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Brief paper

A new approach to robust guaranteed cost control for uncertain multimodeling systems☆

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Abstract

The guaranteed cost control problem for multimodeling systems with norm bounded uncertainty is investigated. The main contribution in this paper is that a new ε-independent controller is derived by solving the reduced-order slow and fast algebraic Riccati equations (AREs) whose dimension is smaller than the dimension of full-order multiparameter algebraic Riccati equation (MARE). It is shown that if these AREs have a positive definite stabilizing solution then the closed-loop system is quadratically stable and has the cost bound.

Keywords: Multiparameter singularly perturbed systems (MSPS); Multiparameter algebraic Riccati equation (MARE); Uncertainty; Guaranteed cost control; ε-Independent controller

1. Introduction

When several small singular perturbation parameters of the same order of magnitude are present in the dynamic model of a physical system, the control problem is usually approached as single parameter perturbation problems. Although this is done by scaling the coefficients, these parameters are often not known exactly. Thus, it is not applicable to a wider class of problems for such case (Khalil and Kokotović, 1979). One solution to this problem is the so-called multimodeling systems approach. The control problem of the multimodeling systems has been widely studied during the past few decades (see e.g., Khalil & Kokotović, 1978, 1979; Khalil, 1979; Coumarbatch and Gajić, 2000; Gajić and Khalil, 1986; Gajić, 1988; Mukaidani et al., 2002b; Mukaidani et al., 2002a). A popular approach to deal with the multiparameter singularly perturbed systems (MSPS) and the singularly perturbed systems (SPS) are the two-time-scale design method (see e.g., Khalil & Kokotović, 1978, 1979; Khalil, 1979; Wang et al., 1994). When ε_j is very small or unknown, the previously used technique is very efficient. However, as long as the stabilizing problems of the uncertain SPS and MSPS are considered, the assumption that the fast state uncertain matrices $A_{jj}+\Delta A_{jj}(t)$ are Hurwitz is needed (see e.g., Corless et al., 1993). Particularly, in order to decompose the SPS, further assumptions for the uncertainties $\Delta A_{jj}(t)$ have been needed (Corless et al., 1993).

In order to construct the controller, the solution of the multiparameter algebraic Riccati equation (MARE) is needed. Although various reliable approaches for solving the MARE have been established (see e.g., Coumarbatch and Gajić, 2000; Mukaidani et al., 2002a, b), a limitation of these approaches is that the small parameters are assumed to be known. In practice, the small perturbation parameters ε_j are often not known. Thus, it is not applicable to a large class of problems where the parameters represent small unknown perturbations whose values are not known exactly.

It is well-known that the guaranteed cost control approach (Petersen and McFarlane, 1994) which satisfies not only the

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robust stability, but also an adequate level of performance is very useful. This approach has the advantage of providing an upper bound on a given performance index. Although there exist various studies of the guaranteed cost-control problem of the SPS (see e.g., Mukaidani and Mizukami, 2000; Mukaidani and Xu, 2001 and reference therein), these approaches need information of the small parameters ε_j. It should be noted that the stabilization problem and the guaranteed cost control problem of the MSPS with uncertain parameters in cases, where the small parameters are not known exactly have not been investigated so far.

In this paper, the guaranteed cost control problem of the MSPS is newly investigated. Firstly, the bounded solution of the MARE with an indefinite sign quadratic term and its asymptotic structure are established. Secondly, using the asymptotic structure, a new guaranteed cost controller which does not depend on the values of the small parameters ε_j is obtained. Therefore, even though the small perturbation parameters ε_j are unknown, the proposed controller can be constructed. As another significant feature, the new method of calculation for the guaranteed cost is proposed to obtain the ε-independent controller. In particular, since the proposed method is based on the reduced-order Algebraic Riccati equations (AREs) with the smaller state dimension, the amount of computation required to get the ε-independent controller becomes small in contrast with the case of solving the full-order MARE.

Notation. The superscript T denotes matrix transpose. det L denotes the determinant of the square matrix L. I_p denotes the $p \times p$ identity matrix. **block diag** denotes the block diagonal matrix. vec M denotes the column vector of the matrix M (Magnus and Neudecker, 1999). $\otimes$ denotes Kronecker product. U_{pq} denotes a permutation matrix in Kronecker matrix sense (Magnus and Neudecker, 1999) such that $U_{pq}\,\mathrm{vec}\,M = \mathrm{vec}\,M^{\mathrm{T}}$, $M \in \mathbf{R}^{p\times q}$. $E[\cdot]$ denotes the expectation.

2. Problem statement

Consider the uncertain MSPS

$$\dot{x}(t) = [A_e + D_e F(t) E_a]x(t) + [B_e + D_e F(t) E_b]u(t), \quad (1)$$

where ε_j, $j = 1, \ldots, N$ are the small positive parameters,

$$x(t) := [x_0^{\mathrm{T}}(t) \quad x_1^{\mathrm{T}}(t) \quad \cdots \quad x_N^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbf{R}^n,$$

$$n := \sum_{i=0}^{N} n_i,$$

where $x_j(t) \in \mathbf{R}^{n_j}$, $j = 0, \ldots, N$ are the state vectors.

$$u(t) := [u_1^{\mathrm{T}}(t) \quad \cdots \quad u_N^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbf{R}^m, \quad m := \sum_{i=1}^{N} m_i,$$

where $u_j(t) \in \mathbf{R}^{m_j}$, $j = 1, \ldots, N$ are the control input. Moreover, $F_{jj}(t) \in \mathbf{R}^{k_j \times s_j}$ are Lebesgue measurable matrix of uncertain parameters satisfying $F_{jj}^{\mathrm{T}}(t) F_{jj}(t) \leqslant I_{s_j}$. All the matrices are the constant matrices of appropriate dimensions. The partitioned matrices are:

$$\Pi_e := \textbf{block diag}\,(\varepsilon_1 I_{n_1} \quad \cdots \quad \varepsilon_N I_{n_N}),$$

$$A_e := \begin{bmatrix} A_{00} & A_{0f} \\ \Pi_e^{-1} A_{f0} & \Pi_e^{-1} A_f \end{bmatrix},$$
$$A_{0f} := [A_{01} \quad \cdots \quad A_{0N}],$$
$$A_{f0} := [A_{10}^{\mathrm{T}} \quad \cdots \quad A_{N0}^{\mathrm{T}}]^{\mathrm{T}},$$
$$A_f := \textbf{block diag}\,(A_{11} \quad \cdots \quad A_{NN}),$$

