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A revised Kleinman algorithm to solve algebraic Riccati equation of singularly perturbed systems☆

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Abstract

In this paper, we show that the Kleinman algorithm can be used well to solve the algebraic Riccati equation (ARE) of singularly perturbed systems, where the quadratic term of the ARE may be indefinite. The quadratic convergence property of the Kleinman algorithm is proved by using the Newton–Kantorovich theorem when the initial condition is chosen appropriately. In addition, the numerical method to solve the generalized algebraic Lyapunov equation (GALE) appearing in the Kleinman algorithm is given.

Keywords: Kleinman algorithm; Singular perturbed systems; Newton–Kantorovich theorem; Generalized algebraic Lyapunov equation

1. Introduction

Many feedback control problems of singularly perturbed systems have been investigated extensively (see, e.g., Kokotović, Khalil, & O'Reilly, 1999; Gajić & Lim, 2001; Gajić, Petkovski, & Shen, 1990). In order to obtain the controller, we must solve the following *algebraic Riccati equation* (ARE) (1) with small positive constant parameter $\varepsilon > 0$.

$$A_\varepsilon^{\mathrm{T}} P_\varepsilon + P_\varepsilon A_\varepsilon - P_\varepsilon S_\varepsilon P_\varepsilon + Q = 0, \tag{1}$$

where

$$\varepsilon > 0, \quad P_\varepsilon = \begin{bmatrix} P_{11} & \varepsilon P_{21}^{\mathrm{T}} \\ \varepsilon P_{21} & \varepsilon P_{22} \end{bmatrix} \in \mathbf{R}^{n\times n}, \quad n = n_1 + n_2,$$

$$A_\varepsilon = \begin{bmatrix} A_{11} & A_{12} \\ \varepsilon^{-1} A_{21} & \varepsilon^{-1} A_{22} \end{bmatrix} \in \mathbf{R}^{n\times n},$$

$$S_\varepsilon = S_\varepsilon^{\mathrm{T}} = \begin{bmatrix} S_{11} & \varepsilon^{-1} S_{12} \\ \varepsilon^{-1} S_{12}^{\mathrm{T}} & \varepsilon^{-2} S_{22} \end{bmatrix} \in \mathbf{R}^{n\times n},$$

$$Q = Q^{\mathrm{T}} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_{22} \end{bmatrix} \in \mathbf{R}^{n\times n},$$

$$P_{11} = P_{11}^{\mathrm{T}},\ A_{11},\ S_{11} = S_{11}^{\mathrm{T}},\ Q_{11} = Q_{11}^{\mathrm{T}} \in \mathbf{R}^{n_1\times n_1},$$

$$P_{22} = P_{22}^{\mathrm{T}},\ A_{22},\ S_{22} = S_{22}^{\mathrm{T}},\ Q_{22} = Q_{22}^{\mathrm{T}} \in \mathbf{R}^{n_2\times n_2}.$$

No assumption is made on the definiteness of S_ε. It is well known that the ARE (1) occurs in the H_∞ control problem (Zhou, 1998), the robust stabilizing problem (Petersen & McFarlane, 1994) and so on. Note that it is very difficult to solve directly the singularly perturbed ARE due to high dimension and numerical stiffness (Kokotović et al., 1999; Gajić et al., 1990).

The recursive algorithm for various control problems of singularly perturbed systems have been developed in many literatures (see, e.g., Gajić et al., 1990; Mukaidani, Xu, & Mizukami, 1999). It is important to point out that the recursive algorithm based on the fixed point algorithm is very useful because of the reduced-order calculation corresponding to the slow and fast systems and that in such a case a parallel processing can be used.

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However, the recursive approach has property of linear convergence. On the other hand, the exact slow–fast decomposition method for solving the ARE of singularly perturbed systems has been proposed (see, e.g., Gajić & Lim, 2001; Fridman, 1996). However, in order to obtain the exact solution, one needs the same workspace as the full-order ARE for calculating the inverse matrix.

In this paper, we show that the Kleinman algorithm (Kleinman, 1968) based on the Newton method can be used well to solve the ARE of singularly perturbed systems. The idea of using the Newton method for solving the ARE of singularly perturbed systems originated in Rutkowski and Gajić (1993). However, the presented algorithm in the work of Rutkowski and Gajić (1993) still involves the fixed point algorithm, while our new iterative algorithm does not include any other iterative method. Thus, we can get the desired solution directly, owing to its quadratic speed of convergence.

