15th Triennial World Congress, Barcelona, Spain

RECURSIVE APPROACH OF H_∞ OPTIMAL FILTERING FOR MULTIPARAMETER SINGULARLY PERTURBED SYSTEMS

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Abstract: In this paper, we study the H_∞ optimal filtering for multiparameter singularly perturbed system (MSPS). In order to obtain the solution, we must solve the multiparameter algebraic Riccati equations (MARE) with indefinite sign quadratic term. First, the existence of a unique and bounded solution of such MARE is newly proven. The main results in this paper are to propose a new recursive algorithm for solving the MARE and to find sufficient conditions regarding the convergence of our proposed algorithm. Using the recursive algorithm, we show that the solution of the MARE converges to a positive semi–definite stabilizing solution with the rate of convergence of $O(\|\mu\|^{i+1})$.

Keywords: Multiparameter singularly perturbed system (MSPS), Multiparameter algebraic Riccati equations (MARE), H_∞ optimal filtering, Recursive algorithm

1. INTRODUCTION

Filtering problems for the multiparameter singularly perturbed system (MSPS) have been investigated extensively (see e.g., Coumarbatch and Gajić, 2000 and reference therein). The multimodeling problems arise in large scale dynamic systems. For example, the multimodel situation in practice is illustrated by the passenger car model (Coumarbatch and Gajić, 2000). In order to obtain the optimal solution to the multimodeling problems, we must solve the multiparameter algebraic Riccati equation (MARE). Various reliable approaches to the theory of the algebraic Riccati equation (ARE) have been well documented in many literatures (see e.g., Laub, 1979). One of the approaches is the invariant subspace approach which is based on the Hamiltonian matrix (Laub, 1979). However, there is no guarantee of symmetry for the computed solution if the ARE is ill-conditioned (Laub, 1979). Note that it is very difficult to solve the MARE due to high dimension and numerical stiffness (Coumarbatch and Gajić, 2000).

A popular approach to deal with the MSPS is the two–time–scale design method (see e.g., Khalil and Kokotović, 1979; Kokotović et al., 1986). However, it is known from Coumarbatch and Gajić (2000) that an $O(\|\mu\|)$ (where $\mu = \left[\ \varepsilon_1\ \varepsilon_2\ \right]$) accuracy is very often not sufficient. Recently, the exact slow–fast decomposition method for solving the MARE of the MSPS has been proposed (Coumarbatch and Gajić, 2000). However, these results are restricted to the MSPS such that the Hamiltonian matrices for the fast subsystems have no eigenvalues in common (Assumption 5, Coumarbatch and Gajić, 2000). More recently, in Mukaidani et al., (2001), the recursive algorithm (see e.g., Gajić et al., 1990) for the solution of the regulator type MARE which has the positive semidefinite sign quadratic term has been proposed. However, the recursive algorithm for solving the filter type MARE which has the indefinite sign quadratic term appearing in H_∞ filtering problems has not been investigated.

In this paper, we study the H_∞ optimal filtering for the MSPS. The advantage of the H_∞ filter over the standard Kalman filter is that former does not require knowledge of the system and measurement noise intensity matrices. The difficulty encountered with the H_∞ filter for the MSPS is that the MARE contains an indefinite sign quadratic term. Therefore, we first investigate the uniqueness and boundedness of the solution to such MARE and establish its asymptotic structure. The proof of the existence of the solution to the MARE with asymptotic expansion is obtained by an implicit function theorem (Gajić et al., 1990). The main contribution of this paper is to propose a new recursive algorithm for solving the MARE and to find the sufficient conditions regarding the convergence of the recursive algorithm by using the reduced–order ARE. It is important to note that the sufficient conditions derived here are independent of the small perturbation parameter μ. We also prove that the solution of the MARE converges to a positive semi–definite stabilizing solution with the rate of convergence of $O(\|\mu\|^{i+1})$, where i is the iteration number. As another important feature, we do not assume here that the Hamiltonian matrices $Z_{jj},\ j = 1,\ 2$ for the fast fast subsystems have no eigenvalues in common. Thus, our new results are applicable to more realistic MSPS.