$$B_e := \begin{bmatrix} B_0 \\ \Pi_e^{-1} B_f \end{bmatrix}, \quad B_0 := [B_{01} \quad \cdots \quad B_{0N}],$$
$$B_f := \textbf{block diag}\,(B_{11} \quad \cdots \quad B_{NN}),$$

$$D_e := \begin{bmatrix} D_0 \\ \Pi_e^{-1} D_f \end{bmatrix}, \quad D_0 := [D_{01} \quad \cdots \quad D_{0N}],$$
$$D_f := \textbf{block diag}\,(D_{11} \quad \cdots \quad D_{NN}),$$
$$F(t) := \textbf{block diag}\,(F_{11}(t) \quad \cdots \quad F_{NN}(t)),$$

$$E_a := [E_{a0} \quad E_{af}], \quad E_{a0} := [E_{a10}^{\mathrm{T}} \quad \cdots \quad E_{aN0}^{\mathrm{T}}]^{\mathrm{T}},$$
$$E_{af} := \textbf{block diag}\,(E_{a11} \quad \cdots \quad E_{aNN}),$$
$$E_b := \textbf{block diag}\,(E_{b11} \quad \cdots \quad E_{bNN}).$$

We assume that the ratios of the small positive parameter ε_j are bounded by some positive constants $\underline{k}_{ij}$ and $\bar{k}_{ij}$ (see e.g., Khalil & Kokotović, 1978, 1979; Khalil, 1979),

$$0 < \underline{k}_{ij} \leqslant \alpha_{ij} \leqslant \bar{k}_{ij} < \infty, \quad (2)$$

where

$$\alpha_{ij} := \frac{\varepsilon_j}{\varepsilon_i}. \quad (3)$$

Associated with system (1) is the cost function

$$\mathscr{J} = \int_0^{\infty} [x^{\mathrm{T}}(t) Q x(t) + u^{\mathrm{T}}(t) R u(t)]\,\mathrm{d}t, \quad (4)$$

where Q and R are the positive definite symmetric matrices.

Definition 1. A control law $u(t) = Kx(t)$ is said to define a quadratic guaranteed cost control with the associated cost matrix $X_e > 0$ for the MSPS (1) and the cost function (4) if

$$\frac{\mathrm{d}}{\mathrm{d}t} x^{\mathrm{T}}(t) X_e x(t) + x^{\mathrm{T}}(t)[Q + K^{\mathrm{T}} R K]x(t) \leqslant 0 \quad (5)$$

for all nonzero $x(t)$ and all uncertain matrix $F(t)$.

The following result is already known in Moheimani and Petersen (1996).

Lemma 2. *Consider the closed-loop uncertain MSPS* (1) *with robust control law* $u(t) = Kx(t)$. *Suppose there exist a symmetric positive-definite matrix* $P_e > 0$ *and a positive scalar parameter* μ *such that for all uncertain matrices* $F_{jj}(t)$, *the following MARE satisfies*

$$P_e\bar{A}_e + \bar{A}_e^{\mathrm{T}}P_e + \mu P_e D_e D_e^{\mathrm{T}} P_e + \mu^{-1}\bar{E}^{\mathrm{T}}\bar{E} + \bar{Q} = 0, \tag{6}$$

where

$$\begin{aligned}
&\bar{A}_e := A_e + B_e K, \quad \bar{E} := E_a + E_b K,\\
&\bar{Q} := Q + K^{\mathrm{T}}RK,\\
&Q := \begin{bmatrix} Q_{00} & Q_{0f} \\ Q_{0f}^{\mathrm{T}} & Q_f \end{bmatrix} > 0, \quad Q_{0f} := [Q_{01} \quad \cdots \quad Q_{0N}],\\
&Q_f := \textbf{block diag}\,(Q_{11} \quad \cdots \quad Q_{NN}),\\
&R := \textbf{block diag}\,(R_{11} \quad \cdots \quad R_{NN}) > 0,\\
&P_e := \begin{bmatrix} P_{00} & P_{f0}^{\mathrm{T}}\Pi_e \\ \Pi_e P_{f0} & \Pi_e P_f \end{bmatrix}, \quad P_{00} = P_{00}^{\mathrm{T}} > 0,\\
&\Pi_e P_f = P_f^{\mathrm{T}}\Pi_e, \quad P_{f0} := [P_{10}^{\mathrm{T}} \quad \cdots \quad P_{N0}^{\mathrm{T}}]^{\mathrm{T}},\\
&P_f := \begin{bmatrix} P_{11} & \alpha_{12}P_{21}^{\mathrm{T}} & \alpha_{13}P_{31}^{\mathrm{T}} & \cdots & \alpha_{1N}P_{N1}^{\mathrm{T}} \\ P_{21} & P_{22} & \alpha_{23}P_{32}^{\mathrm{T}} & \cdots & \alpha_{2N}P_{N2}^{\mathrm{T}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{N1} & P_{N2} & P_{N3} & \cdots & P_{NN} \end{bmatrix}.
\end{aligned}$$

Then the control law $u(t)=Kx(t)$ *is said to be the quadratic guaranteed cost control with the cost matrix* P_e.

The following result is well-known (Petersen and McFarlane, 1994).

Lemma 3. *Assume that there exists a matrix* X *such that* $X_e = X_e^{\mathrm{T}} := \Phi_e X > 0$ *is a symmetric positive definite matrix. Assume also that there exists the positive scalar parameter* ν *such that for all uncertain matrices* $F_{jj}(t)$, *the following generalized multiparameter algebraic Riccati equation* (*GMARE*) *satisfies*

$$\begin{aligned}
&X^{\mathrm{T}}(A - B\mathscr{R}^{-1}E_b^{\mathrm{T}}E_a) + (A - B\mathscr{R}^{-1}E_b^{\mathrm{T}}E_a)^{\mathrm{T}}X\\
&\quad + \nu X^{\mathrm{T}}(DD^{\mathrm{T}} - B\mathscr{R}^{-1}B^{\mathrm{T}})X\\
&\quad + \nu^{-1}E_a^{\mathrm{T}}(I_{\bar{s}} - E_b\mathscr{R}^{-1}E_b^{\mathrm{T}})E_a + Q = 0,
\end{aligned} \tag{7}$$