Firstly, we define zero-order equations regarding the ARE (1). Now, let us define the following matrices:

$$T_1=\begin{bmatrix} A_{11} & -S_{11} \\ -Q_{11} & -A_{11}^{\mathrm{T}} \end{bmatrix}, \quad T_2=\begin{bmatrix} A_{12} & -S_{12} \\ -Q_{12} & -A_{21}^{\mathrm{T}} \end{bmatrix},$$

$$T_3=\begin{bmatrix} A_{21} & -S_{12}^{\mathrm{T}} \\ -Q_{12}^{\mathrm{T}} & -A_{12}^{\mathrm{T}} \end{bmatrix}, \quad T_4=\begin{bmatrix} A_{22} & -S_{22} \\ -Q_{22} & -A_{22}^{\mathrm{T}} \end{bmatrix}.$$

We shall make the following basic assumption without loss of generality (Gajić & Lim, 2001; Fridman, 1996):

Assumption 1. The Hamiltonian matrix T_4 is nonsingular.

Substituting P_ε into (1) and letting $\varepsilon=0$, we obtain the following zero-order equations

$$A_0^{\mathrm{T}}\bar{P}_{11}+\bar{P}_{11}A_0-\bar{P}_{11}S_0\bar{P}_{11}+Q_0=0, \tag{2a}$$

$$A_{22}^{\mathrm{T}}\bar{P}_{22}+\bar{P}_{22}A_{22}-\bar{P}_{22}S_{22}\bar{P}_{22}+Q_{22}=0, \tag{2b}$$

where $\bar{P}_{11}$ and $\bar{P}_{22}$ are zero-order solutions of the ARE (1) and

$$T_0=T_1-T_2T_4^{-1}T_3=\begin{bmatrix} A_0 & -S_0 \\ -Q_0 & -A_0^{\mathrm{T}} \end{bmatrix}.$$

Remark 2. The matrices A_0, S_0 and Q_0 do not depend on $\bar{P}_{22}$ since their matrices can be computed by using T_m, $m=1,\ldots,4$ which is independent of $\bar{P}_{22}$ (see, e.g., Tan, Leung, & Tu, 1998).

Let us now assume that

Assumption 3. The AREs (2a) and (2b) have the positive semidefinite stabilizing solutions.

The matrix $A_{22}-S_{22}\bar{P}_{22}$ for the ARE (2b) is nonsingular because of the Assumption 3. Then we have

$$\bar{P}_{21}=-N_2^{\mathrm{T}}+N_1^{\mathrm{T}}\bar{P}_{11}, \tag{2c}$$

where $\bar{P}_{21}$ is also zero-order solution of the ARE (1) and

$$N_2^{\mathrm{T}}=\Lambda_4^{-\mathrm{T}}q_{12}^{\mathrm{T}},\ N_1^{\mathrm{T}}=-\Lambda_4^{-\mathrm{T}}\Lambda_2^{\mathrm{T}},$$

$$\Lambda_1=A_{11}-S_{11}\bar{P}_{11}-S_{12}\bar{P}_{21},$$

$$\Lambda_3=A_{21}-S_{12}^{\mathrm{T}}\bar{P}_{11}-S_{22}\bar{P}_{21},$$

$$\Lambda_2=A_{12}-S_{12}\bar{P}_{22},\ \Lambda_4=A_{22}-S_{22}\bar{P}_{22},$$

$$\Lambda_0=\Lambda_1-\Lambda_2\Lambda_4^{-1}\Lambda_3,\ q_{12}=Q_{12}+A_{21}^{\mathrm{T}}\bar{P}_{22}.$$

The following Lemma was shown by Mukaidani, Xu, and Mizukami (1999).

