A New Algorithm for Solving Cross–Coupled Algebraic Riccati Equations of Singularly Perturbed Nash Games

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Abstract

In this paper, we study the linear quadratic Nash games for infinite horizon singularly perturbed systems. In order to solve the problem, we must solve a pair of cross–coupled algebraic Riccati equations with a small positive parameter ε. As a matter of fact, we propose a new algorithm, which combines Lyapunov iterations and the generalized Lyapunov equation direct method, to solve the cross–coupled algebraic Riccati equations. The new algorithm ensures that the solution of the cross–coupled algebraic Riccati equations converges to a positive semidefinite stabilizing solution. Furthermore, in order to solve the cross–coupled algebraic Riccati equations, we propose a new Riccati iterations method different from existing method. As another important feature of this paper, our method is applicable to both standard and nonstandard singularly perturbed systems.

1 Introduction

The properties of closed–loop Nash games have been intensively studied in many papers [1]–[4]. For example, Starr and Ho [1] obtained the closed–loop perfect–state linear Nash equilibrium strategies for a class of analytic differential games. In [2], a state feedback mixed H_2/H_∞ control problem has been formulated as dynamic Nash games. In general, note that the cross–coupled algebraic Riccati equations play an important role in problems of the differential Nash Games (see, e.g., [3, 4]). It is well known that in order to obtain the Nash equilibrium strategies, we must solve the cross–coupled algebraic Riccati equations. Li and Gajić [3] proposed an algorithm, called the Lyapunov iterations, to solve the linear quadratic Nash games. Freiling *et al.* [4] found the solutions to the cross–coupled algebraic Riccati equations of the mixed H_2/H_∞ type by using the Riccati iterations based on the coupled algebraic Riccati equations.

Linear time–invariant models of many physical systems contain slow and fast modes. Linear quadratic Nash games for such models, that is, singularly perturbed systems have been studied by using composite controller design [5, 6]. However, the composite Nash equilibrium solution achieves only a performance which is $O(\varepsilon)$ close to the full–order performance.

In recent years, the recursive algorithm for various control problems of not only singularly perturbed but also weakly coupled systems have been developed in many literatures (see, e.g., [7]–[9]). It has been shown that the recursive algorithm are very effective to solve the algebraic Riccati equations when the system matrices are functions of a small perturbation parameter ε. So far, dynamic Nash games of the weakly coupled systems have been studied in Gajić *et al.* [7] and Gajić and Shen [8] by means of the recursive algorithm. However, the recursive algorithm for solving the cross–coupled algebraic Riccati equations corresponding to the dynamic Nash games of the singularly perturbed systems have not been investigated.

In this paper, we study the linear quadratic Nash games for infinite horizon singularly perturbed systems from a viewpoint of solving the cross–coupled algebraic Riccati equations. We apply the Lyapunov iterations to solve the cross–coupled algebraic Riccati equations. However, since the singularly perturbed systems contain a small positive perturbation parameter ε, it is difficult to solve the Lyapunov equations corresponding to the Lyapunov iterations. Therefore, we propose a new algorithm, which combines the Lyapunov iterations and the generalized Lyapunov equation, to solve the cross–coupled algebraic Riccati equations. Using the new composite algorithm, we will overcome many difficulties in computing caused by high dimensions and numerical stiffness in the Lyapunov iteration method. Thus, since our new method is not based on the singular perturbation method [10], full–order Nash equilibrium solution achieves a performance which is more close to the exact performance in comparison with Khalil and Kokotovic [5] and Xu *et al.* [6]. It is worth to note that the numerical approach to solve the linear quadratic Nash games for singularly perturbed systems have never been studied. Furthermore, in order to solve the cross–coupled algebraic Riccati equations, we propose a new Riccati iterations method different from existing method [4]. Since the proposed algorithm is based on the separated algebraic Riccati equation, we expect that the convergence is more fast in comparison with Freiling *et al.* [4]. As another important feature of this paper, we do not assume that A_{22} is non-singular. Therefore, our new algorithm is applicable to both standard and nonstandard singularly perturbed systems.

Notation: The notations used in this paper are fairly standard. The superscript T denotes matrix transpose. I_n denotes the $n \times n$ identity matrix. $\|\cdot\|$ denotes its Euclidean norm for a matrix. ${}_{\mathrm{CS}}M$ denotes the column vector of M. $|M|$ denotes the determinant of the square matrix M.

