil, H. K., & O'Reilly, J. (1986). *Singular perturbation methods in control: Analysis and design*. New York: Academic Press.

Magnus, J. R., & Neudecker, H. (1999). *Matrix differential calculus with applications in statistics and econometrics*. New York: Wiley.

Mukaidani, H. (2001). Pareto near-optimal strategy of multimodeling systems. In *Proceedings of the IEEE international Conference on Industrial Electronics, Control and Instrumentation*, Colorado (pp. 500–505).

Mukaidani, H. (2003). Near-optimal Kalman filters for multiparameter singularly perturbed linear systems. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, *50*, 717–721.

Mukaidani, H., & Mizukami, K. (2000). Robust stabilization for singularly perturbed systems with structured state space uncertainties. *Electrical Engineering in Japan*, *132*(4), 62–72.

Mukaidani, H., & Mizukami, K. (2001). Control of linear multiparameter singularity pertubed systems. In *Proceedings of the IFAC workshop on singular solutions and perturbations*, Bucharest (pp. 13–18).

Mukaidani, H., Xu, H., & Mizukami, K. (2001). New iterative algorithm for algebraic Riccati equation related to H_∞ control problem of singularly perturbed systems. *IEEE Transactions on Automatic Control*, *43*, 1659–1666.

Mukaidani, H., Xu, H., & Mizukami, K. (2002). Recursive computation of Pareto optimal strategy for multiparameter singularly perturbed systems. *Dynamics of Continuous, Discrete and Impulsive Systems*, *9b*, 175–200.

Özgüner, Ü. (1979). Near-optimal control of composite systems: The multi time-scale approach. *IEEE Transactions on Automatic Control*, *24*, 652–655.

Salman, M. A., Lee, A. Y., & Boustany, N. M. (1990). Reduced order design of active suspension control. *Transactions on ASME, Journal of Dynamic Systems, Measurement, and Control*, *122*, 542–550.

Wang, Y-Y, Paul, M., & Wu, N. E. (1994). Near-optimal control of nonstandard singularly perturbed systems. *Automatica*, *30*, 277–292.

Xu, H., Mukaidani, H., & Mizukami, K. (1997). New method for composite optimal control of singularly perturbed systems. *International Journal of Systems Sciences*, *28*, 161–172.

Hiroaki Mukaidani received his B.S. degree in integrated arts and sciences from Hiroshima University, Japan, in 1992 and his M. Eng. and Dr. Eng. degrees in information engineering from Hiroshima University, Japan, in 1994 and 1997, respectively. He worked with Hiroshima City University as a Research Associate from 1998 to 2002. Since 2002, he has been with the Graduate School of Education, Hiroshima University, Japan, as an Assistant Professor. His current research interests include robust control and its application of singularly perturbed systems. He is a member of the Institute of Electrical and Electronics Engineers (IEEE).

Hua Xu received his B.Eng. and M.Eng. degrees in electrical engineering from Northeastern University, China, in 1982 and 1985, respectively, and his Dr.Eng. degree in information engineering from Hiroshima University, Japan, in 1993. He worked with Hiroshima University as a Research Associate and Associate Professor from 1993 to 1998. Since 1998, he has been with the Graduate School of Business Sciences, the University of Tsukuba, Tokyo, Japan, as an Associate Professor, and Professor. His current research interests include dynamic optimization, dynamic games and their applications in business aspects of systems management, marketing, economics and finance. He is a member of International Society of Dynamic Games and some other academic societies of Japan.

Koichi Mizukami received his B. Eng. degree in electrical engineering from Hiroshima University, Japan in 1958 and his M.S. and Dr. Eng. degrees in electrical engineering from Kyoto University in 1960 and 1965. In 1963, he joined the Department of Electrical Engineering of Kyoto University as a Research Associate. He was a postdoctoral fellow of electrical engineering of the University of Toronto from 1966 to 1968. He then became an Associate Professor on the Faculty of Engineering of Hiroshima University, and served a Professor on the Faculty of Integrated Arts and Sciences at Hiroshima University from 1977 to 1999. Since then, he has been a Professor on the Faculty of Engineering at Hiroshima Kokusai Gakuin University. His research interests are in the area of control and information science, especially functional analysis, optimal control and information theory, fuzzy and neural network control, information retrieval and computer graphics. He was on the editorial board of the international Journal of Dynamics and Control, and the Journal of Mathematical Analysis and Applications. He is a member of the Institute of Electrical Engineers of Japan, the Information Processing Society of Japan, the Society of Instrument and Control Engineers of Japan, and the Institute of Systems, Control and Information Engineers of Japan. He is a Fellow of the Society of Instrument and Control Engineers of Japan, and an Emeritus Professor of Hiroshima University.