Lemma 4. *Under the assumptions* 1 *and* 3, *if the AREs* (2a) *and* (2b) *have the positive semidefinite stabilizing solutions, then there exists small* $\bar{\varepsilon}>0$ *such that for all* $\varepsilon\in(0,\ \bar{\varepsilon})$, *the ARE* (1) *admits a positive semidefinite solution, which can be written as*

$$P_\varepsilon=\begin{bmatrix} \bar{P}_{11}+O(\varepsilon) & \varepsilon\bar{P}_{21}^{\mathrm{T}}+O(\varepsilon^2) \\ \varepsilon\bar{P}_{21}+O(\varepsilon^2) & \varepsilon\bar{P}_{22}+O(\varepsilon^2) \end{bmatrix}. \tag{3}$$

2. The new iterative algorithm

We propose the following algorithm based on the Kleinman algorithm for solving the ARE (1):

$$(A-SP^{(i)})^{\mathrm{T}}P^{(i+1)}+P^{(i+1)\mathrm{T}}(A-SP^{(i)})$$
$$+P^{(i)\mathrm{T}}SP^{(i)}+Q=0,\quad i=0,\ 1,\ 2,\ldots, \tag{4a}$$

with the initial condition obtained from

$$P^{(0)}=\begin{bmatrix} \bar{P}_{11} & \varepsilon\bar{P}_{21}^{\mathrm{T}} \\ \bar{P}_{21} & \bar{P}_{22} \end{bmatrix}, \tag{4b}$$

where $A=\Pi_\varepsilon A_\varepsilon$, $S=\Pi_\varepsilon S_\varepsilon\Pi_\varepsilon$,

$$P^{(i)}=\begin{bmatrix} P_{11}^{(i)} & \varepsilon P_{21}^{(i)\mathrm{T}} \\ P_{21}^{(i)} & P_{22}^{(i)} \end{bmatrix},$$

$$\Pi_\varepsilon=\begin{bmatrix} I_{n_1} & 0 \\ 0 & \varepsilon I_{n_2} \end{bmatrix},$$

and $\bar{P}_{11}$, $\bar{P}_{21}$, $\bar{P}_{22}$ are defined by Eq. (2).

In order to avoid the ill-condition caused by ε^{-1}, the resulting Algorithm (4a) will be used as the asymmetric matrices $P^{(i)}$. It is worth pointing out that $P^{(i)}$ is not symmetric, but $P_\varepsilon^{(i)}=\Pi_\varepsilon P^{(i)}=P^{(i)\mathrm{T}}\Pi_\varepsilon$ is.

Remark 5. By directly applying the Kleinman algorithm to the ARE (1), we get

$$(A_\varepsilon - S_\varepsilon P_\varepsilon^{(i)})^{\mathrm{T}} P_\varepsilon^{(i+1)} + P_\varepsilon^{(i+1)\mathrm{T}}(A_\varepsilon - S_\varepsilon P_\varepsilon^{(i)}) + P_\varepsilon^{(i)\mathrm{T}} S_\varepsilon P_\varepsilon^{(i)} + Q = 0, \quad i = 0, 1, 2, \ldots,$$

with the initial condition obtained from

$$P_\varepsilon^{(0)} = \begin{bmatrix} \bar{P}_{11} & \varepsilon \bar{P}_{21}^{\mathrm{T}} \\ \varepsilon \bar{P}_{21} & \varepsilon \bar{P}_{22} \end{bmatrix}.$$

Fortunately, using the scaling $A_\varepsilon = \Pi_\varepsilon^{-1} A$, $S_\varepsilon = \Pi_\varepsilon^{-1} S \Pi_\varepsilon^{-1}$ and $P_\varepsilon^{(i)} = \Pi_\varepsilon P^{(i)}$, we can change the above standard form into the asymmetric form (4a). However, it is important to note that we cannot apply the ordinary approach such as Hessenberg–Schur methods (Golub, Nash, & Loan, 1979) to the Algorithm (4a).

We now give the basic quadratic convergence theorem for the Algorithm (4a).

Theorem 6. *Define the following generalized algebraic Riccati equation* (*GARE*)

$$A^{\mathrm{T}} P + P^{\mathrm{T}} A - P^{\mathrm{T}} S P + Q = 0, \tag{5a}$$

$$\Pi_\varepsilon P = P^{\mathrm{T}} \Pi_\varepsilon. \tag{5b}$$

Under assumptions 1 *and* 3, *if the AREs* (2a) *and* (2b) *have the positive semidefinite stabilizing solutions, then the iterative Algorithm* (4a) *converges to the exact solution* P^* *of the GARE* (5a) *with the rate of quadratic convergence. Then the unique solution* P^* *of the GARE* (5a) *is in the neighborhood of the initial condition* $P^{(0)}$, *i.e.*