where $A := \Phi_e^{-1}A_e$, $B := \Phi_e^{-1}B_e$, $D := \Phi_e^{-1}D_e$,

$$\begin{aligned}
&\bar{s} := \sum_{i=1}^{N} s_i, \quad \mathscr{R} := \nu R + E_b^{\mathrm{T}}E_b, \quad \Phi_e := \begin{bmatrix} I_{n_0} & 0 \\ 0 & \Pi_e \end{bmatrix},\\
&X := \begin{bmatrix} X_{00} & X_{f0}^{\mathrm{T}}\Pi_e \\ X_{f0} & X_f \end{bmatrix}, \quad X_{00} = X_{00}^{\mathrm{T}} > 0,\\
&\Pi_e X_f = X_f^{\mathrm{T}}\Pi_e, \quad X_{f0} := [X_{10}^{\mathrm{T}} \quad \cdots \quad X_{N0}^{\mathrm{T}}]^{\mathrm{T}},\\
&X_f\\
&\quad := \begin{bmatrix} X_{11} & \alpha_{12}X_{21}^{\mathrm{T}} & \alpha_{13}X_{31}^{\mathrm{T}} & \cdots & \alpha_{1N}X_{N1}^{\mathrm{T}} \\ X_{21} & X_{22} & \alpha_{23}X_{32}^{\mathrm{T}} & \cdots & \alpha_{2N}X_{N2}^{\mathrm{T}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{N1} & X_{N2} & X_{N3} & \cdots & X_{NN} \end{bmatrix}.
\end{aligned}$$

Then the closed-loop uncertain MSPS (1) *with a linear state feedback control law* (8) *is the guaranteed cost control*

$$u(t) = K_{\mathrm{exa}}x(t) = -\mathscr{R}^{-1}[\nu B^{\mathrm{T}}X + E_b^{\mathrm{T}}E_a]x(t). \tag{8}$$

Moreover, the corresponding value of the cost function (4) *satisfies the following inequality* (9):

$$\mathscr{J} \leqslant x^{\mathrm{T}}(0)\Phi_e X x(0). \tag{9}$$

The objective of this paper is to design an ε-independent guaranteed cost control law $u(t) = K_{\mathrm{app}}x(t)$ for the uncertain MSPS (1).

3. Asymptotic structure for the GMARE

In this section, the existence condition and the asymptotic structure of the GMARE (7) are studied. The partitioned matrices are:

$$\begin{aligned}
&\Xi := A - B\mathscr{R}^{-1}E_b^{\mathrm{T}}E_a = \begin{bmatrix} \Xi_{00} & \Xi_{0f} \\ \Xi_{f0} & \Xi_f \end{bmatrix},\\
&\Xi_{00} := A_{00} - B_0\mathscr{R}^{-1}E_b^{\mathrm{T}}E_{a0},\\
&\Xi_{0f} := A_{0f} - B_0\mathscr{R}^{-1}E_b^{\mathrm{T}}E_{af} = [\Xi_{01} \quad \cdots \quad \Xi_{0N}],\\
&\Xi_{f0} := A_{f0} - B_f\mathscr{R}^{-1}E_b^{\mathrm{T}}E_{a0} = [\Xi_{10}^{\mathrm{T}} \quad \cdots \quad \Xi_{N0}^{\mathrm{T}}]^{\mathrm{T}},\\
&\Xi_f := A_f - B_f\mathscr{R}^{-1}E_b^{\mathrm{T}}E_{af}\\
&\qquad = \textbf{block diag}\,(\Xi_{11} \quad \cdots \quad \Xi_{NN}),\\
&S := \nu(DD^{\mathrm{T}} - B\mathscr{R}^{-1}B^{\mathrm{T}}) = \begin{bmatrix} S_{00} & S_{0f} \\ S_{0f}^{\mathrm{T}} & S_f \end{bmatrix},\\
&S_{00} := \nu(D_0D_0^{\mathrm{T}} - B_0\mathscr{R}^{-1}B_0^{\mathrm{T}}),\\
&S_{0f} := \nu(D_0D_f^{\mathrm{T}} - B_0\mathscr{R}^{-1}B_f^{\mathrm{T}}) = [S_{01} \quad \cdots \quad S_{0N}],\\
&S_f := \nu(D_fD_f^{\mathrm{T}} - B_f\mathscr{R}^{-1}B_f^{\mathrm{T}})\\
&\qquad = \textbf{block diag}\,(S_{11} \quad \cdots \quad S_{NN}),\\
&W := \nu^{-1}E_a^{\mathrm{T}}(I_{\bar{s}} - E_b\mathscr{R}^{-1}E_b^{\mathrm{T}})E_a + Q = \begin{bmatrix} W_{00} & W_{0f} \\ W_{0f}^{\mathrm{T}} & W_f \end{bmatrix},\\
&W_{00} := \nu^{-1}E_{a0}^{\mathrm{T}}(I_{\bar{s}} - E_b\mathscr{R}^{-1}E_b^{\mathrm{T}})E_{a0} + Q_{00},\\
&W_{0f} := \nu^{-1}E_{a0}^{\mathrm{T}}(I_{\bar{s}} - E_b\mathscr{R}^{-1}E_b^{\mathrm{T}})E_{af} + Q_{0f}\\
&\qquad = [W_{01} \quad \cdots \quad W_{0N}],\\
&W_f := \nu^{-1}E_{af}^{\mathrm{T}}(I_{\bar{s}} - E_b\mathscr{R}^{-1}E_b^{\mathrm{T}})E_{af} + Q_f\\
&\qquad = \textbf{block diag}\,(W_{11} \quad \cdots \quad W_{NN}).
\end{aligned}$$

In order to guarantee the existence of the solution of the GMARE (7), without loss of generality, it is assumed that the limit of α_{ij} exists as ε_i and ε_j tend to zero (see e.g., Khalil & Kokotović, 1978, 1979; Khalil, 1979). That is, the small singular perturbation parameters have the same order of magnitude and

$$\bar{\alpha}_{ij} = \lim_{\substack{\varepsilon_j \to +0 \\ \varepsilon_i \to +0}} \alpha_{ij}. \tag{10}$$

It should be noted that the limit in Eq. (10) may not exist at all without the above assumption.