2 Problem Formulation

Consider a linear time–invariant singularly perturbed system

$$\dot{x} = A_{11}x + A_{12}z + B_{11}u_1 + B_{12}u_2,\ x(0) = x_0, \quad (1a)$$

$$\varepsilon\dot{z} = A_{21}x + A_{22}z + B_{21}u_1 + B_{22}u_2,\ z(0) = z_0, \quad (1b)$$

with a quadratic cost function

$$J_i(u_i,\ u_j) = \frac{1}{2}\int_0^\infty (y^T Q_i y + u_i^T R_{ii} u_i + u_j^T R_{ij} u_j)dt, \quad (2)$$
$$(i, j = 1, 2,\ i \neq j),$$

where

$$y(t) = \begin{bmatrix} x(t) \\ z(t) \end{bmatrix},\quad Q_i = \begin{bmatrix} Q_{i11} & Q_{i12} \\ Q_{i12}^T & Q_{i22} \end{bmatrix} \geq 0,$$
$$R_{ii} > 0,\quad R_{ij} \geq 0$$

and ε is a small positive parameter, $x(t) \in \mathrm{R}^{n_1}$, $z(t) \in \mathrm{R}^{n_2}$ and $y(t) \in \mathbf{R}^n, (n = n_1 + n_2)$ are states, $u_i(t) \in \mathbf{R}^{m_i}$, $(i = 1,\ 2)$ is the control input. All matrices above are of appropriate dimensions. The system (1) is said to be in the standard form if the matrix A_{22} is nonsingular. Otherwise, it is called the nonstandard singularly perturbed systems [10].

Let us introduce the partitioned matrices

$$\begin{aligned}
A_\varepsilon &= \begin{bmatrix} A_{11} & A_{12} \\ \varepsilon^{-1}A_{21} & \varepsilon^{-1}A_{22} \end{bmatrix},\ A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \\
B_{i\varepsilon} &= \begin{bmatrix} B_{1i} \\ \varepsilon^{-1}B_{2i} \end{bmatrix},\ B_i = \begin{bmatrix} B_{1i} \\ B_{2i} \end{bmatrix}, \\
S_{i\varepsilon} &= B_{i\varepsilon}R_{ii}^{-1}B_{i\varepsilon}^T,\ G_{j\varepsilon} = B_{j\varepsilon}R_{jj}^{-1}R_{ij}R_{jj}^{-1}B_{j\varepsilon}^T, \\
S_i &= B_iR_{ii}^{-1}B_i^T = \begin{bmatrix} S_{11i} & S_{12i} \\ S_{12i}^T & S_{22i} \end{bmatrix}, \\
G_j &= B_jR_{jj}^{-1}R_{ij}R_{jj}^{-1}B_j^T = \begin{bmatrix} G_{11j} & G_{12j} \\ G_{12j}^T & G_{22j} \end{bmatrix}, \\
&\quad (i,\ j = 1, 2,\quad i \neq j).
\end{aligned}$$

We now consider the linear quadratic Nash games for infinite horizon singularly perturbed systems (1) under the following basic assumption [7, 8].

Assumption 1 *The triplet* $(A_\varepsilon,\ B_{1\varepsilon},\ \sqrt{Q_1})$ *and* $(A_\varepsilon,\ B_{2\varepsilon},\ \sqrt{Q_2})$ *are stabilizable and detectable for all* $\varepsilon \in (0, \varepsilon^*]$ $(\varepsilon^* > 0)$.

Assumption 2 *The triplet* $(A_{22},\ B_{21},\ \sqrt{Q_{122}})$ *and* $(A_{22},\ B_{22},\ \sqrt{Q_{222}})$ *are stabilizable and detectable.*

These conditions are quite natural since at least one control agent has to be able to control and observe unstable modes. The purpose is to find a linear feedback controller $(u_1^*,\ u_2^*)$ such that

$$J_i(u_i^*,\ u_j^*) \leq J_i(u_i,\ u_j^*).\ (i,\ j = 1, 2,\ \ i \neq j) \quad (3)$$

The Nash inequality shows that $u_i^*(t)$ regulates the state to zero with minimum output energy. The following lemma is already known [1].