$$\|P^{(i)} - P^*\| \leqslant \frac{O(\varepsilon^{2^i})}{2^i \bar{\beta} \mathscr{L}}, \quad i = 1, 2, \ldots, \tag{6a}$$

$$\|P^{(0)} - P^*\| \leqslant \frac{1}{\bar{\beta}\mathscr{L}}[1 - \sqrt{1 - 2\bar{\alpha}}], \tag{6b}$$

where

$$F(P) = A^{\mathrm{T}} P + P^{\mathrm{T}} A - P^{\mathrm{T}} S P + Q,$$

$$\mathscr{L} = 2\|S\| < \infty, \ \bar{\beta} = \|[\nabla F(P^{(0)})]^{-1}\|,$$

$$\bar{\eta} = \bar{\beta} \cdot \|F(P^{(0)})\|, \quad \bar{\alpha} = \bar{\beta}\bar{\eta}\mathscr{L},$$

$$\nabla F = \frac{\partial \operatorname{vec} F}{\partial (\operatorname{vec} P)^{\mathrm{T}}}, \quad P = P^* = \begin{bmatrix} P_{11}^* & \varepsilon P_{21}^{*\mathrm{T}} \\ P_{21}^* & P_{22}^* \end{bmatrix} \tag{7}$$

and vec *denotes an ordered stack of the columns of its matrix* (*Magnus & Neudecker, 1999*). *Moreover, let* $P_{11}^{(\infty)}$, $P_{21}^{(\infty)}$ *and* $P_{22}^{(\infty)}$ *be the limit points of the iterative Algorithm* (4a). *As a results we have*

$$A^{\mathrm{T}} P^{(\infty)} + P^{(\infty)\mathrm{T}} A - P^{(\infty)\mathrm{T}} S P^{(\infty)} + Q = 0, \tag{8}$$

where

$$P^* = P^{(\infty)} = \begin{bmatrix} P_{11}^{(\infty)} & \varepsilon P_{21}^{(\infty)\mathrm{T}} \\ P_{21}^{(\infty)} & P_{22}^{(\infty)} \end{bmatrix}.$$

Proof. The proof is given directly by applying the Newton–Kantorovich theorem (Ortega, 1990) for the GARE (5a). Taking the partial derivative of the function $F(P)$ with respect to P yields

$$\nabla F(P) = (A - SP)^{\mathrm{T}} \otimes I_n + I_n \otimes (A - SP)^{\mathrm{T}}, \tag{9}$$

where $\otimes$ denotes Kronecker product (Magnus & Neudecker, 1999). It is obvious that $\nabla F(P)$ is continuous at for all P. Thus, it is obtained immediately from the above equation that

$$\|\nabla F(P_1) - \nabla F(P_2)\| \leqslant \mathscr{L}\|P_1 - P_2\|. \tag{10}$$

Moreover, using the fact that

$$\nabla F(P^{(0)}) = \begin{bmatrix} \Lambda_1 & \Lambda_2 + O(\varepsilon) \\ \Lambda_3 & \Lambda_4 + O(\varepsilon) \end{bmatrix}^{\mathrm{T}} \otimes I_n + I_n \otimes \begin{bmatrix} \Lambda_1 & \Lambda_2 + O(\varepsilon) \\ \Lambda_3 & \Lambda_4 + O(\varepsilon) \end{bmatrix}^{\mathrm{T}} \tag{11}$$

it follows that $\nabla F(P^{(0)})$ is nonsingular because Λ_4 and Λ_0 are stable (Kokotović et al., 1999; Gajić et al., 1990). Therefore, there exists $\bar{\beta}$ such that $\|[\nabla F(P^{(0)})]^{-1}\| \equiv \bar{\beta}$. On the other hand, since $F(P^{(0)}) = O(\varepsilon)$, there exists $\bar{\eta}$ such that, $\|[\nabla F(P^{(0)})]^{-1}\| \cdot \|F(P^{(0)})\| = O(\varepsilon) \equiv \bar{\eta}$. Thus, there exists $\bar{\alpha}$ such that $\bar{\alpha} \equiv \bar{\beta}\bar{\eta}\mathscr{L} < 2^{-1}$ because of $\bar{\eta} = O(\varepsilon)$. Now, let us define

$$t^* \equiv \frac{1}{\bar{\beta}\mathscr{L}}[1 - \sqrt{1 - 2\bar{\alpha}}] = \frac{1}{2\|S\| \cdot \|[\nabla F(P^{(0)})]^{-1}\|}[1 - \sqrt{1 - 2\bar{\alpha}}]. \tag{12}$$