Let $\bar{X}_{00}$, $\bar{X}_{f0}$ and $\bar{X}_f$ be the limiting solutions of the above GMARE (7) as $\varepsilon_j \to +0$, $j=1,\dots,N$. Then we get the following parameter independent AREs:

$$\bar{X}_{00}^* \mathscr{A} + \mathscr{A}^{\mathrm{T}} \bar{X}_{00}^* + \bar{X}_{00}^* \mathscr{S} \bar{X}_{00}^* + \mathscr{W} = 0, \tag{11a}$$

$$\bar{X}_{j0}^* = [\bar{X}_{jj}^* - I_{n_j}] T_{jj}^{-1} T_{j0} \begin{bmatrix} I_{n_0} \\ \bar{X}_{00}^* \end{bmatrix}, \tag{11b}$$

$$\bar{X}_{jj}^* \Xi_{jj} + \Xi_{jj}^{\mathrm{T}} \bar{X}_{jj}^* + \bar{X}_{jj}^* S_{jj} \bar{X}_{jj}^* + W_{jj} = 0, \tag{11c}$$

where

$$\begin{bmatrix} \mathscr{A} & \mathscr{S} \\ -\mathscr{W} & -\mathscr{A}^{\mathrm{T}} \end{bmatrix} := T_{00} - \sum_{j=1}^{N} T_{0j} T_{jj}^{-1} T_{j0},$$
$$T_{00} := \begin{bmatrix} \Xi_{00} & S_{00} \\ -W_{00} & -\Xi_{00}^{\mathrm{T}} \end{bmatrix}, \quad T_{0j} := \begin{bmatrix} \Xi_{0j} & S_{0j} \\ -W_{0j} & -\Xi_{j0}^{\mathrm{T}} \end{bmatrix},$$
$$T_{j0} := \begin{bmatrix} \Xi_{j0} & S_{0j}^{\mathrm{T}} \\ -W_{0j}^{\mathrm{T}} & -\Xi_{0j}^{\mathrm{T}} \end{bmatrix}, \quad T_{jj} := \begin{bmatrix} \Xi_{jj} & S_{jj} \\ -W_{jj} & -\Xi_{jj}^{\mathrm{T}} \end{bmatrix}. \tag{12}$$

Now, let us define the admissible design parameters (Pan and Basar, 1993).

$v_f := \min\{v_{f_1},\dots,v_{f_N}\}$, where $v_{f_j} := \sup\{v \mid v \in \Lambda_{f_j}\}$ and $\Lambda_{f_j} := \{0 < v \mid$ The AREs $\bar{X}_{jj}\Xi_{jj} + \Xi_{jj}^{\mathrm{T}}\bar{X}_{jj} + \bar{X}_{jj}S_{jj}\bar{X}_{jj} + W_{jj} = 0$ have a positive definite stabilizing solution, respectively.$\}$, $j=1,\dots,N$.

Using the similar technique used by Mukaidani et al. (2003), it is easy to verify that if we select a parameter $0<\mu<\mu_f := \min\{\mu_{f_1},\dots,\mu_{f_N}\}$, then the solution $\bar{X}_f$ has the following form

$$\bar{X}_f^* := \textbf{block diag}\,(\bar{X}_{11}^* \quad \cdots \quad \bar{X}_{NN}^*). \tag{13}$$

Moreover, the following set is defined (Pan and Basar, 1993).

$v_s := \sup\{v \mid v \in \Lambda_s\}$, where $\Lambda_s := \{0<v \mid$ The ARE (11a) has a positive definite stabilizing solution$\}$.

As a result, for every $0<v<\bar{v} = \min\{v_s,\ v_f\}$, the AREs (11a) and (11c) have the positive definite stabilizing solutions. Hence, the limiting behavior of X_e as the parameter $\|\varepsilon\| := \sqrt{\varepsilon_1^2 + \cdots + \varepsilon_N^2} \to +0$ is described by the following lemma.

Lemma 4. *Assume that there exists a positive scalar* $\bar{v} := \min\{v_s,\ v_f\}$ *such that for all* $0<v<\bar{v}$, *the AREs* (11a) *and* (11c) *have the positive definite stabilizing solutions. Then there exists a small* σ^* *such that for all* $\|\varepsilon\| \in (0,\ \sigma^*)$, *for any* $v(<\bar{v})$ *the GMARE* (7) *admits the solution* X *which can be written as*

$$X = \begin{bmatrix} \bar{X}_{00}^* + O(\|\varepsilon\|) & [\bar{X}_{f0}^* + O(\|\varepsilon\|)]^{\mathrm{T}} \Pi_e \\ \bar{X}_{f0}^* + O(\|\varepsilon\|) & \bar{X}_f^* + O(\|\varepsilon\|) \end{bmatrix},$$
$$X_e = \Phi_e X > 0, \quad \bar{X}_{f0}^* = [\bar{X}_{10}^{*\mathrm{T}} \quad \cdots \quad \bar{X}_{N0}^{*\mathrm{T}}]^{\mathrm{T}}. \tag{14}$$

Using the Newton–Kantorovich theorem (Ortega, 1990) instead of the implicit function theorem, it is also possible to show the asymptotic structure of the solution for the GMARE (7) which is given by (14). It should be noted that the solution $\bar{P}_{jj}^*$ of the ARE (11c) exists for all $0<v<v_{f_j}$ by using the result of Petersen and McFarlane (1994). On the other hand, it is shown that the ARE (11a) admits a solution for all $0<v<v_s$ by exploiting the following lemma and the similar technique used in Petersen and McFarlane (1994). Since this lemma can be proved by combining the techniques that have been established in Ran and Vreugdenhil (1988) and Takaba et al. (1995), it is omitted.

Lemma 5. *Let* $\tilde{A} \in \mathbf{R}^{n\times n}$, $\tilde{S} = \tilde{S}^{\mathrm{T}} > 0 \in \mathbf{R}^{n\times n}$ *and* $\tilde{Q} = \tilde{Q}^{\mathrm{T}} \in \mathbf{R}^{n\times n}$, $n := n_0 + \hat{n}$, $\hat{n} := \sum_{i=1}^{N} n_i$ *be given as the matrices. Furthermore, assume that*

$$\tilde{P} = \begin{bmatrix} \tilde{P}_{00} & 0 \\ \tilde{P}_{f0} & \tilde{P}_f \end{bmatrix},$$