Lemma 1 *Under Assumptions 1 and 2, there exists an admissible controller such that* (3) *hold iff the following full–order cross–coupled algebraic Riccati equations*

$$\begin{aligned}
&A_\varepsilon^T X_\varepsilon + X_\varepsilon A_\varepsilon + Q_1 - X_\varepsilon S_{1\varepsilon} X_\varepsilon - X_\varepsilon S_{2\varepsilon} Y_\varepsilon \\
&\quad - Y_\varepsilon S_{2\varepsilon} X_\varepsilon + Y_\varepsilon G_{2\varepsilon} Y_\varepsilon = 0, \quad (4a) \\
&A_\varepsilon^T Y_\varepsilon + Y_\varepsilon A_\varepsilon + Q_2 - Y_\varepsilon S_{2\varepsilon} Y_\varepsilon - Y_\varepsilon S_{1\varepsilon} X_\varepsilon \\
&\quad - X_\varepsilon S_{1\varepsilon} Y_\varepsilon + X_\varepsilon G_{1\varepsilon} X_\varepsilon = 0, \quad (4b)
\end{aligned}$$

have stabilizing solutions $X_\varepsilon \geq 0$ *and* $Y_\varepsilon \geq 0$ *where*

$$X_\varepsilon = \begin{bmatrix} X_{11} & \varepsilon X_{21}^T \\ \varepsilon X_{21} & \varepsilon X_{22} \end{bmatrix},\quad Y_\varepsilon = \begin{bmatrix} Y_{11} & \varepsilon Y_{21}^T \\ \varepsilon Y_{21} & \varepsilon Y_{22} \end{bmatrix}.$$

Then, the closed–loop Nash equilibrium solution to the full–order problem is given by

$$u_1^*(t) = -R_{11}^{-1}B_{1\varepsilon}^T X_\varepsilon y(t), \quad (5a)$$

$$u_2^*(t) = -R_{22}^{-1}B_{2\varepsilon}^T Y_\varepsilon y(t). \quad (5b)$$

However, it is difficult to solve the cross–coupled algebraic Riccati equations (4a) and (4b) because of the different magnitudes of their coefficients caused by the small perturbed parameter ε and high dimensions.

3 The Cross–Coupled Generalized Algebraic Riccati Equations

To obtain the solutions of the cross–coupled algebraic Riccati equations (4a) and (4b), we first define

$$\Pi_\varepsilon = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \varepsilon I_{n_2} \end{bmatrix}.$$

Then, we introduce the following useful lemma.

Lemma 2 *The cross–coupled algebraic Riccati equations* (4a) *and* (4b) are equivalent to the following cross–coupled generalized algebraic Riccati equations (6a) and (6b) respectively.

$$(A - S_1X - S_2Y)^TX + X^T(A - S_1X - S_2Y) + Q_1 + X^TS_1X + Y^TG_2Y = 0, \quad (6a)$$
$$(A - S_1X - S_2Y)^TY + Y^T(A - S_1X - S_2Y) + Q_2 + Y^TS_2Y + X^TG_1X = 0, \quad (6b)$$

where

$$X_\varepsilon = \Pi_\varepsilon X = X^T\Pi_\varepsilon, \quad Y_\varepsilon = \Pi_\varepsilon Y = Y^T\Pi_\varepsilon,$$
$$X = \begin{bmatrix} X_{11} & \varepsilon X_{21}^T \\ X_{21} & X_{22} \end{bmatrix}, \quad Y = \begin{bmatrix} Y_{11} & \varepsilon Y_{21}^T \\ Y_{21} & Y_{22} \end{bmatrix}.$$

Proof: The proof is identical to the proof of Lemma 3 in Mukaidani [11]. ■

In Li and Gajić [3], the Lyapunov iterations for solving cross–coupled algebraic Riccati equations of the linear quadratic Nash differential games have been presented. In this paper, we give the Lyapunov iterations to solve the cross–coupled algebraic Riccati equations. An algorithm for the numerical solutions of (6) is defined as follows [3].

$$(A - S_1X^{(n)} - S_2Y^{(n)})^TX^{(n+1)} + X^{(n+1)T}(A - S_1X^{(n)} - S_2Y^{(n)}) + Q_1 + X^{(n)T}S_1X^{(n)} + Y^{(n)T}G_2Y^{(n)} = 0, \quad (7a)$$
$$(A - S_1X^{(n)} - S_2Y^{(n)})^TY^{(n+1)} + Y^{(n+1)T}(A - S_1X^{(n)} - S_2Y^{(n)}) + Q_2 + X^{(n)T}G_1X^{(n)} + Y^{(n)T}S_2Y^{(n)} = 0, \quad (7b)$$

where $n = 0,\ 1,\ 2,\ 3,\ \cdots$ and the initial conditions $X^{(0)}$ and $Y^{(0)}$ are obtained as solutions of the following auxiliary algebraic Riccati equations (8)

$$A^TX^{(0)} + X^{(0)T}A + Q_1 - X^{(0)T}S_1X^{(0)} = 0, \quad (8a)$$
$$(A - S_1X^{(0)})^TY^{(0)} + Y^{(0)T}(A - S_1X^{(0)}) + Q_2 + X^{(0)T}G_1X^{(0)} - Y^{(0)T}S_2Y^{(0)} = 0, \quad (8b)$$