Using, Newton–Kantorovich theorem, we can show that P^* is the unique solution in the subset $\mathscr{S} \equiv \{P: \|P^{(0)} - P\| \leqslant t^*\}$. Moreover, using Newton–Kantorovich theorem, the error estimate is given by

$$\|P^{(i)} - P^*\| \leqslant \frac{(2\bar{\alpha})^{2^i}}{2^i \bar{\beta}\mathscr{L}}, \quad i = 1, 2, \ldots . \tag{13}$$

Substituting $2\bar{\alpha} = O(\varepsilon)$ into (13), we have (6a). Furthermore, substituting P^* into P of the subset $\mathscr{S}$, we can also get (6b). Therefore, (6) holds for the small ε. □

In the rest of this section, we explain the method for solving the generalized algebraic Lyapunov equation (GALE) (4a). So far, there is little argument as to the numerical method for solving the GALE. Firstly, we convert (4a) into the following form:

$$\mathscr{U}\mathscr{P} = -\mathscr{Q}, \tag{14}$$

where $\mathscr{U}$ is $N \times N$, $(N = n(n+1)/2)$ matrix, $\mathscr{P}$ and $\mathscr{Q}$ are N column vectors given by

$$\begin{aligned}\mathscr{P} = {} & [p_{11}^{11}\ p_{12}^{11}\ \cdots\ p_{1n_1}^{11}\ p_{11}^{21}\ \cdots\ p_{1n_2}^{21}\ p_{22}^{11}\ \cdots\ p_{2n_1}^{11}\ p_{21}^{21}\ \cdots \\ & p_{2n_2}^{21}\ \cdots\ p_{n_1n_1}^{11}\ p_{n_11}^{21}\ \cdots\ p_{n_1n_2}^{21}\ \cdots\ p_{11}^{22}\ p_{12}^{22}\ \cdots\ p_{1n_2}^{22} \\ & p_{22}^{22}\ \cdots\ p_{2n_2}^{22}\ \cdots\ p_{(n_2-1)(n_2-1)}^{22}\ p_{(n_2-1)n_2}^{22}\ p_{n_2n_2}^{22}]^{\mathrm{T}}\end{aligned}$$

$$\begin{aligned}\mathscr{Q} = {} & [q_{11}\ q_{12}\ \cdots\ q_{1n}\ q_{22}\ q_{23}\ \cdots\ q_{2n} \\ & \cdots\ q_{(n-1)(n-1)}\ q_{(n-1)n}\ \cdots\ q_{nn}]^{\mathrm{T}}\end{aligned}$$

and since $P_{11}^{(i)} = P_{11}^{(i)\mathrm{T}}$, $P_{22}^{(i)} = P_{22}^{(i)\mathrm{T}}$,

$$P^{(i+1)} = \begin{bmatrix} P_{11}^{(i+1)} & \varepsilon P_{21}^{(i+1)\mathrm{T}} \\ P_{21}^{(i+1)} & P_{22}^{(i+1)} \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc} p_{11}^{11} & p_{12}^{11} & \cdots & p_{1n_1}^{11} & \varepsilon p_{11}^{21} & \varepsilon p_{12}^{21} & \cdots & \varepsilon p_{1n_2}^{21} \\ \cdot & p_{22}^{11} & \cdots & p_{2n_1}^{11} & \varepsilon p_{21}^{21} & \varepsilon p_{22}^{21} & \cdots & \varepsilon p_{2n_2}^{21} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \cdots & p_{n_1n_1}^{11} & \varepsilon p_{n_11}^{21} & \varepsilon p_{n_12}^{21} & \cdots & \varepsilon p_{n_1n_2}^{21} \\ \hline \cdot & \cdot & \cdots & \cdot & p_{11}^{22} & p_{12}^{22} & \cdots & p_{1n_2}^{22} \\ \cdot & \cdot & \cdots & \cdot & \cdot & p_{22}^{22} & \cdots & p_{2n_2}^{22} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & p_{n_2n_2}^{22} \end{array}\right],$$

$$P^{(i)\mathrm{T}}SP^{(i)} + Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} & \cdots & q_{1n} \\ \cdot & q_{22} & q_{23} & \cdots & q_{2n} \\ \cdot & \cdot & q_{33} & \cdots & q_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \cdot & \cdots & q_{nn} \end{bmatrix}.$$

