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A New Method for H_2 Guaranteed Cost Control Problem of Singularly Perturbed Uncertain Systems

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Abstract

This paper deals with the H_2 guaranteed cost control problem for a singularly perturbed norm–bounded uncertain system. This is an extension of the work by Garcia et al. in the sense that the state matrix and input matrix are allowed to be uncertain and the state matrix A_{22} for the fast subsystem may be singular. Firstly, we construct a high–order accurate controller which is based on the exact decomposition technique. Secondly, we propose an ε–independent controller by solving the reduced–order algebraic Riccati equation without knowledge of the parameter ε.

1 Problem Formulation

Consider the following linear singularly perturbed uncertain systems

$$\begin{aligned}\dot{x}_1 &= (A_{11} + D_1 F E_{a1})x_1 + (A_{12} + D_1 F E_{a2})x_2 \\ &\quad + G_1 w + (B_1 + D_1 F E_b)u, \quad (1a)\\ \varepsilon\dot{x}_2 &= (A_{21} + D_2 F E_{a1})x_1 + (A_{22} + D_2 F E_{a2})x_2 \\ &\quad + G_2 w + (B_2 + D_2 F E_b)u, \quad (1b)\\ z &= C_{11}x_1 + C_{12}x_2 + D_{12}u, \quad (1c)\end{aligned}$$

where ε is a small positive parameter, $x_1 \in \mathbf{R}^{n_1}$ and $x_2 \in \mathbf{R}^{n_2}$ are state vectors, $u \in \mathbf{R}^m$ is the control input, $w \in \mathbf{R}^l$ is the exogenous disturbance, $z \in \mathbf{R}^p$ is the controlled output, $F \in \mathbf{R}^{k\times j}$ is the uncertainty matrix of norm bounded such that $F^T F \le I_j$. Moreover, all matrices above are of appropriate dimensions.

Let us introduce the partitioned matrices

$$\begin{aligned} A_\varepsilon &= \begin{bmatrix} A_{11} & A_{12} \\ \varepsilon^{-1}A_{21} & \varepsilon^{-1}A_{22}\end{bmatrix},\ B_\varepsilon = \begin{bmatrix} B_1 \\ \varepsilon^{-1}B_2\end{bmatrix}, \\ G_\varepsilon &= \begin{bmatrix} G_1 \\ \varepsilon^{-1}G_2\end{bmatrix},\ C_1 = \begin{bmatrix} C_{11} & C_{12}\end{bmatrix}, \\ D_\varepsilon &= \begin{bmatrix} D_1 \\ \varepsilon^{-1}D_2\end{bmatrix},\ E_a = \begin{bmatrix} E_{a1} & E_{a2}\end{bmatrix}.\end{aligned}$$

Now, let us consider the H_2 guaranteed cost control problem of such singularly perturbed uncertain systems (1) by using linear state feedback controller under the following basic assumption [1].

Assumption 1 *1) The pair $(A_\varepsilon,\ B_\varepsilon)$ is stabilizable for $\varepsilon \in (0,\ \varepsilon^*]$ $(\varepsilon^* > 0)$. 2) The pair $(A_{22},\ B_2)$ is stabilizable. 3) $C^T D_{12} = O$, $D_{12}^T D_{12} > O$.*

With (1) we associate the algebraic Riccati equation (ARE) [3]

$$\begin{aligned}&[A_\varepsilon - B_\varepsilon \bar{R} E_b^T E_a]^T P_\varepsilon + P_\varepsilon [A_\varepsilon - B_\varepsilon \bar{R} E_b^T E_a] \\ &+ \mu P_\varepsilon D_\varepsilon D_\varepsilon^T P_\varepsilon - \mu P_\varepsilon B_\varepsilon \bar{R} B_\varepsilon^T P_\varepsilon \\ &+ \frac{1}{\mu} E_a^T [I_j - E_b \bar{R} E_b^T] E_a + R_1 = 0, \quad (2)\end{aligned}$$

for the matrix function

$$P_\varepsilon = P_\varepsilon(\mu) = \begin{bmatrix} P_{11}(\varepsilon,\ \mu) & \varepsilon P_{21}(\varepsilon,\ \mu)^T \\ \varepsilon P_{21}(\varepsilon,\ \mu) & \varepsilon P_{22}(\varepsilon,\ \mu)\end{bmatrix},$$

where μ is positive scalar and $R_1 = C_1^T C_1$, $R_2 = D_{12}^T D_{12} > 0$ and $\bar{R} = (\mu R_2 + E_b^T E_b)^{-1}$. For each ε, a controller that guarantees the quadratically stable for all $F:\ F^T F \le I_j$ exists if and only if there exist $\mu > 0$ and (2) has a positive definite solution [3]. If such conditions are met, a controller is determined by the formula

$$u = K(\mu)x = -\bar{R}[\mu B_\varepsilon^T P_\varepsilon(\mu) + E_b^T E_a]x. \quad (3)$$

