**Proceedings of the 41st IEEE
Conference on Decision and Control
Las Vegas, Nevada USA, December 2002**

FrM06-2

Numerical Algorithm for Solving Cross–Coupled Algebraic Riccati Equations Related to Nash Games of Multimodeling Systems

Hiroaki Mukaidani[1] Tetsu Shimomura[2] Hua Xu[3]

Abstract

In this paper, the numerical design of a Nash equilibrium for infinite horizon multiparameter singularly perturbed systems (MSPS) is analyzed. A new algorithm which is based on the Newton's method for solving the generalized cross–coupled multiparameter algebraic Riccati equations (GCMARE) is proposed. It is proven that the proposed algorithm guarantees the quadratic convergence. As a result, it is shown the proposed algorithm succeed in improving the convergence rate dramatically compared with the existing results.

1 Introduction

The linear quadratic Nash games have been investigated extensively by several researchers [1]–[3]. In order to obtain the Nash equilibrium strategies, we must solve the cross–coupled algebraic Riccati equations (CARE). In [2], an algorithm called the Lyapunov iterations for solving the CARE has been proposed. However, there are no results for the convergence rate of the Lyapunov iterations. Furthermore, it is easy to verify that the convergence speed is very slow when we run the numerical example. In order to improve the convergence rate of the Lyapunov iterations, the Riccati iterations which is based on the algebraic Riccati equation (ARE) have been proposed [8]. On the other hand, the Riccati iterations which is different from the previous algorithm [8] have been derived in [3]. However, the convergence property of these Riccati iterations were not proved exactly.

Multimodeling stability, control and filtering problems have been investigated extensively (see e.g., [4]–[6]). The multimodeling problems arise in large–scale dynamic systems. Linear quadratic Nash games for the multiparameter singularly perturbed systems (MSPS) have been studied by using composite controller design [4]. When the parameters represent the small unknown perturbations whose values are not known exactly, the composite design is very useful. However, there exist two drawbacks for the composite design. Firstly, the composite Nash equilibrium solution achieves only a performance which is $O(\|\mu\|)$ (where $\|\mu\|$ denotes the norm of the vector $\mu := [\varepsilon_1\ \varepsilon_2]$) close to the full–order performance. Secondly, since the closed–loop solution of the reduced Nash games depends on the path along $\varepsilon_1/\varepsilon_2$ as $\|\mu\| \to 0$, we cannot expect that the closed–loop solution of the full problem converges to the closed–loop solution of the reduced problem [5]. Therefore, in order to avoid the dependance of the path along $\varepsilon_1/\varepsilon_2$, as long as the small perturbation parameters ε_j are known, much effort should be made towards finding the exact strategies which guarantees the Nash equilibrium without the ill–conditioning.

In this paper, we study the linear quadratic Nash games for infinite horizon MSPS from a viewpoint of solving the CARE. The main contribution of this paper is to propose a new iterative algorithm for solving the GCMARE. Since the new algorithm is based on the Newton's method, the new algorithm achieves a quadratic convergence property. Using the new algorithm, we will improve the convergence speed compared with the previous results [3, 8]. The idea that the Newton's method is applied to the CARE has no novelty. However, it is worth pointing out that even if there exists the fact that the Newton's method has been applied to the CARE without the perturbation parameters, for the MSPS the proof of the quadratic convergence property of the resulting algorithm by means of the Newton–Kantorovich theorem has not been studied so far. As another important features, the strategies u_i and u_j that are included to the cost functions are added compared with the existing result [5]. Therefore, our results can be implemented for more realistic systems. Finally, the simulation results show that the proposed algorithm succeed in improving the convergence rate dramatically compared with the existing Lyapunov iterations [2].

Notation: The notations used in this paper are fairly standard. The superscript T denotes matrix transpose. I_n denotes the $n \times n$ identity matrix. $\| \cdot \|$ denotes its Euclidean norm for a matrix. $\det M$ denotes the determinant of M. $\mathbf{Re}\lambda[M]$ denotes the real part of

[1]Graduate School of Education, Hiroshima University, 1–1–1, Kagamiyama, Higashi–Hiroshima, Hiroshima, 739–8524 Japan. e-mail: mukaida@hiroshima-u.ac.jp

[2]Graduate School of Education, Hiroshima University, 1–1–1, Kagamiyama, Higashi–Hiroshima, Hiroshima, 739–8524 Japan. e-mail: tshimo@hiroshima-u.ac.jp

[3]Graduate School of Systems Management, The University of Tsukuba, 3–29–1, Otsuka, Bunkyou–ku, Tokyo, 112–0012 Japan. e-mail: xuhua@gssm.otsuka.tsukuba.ac.jp

the eigenvalue of M. vec$M$ denotes an ordered stack of the columns of M [10]. $\otimes$ denotes Kronecker product. U_{lm} denotes a permutation matrix in Kronecker matrix sense [10] such that $U_{lm}\mathrm{vec}M = \mathrm{vec}M^T,\ (M \in \mathbf{R}^{l\times m})$.

