Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Guaranteed Cost Control of Uncertain Singularly Perturbed Systems via Static Output Feedback

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Abstract— **In this paper, the guaranteed cost output feedback control problem for singularly perturbed systems (SPS) with uncertainties is investigated. In order to solve this problem, we must solve a set of cross-coupled algebraic Lyapunov equations and algebraic Riccati equations (CALRE). In this paper, a new algorithm to solve the CALRE is provided which is based on Newton's method. The quadratic local convergence of the algorithm is proved. A numerical example is solved to show a 18.4% reduction of the average CPU time compared with the existing result.**

I. INTRODUCTION

Robust control problems for uncertain singularly perturbed systems (SPS) have been studied extensively [1]-[4]. So far, much effort has been made towards finding a state feedback controller. However, it is not always possible to have access to a state information in the control. Sometimes a measurement of the output information must be used to find the controller. Therefore, the robust static output feedback problem for uncertain SPS is an important issue.

It is well-known that the implementation of the static output feedback controller is hard problem in the control system design. It has been shown in [6] that the two-time-scale design method can not guarantee the robust stability of the closed-loop SPS. Therefore, if the singularly perturbed parameter ε is known, it is better to use the numerical solution of the cross-coupled algebraic equation to obtain the output feedback gain.

In recent years, the guaranteed cost control approach [7], [8] has been studied by several researchers. Since the guaranteed cost control guarantees not only the robust stability, but also an adequate degree of performance, it is very useful for the uncertain SPS. In order to obtain the output feedback gain of the guaranteed cost control, it is necessary to solve a set of cross-coupled algebraic Lyapunov equations and algebraic Riccati equations (CALRE). Although the algorithm for solving the CALRE has been introduced in [8], there is no proof on the convergence of the algorithm.

In this paper, the guaranteed cost output feedback control problem of the uncertain SPS is investigated. This paper is an extension of [5] in the sense that the guaranteed cost output feedback control is applied to the uncertain SPS. The main contribution of this paper is to provide a new algorithm to solve the CALRE. Since the proposed algorithm is based on Newton's method, it is quite different from the existing algorithm [8]. The quadratic local convergence of the algorithm is proved. A numerical example is solved to show the validity of the algorithm from the view point of improving the convergence speed. As a result, about 18.4% reduction of the average CPU time is attained compared with the existing result. As another important result, the uniqueness and the boundedness of the solution to the CALRE are also established. It is worth pointing out that, since the resulting controller is not based on the two-time-scale design method [6], the robust stability of the closed-loop SPS via output feedback is guaranteed.

Notation: The notations used in this paper are fairly standard. The superscript T denotes matrix transpose. Trace denotes the trace of the matrix. I_p denotes the $p \times p$ identity matrix. vec denotes the column vector of the matrix [10]. U_{lm} denotes a permutation matrix in the Kronecker matrix sense [10] such that $U_{lm}\mathrm{vec}M = \mathrm{vec}M^T,\ M \in \mathbf{R}^{l\times m}$. $\otimes$ denotes the Kronecker product. $E[\cdot]$ denotes the expectation.

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II. PROBLEM STATEMENT

Consider the following uncertain SPS

$$
\begin{aligned}
\dot{x}_1 &= [A_{11} + D_1 F E_{a1}]x_1 + [A_{12} + D_1 F E_{a2}]x_2 \\
&\quad + [B_1 + D_1 F E_b]u,\ x_1(0) = x_1^0, \qquad (1a)\\
\varepsilon\dot{x}_2 &= [A_{21} + D_2 F E_{a1}]x_1 + [A_{22} + D_2 F E_{a2}]x_2 \\
&\quad + [B_2 + D_2 F E_b]u,\ x_2(0) = x_2^0, \qquad (1b)\\
y &= C_1 x_1 + C_2 x_2, \qquad (1c)\\
F^T F &\le I_s, \qquad (1d)
\end{aligned}
$$

where ε is a small positive parameter, $x_1 \in \mathbf{R}^{n_1}$ and $x_2 \in \mathbf{R}^{n_2}$ are state vectors, $u \in \mathbf{R}^m$ is the control input, $y \in \mathbf{R}^r$ is the output. Moreover, $F := F(t) \in \mathbf{R}^{k\times s}$ is a Lebesgue measurable matrix of uncertain parameters satisfying (1d). All matrices above are of appropriate dimensions.

Let us introduce the partitioned matrices

$$
A_\varepsilon = \begin{bmatrix} A_{11} & A_{12} \\ \varepsilon^{-1}A_{21} & \varepsilon^{-1}A_{22} \end{bmatrix},\ A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},
$$

$$
B_\varepsilon = \begin{bmatrix} B_1 \\ \varepsilon^{-1}B_2 \end{bmatrix},\ B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},
$$

$$
C = \begin{bmatrix} C_1 & C_2 \end{bmatrix},\ D_\varepsilon = \begin{bmatrix} D_1 \\ \varepsilon^{-1}D_2 \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix},
$$

$$
E_a = \begin{bmatrix} E_{a1} & E_{a2} \end{bmatrix},\ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbf{R}^n,\ n = n_1 + n_2.
$$

By using above relations, the uncertain SPS (1) can be changed to

$$\dot{x} = [A_\varepsilon + D_\varepsilon F E_a]x + [B_\varepsilon + D_\varepsilon F E_b]u, \ x(0) = x^0, \quad (2a)$$
$$y = Cx. \quad (2b)$$

The initial state x^0 is assumed to be a random variable with a covariance matrix $E[x^0 x^{0T}] = I_n$. Associated with the uncertain SPS (2) is the cost function

$$J = E\left[\int_0^\infty (x^T Q x + u^T R u)dt\right], \quad (3)$$

where $Q := \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} > 0$ and R is positive definite symmetric matrix.

Definition 1: A static output feedback control law $u = Ky = KCx$ is said to be a quadratic guaranteed cost control with the associated cost matrix $P_\varepsilon > 0$ for the SPS (2) and the cost function (3), if the closed-loop uncertain SPS is quadratically stable and the closed-loop value of the cost function (3) satisfies the bound $J \le J^*$ for all admissible uncertainties, that is,

$$\frac{d}{dt} x^T P_\varepsilon x + x^T [Q + K^T C^T R C K] x \le 0, \quad (4)$$

where J^* is the guaranteed cost.

The objective of this paper is to design an guaranteed cost output feedback control law $u = Ky$ for the uncertain SPS (2) by means of the numerical algorithm. The following result is already known in [8].