where

$$X^{(n)} = \begin{bmatrix} X_{11}^{(n)} & \varepsilon X_{21}^{(n)T} \\ X_{21}^{(n)} & X_{22}^{(n)} \end{bmatrix},$$
$$Y^{(n)} = \begin{bmatrix} Y_{11}^{(n)} & \varepsilon Y_{21}^{(n)T} \\ Y_{21}^{(n)} & Y_{22}^{(n)} \end{bmatrix},$$
$$X_\varepsilon^{(n)} = \Pi_\varepsilon X^{(n)}, \quad Y_\varepsilon^{(n)} = \Pi_\varepsilon Y^{(n)}.$$

We note that the unique positive semidefinite stabilizing solution of (8) exist under Assumptions 1 and 2 [3, 7]–[9].

4 The New Algorithm on the Basis of the Generalized Lyapunov Equation

In this section, we will derive the new algorithm for solving the generalized Lyapunov equation. We first introduce the notation

$$A - S_1X^{(n)} - S_2Y^{(n)} = \bar{A}^{(n)} = \begin{bmatrix} \bar{A}_{11}^{(n)} & \bar{A}_{12}^{(n)} \\ \bar{A}_{21}^{(n)} & \bar{A}_{22}^{(n)} \end{bmatrix},$$
$$Q_1 + X^{(n)T}S_1X^{(n)} + Y^{(n)T}G_2Y^{(n)} = \bar{Q}^{(n)} = \begin{bmatrix} \bar{Q}_{11}^{(n)} & \bar{Q}_{12}^{(n)} \\ \bar{Q}_{12}^{(n)T} & \bar{Q}_{22}^{(n)} \end{bmatrix},$$
$$Q_2 + X^{(n)T}G_1X^{(n)} + Y^{(n)T}S_2Y^{(n)} = \hat{Q}^{(n)} = \begin{bmatrix} \hat{Q}_{11}^{(n)} & \hat{Q}_{12}^{(n)} \\ \hat{Q}_{12}^{(n)T} & \hat{Q}_{22}^{(n)} \end{bmatrix}.$$

The following set of equation (9) can be produced by substituting above relation into (7).

$$\bar{A}^{(n)T}X^{(n+1)} + X^{(n+1)T}\bar{A}^{(n)} + \bar{Q}^{(n)} = 0, \quad (9a)$$
$$\bar{A}^{(n)T}Y^{(n+1)} + Y^{(n+1)T}\bar{A}^{(n)} + \hat{Q}^{(n)} = 0. \quad (9b)$$

The algorithm (9) is the generalized Lyapunov iterations. The proof for the convergence has been given in [3]. Thus, we can obtain the solution of the cross–coupled algebraic Riccati equations by performing Lyapunov iterations (9) directly. However, the Lyapunov iterations (9) contain the small positive parameter ε for the singularly perturbed system. To remedy this, we propose a new method to find the solution to the generalized Lyapunov iterations (9). The method studied here is a variant of the classical numerical approach to Lyapunov equation, the essentials of which date back to at least Bingulac in 1970 [9].

Since the equation (9a) and (9b) are identical, we explain the method for solving the generalized Lyapunov equation (9a) only. Firstly, we consider the simultaneous linear equation (10) by rearranging the generalized Lyapunov equation (9a)

$$[I_n \otimes \bar{A}^{(n)T}] \cdot_{\mathrm{CS}} [X^{(n+1)}] + [\bar{A}^{(n)T} \otimes I_n] \cdot_{\mathrm{CS}} [X^{(n+1)T}] +_{\mathrm{CS}} [\bar{Q}^{(n)}] = 0. \quad (10)$$

The Kronecker product method [9] on the basis of (10) is very simple and elegant. However, for large $n = n_1 + n_2$, the difficulty in solving n^2 linear equations make it impractical. Furthermore, it is difficult to solve the equation (10) because $_{\mathrm{CS}}[X^{(n+1)}]$ contain a small positive perturbation parameter ε. In order to reduce the number of the linear equations (10), the property of the matrix $X^{(n+1)}$ can be exploited. It can be readily seen that since the matrix $\bar{Q}^{(n)}$ is symmetric, the unknown values of the matrix $X^{(n+1)}$ are only $n(n+1)/2$. Hence we convert (10) into the following form

$$\mathcal{U}\mathcal{X} = -\mathcal{Q}, \quad (11)$$

where $\mathcal{U}$ is $n(n+1)/2 \times n(n+1)/2$ matrix, $\mathcal{X}$ and $\mathcal{Q}$ are $n(n+1)/2$ column vectors given by