2 Problem Formulation

A linear time–invariant MSPS is given by

$$\begin{aligned} \dot{x}_0(t) &= A_{00}x_0(t) + A_{01}x_1(t) + A_{02}x_2(t) \\ &\quad + B_{01}u_1(t) + B_{02}u_2(t), \quad (1a) \\ \varepsilon_1\dot{x}_1(t) &= A_{10}x_0(t) + A_{11}x_1(t) + B_{11}u_1(t), \quad (1b) \\ \varepsilon_2\dot{x}_2(t) &= A_{20}x_0(t) + A_{22}x_2(t) + B_{22}u_2(t), \quad (1c) \\ x_j(0) &= x_j^0,\ j = 0,\ 1,\ 2, \end{aligned}$$

with the quadratic cost functions

$$J_i(u_i,\ u_j) = \frac{1}{2}\int_0^{\infty} [y_i^T y_i + u_i^T R_{ii}u_i + u_j^T R_{ij}u_j]dt, \quad (2a)$$

$$y_i(t) = C_{i0}x_0(t) + C_{ii}x_i(t) = C_i x(t), \quad (2b)$$

$$R_{ii} > 0,\quad R_{ij} \geq 0,\ x = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix},\quad i,\ j = 1,\ 2,\ i \neq j,$$

where $x_j \in \mathbf{R}^{n_j},\ j = 0,\ 1,\ 2$ are the state vector, $u_j \in \mathbf{R}^{m_j},\ j = 1,\ 2$ are the control input. All the matrices are constant matrices of appropriate dimensions. Note that u_i and u_j that are the strategies relating to the cost functions (2a) are included compared with the existing result [5]. ε_1 and ε_2 are two small positive singular parameters of the same order of magnitude such that

$$0 < k_1 \leq \alpha \equiv \frac{\varepsilon_1}{\varepsilon_2} \leq k_2 < \infty. \quad (3)$$

Note that the fast state matrices $A_{jj}, j = 1,\ 2$ may be singular.

Let us introduce the partitioned matrices

$$\begin{aligned} &A_e = \Pi_e^{-1}A,\ B_{1e} = \Pi_e^{-1}B_1,\ B_{2e} = \Pi_e^{-1}B_2, \\ &S_{ie} = B_{ie}R_{ii}^{-1}B_{ie}^T = \Pi_e^{-1}S_i\Pi_e^{-1}, \\ &G_{je} = B_{je}R_{jj}^{-1}R_{ij}R_{jj}^{-1}B_{je}^T = \Pi_e^{-1}G_j\Pi_e^{-1}, \\ &Q_i = C_i^T C_i,\ i,\ j = 1,\ 2,\ i \neq j, \end{aligned}$$

$$\Pi_e = \begin{bmatrix} I_{n_0} & 0 & 0 \\ 0 & \varepsilon_1 I_{n_1} & 0 \\ 0 & 0 & \varepsilon_2 I_{n_2} \end{bmatrix},$$

$$A = \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & 0 \\ A_{20} & 0 & A_{22} \end{bmatrix},$$

$$B_1 = \begin{bmatrix} B_{01} \\ B_{11} \\ 0 \end{bmatrix},\ B_2 = \begin{bmatrix} B_{02} \\ 0 \\ B_{22} \end{bmatrix},$$

$$S_1 = B_1R_{11}^{-1}B_1^T = \begin{bmatrix} S_{001} & S_{011} & 0 \\ S_{011}^T & S_{111} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$S_2 = B_2R_{22}^{-1}B_2^T = \begin{bmatrix} S_{002} & 0 & S_{022} \\ 0 & 0 & 0 \\ S_{022}^T & 0 & S_{222} \end{bmatrix},$$

$$G_1 = B_1R_{11}^{-1}R_{21}R_{11}^{-1}B_1^T = \begin{bmatrix} G_{001} & G_{011} & 0 \\ G_{011}^T & G_{111} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$G_2 = B_2R_{22}^{-1}R_{12}R_{22}^{-1}B_2^T = \begin{bmatrix} G_{002} & 0 & G_{022} \\ 0 & 0 & 0 \\ G_{022}^T & 0 & G_{222} \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} Q_{001} & Q_{011} & 0 \\ Q_{011}^T & Q_{111} & 0 \\ 0 & 0 & 0 \end{bmatrix},\ Q_2 = \begin{bmatrix} Q_{002} & 0 & Q_{022} \\ 0 & 0 & 0 \\ Q_{022}^T & 0 & Q_{222} \end{bmatrix}.$$

We now consider the linear quadratic Nash games for infinite horizon MSPS (1) under the following basic assumptions [2].

Assumption 1 *There exists an* $\|\mu\|^* > 0$ *such that the triplet* $(A_e,\ B_{je},\ C_j),\ j = 1,\ 2$ *are stabilizable and detectable for all* $\|\mu\| \in (0, \|\mu\|^*]$, *where* $\|\mu\| := \sqrt{\varepsilon_1\varepsilon_2}$.

Assumption 2 *The triplet* $(A_{jj},\ B_{jj},\ C_{jj}),\ j = 1,\ 2$ *are stabilizable and detectable.*

These conditions are quite natural since at least one control agent has to be able to control and observe unstable modes. The purpose is to find a linear Nash equilibrium strategy $(u_1^*,\ u_2^*)$ such that

$$J_i(u_i^*,\ u_j^*) \leq J_i(u_i,\ u_j^*),\quad i,\ j = 1, 2,\quad i \neq j. \quad (4)$$

The Nash inequality shows that u_i^* regulates the state to zero with minimum output energy. The following lemma is already known [1].