Lemma 1: Consider system (2) and suppose that the static output feedback control law $u = Ky$ is a quadratic guaranteed cost control with the cost matrix $P_\varepsilon > 0$. Then the closed-loop uncertain SPS

$$\dot{x} = [A_\varepsilon + D_\varepsilon F(E_a + E_b KC) + B_\varepsilon KC]x, \ F^T F \le I_s \quad (5)$$

is quadratically stable. Furthermore, the corresponding value of the cost function J satisfies the following inequality for all admissible uncertain F.

$$J \le \text{Trace } P_\varepsilon. \quad (6)$$

The existing results can be stated in the following lemma [8].

Lemma 2: In order for K to be an optimal quadratic guaranteed cost controller, it is necessary that there exists a positive parameter μ such that the following CALRE $F_k := F_k(P_\varepsilon, \ K, \ S) = 0, \ k = 1, \ 2, \ 3$ have solutions P_ε, K and S

$$\begin{aligned} F_1 &:= (R + \mu^{-1} E_b^T E_b) K C S C^T \\ &\quad + (B_\varepsilon^T P_\varepsilon + \mu^{-1} E_b^T E_a) S C^T = 0, &(7a) \\ F_2 &:= (A_\varepsilon + B_\varepsilon KC)^T P_\varepsilon + P_\varepsilon (A_\varepsilon + B_\varepsilon KC) \\ &\quad + \mu P_\varepsilon D_\varepsilon D_\varepsilon^T P_\varepsilon + \mu^{-1} (E_a + E_b KC)^T (E_a + E_b KC) \\ &\quad + C^T K^T R K C + Q = 0, &(7b) \\ F_3 &:= (A_\varepsilon + B_\varepsilon KC + \mu D_\varepsilon D_\varepsilon^T P_\varepsilon) S \\ &\quad + S(A_\varepsilon + B_\varepsilon KC + \mu D_\varepsilon D_\varepsilon^T P_\varepsilon)^T + I_n = 0, &(7c) \end{aligned}$$

where

$$P_\varepsilon = P_\varepsilon(\mu) = \begin{bmatrix} P_{11}(\varepsilon, \ \mu) & \varepsilon P_{21}^T(\varepsilon, \ \mu) \\ \varepsilon P_{21}(\varepsilon, \ \mu) & \varepsilon P_{22}(\varepsilon, \ \mu) \end{bmatrix},$$
$$K = K(\mu), \ S = S(\mu) = \begin{bmatrix} S_{11}(\varepsilon, \ \mu) & S_{21}^T(\varepsilon, \ \mu) \\ S_{21}(\varepsilon, \ \mu) & S_{22}(\varepsilon, \ \mu) \end{bmatrix}.$$

Moreover, if CSC^T is nonsingular then (7a) may be solved for K to obtain

$$\begin{aligned} K = &-(R + \mu^{-1} E_b^T E_b)^{-1} (B_\varepsilon^T P_\varepsilon + \mu^{-1} E_b^T E_a) \\ &\times S C^T (C S C^T)^{-1}. &(8) \end{aligned}$$

III. PRELIMINARY

In order to investigate the solvability condition of the CALRE (7), they can be partitioned into (9).

$$\begin{aligned}
&(R + \mu^{-1} E_b^T E_b)^{-1} K (C_1 S_{11} C_1^T + C_1 S_{21}^T C_2^T \\
&\quad + C_2 S_{21} C_1^T + C_2 S_{22} C_2^T) \\
&\quad + (B_1^T P_{11} + B_2^T P_{21} + \mu^{-1} E_b^T E_{a1})(S_{11} C_1^T + S_{21}^T C_2^T) \\
&\quad + (B_2^T P_{22} + \varepsilon B_1^T P_{21}^T + \mu^{-1} E_b^T E_{a2}) \\
&\quad \times (S_{21} C_1^T + S_{22} C_2^T) = 0, &(9a) \\
&P_{11}(A_{11} + B_1 K C_1) + (A_{11} + B_1 K C_1)^T P_{11} \\
&\quad + P_{21}^T (A_{21} + B_2 K C_1) + (A_{21} + B_2 K C_1)^T P_{21} \\
&\quad + \mu (P_{11} D_1 D_1^T P_{11} + P_{21}^T D_2 D_1^T P_{11} \\
&\quad + P_{11} D_1 D_2^T P_{21} + P_{21}^T D_2 D_2^T P_{21}) \\
&\quad + \mu^{-1} (E_{a1} + E_b K C_1)^T (E_{a1} + E_b K C_1) \\
&\quad + Q_{11} + C_1^T K^T R K C_1 = 0, &(9b) \\
&P_{11}(A_{12} + B_1 K C_2) + (A_{21} + B_2 K C_1)^T P_{22} \\
&\quad + P_{21}^T (A_{22} + B_2 K C_2) + \varepsilon (A_{11} + B_1 K C_1)^T P_{21}^T \\
&\quad + \mu (P_{11} D_1 D_2^T P_{22} + P_{21}^T D_2 D_2^T P_{22} \\
&\quad + \varepsilon P_{11} D_1 D_1^T P_{21}^T + \varepsilon P_{21}^T D_2 D_1^T P_{21}^T) \\
&\quad + \mu^{-1} (E_{a1} + E_b K C_1)^T (E_{a2} + E_b K C_2) \\
&\quad + Q_{12} + C_1^T K^T R K C_2 = 0, &(9c) \\
&P_{22}(A_{22} + B_2 K C_2) + (A_{22} + B_2 K C_2)^T P_{22} \\
&\quad + \varepsilon P_{21} (A_{12} + B_1 K C_2) + \varepsilon (A_{12} + B_1 K C_2)^T P_{21}^T \\
&\quad + \mu (P_{22} D_2 D_2^T P_{22} + \varepsilon P_{22} D_2 D_1^T P_{21}^T \\
&\quad + \varepsilon P_{21} D_1 D_2^T P_{22} + \varepsilon^2 P_{21} D_1 D_1^T P_{21}^T) \\
&\quad + \mu^{-1} (E_{a2} + E_b K C_2)^T (E_{a2} + E_b K C_2) \\
&\quad + Q_{22} + C_2^T K^T R K C_2 = 0, &(9d) \\
&S_{11}(A_{11} + B_1 K C_1 + \mu D_1 D_1^T P_{11} + \mu D_1 D_2^T P_{21})^T \\
&\quad + (A_{11} + B_1 K C_1 + \mu D_1 D_1^T P_{11} + \mu D_1 D_2^T P_{21}) S_{11} \\
&\quad + S_{21}^T (A_{12} + B_1 K C_2 + \mu D_1 D_2^T P_{22} + \mu \varepsilon D_1 D_1^T P_{21}^T)^T \\
&\quad + (A_{12} + B_1 K C_2 + \mu D_1 D_2^T P_{22} + \mu \varepsilon D_1 D_1^T P_{21}^T) S_{21} \\
&\quad + I_{n_1} = 0, &(9e) \\
&S_{11}(A_{21} + B_2 K C_1 + \mu D_2 D_1^T P_{11} + \mu D_2 D_2^T P_{21})^T \\
&\quad + \varepsilon (A_{11} + B_1 K C_1 + \mu D_1 D_1^T P_{11} + \mu D_1 D_2^T P_{21}) S_{21}^T \\
&\quad S_{21}^T (A_{22} + B_2 K C_2 + \mu D_2 D_2^T P_{22} + \mu \varepsilon D_2 D_1^T P_{21}^T)^T \\
&\quad + \varepsilon (A_{12} + B_1 K C_2 + \mu D_1 D_2^T P_{22} + \mu \varepsilon D_2 D_1^T P_{21}^T) S_{22} \\
&\quad = 0, &(9f)
\end{aligned}$$