2. H_∞ OPTIMAL FILTERING

We consider the linear time–invariant MSPS

$$\begin{aligned} \dot{x}_0 &= A_{00}x_0 + A_{01}x_1 + A_{02}x_2 \\ &\quad + D_{01}w_1 + D_{02}w_2, && \text{(1a)} \\ \varepsilon_1\dot{x}_1 &= A_{10}x_0 + A_{11}x_1 + D_{11}w_1, && \text{(1b)} \\ \varepsilon_2\dot{x}_2 &= A_{20}x_0 + A_{22}x_2 + D_{22}w_2, && \text{(1c)} \end{aligned}$$

with

$$y_j = C_{j0}x_0 + C_{jj}x_j + v_j,\ j = 1,\ 2, \qquad (2)$$

where $x_j \in \mathbf{R}^{n_j},\ j = 0,\ 1,\ 2$ are state vectors, $y_j \in \mathbf{R}^{p_j},\ j = 0,\ 1,\ 2$ are system measurements, $w_j \in \mathbf{R}^{q_j},\ j = 1,\ 2$ and $v_j \in \mathbf{R}^{r_j},\ j = 1,\ 2$ are system and measurement disturbances, respectively. All the matrices are constant matrices of appropriate dimensions.

ε_1 and ε_2 are two small positive singular parameters of the same order of magnitude such that

$$0 < k_1 \le \alpha \equiv \frac{\varepsilon_1}{\varepsilon_2} \le k_2 < \infty. \qquad (3)$$

That is, we assume that the ratio of ε_1 and ε_2 is bounded by some positive constants $k_j,\ j = 1,\ 2$.

In this paper we design a filter to estimate system states x_j. The states to be estimated are given by a linear combination

$$z_j = G_{j0}x_0 + G_{jj}x_j + v_j,\ j = 1,\ 2, \qquad (4)$$

where $z_j \in \mathbf{R}^{s_j},\ j = 1,\ 2$. The estimation problem is to obtain an estimate $\hat{z}_i$ of z_j using the measurements y_j (Lim and Gajić, 2000). The measure of the infinite horizon estimation problem is defined as a disturbance attenuation function

$$J = \int_0^\infty \|z - \hat{z}\|_R^2 dt \cdot \left\{ \int_0^\infty (\|w\|_{W^{-1}}^2 + \|v\|)dt \right\}^{-1}, \qquad (5)$$

where $z = \begin{bmatrix} z_1^T & z_2^T \end{bmatrix}^T$, $\hat{z} = \begin{bmatrix} \hat{z}_1^T & \hat{z}_2^T \end{bmatrix}^T$, $w = \begin{bmatrix} w_1^T & w_2^T \end{bmatrix}^T$ and $v = \begin{bmatrix} v_1^T & v_2^T \end{bmatrix}^T$, and where $R \ge 0$ and $W > 0$ are weighting matrices to be chosen by designer. The H_∞ filter is ensure that the energy gain from the disturbances to estimation errors $z - \hat{z}$ is less that a attenuation level γ^2. That is,

$$\sup_{w,\ v} J < \gamma^2. \qquad (6)$$

The H_∞ filter of (1) and (2) is given by (Lim and Gajić, 2000)

$$\begin{aligned} \dot{\xi}_0 &= A_{00}\xi_0 + A_{01}\xi_1 + A_{02}\xi_2 \\ &\quad + F_{01}\eta_1 + F_{02}\eta_2, && \text{(7a)} \\ \varepsilon_1\dot{\xi}_1 &= A_{10}\xi_0 + A_{11}\xi_1 + F_{11}\eta_1 + F_{12}\eta_2, && \text{(7b)} \\ \varepsilon_2\dot{\xi}_2 &= A_{20}\xi_0 + A_{22}\xi_2 + F_{21}\eta_1 + F_{22}\eta_2, && \text{(7c)} \\ \eta_j &= y_j - C_{j0}x_0 - C_{jj}x_j,\ j = 1,\ 2, && \text{(7d)} \end{aligned}$$