In this paper, we have improved a procedure which is given in Bingulac (1970) for obtaining the matrix $\mathscr{U}$. Using the similar technique, the algorithm has as a starting point the computation of the matrix $\mathscr{U}$, where the resulting algorithm requires an auxiliary $n \times n$ matrix L with integer entries.

Construction of matrix $\mathscr{U}$ is done through the following steps:

Step 1: Construct the $n \times n$ matrix L, given by

$$L = \begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & n-1 & n \\ 2 & n+1 & n+2 & n+3 & \cdots & 2n-2 & 2n-1 \\ 3 & n+2 & 2n & 2n+1 & \cdots & 3n-4 & 3n-3 \\ 4 & n+3 & 2n+1 & 3n-2 & \cdots & 4n-7 & 4n-6 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n-1 & 2n-2 & 3n-4 & 4n-7 & \cdots & N-2 & N-1 \\ n & 2n-1 & 3n-3 & 4n-6 & \cdots & N-1 & N \end{bmatrix}.$$

Step 2: Let us define

$$A - SP^{(i)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}.$$

Construct the $N \times N$ matrix $V = \{v_{i'j'}\}$ with $v_{i'j'} = a_{l'k'}$ where the indices i' and j', $i', j' = 1, 2, 3, \ldots, N$ are given by the following elements of the auxiliary matrix L:

$$i' = L_{k'h'}, \quad j' = L_{l'h'}, \quad k', l', h' = 1, 2, 3, \ldots, n.$$

Moreover, if $1 \leqslant f' \leqslant n_1$, $n_1 + 1 \leqslant g' \leqslant n$ and the columns of V correspond to the number of matrix $L = \{l_{f'g'}\}$, then multiply the ε on the $a_{m'n'}$, $(1 \leqslant m' \leqslant n_1$, $1 \leqslant n' \leqslant n)$ in the matrix V.

Step 3: The required matrix $\mathscr{U}$ is obtained by multiplying by 2 all the elements of V whose row indices correspond to diagonal elements of the matrix L, that is, $1, n+1, 2n, 3n-2, \ldots, N-2$, and N.

We now summarize a perturbation analysis of the GALE (4a). Letting $\varepsilon = 0$ and using Kronecker products, the GALE (4a) can be written as

$$\mathscr{V} \begin{bmatrix} \operatorname{vec} \bar{P}_{11}^{(i+1)} \\ \operatorname{vec} \bar{P}_{21}^{(i+1)} \\ \operatorname{vec} \bar{P}_{22}^{(i+1)} \end{bmatrix} = \begin{bmatrix} \operatorname{vec} \bar{Q}_{11} \\ \operatorname{vec} \bar{Q}_{12}^{\mathrm{T}} \\ \operatorname{vec} \bar{Q}_{22} \end{bmatrix},$$

$$\begin{bmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_3 & \Lambda_4 \end{bmatrix} = A - S\bar{P}^{(i)},$$

$$\begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{12}^{\mathrm{T}} & \bar{Q}_{22} \end{bmatrix} = \bar{P}^{(i)\mathrm{T}} S \bar{P}^{(i)} + Q,$$

$$\bar{P}^{(i)} = \begin{bmatrix} \bar{P}_{11}^{(i)} & 0 \\ \bar{P}_{21}^{(i)} & \bar{P}_{22}^{(i)} \end{bmatrix},$$