$$
\begin{aligned}
&S_{22}(A_{22}+B_2KC_2+\mu D_2D_2^TP_{22}+\mu\varepsilon D_2D_1^TP_{21}^T)^T\\
&\quad+(A_{22}+B_2KC_2+\mu D_2D_2^TP_{22}+\mu\varepsilon D_2D_1^TP_{21}^T)S_{22}\\
&\quad S_{21}(A_{21}+B_2KC_1+\mu D_2D_1^TP_{11}+\mu D_2D_2^TP_{21})^T\\
&\quad+(A_{21}+B_2KC_1+\mu D_2D_1^TP_{11}+\mu D_2D_2^TP_{21})S_{21}^T\\
&\quad+\varepsilon I_{n_2}=0. \qquad (9g)
\end{aligned}
$$

Setting $\varepsilon=0$ for the previous equations (9), the following equations hold.

$$
\begin{aligned}
&(R+\mu^{-1}E_b^TE_b)^{-1}\bar{K}(C_1\bar{S}_{11}C_1^T+C_1\bar{S}_{21}^TC_2^T\\
&\quad+C_2\bar{S}_{21}C_1^T+C_2\bar{S}_{22}C_2^T)\\
&\quad+(B_1^T\bar{P}_{11}+B_2^T\bar{P}_{21}+\mu^{-1}E_b^TE_{a1})(\bar{S}_{11}C_1^T+\bar{S}_{21}^TC_2^T)\\
&\quad+(B_2^T\bar{P}_{22}+\mu^{-1}E_b^TE_{a2})(\bar{S}_{21}C_1^T+\bar{S}_{22}C_2^T)=0, \qquad (10a)\\
&\bar{P}_{11}(A_{11}+B_1\bar{K}C_1)+(A_{11}+B_1\bar{K}C_1)^T\bar{P}_{11}\\
&\quad+\bar{P}_{21}^T(A_{21}+B_2\bar{K}C_1)+(A_{21}+B_2\bar{K}C_1)^T\bar{P}_{21}\\
&\quad+\mu(\bar{P}_{11}D_1D_1^T\bar{P}_{11}+\bar{P}_{21}^TD_2D_1^T\bar{P}_{11}\\
&\quad+\bar{P}_{11}D_1D_2^T\bar{P}_{21}+\bar{P}_{21}^TD_2D_2^T\bar{P}_{21})\\
&\quad+\mu^{-1}(E_{a1}+E_b\bar{K}C_1)^T(E_{a1}+E_b\bar{K}C_1)\\
&\quad+Q_{11}+C_1^T\bar{K}^TR\bar{K}C_1=0, \qquad (10b)\\
&\bar{P}_{11}(A_{12}+B_1\bar{K}C_2)+(A_{21}+B_2\bar{K}C_1)^T\bar{P}_{22}\\
&\quad+\bar{P}_{21}^T(A_{22}+B_2\bar{K}C_2)\\
&\quad+\mu(\bar{P}_{11}D_1D_2^T\bar{P}_{22}+\bar{P}_{21}^TD_2D_2^T\bar{P}_{22})\\
&\quad+\mu^{-1}(E_{a1}+E_b\bar{K}C_1)^T(E_{a2}+E_b\bar{K}C_2)\\
&\quad+Q_{12}+C_1^T\bar{K}^TR\bar{K}C_2=0, \qquad (10c)\\
&\bar{P}_{22}(A_{22}+B_2\bar{K}C_2)+(A_{22}+B_2\bar{K}C_2)^T\bar{P}_{22}\\
&\quad+\mu\bar{P}_{22}D_2D_2^T\bar{P}_{22}\\
&\quad+\mu^{-1}(E_{a2}+E_b\bar{K}C_2)^T(E_{a2}+E_b\bar{K}C_2)\\
&\quad+Q_{22}+C_2^T\bar{K}^TR\bar{K}C_2=0, \qquad (10d)\\
&\bar{S}_{11}(A_{11}+B_1\bar{K}C_1+\mu D_1D_1^T\bar{P}_{11}+\mu D_1D_2^T\bar{P}_{21})^T\\
&\quad+(A_{11}+B_1\bar{K}C_1+\mu D_1D_1^T\bar{P}_{11}+\mu D_1D_2^T\bar{P}_{21})\bar{S}_{11}\\
&\quad+\bar{S}_{21}^T(A_{12}+B_1\bar{K}C_2+\mu D_1D_2^T\bar{P}_{22})^T\\
&\quad+(A_{12}+B_1\bar{K}C_2+\mu D_1D_2^T\bar{P}_{22})\bar{S}_{21}\\
&\quad+I_{n_1}=0, \qquad (10e)\\
&\bar{S}_{11}(A_{21}+B_2\bar{K}C_1+\mu D_2D_1^T\bar{P}_{11}+\mu D_2D_2^T\bar{P}_{21})^T\\
&\quad\bar{S}_{21}^T(A_{22}+B_2\bar{K}C_2+\mu D_2D_2^T\bar{P}_{22})^T=0, \qquad (10f)\\
&\bar{S}_{22}(A_{22}+B_2\bar{K}C_2+\mu D_2D_2^T\bar{P}_{22})^T\\
&\quad+(A_{22}+B_2\bar{K}C_2+\mu D_2D_2^T\bar{P}_{22})\bar{S}_{22}\\
&\quad\bar{S}_{21}(A_{21}+B_2\bar{K}C_1+\mu D_2D_1^T\bar{P}_{11}+\mu D_2D_2^T\bar{P}_{21})^T\\
&\quad+(A_{21}+B_2\bar{K}C_1+\mu D_2D_1^T\bar{P}_{11}+\mu D_2D_2^T\bar{P}_{21})\bar{S}_{21}^T\\
&\quad=0, \qquad (10g)
\end{aligned}
$$

where $\bar{P}_{11}$, $\bar{P}_{21}$, $\bar{P}_{22}$, $\bar{K}$, $\bar{S}_{11}$, $\bar{S}_{21}$ and $\bar{S}_{22}$ are zeroth-order solutions of the equations (9).