where the filter gain F_{0j} and $F_{jj},\ j = 1,\ 2$ are obtained from

$$F = \begin{bmatrix} F_{01} & F_{02} \\ F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} = CY_e, \qquad (8)$$

where Y_e satisfies the MARE

$$A_eY_e + Y_eA_e^T - Y_eVY_e + U_e = 0, \qquad (9)$$

with

$$A_e = \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ \varepsilon_1^{-1}A_{10} & \varepsilon_1^{-1}A_{11} & 0 \\ \varepsilon_2^{-1}A_{20} & 0 & \varepsilon_2^{-1}A_{22} \end{bmatrix},$$

$$D_e = \begin{bmatrix} D_{01} & D_{02} \\ \varepsilon_1^{-1} D_{11} & 0 \\ 0 & \varepsilon_2^{-1} D_{22} \end{bmatrix},$$

$$C = \begin{bmatrix} C_{01} & C_{11} & 0 \\ C_{02} & 0 & C_{22} \end{bmatrix}, \; G = \begin{bmatrix} G_{01} & G_{11} & 0 \\ G_{02} & 0 & G_{22} \end{bmatrix},$$

$$V = C^T C - \gamma^{-2} G^T R G = \begin{bmatrix} V_{00} & V_{01} & V_{02} \\ V_{01}^T & V_{11} & 0 \\ V_{02}^T & 0 & V_{22} \end{bmatrix},$$

$$U_e = D_e W D_e^T = \begin{bmatrix} U_{00} & \varepsilon_1^{-1} U_{01} & \varepsilon_2^{-1} U_{02} \\ \varepsilon_1^{-1} U_{01}^T & \varepsilon_1^{-2} U_{11} & 0 \\ \varepsilon_2^{-1} U_{02}^T & 0 & \varepsilon_2^{-2} U_{22} \end{bmatrix}.$$

Since the matrices A_e and D_e contain terms of order ε_j^{-1}, $j = 1, 2$, a solution Y_e of (9), if it exists, must contain terms of order ε_j. Taking into consideration of this fact, we look for a solutions Y_e to the MARE (9) with the structure

$$Y_e = \begin{bmatrix} Y_{00} & Y_{10}^T & Y_{20}^T \\ Y_{10} & \varepsilon_1^{-1} Y_{11} & \sqrt{\varepsilon_1 \varepsilon_2}^{-1} Y_{21}^T \\ Y_{20} & \sqrt{\varepsilon_1 \varepsilon_2}^{-1} Y_{21} & \varepsilon_2^{-1} Y_{22} \end{bmatrix} \in \mathbf{R}^{N \times N},$$

where $N = n_0 + n_1 + n_2$, $Y_{00} = Y_{00}^T$, $Y_{11} = Y_{11}^T$, $Y_{22} = Y_{22}^T$.

If the sign of the MARE (9) is positive semidefinite, then the equation (9) is known as the optimal Kalman filter, appearing in the multimodeling (Gajić et al., 1990). However, we do not assume in this paper that the sign of the MARE (9) is positive semidefinite. That is, no assumption is made on the definiteness of V.

In order to avoid the ill–conditioned due to the large parameter ε_j^{-1} which is included in the MARE (9), we introduce the following useful lemma.

Lemma 1: The MARE (9) is equivalent to the following generalized multiparameter algebraic Riccati equation (GMARE) (10a)

$$\mathcal{F}(Y) := AY^T + YA^T - YVY^T + U = 0, \quad (10a)$$
$$Y_e = Y^T \Phi_e^{-1} = \Phi_e^{-1} Y, \quad (10b)$$

where

$$\Phi_e = \begin{bmatrix} I_{n_0} & 0 & 0 \\ 0 & \varepsilon_1 I_{n_1} & 0 \\ 0 & 0 & \varepsilon_2 I_{n_2} \end{bmatrix}, \; A = \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & 0 \\ A_{20} & 0 & A_{22} \end{bmatrix},$$

$$U = \begin{bmatrix} U_{00} & U_{01} & U_{02} \\ U_{01}^T & U_{11} & 0 \\ U_{02}^T & 0 & U_{22} \end{bmatrix},$$

$$Y = \begin{bmatrix} Y_{00} & Y_{10}^T & Y_{20}^T \\ \varepsilon_1 Y_{10} & Y_{11} & \sqrt{\alpha} Y_{21}^T \\ \varepsilon_2 Y_{20} & \sqrt{\alpha}^{-1} Y_{21} & Y_{22} \end{bmatrix}.$$