$$\mathscr{V} = \begin{bmatrix} (I_{n_1} \otimes \Lambda_1^{\mathrm{T}})U_{n_1 n_1} + \Lambda_1^{\mathrm{T}} \otimes I_{n_1} & (I_{n_1} \otimes \Lambda_3^{\mathrm{T}})U_{n_1 n_2} + \Lambda_3^{\mathrm{T}} \otimes I_{n_1} & 0 \\ (I_{n_1} \otimes \Lambda_2^{\mathrm{T}})U_{n_1 n_1} & (I_{n_1} \otimes \Lambda_4^{\mathrm{T}})U_{n_1 n_2} & \Lambda_3^{\mathrm{T}} \otimes I_{n_2} \\ 0 & 0 & (I_{n_2} \otimes \Lambda_4^{\mathrm{T}})U_{n_2 n_2} + \Lambda_4^{\mathrm{T}} \otimes I_{n_2} \end{bmatrix}.$$

where $U_{n_1 n_1}$ denotes a permutation matrix in Kronecker matrix sense (Magnus & Neudecker, 1999) and $\mathrm{vec}\,\bar{P}_{lm}^{(i+1)}$ denotes an ordered stack of the columns of $P_{lm}^{(i+1)}$ when $\varepsilon = 0$. It can be shown, after some algebra, that the determinant of $\mathscr{V}$ is expressed as

$$\begin{aligned} \det \mathscr{V} = {} & \det\left[(I_{n_2} \otimes \Lambda_4^{\mathrm{T}})U_{n_2 n_2} + \Lambda_4^{\mathrm{T}} \otimes I_{n_2}\right] \\ & \cdot \det(I_{n_1} \otimes \Lambda_4^{\mathrm{T}}) \\ & \cdot \det[(I_{n_1} \otimes \Lambda_0^{\mathrm{T}})U_{n_1 n_1} + \Lambda_0^{\mathrm{T}} \otimes I_{n_1}]. \end{aligned}$$

Obviously, Λ_4 and Λ_0 are nonsingular matrices. Thus, there exists $\mathscr{V}^{-1}$. Therefore, the condition number (Ortega, 1990) of $\mathscr{V}$, that is, $K(\mathscr{V}) = \|\mathscr{V}\| \cdot \|\mathscr{V}^{-1}\|$ is given by $K(\mathscr{V}) = O(1)$. Since $K(\mathscr{V})$ is not large, the matrix $\mathscr{V} + O(\varepsilon)$ is well-conditioned for small ε. It is very important to note that by compressing the same rows and columns the matrix $\mathscr{V} + O(\varepsilon)$ can be changed as $\mathscr{U}$. Consequently, in words the perturbation theorem says that small changes in $\mathscr{V}$ i.e., $\mathscr{U}$ can induce a small relative error in $P^{(i+1)}$ for the GALE (4a).

In iterative calculations, where the solution of (14) and construction of the matrix $\mathscr{U}$ are in an iterative loop, the presented method is more suitable from computing time point of view. Moreover, since the matrix $\mathscr{U}$ is compressed into the matrix $\mathscr{V} + O(\varepsilon)$, the matrix $\mathscr{U}$ is well-conditioned for the small parameter ε. That is, well-conditioning of Eq. (14) is preserved. Therefore, the algorithm of Bingulac (1970) is most suitable for solving the GALE (4a). On the other hand, for Eq. (14), we can exactly get the required solution by using Gaussian elimination with partial pivoting which is most favorably computed in term of a system of linear equations and has been investigated extensively in many literature (see, e.g., Ortega, 1990).

We must solve Eq. (14) with the dimension N larger than n_1 or n_2 in comparison with the exact decomposition technique (Gajić & Lim, 2001; Fridman, 1996). In order to reduce the dimension of the workspace, there exists an algorithm for solving the GALE which is based on the generalized Schur method (Kågström & Westin, 1989; Mukaidani, Xu, & Mizukami, 2000). The reader is referred to above references regarding the algorithm.

3. Conclusions

In this paper, we have discussed the iterative method for solving the ARE with an indefinite quadratic term of the singularly perturbed control systems. The main result of this paper is the proof of the quadratic convergence property for the Kleinman algorithm. The proof has been done by using the Newton–Kantorovich theorem different from the successive approximation method. Consequently, we need no assumption for the definiteness regarding the quadratic term. Based on the classical numerical approach, we have also presented the numerical method for solving the GALE appearing in the Kleinman algorithm. It can be therefore applied to a control law synthesis involving a solution of an ARE such as the H_2 and H_∞ control problem (Zhou, 1998) and the guaranteed cost control problem (Petersen & McFarlane, 1994).

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