Then, taking the partial derivative of the function $F_k(P_\varepsilon,\ K,\ S)$, $k=1,\ 2,\ 3$ with respect to P_ε, K, S results in (11).

Using (11), the following asymptotic structure of the CALRE (7) is established.

Theorem 1: Assume that zeroth-order equations (10) have the solutions such that

$$
\begin{aligned}
&\det\bar{\mathcal{J}}(0,\ \bar{P},\ \bar{K},\ \bar{S})\\
&=\det\begin{bmatrix}(C\bar{S})\otimes B^T & (C\bar{S}C^T)\otimes\tilde{R} & \bar{\Pi}_{13}\\ \bar{\Pi}_{21} & \bar{\Pi}_{22} & 0\\ \bar{\Pi}_{31} & \bar{\Pi}_{32} & \bar{\Pi}_{33}\end{bmatrix}\neq 0, \qquad (12)
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{\Pi}_{13}&:=C\otimes[(R+\mu^{-1}E_b^TE_b)\bar{K}C+B^T\bar{P}+\mu^{-1}E_b^TE_a]\\
\bar{\Pi}_{21}&:=I_n\otimes(A+B\bar{K}C+\mu DD^T\bar{P})^T\\
&\quad+\big((A+B\bar{K}C+\mu DD^T\bar{P})^T\otimes I_n\big)U_{nn}\\
\bar{\Pi}_{22}&:=\big[(\mu^{-1}E_a^TE_b+\bar{P}^TB+\mu^{-1}C^T\bar{K}^TE_b^TE_b\\
&\quad+C^T\bar{K}^TR)\otimes C^T\big]U_{nn}+C^T\otimes(\mu^{-1}E_a^TE_b\\
&\quad+\bar{P}^TB+\mu^{-1}C^T\bar{K}^TE_b^TE_b+C^T\bar{K}^TR)\\
\bar{\Pi}_{31}&:=(\Phi_0\bar{S})\otimes(\mu DD^T)+\big((\mu DD^T)\otimes(\Phi_0\bar{S})\big)U_{nn}\\
\bar{\Pi}_{32}&:=\big(B\otimes(\Phi_0\bar{S}C^T)\big)U_{nn}+(\Phi_0\bar{S}C^T)\otimes B\\
\bar{\Pi}_{33}&:=\Phi_0\otimes(A+B\bar{K}C+\mu DD^T\bar{P})\\
&\quad+(A+B\bar{K}C+\mu DD^T\bar{P}_\varepsilon)\otimes\Phi_0\\
\bar{P}&:=\begin{bmatrix}\bar{P}_{11} & 0\\ \bar{P}_{21} & \bar{P}_{22}\end{bmatrix},\ \bar{S}:=\begin{bmatrix}\bar{S}_{11} & \bar{S}_{21}^T\\ \bar{S}_{21} & \bar{S}_{22}\end{bmatrix},\ \Phi_0:=\begin{bmatrix}I_{n_1} & 0\\ 0 & 0\end{bmatrix}.
\end{aligned}
$$

Then there exists small $\bar{\varepsilon}>0$ such that for all $\varepsilon\in(0,\ \bar{\varepsilon})$, the CALRE (7) admits the solutions $P_\varepsilon>0$, $S>0$ and K, which can be written as

$$
\begin{aligned}
P_\varepsilon&=\begin{bmatrix}\bar{P}_{11}+O(\varepsilon) & \varepsilon\bar{P}_{21}^T+O(\varepsilon^2)\\ \varepsilon\bar{P}_{21}+O(\varepsilon^2) & \varepsilon\bar{P}_{22}+O(\varepsilon^2)\end{bmatrix}, \qquad (13a)\\
K&=\bar{K}+O(\varepsilon), \qquad (13b)\\
S&=\begin{bmatrix}\bar{S}_{11}+O(\varepsilon) & \bar{S}_{21}^T+O(\varepsilon)\\ \bar{S}_{21}+O(\varepsilon) & \bar{S}_{22}+O(\varepsilon)\end{bmatrix}. \qquad (13c)
\end{aligned}
$$

Proof: It can be done by applying the implicit function theorem to the CALRE (7) or the partitioned equations (9). To do so, it is enough to show that the corresponding Jacobian is nonsingular at $\varepsilon=0$. After some tedious algebra, we get (11). Setting $\varepsilon=0$, the condition (12) is obtained. Finally, applying the implicit function theorem results in the desired result. ■

IV. NEWTON'S METHOD

In order to obtain the solutions of the CALRE (7), the following new algorithm is given.

$$
\begin{aligned}
&B_\varepsilon^TP_\varepsilon^{(i+1)}S^{(i)}C^T+(R+\mu^{-1}E_b^TE_b)K^{(i+1)}CS^{(i)}C^T\\
&\quad+[(R+\mu^{-1}E_b^TE_b)K^{(i)}C+B_\varepsilon^TP_\varepsilon^{(i)}\\
&\quad+\mu^{-1}E_b^TE_a]S^{(i+1)}C^T\\
&\quad-\Big[(R+\mu^{-1}E_b^TE_b)K^{(i)}C+B_\varepsilon^TP_\varepsilon^{(i)}\Big]S^{(i)}C^T=0, \qquad (14a)\\
&(A_\varepsilon+B_\varepsilon K^{(i)}C+\mu D_\varepsilon D_\varepsilon^TP_\varepsilon^{(i)})^TP_\varepsilon^{(i+1)}\\
&\quad+P_\varepsilon^{(i+1)}(A_\varepsilon+B_\varepsilon K^{(i)}C+\mu D_\varepsilon D_\varepsilon^TP_\varepsilon^{(i)})\\
&\quad+C^TK^{(i+1)T}(\mu^{-1}E_a^TE_b+P_\varepsilon^{(i)}B_\varepsilon\\
&\quad+\mu^{-1}C^TK^{(i)T}E_b^TE_b+C^TK^{(i)T}R)^T\\
&\quad+(\mu^{-1}E_a^TE_b+P_\varepsilon^{(i)}B_\varepsilon
\end{aligned}
$$