$$\begin{aligned}
\mathcal{X} &= [\; x_{11}^{11} \;\cdots\; x_{1n_1}^{11} \;\; x_{11}^{21} \;\cdots\; x_{1n_2}^{21} \\
&\quad x_{22}^{11} \;\cdots\; x_{n_1n_1}^{11} \\
&\quad x_{n_11}^{21} \;\cdots\; x_{n_1n_2}^{21} \;\; x_{11}^{22} \;\; x_{12}^{22} \;\cdots\; x_{1n_2}^{22} \\
&\quad \cdots \;\; x_{(n_2-1)(n_2-1)}^{22} \;\; x_{(n_2-1)n_2}^{22} \;\; x_{n_2n_2}^{22} \;]^T \\
\mathcal{Q} &= [\; q_{11} \;\; q_{12} \;\cdots\; q_{1n} \;\; q_{22} \;\; q_{23} \;\cdots\; q_{2n} \\
&\quad \cdots \;\; q_{(n-1)(n-1)} \;\; q_{(n-1)n} \;\; q_{nn} \;]^T
\end{aligned}$$

and since $X_{11}^{(n)} = X_{11}^{(n)T}$, $X_{22}^{(n)} = X_{22}^{(n)T}$,

$$X^{(n+1)} = \begin{bmatrix} X_{11}^{(n)} & \varepsilon X_{21}^{(n)T} \\ X_{21}^{(n)} & X_{22}^{(n)} \end{bmatrix}$$

$$= \left[\begin{array}{ccc|ccc} x_{11}^{11} & \cdots & x_{1n_1}^{11} & \varepsilon x_{11}^{21} & \cdots & \varepsilon x_{1n_2}^{21} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdots & x_{n_1n_1}^{11} & \varepsilon x_{n_11}^{21} & \cdots & \varepsilon x_{n_1n_2}^{21} \\ \hline \cdot & \cdots & \cdot & x_{11}^{22} & \cdots & x_{1n_2}^{22} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdots & \cdot & \cdot & \cdots & x_{n_2n_2}^{22} \end{array}\right],$$

$$\bar{Q}^{(n)} = \begin{bmatrix} q_{11} & q_{12} & q_{13} & \cdots & q_{1n} \\ \cdot & q_{22} & q_{23} & \cdots & q_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \cdot & \cdots & q_{nn} \end{bmatrix}.$$

In this paper, we have improved a procedure, which is given in Bingulac [12] for obtaining the matrix $\mathcal{U}$. Similar to the previous procedure (see [9, 12]), the algorithm in order to compute $\mathcal{U}$ requires an auxiliary $n \times n$ matrix L with integer entries. Construction of matrix $\mathcal{U}$ is done through the following steps.

Step 1. Construct the $n \times n$ matrix L, given by

$$L = \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & n+1 & n+2 & \cdots & 2n-1 \\ 3 & n+2 & 2n & \cdots & 3n-3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n-1 & 2n-2 & 3n-4 & \cdots & N-1 \\ n & 2n-1 & 3n-3 & \cdots & N \end{bmatrix}$$

where $N = n(n+1)/2$.

Step 2. Let us define

$$\bar{A}^{(n)} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Construct the $N \times N$ matrix $V = \{v_{i'j'}\}$ with $v_{i'j'} = a_{l'k'}$ where the indices i' and j' (i', $j' = 1, 2, 3, \ldots, N$) are given by the following elements of the auxiliary matrix L:

$$i' = L_{k'h'},\ j' = L_{l'h'},\ k',\ l',\ h' = 1, \ldots, n.$$

Moreover, if $1 \le f' \le n_1$, $n_1 + 1 \le g' \le n$ and the columns of V correspond to the number of matrix $L = \{l_{f'g'}\}$, then multiply the ε on the $a_{m'n'}$, ($1 \le m' \le n_1$, $1 \le n' \le n$) in the matrix V.

Step 3. The required matrix $\mathcal{U}$ is obtained by multiplying by 2 all the elements of V whose row indices correspond to diagonal elements of the matrix L, that is, $1,\ n+1,\ 2n,\ 3n-2, \ldots,\ N-2$ and N.

An algorithm which solves the cross–coupled algebraic Riccati equation (6) with small positive parameter ε is as follows.

Step 1. Solve the algebraic Riccati equations (8) by using the recursive technique proposed by Mukaidani *et al.* [11]. Starting with the initial matrices of $X^{(0)}$ and $Y^{(0)}$.