Proof: Firstly, by direct calculation we verify that $Y_e = \Phi_e^{-1} Y$. Secondly, it is easy to verify that $A = \Phi_e A_e$, $U_e = \Phi_e^{-1} U \Phi_e^{-1}$. Hence,

$$A_e Y_e = \Phi_e^{-1} A Y^T \Phi_e^{-1}.$$

By using the similar calculation, we can immediately rewrite (9) as (10a). □

3. THE MARE

Firstly, it is assumed that the limit of α exists as ε_1 and ε_2 tend to zero (see e.g., Khalil and Kokotović, 1979; Gajić, 1988), that is

$$\bar{\alpha} = \lim_{\substack{\varepsilon_1 \to +0 \\ \varepsilon_2 \to +0}} \alpha. \quad (11)$$

Let $\bar{Y}_{00}, \bar{Y}_{10}, \bar{Y}_{20}, \bar{Y}_{11}, \bar{Y}_{21}$ and $\bar{Y}_{22}$ be the limiting solutions of the equations (10) as $\varepsilon_j \to +0, j = 1, 2$, then we obtain the following zeroth order equations by partitioning the GMARE (10a).

$$A_s \bar{Y}_{00} + \bar{Y}_{00} A_s^T - \bar{Y}_{00} V_s \bar{Y}_{00} + U_s = 0, \quad (12a)$$
$$\bar{Y}_{j0}^T = \bar{Y}_{00} S_{0j} - R_{0j}, \; j = 1, 2, \quad (12b)$$
$$A_{jj} \bar{Y}_{jj} + \bar{Y}_{jj} A_{jj}^T - \bar{Y}_{jj} V_{jj} \bar{Y}_{jj} + U_{jj} = 0, \quad (12c)$$
$$j = 1, 2,$$

where

$$\begin{aligned} A_s &= A_{00} + A_{01} S_{01}^T + A_{02} S_{02}^T + R_{01} V_{01}^T \\ &\quad + R_{02} V_{02}^T + R_{01} V_{11} S_{01}^T + R_{02} V_{22} S_{02}^T, \\ V_s &= V_{00} + S_{01} V_{01}^T + V_{01} S_{01}^T + S_{02} V_{02}^T + V_{02} S_{02}^T \\ &\quad + S_{01} V_{11} S_{01}^T + S_{02} V_{22} S_{02}^T, \\ U_s &= U_{00} - R_{01} A_{01}^T - A_{01} R_{01}^T - R_{02} A_{02}^T \\ &\quad - A_{02} R_{02}^T - R_{01} V_{11} R_{01}^T - R_{02} V_{22} R_{02}^T, \\ S_{0j} &= -H_{j0}^T H_{jj}^{-T}, \; R_{0j} = \hat{Q}_{0j} H_{jj}^{-T}, \\ \hat{Q}_{0j} &= A_{0j} \bar{Y}_{jj} + U_{0j}, \; H_{jj} = A_{jj} - \bar{Y}_{jj} V_{jj}, \\ H_{00} &= A_{00} - \bar{Y}_{00} V_{00} - \bar{Y}_{10}^T V_{01}^T - \bar{Y}_{20}^T V_{02}^T, \\ H_{j0} &= A_{j0} - \bar{Y}_{jj} V_{0j}^T, \; j = 1, 2, \\ H_{0j} &= A_{0j} - \bar{Y}_{00} V_{0j} - \bar{Y}_{j0}^T V_{jj}. \end{aligned}$$