$$\mathcal{J}(\varepsilon,\ P_\varepsilon,\ K,\ S) := \begin{bmatrix} \dfrac{\partial F_1}{\partial P_\varepsilon} & \dfrac{\partial F_1}{\partial K} & \dfrac{\partial F_1}{\partial S} \\ \dfrac{\partial F_2}{\partial P_\varepsilon} & \dfrac{\partial F_2}{\partial K} & \dfrac{\partial F_2}{\partial S} \\ \dfrac{\partial F_3}{\partial P_\varepsilon} & \dfrac{\partial F_3}{\partial K} & \dfrac{\partial F_3}{\partial S} \end{bmatrix} = \begin{bmatrix} (CS)\otimes B_\varepsilon^T & (CSC^T)\otimes(R+\mu^{-1}E_b^TE_b) & \Xi_{13} \\ \Xi_{21} & \Xi_{22} & O \\ S\otimes(\mu D_\varepsilon D_\varepsilon^T)+(\mu D_\varepsilon D_\varepsilon^T)\otimes S & B_\varepsilon\otimes(SC^T)+(SC^T)\otimes B_\varepsilon & \Xi_{33} \end{bmatrix}, \tag{11}$$

where

$$\begin{aligned}
\Xi_{13} &:= C\otimes[(R+\mu^{-1}E_b^TE_b)KC+B_\varepsilon^TP_\varepsilon+\mu^{-1}E_b^TE_a],\\
\Xi_{21} &:= I_n\otimes(A_\varepsilon+B_\varepsilon KC+\mu D_\varepsilon D_\varepsilon^TP_\varepsilon)^T+(A_\varepsilon+B_\varepsilon KC+\mu D_\varepsilon D_\varepsilon^TP_\varepsilon)^T\otimes I_n,\\
\Xi_{22} &:= (\mu^{-1}E_a^TE_b+P_\varepsilon B_\varepsilon+\mu^{-1}C^TK^TE_b^TE_b+C^TK^TR)\otimes C^T+C^T\otimes(\mu^{-1}E_a^TE_b+P_\varepsilon B_\varepsilon+\mu^{-1}C^TK^TE_b^TE_b+C^TK^TR),\\
\Xi_{33} &:= I_n\otimes(A_\varepsilon+B_\varepsilon KC+\mu D_\varepsilon D_\varepsilon^TP_\varepsilon)+(A_\varepsilon+B_\varepsilon KC+\mu D_\varepsilon D_\varepsilon^TP_\varepsilon)\otimes I_n.
\end{aligned}$$

$$\begin{aligned}
&+\mu^{-1}C^TK^{(i)T}E_b^TE_b+C^TK^{(i)T}R)K^{(i+1)}C\\
&-C^TK^{(i)T}B_\varepsilon^TP_\varepsilon^{(i)}-P_\varepsilon^{(i)}B_\varepsilon K^{(i)}C-\mu P_\varepsilon^{(i)}D_\varepsilon D_\varepsilon^TP_\varepsilon^{(i)}\\
&-C^TK^{(i)T}(\mu^{-1}E_b^TE_b+R)K^{(i)}C\\
&+Q+\mu^{-1}E_a^TE_a=0, \qquad (14b)\\
&\mu D_\varepsilon D_\varepsilon^TP_\varepsilon^{(i+1)}S^{(i)}+\mu S^{(i)}P_\varepsilon^{(i+1)}D_\varepsilon D_\varepsilon^T\\
&+B_\varepsilon K^{(i+1)}CS^{(i)}+S^{(i)}C^TK^{(i+1)T}B_\varepsilon^T\\
&+(A_\varepsilon+B_\varepsilon K^{(i)}C+\mu D_\varepsilon D_\varepsilon^TP_\varepsilon^{(i)})S^{(i+1)}\\
&+S^{(i+1)}(A_\varepsilon+B_\varepsilon K^{(i)}C+\mu D_\varepsilon D_\varepsilon^TP_\varepsilon^{(i)})^T\\
&-B_\varepsilon K^{(i)}CS^{(i)}-S^{(i)}C^TK^{(i)T}B_\varepsilon^T\\
&-\mu D_\varepsilon D_\varepsilon^TP_\varepsilon^{(i)}S^{(i)}-\mu S^{(i)}P_\varepsilon^{(i)}D_\varepsilon D_\varepsilon^T+I_n=0, \qquad (14c)
\end{aligned}$$

where

$$P_\varepsilon^{(i)}=\begin{bmatrix} P_{11}^{(i)} & \varepsilon P_{21}^{(i)T} \\ \varepsilon P_{21}^{(i)} & \varepsilon P_{22}^{(i)} \end{bmatrix},\ S^{(i)}=\begin{bmatrix} S_{11}^{(i)} & S_{21}^{(i)T} \\ S_{21}^{(i)} & S_{22}^{(i)} \end{bmatrix},$$

and the initial condition $K^{(0)}$ is chosen such that the closed-loop uncertain SPS is stable. Moreover, $P_\varepsilon^{(0)}$ and $S^{(0)}$ satisfies the following algebraic Riccati and Lyapunov equations, respectively.

$$\begin{aligned}
&(A_\varepsilon+B_\varepsilon K^{(0)}C)^TP_\varepsilon^{(0)}+P_\varepsilon^{(0)}(A_\varepsilon+B_\varepsilon K^{(0)}C)\\
&\quad+\mu P_\varepsilon^{(0)}D_\varepsilon D_\varepsilon^TP_\varepsilon^{(0)}\\
&\quad+\mu^{-1}(E_a+E_bK^{(0)}C)^T(E_a+E_bK^{(0)}C)\\
&\quad+C^TK^{(0)T}RK^{(0)}C+Q=0,\\
&(A_\varepsilon+B_\varepsilon K^{(0)}C+\mu D_\varepsilon D_\varepsilon^TP_\varepsilon^{(0)})S^{(0)}\\
&\quad+S^{(0)}(A_\varepsilon+B_\varepsilon K^{(0)}C+\mu D_\varepsilon D_\varepsilon^TP_\varepsilon^{(0)})^T+I_n=0.
\end{aligned}$$

The new algorithm (14) can be constructed by setting $P_\varepsilon^{(i+1)} = P_\varepsilon^{(i)} + \Delta P_\varepsilon^{(i)}$, $S^{(i+1)} = S^{(i)} + \Delta S^{(i)}$ and $K^{(i+1)} = K^{(i)} + \Delta K^{(i)}$, and neglecting $O(\Delta^2)$ term. The following theorem indicates that the algorithm (14) is Newton's method.

Theorem 2: Suppose that there exist a solution to the CALRE (7). It can be obtained by performing the algorithm (14) which is equal to Newton's method.