Lemma 1 *Under Assumptions 1 and 2, there exists a linear Nash equilibrium strategy such that* (4) *hold if the following full–order CARE*

$$\begin{aligned} &A_e^T X_e + X_e A_e + Q_1 - X_e S_{1e} X_e \\ &\quad - X_e S_{2e} Y_e - Y_e S_{2e} X_e + Y_e G_{2e} Y_e = 0, \quad (5a) \\ &A_e^T Y_e + Y_e A_e + Q_2 - Y_e S_{2e} Y_e \\ &\quad - Y_e S_{1e} X_e - X_e S_{1e} Y_e + X_e G_{1e} X_e = 0, \quad (5b) \end{aligned}$$

have stabilizing solutions $X_e \geq 0$ *and* $Y_e \geq 0$ *with* $\mathbf{Re}\lambda[A_e - S_{1e}X_e - S_{2e}Y_e] < 0$ *where*

$$X_e = \begin{bmatrix} X_{00} & \varepsilon_1 X_{10}^T & \varepsilon_2 X_{20}^T \\ \varepsilon_1 X_{10} & \varepsilon_1 X_{11} & \sqrt{\varepsilon_1\varepsilon_2}X_{21}^T \\ \varepsilon_2 X_{20} & \sqrt{\varepsilon_1\varepsilon_2}X_{21} & \varepsilon_2 X_{22} \end{bmatrix},$$

$$Y_e = \begin{bmatrix} Y_{00} & \varepsilon_1 Y_{10}^T & \varepsilon_2 Y_{20}^T \\ \varepsilon_1 Y_{10} & \varepsilon_1 Y_{11} & \sqrt{\varepsilon_1\varepsilon_2}Y_{21}^T \\ \varepsilon_2 Y_{20} & \sqrt{\varepsilon_1\varepsilon_2}Y_{21} & \varepsilon_2 Y_{22} \end{bmatrix}.$$

Then, the closed–loop linear Nash equilibrium solutions to the full–order problem are given by

$$u_1^*(t) = -R_{11}^{-1} B_{1e}^T X_e x(t), \tag{6a}$$

$$u_2^*(t) = -R_{22}^{-1} B_{2e}^T Y_e x(t). \tag{6b}$$

3 Asymptotic Structure

In order to obtain the solutions of the CARE (5), we introduce the following useful lemma.

Lemma 2 *The CARE* (5) *is equivalent to the following GCMARE* (7)*, respectively.*

$$\begin{aligned} &A^T X + X^T A + Q_1 - X^T S_1 X \\ &\quad -X^T S_2 Y - Y^T S_2 X + Y^T G_2 Y = 0, \quad (7a) \\ &A^T Y + Y^T A + Q_2 - Y^T S_2 Y \\ &\quad -Y^T S_1 X - X^T S_1 Y + X^T G_1 X = 0, \quad (7b) \end{aligned}$$

where

$$X_e = \Pi_e X = X^T \Pi_e,\ X_{ii} = X_{ii}^T,\ i = 0,\ 1,\ 2,$$

$$X = \begin{bmatrix} X_{00} & \varepsilon_1 X_{10}^T & \varepsilon_2 X_{20}^T \\ X_{10} & X_{11} & \sqrt{\alpha}^{-1} X_{21}^T \\ X_{20} & \sqrt{\alpha} X_{21} & X_{22} \end{bmatrix},$$

$$Y_e = \Pi_e Y = Y^T \Pi_e,\ Y_{ii} = Y_{ii}^T,\ i = 0,\ 1,\ 2,$$

$$Y = \begin{bmatrix} Y_{00} & \varepsilon_1 Y_{10}^T & \varepsilon_2 Y_{20}^T \\ Y_{10} & Y_{11} & \sqrt{\alpha}^{-1} Y_{21}^T \\ Y_{20} & \sqrt{\alpha} Y_{21} & Y_{22} \end{bmatrix}.$$

Proof: The proof is identical to the proof of Lemma 3 in [8]. ■

After substituting X and Y into the GCMARE (7), we obtain the following equations as $\varepsilon_j \to +0,\ j = 1,\ 2$, where $\bar{X}_{lm},\ \bar{Y}_{lm},\ lm = 00,\ 10,\ 20,\ 11,\ 21,\ 22$ are the 0–order solutions of the GCMARE (7).

$$\begin{aligned} &A^T \bar{X} + \bar{X}^T A + Q_1 - \bar{X}^T S_1 \bar{X} \\ &\quad -\bar{X}^T S_2 \bar{Y} - \bar{Y}^T S_2 \bar{X} + \bar{Y}^T G_2 \bar{Y} = 0, \quad (8a) \\ &A^T \bar{Y} + \bar{Y}^T A + Q_2 - \bar{Y}^T S_2 \bar{Y} \\ &\quad -\bar{Y}^T S_1 \bar{X} - \bar{X}^T S_1 \bar{Y} + \bar{X}^T G_1 \bar{X} = 0. \quad (8b) \end{aligned}$$

where

$$\bar{X} = \begin{bmatrix} \bar{X}_{00} & 0 & 0 \\ \bar{X}_{10} & \bar{X}_{11} & 0 \\ \bar{X}_{20} & 0 & \bar{X}_{22} \end{bmatrix},\ \bar{Y} = \begin{bmatrix} \bar{Y}_{00} & 0 & 0 \\ \bar{Y}_{10} & \bar{Y}_{11} & 0 \\ \bar{Y}_{20} & 0 & \bar{Y}_{22} \end{bmatrix},$$

$$A_{11}^T \bar{X}_{11} + \bar{X}_{11} A_{11} - \bar{X}_{11} S_{111} \bar{X}_{11} + Q_{111} = 0,$$

$$A_{22}^T \bar{Y}_{22} + \bar{Y}_{22} A_{22} - \bar{Y}_{22} S_{222} \bar{Y}_{22} + Q_{222} = 0.$$

It is well–known that there exist the positive semidefinite solution $\bar{X}_{11}$ and $\bar{Y}_{22}$ under Assumption 2. The following theorem will establish the relation between the solutions X and Y and the solutions $\bar{X}_{lm}$ and $\bar{Y}_{lm}$ for the reduced–order equations (8).