Step 2. Calculate $\bar{A}^{(n)}$, $\bar{Q}^{(n)}$ and $\hat{Q}^{(n)}$ of relation (9).

Step 3. Compute the solutions $X^{(n+1)}$ and $Y^{(n+1)}$ of the generalized Lyapunov equation (9) by using equation (11).

Step 4.
If $\min\{\|F_1(X_\varepsilon^{(n)},\ Y_\varepsilon^{(n)})\|,\ \|F_2(X_\varepsilon^{(n)},\ Y_\varepsilon^{(n)})\|\} < O(\varepsilon^M)$ for a given integer $M > 0$, go to Step 5. Otherwise, increment $n \to n+1$ and go to Step 2. Here $F_1(\cdot)$ and $F_2(\cdot)$ are defined by equations (18) after.

Step 5. Calculate $u_1^*(t) = -R_{11}^{-1}B_{1\varepsilon}^T X_\varepsilon y(t)$, $u_2^*(t) = -R_{22}^{-1}B_{2\varepsilon}^T Y_\varepsilon y(t)$.

5 The New Algorithm on the Basis of the Generalized Riccati Equation

In the previous section we have derived the algorithm for solving the cross–coupled algebraic Riccati equations based on the Lyapunov iteration. Here we propose the following new algorithm based on the Riccati iterations different from Freiling *et al.* [4].

$$\begin{aligned}
&(A - S_2Y^{(n)})^T X^{(n+1)} + X^{(n+1)T}(A - S_2Y^{(n)}) + Q_1 \\
&\quad - X^{(n+1)T}S_1X^{(n+1)} + Y^{(n)T}G_2Y^{(n)} = 0, \qquad (12a) \\
&(A - S_1X^{(n+1)})^T Y^{(n+1)} + Y^{(n+1)T}(A - S_1X^{(n+1)}) \\
&\quad + Q_2 - Y^{(n+1)T}S_2Y^{(n+1)} \\
&\quad + X^{(n+1)T}G_1X^{(n+1)} = 0, \qquad (12b)
\end{aligned}$$

where $n = 0,\ 1,\ 2,\ 3,\ \cdots$ and the initial condition $Y^{(0)}$ is to take the stabilizing solution of the standard algebraic Riccati equation

$$A_\varepsilon^T Y_\varepsilon^{(0)} + Y_\varepsilon^{(0)} A_\varepsilon + Q_2 - Y_\varepsilon^{(0)} S_{2\varepsilon} Y_\varepsilon^{(0)} = 0. \qquad (12c)$$

The considered algorithm (12) is original. In this paper we derive the algorithm by using successive approximation.

Firstly, we take any stabilizing linear control law $u_2^{(0)} = -R_{22}^{-1}B_{2\varepsilon}Y_\varepsilon^{(0)}y(t)$, where $Y_\varepsilon^{(0)}$ is the positive semidefinite stabilizing solution for auxiliary generalized algebraic Riccati equation (12c). Then, let us consider a following minimization problem.

$$\dot{y} = (A_\varepsilon - S_{2\varepsilon} Y_\varepsilon^{(0)}) y + B_{1\varepsilon} u_1, \tag{13a}$$

$$\min J_u^{(0)} = \frac{1}{2} \int_0^\infty [y^T (Q_1 + Y_\varepsilon^{(0)} G_2 Y_\varepsilon^{(0)}) y + u_1^T R_{11} u_1] dt. \tag{13b}$$

By using the results of the standard Linear Quadratic Regulator (LQR) problem (see, e.g., [10]), we have $u_1^{(1)}(t) = -R_{11}^{-1} B_{1\varepsilon} X_\varepsilon^{(1)} y(t)$ where

$$(A - S_2 Y^{(0)})^T X^{(1)} + X^{(1)T} (A - S_2 Y^{(0)}) - X^{(1)T} S_1 X^{(1)} + Y^{(0)T} G_2 Y^{(0)} + Q_1 = 0. \tag{14}$$

Similarly, performing previous operations for the second control agent, let us consider a similar minimization problem.

$$\dot{y} = (A_\varepsilon - S_{1\varepsilon} X_\varepsilon^{(1)}) y + B_{2\varepsilon} u_2, \tag{15a}$$

$$\min J_u^{(0)} = \frac{1}{2} \int_0^\infty [y^T (Q_2 + X_\varepsilon^{(1)} G_1 X_\varepsilon^{(1)}) y + u_2^T R_{22} u_2] dt. \tag{15b}$$