Now, let us define the sets as $\Gamma_{jf} := \{\gamma > 0 |$ the ARE (12c) has the positive semidefinite stabilizing solutions$\}, j = 1, 2$. If we choose $\gamma_{jf} := \inf\{\gamma \; | \gamma \in \Gamma_{jf}\} < \gamma$, $A_{jj} - \bar{Y}_{jj} V_{jj}$ are stable. Hence, the parameter $\bar{\alpha}$ does not appear in (12) because $\bar{Y}_{21} = 0$. The matrices A_s, V_s and U_s do not depend on $\bar{Y}_{11}$ and $\bar{Y}_{22}$ because their matrices can be computed by using Z_{pq}, p, $q = 0, 1, 2$ which is independent of $\bar{Y}_{11}$ and $\bar{Y}_{22}$ (Coumarbatch and Gajić, 2000), that is,

$$\begin{aligned} Z_s &= Z_{00} - Z_{01} Z_{11}^{-1} Z_{10} - Z_{02} Z_{22}^{-1} Z_{20} \\ &= \begin{bmatrix} A_s^T & -V_s \\ -U_s & -A_s \end{bmatrix}, \; j = 1, 2, \end{aligned}$$

$$Z_{00} = \begin{bmatrix} A_{00}^T & -V_{00} \\ -U_{00} & -A_{00} \end{bmatrix}, \; Z_{0j} = \begin{bmatrix} A_{j0}^T & -V_{0j} \\ -U_{0j} & -A_{0j} \end{bmatrix},$$

$$Z_{j0} = \begin{bmatrix} A_{0j}^T & -V_{0j}^T \\ -U_{0j}^T & -A_{j0} \end{bmatrix}, \ Z_{jj} = \begin{bmatrix} A_{jj}^T & -V_{jj} \\ -U_{jj} & -A_{jj} \end{bmatrix}.$$

The AREs (12c) will produce the unique positive semidefinite stabilizing solution under the following condition if γ is large enough. Moreover, let us define the set as $\Gamma_s := \{\gamma > 0|$ the ARE (12a) has a positive semidefinite stabilizing solution$\}$, $\gamma_s := \inf\{\gamma \ |\gamma \in \Gamma_s\}$. As the results, for every $\gamma > \bar{\gamma} = \max\{\gamma_s, \ \gamma_{1f}, \ \gamma_{2f}\}$, the MARE (9) has the positive semidefinite stabilizing solution if $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are small enough. Then, we have the following result.

Theorem 1: If we select a parameter $\gamma > \bar{\gamma} = \max\{\gamma_s, \ \gamma_{1f}, \ \gamma_{2f}\}$, then there exist small $\tilde{\varepsilon}_1$ and $\tilde{\varepsilon}_2$ such that for all $\varepsilon_1 \in (0, \ \tilde{\varepsilon}_1)$ and $\varepsilon_2 \in (0, \ \tilde{\varepsilon}_2)$, the MARE (9) admits a solution such that Y_e is the symmetric positive semidefinite stabilizing solution, which can be written as (13).

$$Y_e = \begin{bmatrix} \bar{Y}_{00} + O(\|\mu\|) & \bar{Y}_{10}^T + O(\|\mu\|) & \bar{Y}_{20}^T + O(\|\mu\|) \\ \bar{Y}_{10} + O(\|\mu\|) & \varepsilon_1^{-1}(\bar{Y}_{11} + O(\|\mu\|)) & \sqrt{\varepsilon_1\varepsilon_2}^{-1}O(\|\mu\|) \\ \bar{Y}_{20} + O(\|\mu\|) & \sqrt{\varepsilon_1\varepsilon_2}^{-1}O(\|\mu\|) & \varepsilon_2^{-1}(\bar{Y}_{22} + O(\|\mu\|)) \end{bmatrix}, \quad (13)$$

where $\mu = \begin{bmatrix} \varepsilon_1 & \varepsilon_2 \end{bmatrix}$.