Theorem 1 *Let us now assume that*

$$\det \nabla \mathcal{F}(\bar{\mathcal{P}}) \neq 0, \tag{9}$$

where $\bar{X}_{21} = 0,\ \bar{Y}_{21} = 0$ *and*

$$\begin{aligned} \nabla \mathcal{F}(\mathcal{P}) &= \frac{\partial \mathrm{vec} \mathcal{F}(\mathcal{P})}{\partial (\mathrm{vec} \mathcal{P})^T} \\ &= [(\tilde{A} - \tilde{S}\mathcal{P} - \mathcal{J}\tilde{S}\mathcal{P}\mathcal{J})^T \otimes I_N] U_{2N2N} \\ &\quad + I_N \otimes (\tilde{A} - \tilde{S}\mathcal{P} - \mathcal{J}\tilde{S}\mathcal{P}\mathcal{J})^T \\ &\quad - [(\tilde{S}\mathcal{J}\mathcal{P} - \tilde{G}\mathcal{P}\mathcal{J})^T \otimes \mathcal{J}] U_{2N2N} \\ &\quad - \mathcal{J} \otimes (\tilde{S}\mathcal{J}\mathcal{P} - \tilde{G}\mathcal{P}\mathcal{J})^T, \end{aligned} \tag{10}$$

$$\begin{aligned} \mathcal{F}(\mathcal{P}) &:= \tilde{A}^T \mathcal{P} + \mathcal{P}^T \tilde{A} + \tilde{Q} - \mathcal{P}^T \tilde{S} \mathcal{P} \\ &\quad - \mathcal{J}\mathcal{P}^T \tilde{S} \mathcal{J}\mathcal{P} - \mathcal{P}^T \mathcal{J} \tilde{S} \mathcal{P} \mathcal{J} + \mathcal{J}\mathcal{P}^T \tilde{G} \mathcal{P} \mathcal{J}, \end{aligned}$$

$$\bar{\mathcal{P}} := \begin{bmatrix} \bar{X} & 0 \\ 0 & \bar{Y} \end{bmatrix},\ \mathcal{P} = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix},\ \tilde{A} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix},$$

$$\tilde{Q} = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix},\ \tilde{S} = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix},\ \tilde{G} = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix},$$

$$\mathcal{J} = \begin{bmatrix} 0 & I_N \\ I_N & 0 \end{bmatrix},\ N = n_0 + n_1 + n_2.$$

Under Assumptions 1 and 2, the GCMARE (7) *admits the solutions X and Y such that these matrices possess a power series expansion at $\|\mu\| = 0$. That is,*

$$X = \bar{X} + O(\|\mu\|), \tag{11a}$$

$$Y = \bar{Y} + O(\|\mu\|). \tag{11b}$$

Proof: We apply the implicit function theorem [7] to the GCMARE (7). To do so, it is enough to show that the corresponding Jacobian is nonsingular at $\|\mu\| = 0$. It can be shown, after some algebra, that the Jacobian of (7) in the limit as $\|\mu\| \to +0$ is given by

$$J_{\bar{\mathcal{P}}} = \lim_{\|\mu\| \to +0} \frac{\partial \mathrm{vec} \mathcal{F}(\mathcal{P})}{\partial (\mathrm{vec} \mathcal{P})^T} = \nabla \mathcal{F}(\bar{\mathcal{P}}). \tag{12}$$

Therefore, using the assumption (9), $J_{\bar{\mathcal{P}}}$ is nonsingular at $\|\mu\| = 0$. The conclusion of Theorem 1 is obtained directly by using the implicit function theorem. ■

4 Newton's method

In order to improve the convergence rate of the Lyapunov iterations [2], we propose the following new algorithm which is based on the Newton's method [9].

$$\begin{aligned} &\Phi^{(n)T} \mathcal{P}^{(n+1)} + \mathcal{P}^{(n+1)T} \Phi^{(n)} \\ &\quad - \Theta^{(n)T} \mathcal{P}^{(n+1)} \mathcal{J} - \mathcal{J} \mathcal{P}^{(n+1)T} \Theta^{(n)} + \Xi^{(n)} = 0, \end{aligned} \tag{13}$$

$$n = 0,\ 1,\ \cdots,$$

where

$$\Phi^{(n)} := \tilde{A} - \tilde{S}\mathcal{P}^{(n)} - \mathcal{J}\tilde{S}\mathcal{P}^{(n)}\mathcal{J} = \begin{bmatrix} \Phi_1^{(n)} & 0 \\ 0 & \Phi_2^{(n)} \end{bmatrix},$$

$$\Theta^{(n)} := \tilde{S}\mathcal{J}\mathcal{P}^{(n)} - \tilde{G}\mathcal{P}^{(n)}\mathcal{J} = \begin{bmatrix} 0 & \Theta_1^{(n)} \\ \Theta_2^{(n)} & 0 \end{bmatrix},$$