For such a controller, taking $K(\mu) = -\bar{R}[\mu B_\varepsilon^T P_\varepsilon(\mu) + E_b^T E_a]$ and letting $\Pi = C_1 + D_{12}K$, the transfer matrix from w to z is expressed by

$$\begin{aligned}T(s) = \Pi(sI_n - A_\varepsilon - D_\varepsilon F\{E_a + E_b K(\mu)\} \\ - B_\varepsilon K(\mu))^{-1} \cdot G_\varepsilon. \quad (4)\end{aligned}$$

Then the H_2 guaranteed cost control problem for singularly perturbed uncertain systems is given below.

Find $K(\mu) = K(\mu^*)$ and determine ρ as small as possible such that

$$\|T(s)\|_2 \le \rho \quad (5)$$

where

$$\begin{aligned}&\|T(s)\|_2^2 = \mathrm{Trace}[G_\varepsilon^T L_o(F) G_\varepsilon], \\ &[A_\varepsilon + D_\varepsilon F(E_a + E_b K) + B_\varepsilon K]^T L_o(F) \\ &\quad + L_o(F)[A_\varepsilon + D_\varepsilon F(E_a + E_b K) + B_\varepsilon K] + H^T H = 0.\end{aligned}$$

By using a similar technique in [1], we can easily prove the following result.

Theorem 1 *Suppose that the assumption 1 are satisfied. Then we have*

$$L_o(F) \le P_\varepsilon. \tag{6}$$

Consequently, the best H_2 guaranteed cost ρ is given by

$$\rho = \min_{\mu} \sqrt{\mathrm{Trace}[G_\varepsilon^T P_\varepsilon(\mu) G_\varepsilon]}, \tag{7a}$$

$$\mu^* = \arg\min_{\mu} \sqrt{\mathrm{Trace}[G_\varepsilon^T P_\varepsilon(\mu) G_\varepsilon]}. \tag{7b}$$

Moreover, the controller is defined by $K(\mu) = K(\mu^*) = -(\mu^* R_2 + E_b^T E_b)^{-1}[\mu B_\varepsilon^T P_\varepsilon(\mu^*) + E_b^T E_a]$.

Now, let us define the following matrices

$$A^\mu = A - B\bar{R}E_b^T E_a = \begin{bmatrix} A_{11}^\mu & A_{12}^\mu \\ A_{21}^\mu & A_{22}^\mu \end{bmatrix},$$

$$S^\mu = \mu(B\bar{R}B^T - DD^T) = \begin{bmatrix} S_{11}^\mu & S_{12}^\mu \\ S_{12}^{\mu T} & S_{22}^\mu \end{bmatrix},$$

$$Q^\mu = \frac{1}{\mu}E_a^T[I_j - E_b\bar{R}E_b^T]E_a + R_1 = \begin{bmatrix} Q_{11}^\mu & Q_{12}^\mu \\ Q_{12}^{\mu T} & Q_{22}^\mu \end{bmatrix}.$$

Substituting P_ε into the ARE (2) and setting $\varepsilon = 0$, we obtain the zeroth order equations

$$\bar{P}_{11}^T A_0^\mu + A_0^{\mu T}\bar{P}_{11} - \bar{P}_{11}^T S_0^\mu \bar{P}_{11} + Q_0^\mu = 0, \tag{8a}$$

$$\bar{P}_{21} = -N_2^T + N_1^T \bar{P}_{11}, \tag{8b}$$

$$A_{22}^{\mu T}\bar{P}_{22} + \bar{P}_{22}^T A_{22}^\mu - \bar{P}_{22}^T S_{22}^\mu \bar{P}_{22} + Q_{22}^\mu = 0, \tag{8c}$$

where

$$\begin{aligned}
A_0^\mu &= A_{11}^\mu + N_1 A_{21}^\mu + S_{12}^\mu N_2^T + N_1 S_{22}^\mu N_2^T, \\
S_0^\mu &= S_{11}^\mu + N_1 S_{12}^{T\mu} + S_{12}^\mu N_1^T + N_1 S_{22}^\mu N_1^T, \\
Q_0^\mu &= Q_{11}^\mu - N_2 A_{21}^\mu - A_{21}^{\mu T} N_2^T - N_2 S_{22}^\mu N_2^T, \\
N_2^T &= \Lambda_4^{-T}\hat{Q}_{12}^T,\ N_1^T = -\Lambda_4^{-T}\Lambda_2^T, \\
\Lambda_2 &= A_{12}^\mu - S_{12}^\mu \bar{P}_{22},\ \Lambda_4 = A_{22}^\mu - S_{22}^\mu \bar{P}_{22}, \\
\hat{Q}_{12} &= Q_{12} + A_{21}^{\mu T}\bar{P}_{22}.
\end{aligned}$$

Lemma 1 *If the AREs* (8) *have the unique positive definite stabilizing solution, then there exists small* $\bar{\varepsilon} > 0$ *such that for all* $\varepsilon \in (0,\ \bar{\varepsilon})$*, the ARE* (2) *admits a positive definite solution.*

2 The Exact Decomposition Technique

The exact slow–fast decomposition method for solving the ARE of singularly perturbed systems has been proposed [2]. An algorithm which solves the ARE (2) with small positive parameter ε is as follows.