Proof: We apply the implicit function theorem (Gajić et al., 1990;Gajić, 1988) to (10a). To do so, it is enough to show that the corresponding Jacobian is nonsingular at $\varepsilon_j = 0, \ j = 1, \ 2$. It can be shown, after some algebra, that the Jacobian of (10a) in the limit as $\mu \to \bar{\mu} = \begin{bmatrix} 0 & 0 \end{bmatrix}$ is given by

$$J_Y = \begin{bmatrix} J_{00} & J_{01} & J_{02} & 0 & 0 & 0 \\ J_{10} & J_{11} & 0 & J_{13} & J_{14} & 0 \\ J_{20} & 0 & J_{22} & 0 & J_{24} & J_{25} \\ 0 & 0 & 0 & J_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & J_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & J_{55} \end{bmatrix}, \quad (14)$$

and

$$\begin{aligned} J_{00} &= (I_{n_0} \otimes H_{00})U_{n_0n_0} + H_{00} \otimes I_{n_0}, \\ J_{0j} &= (I_{n_0} \otimes H_{0j})U_{n_0n_j} + H_{0j} \otimes I_{n_0}, \\ J_{j0} &= H_{j0} \otimes I_{n_0}, \ J_{jj} = H_{jj} \otimes I_{n_0}, \\ J_{13} &= I_{n_1} \otimes H_{01}, \ J_{14} = \sqrt{\bar{\alpha}}(I_{n_1} \otimes H_{02})U_{n_1n_2}, \\ J_{24} &= \frac{1}{\sqrt{\bar{\alpha}}}(I_{n_2} \otimes H_{01}), \ J_{25} = I_{n_2} \otimes H_{02}, \\ J_{33} &= (I_{n_1} \otimes H_{11})U_{n_1n_1} + H_{11} \otimes I_{n_1}, \\ J_{44} &= \sqrt{\bar{\alpha}}H_{22} \otimes I_{n_1} + \frac{1}{\sqrt{\bar{\alpha}}}I_{n_2} \otimes H_{11}, \\ J_{55} &= (I_{n_2} \otimes H_{22})U_{n_2n_2} + H_{22} \otimes I_{n_2}, \ j = 1, \ 2, \end{aligned}$$

where $\otimes$ denotes Kronecker products and $U_{n_jn_j}$ is the permutation matrix in the Kronecker matrix sense. The Jacobian (14) can be expressed as

$$\begin{aligned} \det J_Y = &\det J_{11} \cdot \det J_{22} \cdot \det J_{33} \cdot \det J_{44} \cdot \det J_{55} \\ &\cdot \det[I_{n_0} \otimes H_0U_{n_0n_0} + H_0 \otimes I_{n_0}], \quad (15) \end{aligned}$$

where $H_0 \equiv H_{00} - H_{01}H_{11}^{-1}H_{10} - H_{02}H_{22}^{-1}H_{20}$. Obviously, $J_{jj}, \ j = 1, \cdots, 5$ are nonsingular because the matrices $H_{jj} = A_{jj} - \bar{Y}_{jj}V_{jj}, \ j = 1, \ 2$ are stable. After some straightforward but tedious algebra, we see that $A_s - \bar{Y}_{00}V_s = H_{00} - H_{01}H_{11}^{-1}H_{10} - H_{02}H_{22}^{-1}H_{20} = H_0$. Therefore, the matrix H_0 is stable if γ is sufficiently large. Thus, $\det J_Y \neq 0$. The conclusion of Theorem 1 is obtained directly by using the implicit function theorem. The remainder of the proof is to show that Y_e is the positive semidefinite stabilizing solution. However, the proof is omitted since it is similar to that of the reference Mukaidani et al,. (2001). □

4. THE RECURSIVE ALGORITHM

Now, let us define $\|\mu\| := \mathcal{E} = \sqrt{\varepsilon_1\varepsilon_2}$. By making use of the zeroth order solutions (12), the solution (13) can be changed as follows.

$$Y_{pq} = \bar{Y}_{pq} + \mathcal{E}F_{pq}, \ pq = 00, 10, 20, 11, 21, 22, \quad (16)$$

where $F_{00} = F_{00}^T$, $F_{11} = F_{11}^T$, $F_{22} = F_{22}^T$, $\bar{Y}_{21} = 0$.