$$\begin{aligned}\Xi^{(n)} &:= \tilde{Q} + \mathcal{P}^{(n)T}\tilde{S}\mathcal{P}^{(n)} + \mathcal{J}\mathcal{P}^{(n)T}\tilde{S}\mathcal{J}\mathcal{P}^{(n)} \\ &\quad + \mathcal{P}^{(n)T}\mathcal{J}\tilde{S}\mathcal{P}^{(n)}\mathcal{J} - \mathcal{J}\mathcal{P}^{(n)T}\tilde{G}\mathcal{P}^{(n)}\mathcal{J} \\ &= \begin{bmatrix} \Xi_1^{(n)} & 0 \\ 0 & \Xi_2^{(n)} \end{bmatrix},\end{aligned}$$

$$\Phi_i^{(n)} := \begin{bmatrix} \Phi_{00i}^{(n)} & \Phi_{01i}^{(n)} & \Phi_{02i}^{(n)} \\ \Phi_{10i}^{(n)} & \Phi_{11i}^{(n)} & \mu\Phi_{12i}^{(n)} \\ \Phi_{20i}^{(n)} & \mu\Phi_{21i}^{(n)} & \Phi_{22i}^{(n)} \end{bmatrix},$$

$$\Theta_i^{(n)} := \begin{bmatrix} \Theta_{00i}^{(n)} & \Theta_{01i}^{(n)} & \Theta_{02i}^{(n)} \\ \Theta_{10i}^{(n)} & \Theta_{11i}^{(n)} & \mu\Theta_{12i}^{(n)} \\ \Theta_{20i}^{(n)} & \mu\Theta_{21i}^{(n)} & \Theta_{22i}^{(n)} \end{bmatrix},$$

$$\Xi_i^{(n)} := \begin{bmatrix} \Xi_{00i}^{(n)} & \Xi_{01i}^{(n)} & \Xi_{02i}^{(n)} \\ \Xi_{01i}^{(n)T} & \Xi_{11i}^{(n)} & \mu\Xi_{21i}^{(n)T} \\ \Xi_{02i}^{(n)T} & \mu\Xi_{21i}^{(n)} & \Xi_{22i}^{(n)} \end{bmatrix}, \ i = 1,\ 2,$$

$$\mathcal{P}^{(n)} = \begin{bmatrix} X^{(n)} & 0 \\ 0 & Y^{(n)} \end{bmatrix},$$

$$X^{(n)} = \begin{bmatrix} X_{00}^{(n)} & \varepsilon_1 X_{10}^{(n)T} & \varepsilon_2 X_{20}^{(n)T} \\ X_{10}^{(n)} & X_{11}^{(n)} & \sqrt{\alpha}^{-1} X_{21}^{(n)T} \\ X_{20}^{(n)} & \sqrt{\alpha} X_{21}^{(n)} & X_{22}^{(n)} \end{bmatrix},$$

$$Y^{(n)} = \begin{bmatrix} Y_{00}^{(n)} & \varepsilon_1 Y_{10}^{(n)T} & \varepsilon_2 Y_{20}^{(n)T} \\ Y_{10}^{(n)} & Y_{11}^{(n)} & \sqrt{\alpha}^{-1} Y_{21}^{(n)T} \\ Y_{20}^{(n)} & \sqrt{\alpha} Y_{21}^{(n)} & Y_{22}^{(n)} \end{bmatrix},$$

and the initial condition $\mathcal{P}^{(0)}$ has the following form

$$\mathcal{P}^{(0)} = \begin{bmatrix} X^{(0)} & 0 \\ 0 & Y^{(0)} \end{bmatrix},$$

$$X^{(0)} = \begin{bmatrix} \bar{X}_{00} & \varepsilon_1 \bar{X}_{10}^T & \varepsilon_2 \bar{X}_{20}^T \\ \bar{X}_{10} & \bar{X}_{11} & 0 \\ \bar{X}_{20} & 0 & \bar{X}_{22} \end{bmatrix},$$

$$Y^{(0)} = \begin{bmatrix} \bar{Y}_{00} & \varepsilon_1 \bar{Y}_{10}^T & \varepsilon_2 \bar{Y}_{20}^T \\ \bar{Y}_{10} & \bar{Y}_{11} & 0 \\ \bar{Y}_{20} & 0 & \bar{Y}_{22} \end{bmatrix}.$$

Note that the considered algorithm (13) is based on the generalized cross–coupled multiparameter algebraic Lyapunov equations (GCMALE). The new algorithm (13) can be constructed setting $\mathcal{P}^{(n+1)} = \mathcal{P}^{(n)} + \Delta\mathcal{P}^{(n)}$ and neglecting $O(\Delta\mathcal{P}^{(n)T}\Delta\mathcal{P}^{(n)})$ term. Newton's method is well–known and is widely used to find a solution of the algebraic equations, and its local convergence properties are well understood.

The main result of this section is as follows.