$\tilde{P}_{00} = \tilde{P}_{00}^{\mathrm{T}} > 0 \in \mathbf{R}^{n_0\times n_0}$, $\tilde{P}_f = \tilde{P}_f^{\mathrm{T}} > 0 \in \mathbf{R}^{\hat{n}\times\hat{n}}$ *is a solution satisfying the GMARE* $\tilde{P}^{\mathrm{T}}\tilde{A} + \tilde{A}^{\mathrm{T}}\tilde{P} + \tilde{P}^{\mathrm{T}}\tilde{S}\tilde{P} + \tilde{Q} = 0$. *If* $\hat{Q} \leqslant \tilde{Q}$, *then there exists a solution* $\hat{P}$ *such that* $\Phi_0\tilde{P} \leqslant \Phi_0\hat{P}$ *and* $\hat{P}^{\mathrm{T}}\tilde{A} + \tilde{A}^{\mathrm{T}}\hat{P} + \hat{P}^{\mathrm{T}}\tilde{S}\hat{P} + \hat{Q} = 0$, *where*

$$\hat{P} = \begin{bmatrix} \hat{P}_{00} & 0 \\ \hat{P}_{f0} & \hat{P}_f \end{bmatrix}, \quad \hat{P}_{00} = \hat{P}_{00}^{\mathrm{T}} > 0 \in \mathbf{R}^{n_0\times n_0},$$

$$\hat{P}_f = \hat{P}_f^{\mathrm{T}} > 0 \in \mathbf{R}^{\hat{n}\times\hat{n}}, \ \Phi_0 := \begin{bmatrix} I_{n0} & 0 \\ 0 & 0 \end{bmatrix}.$$

Similarly, assume that $\tilde{P}$ *is a solution satisfying the GMARE* $\tilde{P}^{\mathrm{T}}\tilde{A} + \tilde{A}^{\mathrm{T}}\tilde{P} + \tilde{P}^{\mathrm{T}}\tilde{S}\tilde{P} + \tilde{Q} = 0$. *If* $0<\hat{S}\leqslant\tilde{S}$, *then there exists a solution* $\hat{P}$ *such that* $\Phi_0\tilde{P} \leqslant \Phi_0\hat{P}$ *and* $\hat{P}^{\mathrm{T}}\tilde{A} + \tilde{A}^{\mathrm{T}}\hat{P} + \hat{P}^{\mathrm{T}}\hat{S}\hat{P} + \tilde{Q} = 0$.

4. An approximate guaranteed cost controller

We now give the new design approach for the construction of the guaranteed cost controller. The new ε-independent guaranteed cost controller can be obtained by solving reduced-order slow and fast AREs (11). The ε-independent guaranteed cost controller is obtained by neglecting the term of $O(\|\varepsilon\|)$ of the guaranteed cost controller (8). If $\|\varepsilon\|$ is very small, it is obvious that the guaranteed cost controller (8) can be approximated as

$$\begin{aligned} &u_{\mathrm{exa}}(t) \\ &\quad = K_{\mathrm{exa}}x(t) = -\mathscr{R}^{-1}[vB^{\mathrm{T}}X + E_b^{\mathrm{T}}E_a]x(t) \\ &\quad \approx u_{\mathrm{app}}(t) = K_{\mathrm{app}}x(t) \\ &\quad = -\mathscr{R}^{-1}\left(v[B_0^{\mathrm{T}} \quad B_f^{\mathrm{T}}]\begin{bmatrix} \bar{X}_{00}^* & 0 \\ \bar{X}_{f0}^* & \bar{X}_f^* \end{bmatrix} + E_b^{\mathrm{T}}E_a\right)x(t). \end{aligned} \tag{15}$$

The main result of this paper is as follows.

Theorem 6. *If we select a parameter* $0 < \nu < \bar{\nu} = \min\{\nu_s,\ \nu_f\}$, *then there exists a small* $0 < \bar{\sigma}$ *such that for all* $\|\varepsilon\| \in (0,\ \bar{\sigma})$, *the uncertain closed-loop MSPS is quadratically stable and cost* (4) *has the upper bound via the* ε*-independent controller* (15). *That is, the approximate controller* (15) *is the guaranteed cost controller*.

Proof. Using the result of Lemma 2, it is enough to show that the GMARE (16) has the positive definite symmetric solution $\Phi_e P$ as $A + BK_{\text{app}} \to \bar{A}$, $E_a + E_b K_{\text{app}} \to \bar{E}$ and $Q + K_{\text{app}}^{\mathrm{T}} R K_{\text{app}} \to \bar{Q}$.

$$\begin{aligned} &P^{\mathrm{T}}(A + BK_{\text{app}}) + (A + BK_{\text{app}})^{\mathrm{T}} P + \lambda P^{\mathrm{T}} D D^{\mathrm{T}} P \\ &\quad + \lambda^{-1}(E_a + E_b K_{\text{app}})^{\mathrm{T}}(E_a + E_b K_{\text{app}}) \\ &\quad + K_{\text{app}}^{\mathrm{T}} R K_{\text{app}} + Q = 0. \end{aligned} \tag{16}$$

The proof of the existence of P is obtained by the implicit function theorem (Gajić, 1988). To do so, it is enough to show that the corresponding Jacobian is nonsingular at $\|\varepsilon\| = 0$. It can be shown, after some algebra, that the Jacobian matrix of the GMARE (16) in the limit as $\|\varepsilon\| \to 0$ is given by

$$\begin{aligned} \mathbf{J} &= \nabla \mathbf{F}|_{\|\varepsilon\|=0,\ P_{00}=\bar{X}_{00}^*,\ P_{f0}=\bar{X}_{f0}^*,\ P_f=\bar{X}_f^*} \\ &= \begin{bmatrix} \mathbf{J}_{00} & \mathbf{J}_{01} & 0 \\ \mathbf{J}_{10} & \mathbf{J}_{11} & \mathbf{J}_{12} \\ 0 & 0 & \mathbf{J}_{22} \end{bmatrix}, \end{aligned} \tag{17}$$