Proof: Taking the vec-operator transformation on both sides of (7) results in

$$\begin{aligned}
&\mathrm{vec}F_1(P_\varepsilon^{(i)},\ K^{(i)},\ S^{(i)})\\
&=[(CS)\otimes B_\varepsilon^T]\mathrm{vec}P_\varepsilon^{(i)}\\
&\quad+[(CSC^T)\otimes(R+\mu^{-1}E_b^TE_b)]\mathrm{vec}K^{(i)}\\
&\quad+\Big[C\otimes[(R+\mu^{-1}E_b^TE_b)K^{(i)}C+B_\varepsilon^TP^{(i)}\\
&\quad+\mu^{-1}E_b^TE_a]\Big]\mathrm{vec}S^{(i)}\\
&\quad-\mathrm{vec}\Big[(R+\mu^{-1}E_b^TE_b)K^{(i)}CS^{(i)}C^T\\
&\quad+B_\varepsilon^TP_\varepsilon^{(i)}S^{(i)}C^T\Big], \qquad (15a)\\
&\mathrm{vec}F_2(P_\varepsilon^{(i)},\ K^{(i)},\ S^{(i)})\\
&=\Big[I_n\otimes(A_\varepsilon+B_\varepsilon K^{(i)}C+\mu D_\varepsilon D_\varepsilon^TP_\varepsilon^{(i)})^T\Big]\mathrm{vec}P_\varepsilon^{(i)}\\
&\quad+\Big[(A_\varepsilon+B_\varepsilon K^{(i)}C+\mu D_\varepsilon D_\varepsilon^TP_\varepsilon^{(i)})^T\otimes I_n\Big]\mathrm{vec}P_\varepsilon^{(i)}\\
&\quad+\Big[(\mu^{-1}E_a^TE_b+P_\varepsilon^{(i)}B_\varepsilon+\mu^{-1}C^TK^{(i)T}E_b^TE_b\\
&\quad+C^TK^{(i)T}R)\otimes(C^TU_{mr})\Big]\mathrm{vec}K^{(i)}\\
&\quad+\Big[C^T\otimes(\mu^{-1}E_a^TE_b+P_\varepsilon^{(i)}B_\varepsilon+\mu^{-1}C^TK^{(i)T}E_b^TE_b\\
&\quad+C^TK^{(i)T}R)\Big]\mathrm{vec}K^{(i)}\\
&\quad-\mathrm{vec}\Big[C^TK^{(i)T}B_\varepsilon^TP_\varepsilon^{(i)}+P_\varepsilon^{(i)}B_\varepsilon K^{(i)}C\\
&\quad+\mu P_\varepsilon^{(i)}D_\varepsilon D_\varepsilon^TP_\varepsilon^{(i)}+C^TK^{(i)T}(\mu^{-1}E_b^TE_b+R)K^{(i)}C\\
&\quad-Q-\mu^{-1}E_a^TE_a\Big], \qquad (15b)\\
&\mathrm{vec}F_3(P_\varepsilon^{(i)},\ K^{(i)},\ S^{(i)})\\
&=\Big[S^{(i)}\otimes(\mu D_\varepsilon D_\varepsilon^T)\Big]\mathrm{vec}P_\varepsilon^{(i)}+\Big[(\mu D_\varepsilon D_\varepsilon^T)\otimes S^{(i)}\Big]\mathrm{vec}P_\varepsilon^{(i)}\\
&\quad+\Big[B_\varepsilon\otimes(S^{(i)}C^TU_{mr})\Big]\mathrm{vec}K^{(i)}\\
&\quad+\Big[(S^{(i)}C^T)\otimes B_\varepsilon\Big]\mathrm{vec}K^{(i)}\\
&\quad+\Big[I_n\otimes(A_\varepsilon+B_\varepsilon K^{(i)}C+\mu D_\varepsilon D_\varepsilon^TP_\varepsilon^{(i)})\Big]\mathrm{vec}S^{(i)}\\
&\quad+\Big[(A_\varepsilon+B_\varepsilon K^{(i)}C+\mu D_\varepsilon D_\varepsilon^TP_\varepsilon^{(i)})\otimes I_n\Big]\mathrm{vec}S^{(i)}\\
&\quad-\mathrm{vec}\Big[B_\varepsilon K^{(i)}CS^{(i)}+S^{(i)}C^TK^{(i)T}B_\varepsilon^T\\
&\quad+\mu D_\varepsilon D_\varepsilon^TP_\varepsilon^{(i)}S^{(i)}+\mu S^{(i)}P_\varepsilon^{(i)}D_\varepsilon D_\varepsilon^T-I_n\Big]. \qquad (15c)
\end{aligned}$$

Moreover, taking the vec-operator transformation on both sides of (14) results in

$$\begin{aligned}
&[(CS)\otimes B_\varepsilon^T]\mathrm{vec}P_\varepsilon^{(i+1)}\\
&\quad+[(CSC^T)\otimes(R+\mu^{-1}E_b^TE_b)]\mathrm{vec}K^{(i+1)}\\
&\quad+\Big[C\otimes[(R+\mu^{-1}E_b^TE_b)K^{(i)}C+B_\varepsilon^TP^{(i)}\\
&\quad+\mu^{-1}E_b^TE_a]\Big]\mathrm{vec}S^{(i+1)}\\
&\quad-\mathrm{vec}\Big[(R+\mu^{-1}E_b^TE_b)K^{(i)}CS^{(i)}C^T
\end{aligned}$$

$$+B_\varepsilon^T P_\varepsilon^{(i)} S^{(i)} C^T\Big] = 0, \quad (16a)$$

$$\Big[I_n \otimes (A_\varepsilon + B_\varepsilon K^{(i)} C + \mu D_\varepsilon D_\varepsilon^T P_\varepsilon^{(i)})^T\Big] \mathrm{vec} P_\varepsilon^{(i+1)}$$
$$+\Big[(A_\varepsilon + B_\varepsilon K^{(i)} C + \mu D_\varepsilon D_\varepsilon^T P_\varepsilon^{(i)})^T \otimes I_n\Big] \mathrm{vec} P_\varepsilon^{(i+1)}$$
$$+\Big[(\mu^{-1} E_a^T E_b + P_\varepsilon^{(i)} B_\varepsilon + \mu^{-1} C^T K^{(i)T} E_b^T E_b$$
$$+C^T K^{(i)T} R) \otimes (C^T U_{mr})\Big] \mathrm{vec} K^{(i+1)}$$
$$+\Big[C^T \otimes (\mu^{-1} E_a^T E_b + P_\varepsilon^{(i)} B_\varepsilon + \mu^{-1} C^T K^{(i)T} E_b^T E_b$$
$$+C^T K^{(i)T} R)\Big] \mathrm{vec} K^{(i+1)}$$
$$-\mathrm{vec}\Big[C^T K^{(i)T} B_\varepsilon^T P_\varepsilon^{(i)} + P_\varepsilon^{(i)} B_\varepsilon K^{(i)} C$$
$$+\mu P_\varepsilon^{(i)} D_\varepsilon D_\varepsilon^T P_\varepsilon^{(i)} + C^T K^{(i)T} (\mu^{-1} E_b^T E_b + R) K^{(i)} C$$
$$-Q - \mu^{-1} E_a^T E_a\Big] = 0, \quad (16b)$$