Theorem 2 *Under Assumptions 1 and 2, the new iterative algorithm* (13) *converges to the exact solution* $\mathcal{P}^*$ *of the GCMARE* (7) *with rate of the quadratic convergence. Moreover, the unique bounded solution* $\mathcal{P}^{(n)}$ *of the GCMARE* (7) *is in the neighborhood of the exact solution* $\mathcal{P}^*$. *That is, the following condition is satisfied.*

$$\|\mathcal{P}^{(n)} - \mathcal{P}^*\| \le O(\|\mu\|^{2^n}),\ n = 0,\ 1,\ \cdots, \tag{14}$$

where

$$\mathcal{P} = \mathcal{P}^* = \begin{bmatrix} X^* & 0 \\ 0 & Y^* \end{bmatrix},\ \mathcal{L} := 6\|\tilde{S}\| + 2\|\tilde{G}\|,$$

$$\beta := \|[\nabla\mathcal{F}(\mathcal{P}^{(0)})]^{-1}\|,\ \eta := \beta \cdot \|\mathcal{F}(\mathcal{P}^{(0)})\|,\ \theta := \beta\eta\mathcal{L}.$$

Proof: The proof is given directly by applying the Newton–Kantorovich theorem [9] for the GCMARE (7). Taking the partial derivative of the function $\mathcal{F}(\mathcal{P})$ with respect to $\mathcal{P}$ yields (10). It is obvious that $\nabla\mathcal{F}(\mathcal{P})$ is continuous at for all $\mathcal{P}$. Thus, it is immediately obtained from the equation (10) that

$$\|\nabla\mathcal{F}(\mathcal{P}_1) - \nabla\mathcal{F}(\mathcal{P}_2)\| \le \mathcal{L}\|\mathcal{P}_1 - \mathcal{P}_2\|. \tag{15}$$

Moreover, using the fact that

$$\nabla\mathcal{F}(\mathcal{P}^{(0)}) = \nabla\mathcal{F}(\bar{\mathcal{P}}) + O(\|\mu\|), \tag{16}$$

it follows that $\nabla\mathcal{F}(\mathcal{P}^{(0)})$ is nonsingular under the condition (9) for sufficiently small $\|\mu\|$. Therefore, there exists β such that $\beta = \|[\nabla\mathcal{F}(\mathcal{P}^{(0)})]^{-1}\|$. On the other hand, since $\mathcal{F}(\mathcal{P}^{(0)}) = O(\|\mu\|)$, there exists η such that $\eta = \|[\nabla\mathcal{F}(\mathcal{P}^{(0)})]^{-1}\| \cdot \|\mathcal{F}(\mathcal{P}^{(0)})\| = O(\|\mu\|)$. Thus, there exists θ such that $\theta = \beta\eta\mathcal{L} < 2^{-1}$ because of $\eta = O(\|\mu\|)$. Now, let us define

$$t^* \equiv \frac{1}{\beta\mathcal{L}}[1 - \sqrt{1 - 2\theta}]. \tag{17}$$

Using the Newton–Kantorovich theorem, we can show that $\mathcal{P}^*$ is the unique solution in the subset $\mathcal{S} \equiv \{\ \mathcal{P} : \|\mathcal{P}^{(0)} - \mathcal{P}\| \le t^*\ \}$. Moreover, using the Newton–Kantorovich theorem, the error estimate is given by

$$\|\mathcal{P}^{(n)} - \mathcal{P}^*\| \le \frac{(2\theta)^{2^n}}{2^n\beta\mathcal{L}},\ n = 1,\ \cdots. \tag{18}$$

Finally, substituting $2\theta = O(\|\mu\|)$ into (18), we have (14). ∎

Remark 1 *It is well–known that the solution of the GCMARE* (7) *is not unique and several non–negative solutions exist. In this paper, it should be pointed out that if the initial conditions* $\Pi_e X^{(0)}$ *and* $\Pi_e Y^{(0)}$ *are the positive semidefinite solutions the new algorithm* (13) *also converge to the positive semidefinite solutions* $\Pi_e X^*$ *and* $\Pi_e Y^*$ *respectively compared with the Lyapunov iterations because of the following inequality.*

$$\begin{aligned}&\|\mathcal{P}^{(n)} - \mathcal{P}^*\| \le O(\|\mu\|^{2^n}) \Leftrightarrow \\ &\|X^{(n)} - X^*\| \le O(\|\mu\|^{2^n}),\ \|Y^{(n)} - Y^*\| \le O(\|\mu\|^{2^n}).\end{aligned}$$

5 High–Oreder Approximate Nash Strategy

In this section, the high–order approximate Nash strategy is given. Such a strategy is obtained by using the iterative solution (13).

$$u_{1\mathrm{app}}^{(n)}(t) = -R_{11}^{-1}B_1^T X^{(n)}x(t),\ n = 0,\ 1,\ \cdots, \quad (19a)$$
$$u_{2\mathrm{app}}^{(n)}(t) = -R_{22}^{-1}B_2^T Y^{(n)}x(t),\ n = 0,\ 1,\ \cdots. \quad (19b)$$

Theorem 3 *Let us assume that* $\mathbf{Re}\lambda[\Pi_e^{-1}(A - S_1X^{(0)} - S_2Y^{(0)})] < 0$. *Under Assumptions 1 and 2, the high–order approximate strategies* (19) *result in*

$$J_{i\mathrm{app}}^{(n)} = J_i^* + O(\|\mu\|^{2^n}),\quad i = 1,\ 2,\ n = 0,\ 1,\ \cdots, \quad (20)$$

where $J_i^*,\ i = 1,\ 2$ *are the equilibrium optimal values of the cost functionals, while* $J_{i\mathrm{app}}^{(n)},\ i = 1,\ 2$ *are the equilibrium suboptimal ones.*