where

$$\begin{aligned} &\mathbf{J}_{00} = I_{n_0} \otimes \Theta_{00}^{\mathrm{T}} + \Theta_{00}^{\mathrm{T}} \otimes I_{n_0}, \\ &\mathbf{J}_{01} = (I_{n_0} \otimes \Theta_{f0}^{\mathrm{T}}) U_{n_0 \hat{n}} + \Theta_{f0}^{\mathrm{T}} \otimes I_{n_0}, \\ &\mathbf{J}_{10} = \Theta_{0f}^{\mathrm{T}} \otimes I_{n_0} = (\Theta_{0f}^{\mathrm{T}} \otimes I_{n_0}) U_{n_0 n_0}, \\ &\mathbf{J}_{11} = \Theta_f^{\mathrm{T}} \otimes I_{n_0}, \quad \mathbf{J}_{22} = I_{\hat{n}} \otimes \Theta_f^{\mathrm{T}} + \Theta_f^{\mathrm{T}} \otimes I_{\hat{n}}, \\ &\Theta_{00} = \Xi_{00} + S_{00} \bar{X}_{00}^* + S_{0f} \bar{X}_{f0}^*, \\ &\Theta_{0f} = \Xi_{0f} + S_{0f} \bar{X}_f^*, \quad \Theta_{f0} = \Xi_{f0} + S_{0f}^{\mathrm{T}} \bar{X}_{00}^* + S_f \bar{X}_{f0}^*, \\ &\Theta_f = \Xi_f + S_f \bar{X}_f^*, \\ &\Theta_f = \textbf{block diag}\,(H_{11} \quad \cdots \quad H_{NN}), \\ &H_{jj} := \Xi_{jj} + S_{jj} \bar{X}_{jj}^*, \quad \hat{n} := \sum_{i=1}^{N} n_i. \end{aligned}$$

Jacobian (17) can be expressed as

$$\det \mathbf{J} = \det \mathbf{J}_{22} \cdot \det \mathbf{J}_{11} \cdot \det[I_{n_0} \otimes \Theta_0^{\mathrm{T}} + \Theta_0^{\mathrm{T}} \otimes I_{n_0}], \tag{18}$$

where $\Theta_0 := \Theta_{00} - \Theta_{0f} \Theta_f^{-1} \Theta_{f0}$. Obviously, $\mathbf{J}_{jj}$, $j = 1, 2$ are nonsingular because the matrix $\Theta_f = \Xi_f + S_f \bar{X}_f^* = \textbf{block diag}\,(H_{11} \quad \cdots \quad H_{NN})$ is stable. After some straightforward but tedious algebra, we see that $\mathscr{A} + \mathscr{S} \bar{X}_{00}^* = \Theta_{00} - \Theta_{0f} \Theta_f^{-1} \Theta_{f0} = \Theta_0$. Therefore, the matrix Θ_0 is also stable because the assumption that the ARE (11a) has the positive definite stabilizing solution is satisfied. Thus, $\det \mathbf{J} \neq 0$, i.e., $\mathbf{J}$ is nonsingular at $\|\varepsilon\| = 0$. The asymptotic structure of P_e is obtained directly by using the implicit function theorem. Hence, it is easy to establish that

$$\begin{aligned} &P = \begin{bmatrix} \bar{X}_{00}^* + O(\|\varepsilon\|) & [\bar{X}_{f0}^* + O(\|\varepsilon\|)]^{\mathrm{T}} \Pi_e \\ \bar{X}_{f0}^* + O(\|\varepsilon\|) & \bar{X}_f^* + O(\|\varepsilon\|) \end{bmatrix} \\ &\Rightarrow \|P - X\| = O(\|\varepsilon\|) \quad \Leftrightarrow \quad \|P_e - X_e\| = O(\|\varepsilon\|), \end{aligned} \tag{19}$$

under $\lambda = \nu$ because the zeroth-order solutions of the GMARE (16) are equal to the zeroth-order solutions of the GMARE (7). The remainder of the proof is to show that $P_e := \Phi_e P$ is the positive definite stabilizing solution. Applying the Schur complement to the matrix P_e, we get

$$\begin{aligned} &P_e := \Phi_e P > 0 \quad \Leftrightarrow \quad \bar{X}_{00}^* + O(\|\varepsilon\|) > 0, \\ &\bar{X}_{00}^* - \bar{X}_{f0}^{*\mathrm{T}} \Pi_e [\bar{X}_f^* + O(\|\varepsilon\|)]^{-1} \bar{X}_{f0}^* + O(\|\varepsilon\|) > 0. \end{aligned} \tag{20}$$

Taking into consideration the fact that the solutions $\bar{X}_{00}^*$, $\bar{X}_f^*$ are the positive definite and $\Pi_e = O(\|\varepsilon\|)$, we have $P_e > 0$ for sufficiently small $\|\varepsilon\|$. Thus, the proof of Theorem 6 is completed. □

It should be noted that the zeroth-order solution of P are the same as the zeroth-order solution of X.

Remark 7. It has been shown from Petersen and McFarlane (1994) that the GMARE (7) has a positive definite solution for each ν in the interval $(0,\ \bar{\nu})$. Moreover, it is known that Trace $\Phi_e X$ is a convex function of ν over $(0,\ \bar{\nu})$. This convexity can be exploited to design the guaranteed cost controllers which minimizes the value of the guaranteed cost for the closed-loop uncertain MSPS.

Remark 8. Using the Newton–Kantorovich theorem (Ortega, 1990), it can be shown that there exists a small $\hat{\sigma} (\leqslant \min\{\sigma^*,\ \bar{\sigma}\})$ such that for all $\|\varepsilon\| \in (0,\ \hat{\sigma})$, the MAREs (7) and (16) have the positive definite solutions in the meaning of the sufficient condition. Since this proof can be done by using the similar technique in Mukaidani and Mizukami (2000), it is omitted.

Using the useful result for the asymptotic structure (19) of the GMARE (16), we show how to select a parameter ν which is addressed in the guaranteed cost control problem. According to the existing results (Petersen and McFarlane, 1994), we need to solve the full-order GMARE (16) to calculate the bound of the cost $x(0)^{\mathrm{T}} P_e x(0) = x(0)^{\mathrm{T}} \Phi_e P x(0)$ for every $0 < \nu < \bar{\nu}$. However, since the numerical stiffness and the high-dimension arise and there is no information of the small parameters ε_j, it is impossible to solve the GMARE (16) directly. So far, the problem of how to calculate the approximate cost bound has never been studied. Motivated by these reasons, we will propose the new approximate method of calculation for the cost bound briefly.

If $\|\varepsilon\|$ is very small, then the guaranteed cost $x(0)^{\mathrm{T}}P_e x(0)$ can be written as (21) because the zeroth-order solutions of the GMARE (16) are equal to the zeroth-order solutions of the GMARE (7):

$$\begin{aligned} x(0)^{\mathrm{T}}P_e x(0) &= x(0)^{\mathrm{T}}X_e x(0) + O(\|\varepsilon\|) \\ &= x_0(0)\bar{X}_{00}x_0(0) + O(\|\varepsilon\|). \end{aligned} \tag{21}$$

Thus, in order to calculate the bound of the cost, our new idea is to use only the solution $\bar{X}_{00}$ of the reduced-order ARE (11a). That is, we can neglect the $O(\|\varepsilon\|)$ term of cost (21) if $\|\varepsilon\|$ is sufficiently small. Therefore, the amount of the computation required to get the ε-independent controller becomes small compared with the case of solving the full-order GMARE (7) because the approximate cost can be computed by the small dimension which is the same as the slow subsystems. Moreover, we do not need the knowledge of the parameters ε_j for calculating the guaranteed cost.