By using the similar steps in LQR problem (13), we get $u_2^{(1)}(t) = -R_{22}^{-1} B_{2\varepsilon} Y_\varepsilon^{(1)} y(t)$ where

$$(A - S_1 X^{(1)})^T Y^{(1)} + Y^{(1)T} (A - S_1 X^{(1)}) - Y^{(1)T} S_2 Y^{(1)T} + X^{(1)T} G_1 X^{(1)} + Q_2 = 0. \tag{16}$$

By repeating steps 1 and 2 now with $u_1^{(1)}(t)$ and $u_2^{(1)}(t)$ we get $u_1^{(2)}(t)$ and $u_2^{(2)}(t)$ as well as $X_\varepsilon^{(2)}$ and $Y_\varepsilon^{(2)}$. Continuing the same procedure, we get the sequences of the solution matrices. Thus, we can get the algorithm (12).

This algorithm is based on the Riccati iterations. Even though it looks like this algorithm has the form of Freiling [4], it is quite easy to show that this is not same algorithm. Note that Freiling's algorithm need the initial conditions $X_\varepsilon^{(0)}$ and $Y_\varepsilon^{(0)}$. On the other hand, we need only $Y_\varepsilon^{(0)}$. Furthermore, for obtaining the $Y_\varepsilon^{(n+1)}$, the Riccati equation (12b) does not need the solutions $X_\varepsilon^{(n)}$ and $Y_\varepsilon^{(n)}$. Therefore, we expect that the convergence is more fast in comparison with Freiling *et al.* [4]. In order to show the effectiveness of the Riccati iterations algorithm, numerical example is discussed in the next section.

6 Numerical Example

In order to demonstrate the efficiency of the proposed algorithm, we have run a simple numerical example. Matrices A, B_1 and B_2 are chosen randomly. These matrices are given by

$$A_{11} = \begin{bmatrix} 0 & 0.4 \\ 0 & 0 \end{bmatrix},\ A_{12} = \begin{bmatrix} 0 & 0 \\ 0.345 & 0 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} 0 & -0.524 \\ 0 & 0 \end{bmatrix},\ A_{22} = \begin{bmatrix} 0 & 0.262 \\ 0 & -1 \end{bmatrix},$$

$$B_{11} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ B_{21} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$B_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ B_{22} = \begin{bmatrix} 0.2 \\ 1 \end{bmatrix}.$$

and a quadratic cost function

$$J_1 = \frac{1}{2} \int_0^\infty (y^T Q_1 y + u_1^2 + 2u_2^2) dt, \tag{17a}$$

$$J_2 = \frac{1}{2} \int_0^\infty (y^T Q_2 y + 2u_1^2 + u_2^2) dt \tag{17b}$$

where $Q_1 = \mathrm{diag}\{1,\ 0,\ 1,\ 0\}$, $Q_2 = \mathrm{diag}\{0,\ 0,\ 1,\ 1\}$.

Since $|A_{22}| = 0$, the system (1) is a nonstandard singularly perturbed system. Thus, it is obvious that the existing method [5] to find the suboptimal solution is not valid for this example. However, it is solvable by using the method of this paper. Moreover, we can get the full–order Nash equilibrium solution which is more close to the exact performance different from existing methods [6]. We show the obtained results for small parameter $\varepsilon = 0.0001$. Firstly, we give the 0–order solutions of the algebraic Riccati equations (8) in Table 1. Secondly, it can be seen that the solutions of the Lyapunov iterations (7) converge to the solutions with accuracy of $O(10^{-12})$ after 44 Lyapunov iterations. In order to verify the exactitude of the solution, we calculate the remainder by substituting $X^{(44)}$ and $Y^{(44)}$ into the cross–coupled generalized algebraic Riccati equations (6a) and (6b) respectively.

$$\|F_1(X^{(44)},\ Y^{(44)})\| = 5.13 \times 10^{-13},$$
$$\|F_2(X^{(44)},\ Y^{(44)})\| = 1.24 \times 10^{-14}$$

where the errors $F_1(X,\ Y)$ and $F_2(X,\ Y)$ are defined as follows

$$A^T X + X^T A + Q_1 - X^T S_1 X - X^T S_2 Y - Y^T S_2 X + Y^T G_2 Y \equiv F_1(X,\ Y), \tag{18a}$$

$$A^T Y + Y^T A + Q_2 - Y^T S_2 Y - Y^T S_1 X - X^T S_1 Y + X^T G_1 X \equiv F_2(X,\ Y). \tag{18b}$$

Therefore, the numerical example illustrates the effectiveness of the proposed algorithm since the solutions $X_\varepsilon^{(n)} = \Pi_\varepsilon X^{(n)}$ and $Y_\varepsilon^{(n)} = \Pi_\varepsilon Y^{(n)}$ converge to the exact solutions $X_\varepsilon = \Pi_\varepsilon X$ and $Y_\varepsilon = \Pi_\varepsilon Y$ which are defined by (4a) and (4b). Indeed, we can obtain the solution of the cross–coupled algebraic Riccati equations (4a) and (4b) even though A_{22} is singular.