Step 1. Using the matrices $T_i,\ i = 1,\ 2,\ 3,\ 4$, solve the following equations (9a) and (9b) for $L = L(\varepsilon)$ and $H = H(\varepsilon)$, respectively.

$$T_4 L - T_3 - \varepsilon L(T_1 - T_2 L) = 0, \tag{9a}$$

$$-H(T_4 + \varepsilon L T_2) + T_2 + \varepsilon(T_1 - T_2 L)H = 0, \tag{9b}$$

where

$$T_1 = \begin{bmatrix} A_{11}^\mu & -S_{11}^\mu \\ -Q_{11}^\mu & -A_{11}^{\mu T} \end{bmatrix},\ T_2 = \begin{bmatrix} A_{12}^\mu & -S_{12}^\mu \\ -Q_{12}^\mu & -A_{21}^{\mu T} \end{bmatrix},$$

$$T_3 = \begin{bmatrix} A_{21}^\mu & -S_{12}^{\mu T} \\ -Q_{12}^{\mu T} & -A_{12}^{\mu T} \end{bmatrix},\ T_4 = \begin{bmatrix} A_{22}^\mu & -S_{22}^\mu \\ -Q_{22}^\mu & -A_{22}^{\mu T} \end{bmatrix}.$$

Step 2. Calculate the coefficients $\Omega_i,\ a_i,\ b_i,\ i = 1,\ 2,\ 3,\ 4$ from (10).

$$\Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_2} & 0 \\ 0 & I_{n_1} & 0 & 0 \\ 0 & 0 & 0 & \varepsilon I_{n_2} \end{bmatrix}$$

$$\cdot \begin{bmatrix} I_{2n_1} & \varepsilon H \\ -L & I_{2n_2} - \varepsilon LH \end{bmatrix} \cdot \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix}, \tag{10}$$

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = T_1 - T_2 L,\ \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = T_4 + \varepsilon L T_2.$$

Step 3. Solve the following equations (11a) and (11b) for P_1 and P_2, respectively.

$$P_1 a_1 - a_4 P_1 - a_3 + P_1 a_2 P_1 = 0, \tag{11a}$$

$$P_2 b_1 - b_4 P_2 - b_3 + P_2 b_2 P_2 = 0. \tag{11b}$$

Step 4. Using the coefficients Ω_i obtained in Step 2 and the following formula (12), we calculate the required solution P_ε of the ARE (2).

$$P_\varepsilon = \left(\Omega_3 + \Omega_4 \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}\right) \cdot \left(\Omega_1 + \Omega_2 \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}\right)^{-1} \tag{12}$$

In order to solve the equation (11), the following Lyapunov iterations (13a) and (13b) with $P_1^{(0)} = \bar{P}_{11}$ and $P_2^{(0)} = \bar{P}_{22}$ is proposed in [2].

$$P_1^{(i+1)}(a_1 + a_2 P_1^{(i)}) - (a_4 - P_1^{(i)} a_2)P_1^{(i+1)} = a_3 + P_1^{(i)} a_2 P_1^{(i)},\ i = 0,\ 1,\ 2,\ \cdots, \tag{13a}$$

$$P_2^{(i+1)}(b_1 + b_2 P_2^{(i)}) - (b_4 - P_2^{(i)} b_2)P_2^{(i+1)} = b_3 + P_2^{(i)} b_2 P_2^{(i)},\ i = 0,\ 1,\ 2,\ \cdots. \tag{13b}$$

However, the uniqueness of the solutions and the quadratic convergence property of the algorithm (13) have not been established so far in the previous literature. Therefore, we newly show that there exist the unique solutions to the reduced–order pure–slow and pure–fast AREs and that the iterative algorithm is quadratic convergence. We now are in a position to state our main result.