Remark 9. It can be noted that the cost bound (21) depends on the initial condition $x(0)$. To remove this dependence on $x(0)$, we assume that $x(0)$ is a zero mean random variable satisfying $E[x(0)x^{\mathrm{T}}(0)] = I_n$. In this case, it is interesting to point out that the guaranteed cost becomes

$$x(0)^{\mathrm{T}}P_e x(0) = \text{Trace } P_e = \text{Trace } \bar{X}_{00} + O(\|\varepsilon\|). \tag{22}$$

Finally, we give an algorithm for the guaranteed cost control problem of the uncertain MSPS.

Step 1: *Search the minimum parameter* $v_f = \min\{v_{f_1}, \dots, v_{f_N}\}$ *such that the reduced-order AREs* (11c) *have the positive definite stabilizing solution* $\bar{X}_{jj}$ *by using the bisection method. If* v_f *is less than some prescribed computational accuracy, then stop and declare that the guaranteed cost control fails. Otherwise, proceed to Step* 2.

Step 2: *Using the relation* (12), *search the minimum parameter* $v_s\,(\leqslant v_f)$ *such that the reduced-order ARE* (11a) *has the positive definite stabilizing solution* $\bar{X}_{00}$ *by means of the bisection method. If* v_s *is less than some prescribed computational accuracy, then stop and declare that the guaranteed cost control fails. Otherwise, proceed to Step* 3.

Step 3: *Choose any parameter* v *such that* $0 < v < \bar{v} = \min\{v_s,\ v_f\}$ *and calculate* $\mathscr{A}$, $\mathscr{S}$ *and* $\mathscr{W}$ *of* (12) *via the matrices* T_{00}, T_{0j}, T_{j0} *and* T_{jj}, $j = 1, \dots, N$.

Step 4: *Compute the positive definite stabilizing solution* $\bar{X}_{00}$ *and calculate the approximate guaranteed cost*

$$f(v) = \text{Trace } \bar{X}_{00}, \tag{23}$$

where we have neglected the term $O(\|\varepsilon\|)$ *of the cost* (21).

Step 5: *Find a* $v = \hat{v}$ *that minimizes* $f(v)$ *for all* $0 < v < \bar{v}$.

Step 6: *For the obtained* $v = \hat{v}$, *design the* ε*-independent controller* (15).

Consequently, solving the proposed above optimization problem allows us to determine the near-optimal cost bound $O(\|\varepsilon\|)$ close to the optimal guaranteed cost performance.

5. Numerical example

In order to demonstrate the efficiency of the proposed algorithm, we have run a simple example. Let us consider the *R–L–C* electric network in Fig. 1. In this network, L and R are the inductance and the resistance, respectively. The capacitances are denoted by C_j, $j = 1, \dots, 9$. Suppose that C_j is a very small positive parameter, that is, $C_j = \varepsilon_j$. $I := x_0$ denotes the electric current in the inductance, $V_j := x_j$, $j = 1, \dots, 9$ denote the voltage across capacitances $C_j := \varepsilon_j$, respectively. Moreover, $E_j := u_j$, $j = 1, \dots, 9$ denote the applied voltages, that is, the control inputs. The nominal values of the element are defined as $L = 100$ mH and $R = 10\,\Omega$. It should be noted that C_j, $j = 1, \dots, 9$ are sufficiently small but these values are unknown.

We suppose that the variation of the resistance R is within 10% of the nominal value taking the saturation into account. Therefore, we assume that the considered uncertainty is represented in the following inequality:

$$\frac{1}{R} = \frac{1}{10 + \delta_R(t)} = 0.0955 + 0.0045 \times \varDelta_R(t),$$
$$0 \leqslant \delta_R(t) \leqslant 1.0, \quad |\varDelta_R(t)| \leqslant 1.0.$$

Then the system matrices are given below:

$$A_{00} = [0], \quad A_{0j} = [-10], \quad A_{j0} = [1],$$

$$A_{jj} = [0.0955], \quad B_{0j} = [0], \quad B_{jj} = [-0.0955],$$

$$D_{0j} = [0], \quad D_{jj} = [0.0045], \quad E_{aj0} = [0],$$

$$E_{ajj} = E_{bjj} = [1], \quad Q_0 = [1], \quad Q_{jj} = [1],$$

$$R_{jj} = [1], \quad F_{jj}(t) = \varDelta_R(t), \quad j = 1, \dots, 9,$$

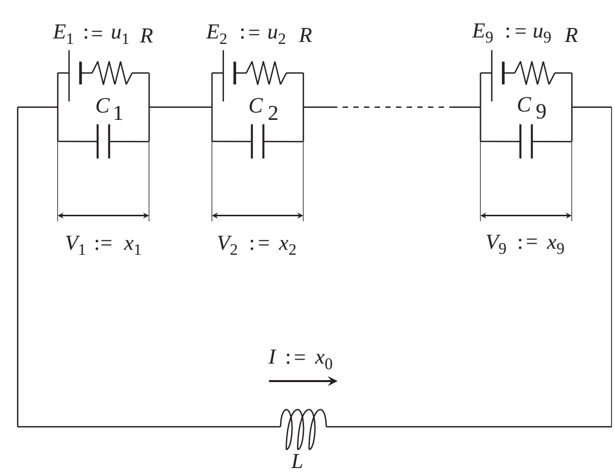

Fig. 1. *R–L–C* electric network.