Proof: When $u_{i\mathrm{app}}^{(n)}$ is used, the value of the performance index is

$$J_{i\mathrm{app}}^{(n)} = \frac{1}{2}x(0)^T W_{ie}^{(n)}x(0), \quad (21)$$

where $W_{ie}^{(n)}$ are the positive semidefinite solutions of the following multiparameter algebraic Lyapunov equations (MALE), respectively

$$\begin{aligned}&(A_e - S_{1e}X_e^{(n)} - S_{2e}Y_e^{(n)})^T W_{1e}^{(n)}\\ &\quad + W_{1e}^{(n)}(A_e - S_{1e}X_e^{(n)} - S_{2e}Y_e^{(n)}) + Q_1\\ &\quad + X_e^{(n)}S_{1e}X_e^{(n)} + Y_e^{(n)}G_{2e}Y_e^{(n)} = 0, \quad (22a)\\ &(A_e - S_{1e}X_e^{(n)} - S_{2e}Y_e^{(n)})^T W_{2e}^{(n)}\\ &\quad + W_{2e}^{(n)}(A_e - S_{1e}X_e^{(n)} - S_{2e}Y_e^{(n)}) + Q_2\\ &\quad + Y_e^{(n)}S_{2e}Y_e^{(n)} + X_e^{(n)}G_{1e}X_e^{(n)} = 0. \quad (22b)\end{aligned}$$

Subtracting (5a) from (22a), we find that $V_{1e}^{(n)} = W_{1e}^{(n)} - X_e$ satisfies the following MALE

$$\begin{aligned}&(A_e - S_{1e}X_e^{(n)} - S_{2e}Y_e^{(n)})^T V_{1e}^{(n)}\\ &\quad + V_{1e}^{(n)}(A_e - S_{1e}X_e^{(n)} - S_{2e}Y_e^{(n)})\\ &\quad + (X_e^{(n)} - X_e)S_{1e}(X_e^{(n)} - X_e)\\ &\quad + Y_eS_{2e}(X_e^{(n)} - X_e) + (X_e^{(n)} - X_e)S_{2e}Y_e\\ &\quad + Y_e^{(n)}G_{2e}Y_e^{(n)} - Y_eG_{2e}Y_e = 0. \quad (23)\end{aligned}$$

Using the relations $X_e^{(n)} - X_e = O(\|\mu\|^{2^n})$ and $Y_e^{(n)} - Y_e = O(\|\mu\|^{2^n})$ from (14), we can change the form of (23) into (24)

$$\begin{aligned}&(A_e - S_{1e}X_e^{(n)} - S_{2e}Y_e^{(n)})^T V_{1e}^{(n)}\\ &\quad + V_{1e}^{(n)}(A_e - S_{1e}X_e^{(n)} - S_{2e}Y_e^{(n)}) + O(\|\mu\|^{2^n}) = 0 \quad (24)\end{aligned}$$

It is easy to verify that $V_{1e}^{(n)} = O(\|\mu\|^{2^n})$ because $A_e - S_{1e}X_e^{(n)} - S_{2e}Y_e^{(n)} = \Pi_e^{-1}[A - S_1X^{(0)} - S_2Y^{(0)} + O(\|\mu\|)]$ is stable for sufficiently small $\|\mu\|$ by using the standard Lyapunov theorem [11]. Consequently, the equality (20) holds. Since the rest of the proof of Theorem 3 corresponding to the equation $V_{2e}^{(n)} = W_{2e}^{(n)} - Y_e = O(\|\mu\|^{2^n})$ is performed by a similar argument, it is omitted. ■

6 Numerical Example

In order to demonstrate the efficiency of our proposed algorithm, we have run a simple numerical example. Let us consider the following MSPS

$$\begin{bmatrix}\dot{x}_0\\ \hline \varepsilon_1\dot{x}_1\\ \varepsilon_2\dot{x}_2\end{bmatrix} = \left[\begin{array}{cc|cc}0 & 1 & 1 & 0\\ -1 & -2 & 0 & 1\\ \hline 2 & 1 & 0 & 0\\ 4 & 1 & 0 & 0\end{array}\right]\begin{bmatrix}x_0\\ \hline x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \hline 1\\ 0\end{bmatrix}u_1 + \begin{bmatrix}0\\ 0\\ \hline 0\\ 5\end{bmatrix}u_2, \quad (25)$$

with the performance index

$$J_1 = \frac{1}{2}\int_0^\infty (x_0^2 + x_1^2 + u_1^2 + u_2^2)dt, \quad (26a)$$
$$J_2 = \frac{1}{2}\int_0^\infty (x_0^2 + 10^{-1}x_2^2 + 2u_1^2 + 4u_2^2)dt. \quad (26b)$$

The small parameters are chosen as $\varepsilon_1 = \varepsilon_2 = 10^{-3}$. Since $A_{11} = 0$ and $A_{22} = 0$, the system is the nonstandard MSPS. Therefore, it should be noted that the existing technique [5] cannot be applied to the MSPS (25).