In order to compare with the Lyapunov iterations (7), we have also run the new Riccati iterations algorithm (12) under the same accuracy, that is, $O(10^{-12})$. As a result, the new algorithms (12) converged to the same solutions $X_\varepsilon^{(44)}$ and $Y_\varepsilon^{(44)}$ after 23 Riccati iterations. It is interesting to point out that same accuracy is obtained after about half number of iterations by using the new algorithm (12) based on the Riccati iterations. Therefore, we got good convergence.

Table 1. Initial conditions for the algorithm (7).

$$X^{(0)} = \begin{bmatrix} 6.28449323 & 2.89877152 & 4.71190863 \times 10^{-4} & 9.99999950 \times 10^{-4} \\ 2.89877152 & 7.28636943 & 2.21089120 \times 10^{-4} & 4.47571105 \times 10^{-6} \\ 4.71190863 & 2.21089120 & 4.71226223 & 1.00007622 \\ 9.99999950 & 4.47571105 \times 10^{-2} & 1.00007622 & 2.34520123 \times 10^{-1} \end{bmatrix},$$

$$Y^{(0)} = \begin{bmatrix} 5.15115389 & 1.30681124 & 3.84208016 \times 10^{-4} & 3.61568014 \times 10^{-5} \\ 1.30681124 & 7.76793757 & 9.84286146 \times 10^{-5} & -4.78419976 \times 10^{-6} \\ 3.84208016 & 9.84286146 \times 10^{-1} & 4.92716548 & 4.58164327 \times 10^{-1} \\ 3.61568014 \times 10^{-1} & -4.78419976 \times 10^{-2} & 4.58164327 \times 10^{-1} & 4.34631603 \times 10^{-1} \end{bmatrix}.$$

Table 2. Simulation result for $\varepsilon = 1.0 \times 10^{-4}$.

$$X^{(44)} = \begin{bmatrix} 5.3735083498 & 3.6423573798 & 3.3845648024 \times 10^{-4} & 5.2601715885 \times 10^{-5} \\ 3.6423573798 & 7.0025729735 & 2.7060454965 \times 10^{-4} & 4.8753202671 \times 10^{-6} \\ 3.3845648024 & 2.7060454965 & 3.2957866169 & 7.5313947531 \times 10^{-1} \\ 5.2601715885 \times 10^{-1} & 4.8753202671 \times 10^{-2} & 7.5313947531 \times 10^{-1} & 2.3376743602 \times 10^{-1} \end{bmatrix},$$

$$Y^{(44)} = \begin{bmatrix} 2.5279245581 & 8.5751736777 \times 10^{-1} & 1.9861004245 \times 10^{-4} & 1.9388912118 \times 10^{-5} \\ 8.5751736777 \times 10^{-1} & 6.3314203609 & 6.5949921391 \times 10^{-5} & -5.8790738357 \times 10^{-6} \\ 1.9861004245 & 6.5949921391 \times 10^{-1} & 4.1834728413 & 4.0064281626 \times 10^{-1} \\ 1.9388912118 \times 10^{-1} & -5.8790738357 \times 10^{-2} & 4.0064281626 \times 10^{-1} & 4.2941503846 \times 10^{-1} \end{bmatrix}.$$

7 Conclusions

We have developed an algorithm for solving the cross–coupled algebraic Riccati equations with a small positive parameter ε for the linear quadratic Nash games. So far, the recursive algorithm for solving the cross–coupled algebraic Riccati equations regarding the dynamic Nash games of the singularly perturbed systems had not been investigated. However, we derived the new algorithm to solve the cross–coupled algebraic Riccati equations with a small positive parameter ε by combining the Lyapunov iterations and the generalized Lyapunov equation technique. By using the new algorithm, we overcame the computational difficulties caused by high dimensions and numerical stiffness in the Lyapunov iterations method. Moreover, in order to solve the cross–coupled algebraic Riccati equations, we proposed a new Riccati iterations method different from existing method by Freiling *et al.* [4]. Since the proposed algorithm is based on the separated algebraic Riccati equation, we expect that the convergence is more fast in comparison with the existing algorithm. In addition, our new results are applicable to both standard and nonstandard singularly perturbed systems and include the existing methods as a special case.

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