$$
u_{\mathrm{exa}}(t) = K_{\mathrm{exa}}x(t) = \begin{bmatrix}
-6.0526\mathrm{e}-3 & 2.5847 & 8.6966\mathrm{e}-6 & 1.2578\mathrm{e}-5 & 1.6130\mathrm{e}-5 & 1.9375\mathrm{e}-5 & 2.2344\mathrm{e}-5 & 2.5065\mathrm{e}-5 & 2.7563\mathrm{e}-5 & 2.9861\mathrm{e}-5 \\
-5.7927\mathrm{e}-3 & 4.3483\mathrm{e}-6 & 2.5847 & 1.2518\mathrm{e}-5 & 1.6169\mathrm{e}-5 & 1.9530\mathrm{e}-5 & 2.2620\mathrm{e}-5 & 2.5462\mathrm{e}-5 & 2.8080\mathrm{e}-5 & 3.0494\mathrm{e}-5 \\
-5.5412\mathrm{e}-3 & 4.1928\mathrm{e}-6 & 8.3452\mathrm{e}-6 & 2.5847 & 1.5988\mathrm{e}-5 & 1.9417\mathrm{e}-5 & 2.2590\mathrm{e}-5 & 2.5524\mathrm{e}-5 & 2.8236\mathrm{e}-5 & 3.0747\mathrm{e}-5 \\
-5.3021\mathrm{e}-3 & 4.0324\mathrm{e}-6 & 8.0846\mathrm{e}-6 & 1.1991\mathrm{e}-5 & 2.5847 & 1.9144\mathrm{e}-5 & 2.2366\mathrm{e}-5 & 2.5361\mathrm{e}-5 & 2.8144\mathrm{e}-5 & 3.0729\mathrm{e}-5 \\
-5.0768\mathrm{e}-3 & 3.8751\mathrm{e}-6 & 7.8120\mathrm{e}-6 & 1.1650\mathrm{e}-5 & 1.5315\mathrm{e}-5 & 2.5847 & 2.2017\mathrm{e}-5 & 2.5049\mathrm{e}-5 & 2.7879\mathrm{e}-5 & 3.0519\mathrm{e}-5 \\
-4.8657\mathrm{e}-3 & 3.7241\mathrm{e}-6 & 7.5400\mathrm{e}-6 & 1.1295\mathrm{e}-5 & 1.4910\mathrm{e}-5 & 1.8347\mathrm{e}-5 & 2.5847 & 2.4639\mathrm{e}-5 & 2.7498\mathrm{e}-5 & 3.0175\mathrm{e}-5 \\
-4.6684\mathrm{e}-3 & 3.5807\mathrm{e}-6 & 7.2749\mathrm{e}-6 & 1.0939\mathrm{e}-5 & 1.4492\mathrm{e}-5 & 1.7892\mathrm{e}-5 & 2.1119\mathrm{e}-5 & 2.5847 & 2.7039\mathrm{e}-5 & 2.9739\mathrm{e}-5 \\
-4.4841\mathrm{e}-3 & 3.4453\mathrm{e}-6 & 7.0199\mathrm{e}-6 & 1.0589\mathrm{e}-5 & 1.4072\mathrm{e}-5 & 1.7424\mathrm{e}-5 & 2.0623\mathrm{e}-5 & 2.3659\mathrm{e}-5 & 2.5847 & 2.9240\mathrm{e}-5 \\
-4.3121\mathrm{e}-3 & 3.3179\mathrm{e}-6 & 6.7765\mathrm{e}-6 & 1.0249\mathrm{e}-5 & 1.3657\mathrm{e}-5 & 1.6955\mathrm{e}-5 & 2.0117\mathrm{e}-5 & 2.3130\mathrm{e}-5 & 2.5991\mathrm{e}-5 & 2.5847
\end{bmatrix} \cdot x(t) \tag{24a}
$$

$$
u_{\mathrm{app}}(t) = K_{\mathrm{app}}x(t) = \begin{bmatrix} -5.1217\mathrm{e}-3 & 2.5847 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -5.1217\mathrm{e}-3 & 0 & \cdots & 2.5847 \end{bmatrix} x(t). \tag{24b}
$$

For every boundary value $0 < \nu < \bar{\nu} = \min\{\nu_f, \ \nu_s\} = 4.4938\mathrm{e}+2$, the AREs (11a) and (11c) have the positive definite stabilizing solution, where $\nu_{f_j} = 4.4938\mathrm{e}+2, \ j = 1, \ldots, 9$ and $\nu_s = 4.4938\mathrm{e}+2$. It is easy to verify that the approximate minimum bound of cost (23) is $f(\hat{\nu}) = 2.8411$ when $\hat{\nu} = 2.8050\mathrm{e}+1$. On the other hand, the exact cost bound $\min \ f(\nu^*) = 2.8540$ is obtained for $\nu^* = 2.8050\mathrm{e}+1$ when $\varepsilon_1 := 1.0000\mathrm{e}-5, \ldots, \varepsilon_9 := 9.0000\mathrm{e}-5$. Thus, for the tested example, our searching algorithm is quite good.

Now, we choose as $\nu = \hat{\nu} = 2.8050\mathrm{e}+1 < \bar{\nu}$ to design the ε-independent controller. The exact guaranteed cost control (8) under $\nu^* = 2.8050\mathrm{e}+1$ and the proposed approximate one (15) are given in (24a) and (24b). It is worth pointing out that the proposed guaranteed cost controller can be constructed without any information for the small parameters. Moreover, the required work space as the dimension is small compared with the dimension of the full-order system. In this example, the dimension for calculation is one smaller than ten.

6. Conclusions

The guaranteed cost control problem for the uncertain MSPS has been studied. By solving the reduced-order slow and fast AREs, the new ε-independent guaranteed cost controller can be obtained. The new technique has the following advantages. (i) The proposed method does not need the information for the small parameters ε_j. (ii) The required work space is the same as the reduced-order slow and fast subsystems. (iii) Our new results would be applied to the MSPS without various assumptions that have been made for the fast subsystems in the existing results although the fast subsystems have the uncertainty. Therefore, we have succeeded in applying the new design approach to more practical uncertain MSPS.

Recently, LMI technique, which can be solved efficiently by convex optimization, has been widely used to get the solutions in the guaranteed cost problem (see e.g., Yu and Chu, 1999; Mukaidani, 2003). Compared with the presented GMARE approach, there exist the important features, in which the LMI control design methodology is a simpler structure and is easier to be implemented. However, to get the new analytic sufficient condition, a new convex optimization algorithm, which is based on the LMI should be developed by avoiding the difficulty of the large dimension and the numerical stiffness due to the small parameters ε_j. This problem will be addressed in future investigations. Finally, it may be possible to design the output feedback if the considered MSPS is limited (Khalil, 1981). Particularly, in case where the small parameters are known, we can construct the output feedback controller by combining the existing results (Moheimani and Petersen, 1996) with Gajić et al. (1989). Such a problem is more realistic than the state feedback case. Since the analysis and the construction can be done by using the straightforward extension of these existing results, it is omitted.

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