Table 1 shows the results of the error $\|\mathcal{F}(\mathcal{P}^{(n)})\|$ per iterations for different ε_j. It is easy to verify that the solutions of the GCMARE (7) converge to the exact solution with accuracy of $\|\mathcal{F}(\mathcal{P}^{(n)})\| < 10^{-12}$ after 3 iterations. Moreover, it is interested in pointing out that the result of Table 1 shows that the algorithms (13) are quadratic convergence. Table 2 shows the results of iterations under the same accuracy of $\|\mathcal{F}(\mathcal{P}^{(n)})\| < 10^{-12}$ for the Lyapunov iterations [2] versus the new algorithm. It can be seen that the convergence rate of the resulting algorithm is stable for all ε_j since the initial conditions $\mathcal{P}^{(0)}$ is quite good. On the other hand, the Lyapunov iterations converge to the exact solutions very slowly.

$$X = X^{(3)} = \begin{bmatrix} 1.3079 & 2.5264 \times 10^{-1} & 3.2919 \times 10^{-3} & 5.8294 \times 10^{-4} \\ 2.5264 \times 10^{-1} & 1.8513 \times 10^{-1} & 1.2451 \times 10^{-3} & 3.4574 \times 10^{-4} \\ 3.2919 & 1.2451 & 1.0033 & 1.1847 \times 10^{-3} \\ 5.8294 \times 10^{-1} & 3.4574 \times 10^{-1} & 1.1847 \times 10^{-3} & 1.6365 \times 10^{-2} \end{bmatrix},$$

$$Y = Y^{(3)} = \begin{bmatrix} 2.0446 & 7.1409 \times 10^{-1} & 7.3002 \times 10^{-3} & 1.5293 \times 10^{-3} \\ 7.1409 \times 10^{-1} & 4.8507 \times 10^{-1} & 2.9446 \times 10^{-3} & 7.6728 \times 10^{-4} \\ 7.3002 & 2.9446 & 1.0105 & 3.1413 \times 10^{-3} \\ 1.5293 & 7.6728 \times 10^{-1} & 3.1413 \times 10^{-3} & 1.2746 \times 10^{-1} \end{bmatrix}.$$

Table 1. $\|\mathcal{F}(\mathcal{P}^{(n)})\|$

n	$\varepsilon_1 = \varepsilon_2 = 10^{-1}$	$\varepsilon_1 = \varepsilon_2 = 10^{-2}$	$\varepsilon_1 = \varepsilon_2 = 10^{-3}$	$\varepsilon_1 = \varepsilon_2 = 10^{-4}$	$\varepsilon_1 = \varepsilon_2 = 10^{-5}$
0	1.7	1.7×10^{-1}	1.7×10^{-2}	1.7×10^{-3}	1.7×10^{-4}
1	1.2	1.5×10^{-2}	1.5×10^{-4}	1.5×10^{-6}	1.5×10^{-8}
2	8.6×10^{-2}	7.0×10^{-5}	1.0×10^{-8}	1.0×10^{-12}	3.7×10^{-14}
3	8.3×10^{-4}	1.6×10^{-9}	7.7×10^{-14}	3.8×10^{-14}	–
4	1.0×10^{-7}	7.8×10^{-14}	–	–	–
5	5.4×10^{-14}	–	–	–	–

Table 2.

Number of iterations such that $\|\mathcal{F}(\mathcal{P}^{(n)})\| < 10^{-12}$.		
$\varepsilon_1 = \varepsilon_2$	Lyapunov iterations	Newton's method
10^{-1}	15	5
10^{-2}	20	4
10^{-3}	20	3
10^{-4}	19	3
10^{-5}	17	2
10^{-6}	15	2
10^{-7}	13	2

7 Conclusions

The linear quadratic Nash games for infinite horizon MSPS have been studied. The new iterations method based on the Newton's method has been proposed. Moreover it was shown that the resulting algorithm has the property of the quadratic convergence. Comparing with Lyapunov iterations [2], even if the singular perturbation parameter is extremely small, we have succeeded in improving the convergence rate dramatically.

References

[1] Starr, A. W., and Y. C. Ho, "Nonzero–sum differential games," *J. Optimization Theory and Application*, vol. 3 (1969), pp.184–206.

[2] Li, T., and Z. Gajić, "Lyapunov iterations for solving coupled algebraic Lyapunov equations of Nash differential games and algebraic Riccati equations of Zero–sum games," *New Trends in Dynamic Games and Applications*, Birkhauser, (1994), pp.333–351.

[3] Freiling, G., G. Jank and H. Abou–Kandil, "On global existence of solutions to coupled matrix Riccati equations in closed–loop Nash games," *IEEE Trans. Automat. Contr.*, vol. 41 (1996), pp.264–269.

[4] Khalil, H. K., "Multimodel design of a Nash strategy," *J. Optimization Theory and Application*, vol. 31, (1980), pp.553–564.

[5] Khalil, H. K., and P. V. Kokotovic, "Control of linear systems with multiparameter singular perturbations," *Automatica*, vol. 15 (1979), pp.197–207.

[6] Coumarbatch, C., and Z. Gajic, "Parallel optimal Kalman filtering for stochastic systems in multimodeling form," *Trans. ASME, Journal of Dynamic Systems, Measurement, and Control*, vol. 122 (2000), pp.542–550.

[7] Gajić, Z., D. Petkovski and X. Shen, *Singularly Perturbed and Weakly Coupled Linear System-a Recursive Approach, Lecture Notes in Control and Information Sciences*, vol. 140, Springer–Verlag, 1990.

[8] Mukaidani, H., H. Xu and K. Mizukami, "A new algorithm for solving cross–coupled algebraic Riccati equations of singularly perturbed Nash games," *Proceedings of the 39th IEEE Conference on Decision and Control*, Sydney, December, 2000, pp.3648–3653.

[9] Ortega, J. M., *Numerical analysis, A second course*, SIAM, 1990.

[10] Magnus, J. R., and H. Neudecker, *Matrix differential calculus with applications in statistics and econometrics*, John Wiley and Sons, 1999.

[11] Zhou, K., *Essentials of Robust Control*, Prentice–Hall, 1998.