$$\Big[S^{(i)} \otimes (\mu D_\varepsilon D_\varepsilon^T)\Big] \mathrm{vec} P_\varepsilon^{(i+1)}$$
$$+\Big[(\mu D_\varepsilon D_\varepsilon^T) \otimes S^{(i)}\Big] \mathrm{vec} P_\varepsilon^{(i+1)}$$
$$+\Big[B_\varepsilon \otimes (S^{(i)} C^T U_{mr})\Big] \mathrm{vec} K^{(i+1)}$$
$$+\Big[(S^{(i)} C^T) \otimes B_\varepsilon\Big] \mathrm{vec} K^{(i+1)}$$
$$+\Big[I_n \otimes (A_\varepsilon + B_\varepsilon K^{(i)} C + \mu D_\varepsilon D_\varepsilon^T P_\varepsilon^{(i)})\Big] \mathrm{vec} S^{(i+1)}$$
$$+\Big[(A_\varepsilon + B_\varepsilon K^{(i)} C + \mu D_\varepsilon D_\varepsilon^T P_\varepsilon^{(i)}) \otimes I_n\Big] \mathrm{vec} S^{(i+1)}$$
$$-\mathrm{vec}\Big[B_\varepsilon K^{(i)} C S^{(i)} + S^{(i)} C^T K^{(i)T} B_\varepsilon^T$$
$$+\mu D_\varepsilon D_\varepsilon^T P_\varepsilon^{(i)} S^{(i)} + \mu S^{(i)} P_\varepsilon^{(i)} D_\varepsilon D_\varepsilon^T - I_n\Big] = 0. \quad (16c)$$

Subtracting (15) from (16) and using (11), it is easy to verify that

$$\begin{bmatrix} \mathrm{vec} P_\varepsilon^{(i+1)} \\ \mathrm{vec} K^{(i+1)} \\ \mathrm{vec} S^{(i+1)} \end{bmatrix} = \begin{bmatrix} \mathrm{vec} P_\varepsilon^{(i)} \\ \mathrm{vec} K^{(i)} \\ \mathrm{vec} S^{(i)} \end{bmatrix} - \mathcal{J}(\varepsilon,\ P_\varepsilon^{(i)},\ K^{(i)},\ S^{(i)})^{-1} \times \begin{bmatrix} \mathrm{vec} F_1(P_\varepsilon^{(i)},\ K^{(i)},\ S^{(i)}) \\ \mathrm{vec} F_2(P_\varepsilon^{(i)},\ K^{(i)},\ S^{(i)}) \\ \mathrm{vec} F_3(P_\varepsilon^{(i)},\ K^{(i)},\ S^{(i)}) \end{bmatrix}, \quad (17)$$

which is the desired result. ■

Newton's method is well-known and is widely used to find a solution of algebraic nonlinear equations. Its local convergence properties are well understood [9]. Although Newton's method guarantees the local convergence, it may not converge to the required solution if the initial condition is not suitable. Particularly, the proposed algorithm with the initial condition (14) may not converge to the required solutions because the local quadratic convergence is not guaranteed. In order to guarantee the local quadratic convergence, the initial condition is chosen as follows that is different from (14).

$$P_\varepsilon^{(0)} = \hat{P}_\varepsilon := \begin{bmatrix} \bar{P}_{11} & \varepsilon \bar{P}_{21}^T \\ \varepsilon \bar{P}_{21} & \varepsilon \bar{P}_{22} \end{bmatrix},\ K^{(0)} = \bar{K},\ S^{(0)} = \bar{S}.$$

It is highly expected that the proposed algorithm can converge to the adequate solutions because the initial conditions are close to the exact solutions with the structure of (13) under the sufficiently small parameter ε.

Theorem 3: Assume that the conditions of Theorem 1 hold. Then, there exists a small σ^* such that for all $\varepsilon \in (0,\ \sigma^*)$, Newton's method (14) converges to the exact solution of P_ε^*, K^* and S^* with the rate of the quadratic convergence. Moreover, the convergence solutions P_ε^*, K^* and S^* are unique solution of the CALRE (7) in the neighborhood of the initial condition $P_\varepsilon^{(0)} = \hat{P}_\varepsilon$, $K^{(0)} = \bar{K}$ and $S^{(0)} = \bar{S}$, respectively. That is, the following relations are satisfied.

$$\|P_\varepsilon^{(i)} - P_\varepsilon^*\| \le O(\varepsilon^{2^i}), \quad (18a)$$
$$\|K^{(i)} - K^*\| \le O(\varepsilon^{2^i}), \quad (18b)$$
$$\|S^{(i)} - S^*\| \le O(\varepsilon^{2^i}),\ i = 0,\ 1,\ \ldots\ . \quad (18c)$$

Proof: The proof of this theorem can be done by using Newton-Kantorovich theorem [9]. It is immediately obtained from the equation (11) that there exists the positive scalar constant $\mathcal{L}$ such that for any P_ε^a, K^a, S^a, P_ε^b, K^b and S^b,

$$\|\mathcal{J}(\varepsilon,\ P_\varepsilon^a,\ K^a,\ S^a) - \mathcal{J}(\varepsilon,\ P_\varepsilon^b,\ K^b,\ S^b)\| \le \mathcal{L}\|(P_\varepsilon^a,\ K^a,\ S^a) - (P_\varepsilon^b,\ K^b,\ S^b)\|. \quad (19)$$

Moreover, using (13), we get

$$\mathcal{J}(\varepsilon,\ P_\varepsilon^{(0)},\ K^{(0)},\ S^{(0)}) = \bar{\mathcal{J}}(0,\ \bar{P},\ \bar{K},\ \bar{S}) + O(\varepsilon). \quad (20)$$

Hence, it follows that $\mathcal{J}(\varepsilon,\ P_\varepsilon^{(0)},\ K^{(0)},\ S^{(0)})$ is nonsingular under the condition (12) for sufficiently small ε. Therefore, there exists β such that $\beta = \|\mathcal{J}(\varepsilon,\ P_\varepsilon^{(0)},\ K^{(0)},\ S^{(0)})^{-1}\|$. On the other hand, since $F_k(P_\varepsilon^{(0)},\ K^{(0)},\ S^{(0)}) = O(\varepsilon)$, there exists η such that $\eta = \|[\mathcal{J}(\varepsilon,\ P_\varepsilon^{(0)},\ K^{(0)},\ S^{(0)})]^{-1}\| \cdot \|F_k(P_\varepsilon^{(0)},\ K^{(0)},\ S^{(0)})\| = O(\varepsilon)$. Thus, there exists θ such that $\theta = \beta\eta\mathcal{L} < 2^{-1}$ because $\eta = O(\varepsilon)$. Finally, using the Newton-Kantorovich theorem, we can show that P_ε^*, K^* and S^* are the unique solution in the subset. Moreover, the error estimate is given by (18). ■

It should be noted that the proposed algorithm converges to the same local minima compared with the other algorithm mentioned [8]. Moreover, since the CALRE (7) is nonlinear and it is considered to be very difficult to find a good solution, it may be noted that the optimality or adequacy of the obtained local solution need to be considered for the computed solutions.