Theorem 2 *Under the assumption 1, if the AREs* (8a) *and* (8c) *have the positive semidefinite stabilizing solutions, then the iterative algorithms* (13a) *and* (13b) *converge to the exact solution with the rate of quadratic convergence. In this case, there exist the unique solutions of the equation* (11a) *and* (11b) *in neighborhood of the initial conditions, respectively. That is,*

$$\|P_1^{(i)} - P_1\| = O(\varepsilon^{2^i}),\ i = 0,\ 1,\ 2,\ \cdots, \quad (14a)$$
$$\|P_2^{(i)} - P_2\| = O(\varepsilon^{2^i}),\ i = 0,\ 1,\ 2,\ \cdots. \quad (14b)$$

Proof: The proof is given directly by applying the Newton–Kantorovich theorem [4] for the AREs (11). The proof is omitted due to the page limitation. ∎

3 ε–independent Controller

In this section, a new method of calculation for the H_2 norm is proposed. It is worth pointing out that the small parameter ε is unknown. It is easy to show that the H_2 norm is mainly determined by the $O(\varepsilon^{-1})$ due to

$$\mathrm{Trace}[G_\varepsilon^T P_\varepsilon(\mu) G_\varepsilon] = \mathrm{Trace}[G_1^T P_{11}(\mu) G_1 + 2G_2^T P_{21}(\mu) G_1 + \varepsilon^{-1} G_2^T P_{22}(\mu) G_2]. \quad (15)$$

In order to avoid $O(\varepsilon^{-1})$, we introduce the scaling parameter:

$$G_2 = \varepsilon^\delta \bar{G}_2,\ \delta > \frac{1}{2}, \quad (16)$$

where $\bar{G}_2$ is constant matrices.

In this case the H_2 norm as well as the proposed guaranteed cost is finite when the parameter ε tends to zero. The idea behind inserting the ε^δ factor multiplying the exogenous disturbance term in the state equation for the variable x_2 is to make this meaningful physical fast variable for control purpose. If this factor is dropped from the equation (1b), the fast variable controller can not be employed meaningfully because the H_2 norm tends to infinity.

We show that an ε–independent stabilizing controller can be obtained by solving the reduced–order slow and fast AREs (8). The ε–independent linear state feedback controller are obtained by neglecting $O(\varepsilon)$–term of the linear state feedback controller (3). If the parameter ε is very small, it is obvious that the linear state feedback controller (3) can be approximated as

$$u \approx u_{app} = -\bar{R}\left\{\mu\left([B_1^T \bar{P}_{11} + B_2^T \bar{P}_{21}\ \ B_2^T \bar{P}_{22}] + D_{12}^T C_1\right) + E_b^T E_a\right\} x. \quad (17)$$

Theorem 3 *If the AREs* (8) *have the unique positive definite stabilizing solution, then there exists small* $\hat{\varepsilon} > 0$ *such that for all* $\varepsilon \in (0,\ \hat{\varepsilon})$*, the uncertain linear singularly perturbed system* (1) *is quadratically stable via the* ε*–independent linear state feedback controller* (17).

In the rest of this section, we show how to select the parameter μ which is included in the controller (17). If ε is very small, then the guaranteed cost can be changed as follows

$$\begin{aligned}\mathrm{Trace}[G_\varepsilon^T P_\varepsilon(\mu) G_\varepsilon] &= \mathrm{Trace}[G_1^T P_{11}(\mu) G_1 + 2\varepsilon^\delta \bar{G}_2^T P_{21}(\mu) G_1 + \varepsilon^{2\delta-1} \bar{G}_2^T P_{22}(\mu) \bar{G}_2] \\ &\approx \mathrm{Trace}[G_1^T \bar{P}_{11}(\mu) G_1],\end{aligned}$$

where $2\delta - 1 > 0$.

Our new idea is to use only the matrix $\bar{P}_{11}$ for the above cost. As a result, the ε–independent controller (17) does not require knowing the value of the small parameter ε because we can determine the parameter μ without information of the small parameter. Moreover, the amount of the computation required to get the ε–independent controller becomes extremely small for problems with small state dimension in contrast with the case of solving the full–order ARE (2).

Finally, we give an algorithm for the H_2 guaranteed cost control problem of singularly perturbed system without information of ε.

Step 1. Firstly, starting for any small μ, calculate $T_i, i = 1,\ 2,\ 3,\ 4.$

Step 2. Search the minimum parameter $\bar{\mu}$ such that the reduced–order AREs (8) have positive definite stabilizing solution $\bar{P}_{11}$ and $\bar{P}_{22}$ respectively by using the bisection method.

Step 3. Choose μ such that $0 < \mu < \bar{\mu}$ and compute the positive definite stabilizing solution $\bar{P}_{11}$.

Step 4. Calculate the approximate guaranteed cost $f(\mu) = \mathrm{Trace}[G_1^T \bar{P}_{11}(\mu) G_1]$.

Step 5. Find a $\mu = \hat{\mu}$ that minimizes $f(\mu)$ for all $0 < \mu < \bar{\mu}$.

Step 6. For the obtained $\mu = \hat{\mu}$, design the ε–independent controller (17).

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