V. NUMERICAL EXAMPLE

In order to demonstrate the efficiency of the proposed algorithm, an illustrative example is given. The system matrices are given below.

$$A_{11} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix},\ A_{12} = \begin{bmatrix} 0.25 \\ 0.15 \end{bmatrix},$$
$$A_{21} = \begin{bmatrix} 1 & 1.2 \end{bmatrix},\ A_{22} = \begin{bmatrix} -0.1 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 \end{bmatrix},\ C_1 = \begin{bmatrix} 0 & 1 \end{bmatrix},\ C_2 = \begin{bmatrix} 1 \end{bmatrix},$$
$$D_1 = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix},\ D_2 = \begin{bmatrix} 0.01 \end{bmatrix},$$
$$E_{a1} = \begin{bmatrix} 0 & 0 \end{bmatrix},\ E_{a2} = \begin{bmatrix} 1 \end{bmatrix},\ E_b = \begin{bmatrix} 1 \end{bmatrix}.$$

Table 1.

k	$\mathcal{E}(1.0e-01)$	$\mathcal{E}(1.0e-02)$	$\mathcal{E}(1.0e-03)$	$\mathcal{E}(1.0e-04)$
0	3.6902	4.2896	4.3573	4.3642
1	4.5250	$3.7930e+01$	$3.6964e+02$	$3.6864e+03$
2	1.9596	$2.1000e+01$	$2.1015e+02$	$2.1015e+03$
3	$3.3736e-01$	6.9416	$7.4190e+01$	$7.4679e+02$
4	$5.9030e-03$	$6.3625e-01$	8.2120	$8.4264e+01$
5	$9.3995e-07$	$3.2872e-03$	$6.5032e-02$	$6.9739e-01$
6	$1.7391e-14$	$5.4174e-08$	$2.5559e-06$	$2.9989e-05$
7	–	$8.6068e-14$	$1.2679e-12$	$1.3269e-11$

Table 2. Number of iterations

ε	Newton's Method	Existing Method [8]
$1.0e-01$	6	36
$1.0e-02$	7	70
$1.0e-03$	7	81
$1.0e-04$	7	96

Table 3. CPU Time [sec]

ε	Newton's Method	Existing Method [8]
$1.0e-01$	0.0290	0.0991
$1.0e-02$	0.0310	0.1933
$1.0e-03$	0.0301	0.1882
$1.0e-04$	0.0311	0.2523

In order to verify the exactitude of the solution, the remainder per iteration is computed for several values ε by substituting $P_\varepsilon^{(i)}$, $K^{(i)}$ and $S^{(i)}$ into the CALRE (7). It should be noted that μ is chosen as one. Table 1 shows the error per iterations for various values of the parameter ε. In the case of $\varepsilon = 1.0e-01$, it should be noted that the algorithm (14) converges to the exact solution with accuracy of $\mathcal{E}(\varepsilon) < 1.0e-10$ after six iterations, where $\mathcal{E}(\varepsilon) := \sum_{k=1}^{3} \|F_k(P_\varepsilon^{(i)},\ K^{(i)},\ S^{(i)})\|$. Hence, it can be seen from Table 1 that the algorithm (14) attains the quadratic convergence.

The required iterations of the proposed algorithm (14) versus the existing algorithm [8] are presented in Table 2. It can be seen from Table 2 that the proposed algorithm (14) succeed in reducing the iterations compared with the existing algorithm [8] for different values of ε. Hence, the resulting algorithm of this paper is very reliable.

In Table 3, the results of the CPU time are given when the new method versus the existing algorithm [8] are carried out. The CPU time represents the average based on the computations of ten runs. From Table 3, it is shown that we succeed in reducing the average CPU time of about 18.4% compared with the existing result [8]. Particularly, when $\varepsilon = 1.0e-04$, the average CPU time could reduce to 12.3%. Therefore, the proposed algorithm is quite useful from the numerical point of view because the CPU time can be reduced dramatically.

Now, we choose as $\varepsilon = 1.0e-02$ to design the controller. It is easy to verify that the exact cost bound $\min\ \mathrm{Trace}\, P_\varepsilon = 3.9899$ is obtained for $\mu^* = 3.9301e+01$. The obtained guaranteed cost output feedback control (8) under $\mu^* = 3.9301e+01$ is as follows.

$$u = Ky = -3.9184y, \tag{21}$$

where

$$P_\varepsilon = \begin{bmatrix} 2.6006 & 9.5497e-01 & 1.1370e-02 \\ 9.5497e-01 & 1.3485 & 4.9298e-02 \\ 1.1370e-02 & 4.9298e-02 & 4.0779e-02 \end{bmatrix},$$

$$S = \begin{bmatrix} 5.0875e-01 & 1.3840e-01 & 3.4992e-02 \\ 1.3840e-01 & 2.8504e-01 & -1.5604e-01 \\ 3.4992e-02 & -1.5604e-01 & 1.1509e-01 \end{bmatrix}.$$

It is worth pointing out that the proposed guaranteed cost controller is numerically attractive because fast convergence speed is attained. Moreover, the numerical efficiency of the proposed algorithm will be claimed as a general result under the appropriate initial guess.

VI. CONCLUSION

The guaranteed cost control problem via the static output feedback for the uncertain SPS has been studied. The main contribution of this paper is to provide the new algorithm for solving the CALRE. Since the new algorithm is based on Newton's method, the quadratic local convergence is guaranteed. As a result, the CALRE can be solved quickly. Moreover, we have shown the success of reducing the average CPU time drastically by demonstrating the numerical example.

When the size of the system increases, the algorithm may not work well as the same size of the reduced-order dimension of the SPS. This problem will be investigated in the near future.

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