Substituting (16) into (10a) and subtracting (12) from (10a), we arrive at the following error equations (17).

$$\begin{aligned} &H_{00}F_{00} + F_{00}H_{00}^T + H_{10}F_{10} + F_{10}^TH_{10}^T + H_{20}F_{20} \\ &\quad + F_{20}^TH_{20}^T - \mathcal{E}(F_{00}V_{00}F_{00} + F_{10}^TV_{01}^TF_{00} \\ &\quad + F_{00}V_{01}F_{10} + F_{20}^TV_{02}^TF_{00} + F_{00}V_{02}F_{20} \\ &\quad + F_{10}^TV_{11}F_{10} + F_{20}^TV_{22}F_{20}) = 0, \quad (17a) \\ &F_{00}H_{10}^T + F_{10}^TH_{11}^T + H_{01}F_{11} + \sqrt{\alpha}H_{02}F_{21} \\ &\quad + \frac{\varepsilon_1}{\mathcal{E}}H_{00}Y_{10}^T - \mathcal{E}(F_{00}V_{01}F_{11} + F_{10}^TV_{11}F_{11}) \\ &\quad - \mathcal{E}\sqrt{\alpha}(F_{00}V_{02}F_{21} + F_{20}^TV_{22}F_{21}) \\ &\quad - \varepsilon_1(F_{00}V_{00} + F_{10}^TV_{01}^T + F_{20}^TV_{02}^T)Y_{10}^T = 0, \quad (17b) \\ &F_{00}H_{20}^T + F_{20}^TH_{22} + H_{02}F_{22} + \frac{1}{\sqrt{\alpha}}H_{01}F_{21}^T \\ &\quad + \frac{\varepsilon_2}{\mathcal{E}}H_{00}Y_{20}^T - \mathcal{E}(F_{00}V_{02}F_{22} + F_{20}^TV_{22}F_{22}) \\ &\quad - \frac{\mathcal{E}}{\sqrt{\alpha}}(F_{00}V_{01}F_{21}^T + F_{10}^TV_{11}^TF_{21}^T) \\ &\quad - \varepsilon_2(F_{00}V_{00} + F_{10}^TV_{01}^T + F_{20}^TV_{02}^T)Y_{20}^T = 0, \quad (17c) \\ &H_{11}F_{11} + F_{11}H_{11}^T + \frac{\varepsilon_1}{\mathcal{E}}(H_{10}\bar{Y}_{10}^T + \bar{Y}_{10}H_{10}^T) \\ &\quad + \varepsilon_1(H_{10}F_{10}^T + F_{10}H_{10}^T) \\ &\quad - \frac{\varepsilon_1^2}{\mathcal{E}}Y_{10}V_{00}Y_{10}^T - \varepsilon_1(F_{11}V_{01}^TY_{10}^T + Y_{10}V_{01}F_{11}) \end{aligned}$$

$$
\begin{aligned}
&-\varepsilon_1\sqrt{\alpha}(F_{21}^T V_{02}^T Y_{10}^T + Y_{10}V_{02}F_{21})\\
&-\mathcal{E}(F_{11}V_{11}F_{11} + \alpha F_{21}^T V_{22}F_{21}) = 0, \qquad (17\text{d})\\
&\sqrt{\alpha}F_{21}^T H_{22} + \frac{1}{\sqrt{\alpha}}H_{11}F_{21}^T + \frac{\varepsilon_1}{\mathcal{E}}\bar{Y}_{10}H_{20}^T\\
&+\frac{\varepsilon_2}{\mathcal{E}}H_{10}\bar{Y}_{20}^T + \varepsilon_1 F_{10}H_{20}^T + \varepsilon_2 H_{10}F_{20}^T\\
&-\varepsilon_1(Y_{10}V_{02}F_{22} + \frac{1}{\sqrt{\alpha}}Y_{10}V_{01}F_{21}^T)\\
&-\varepsilon_2(F_{11}V_{01}^T Y_{20}^T + \sqrt{\alpha}F_{21}^T V_{02}^T Y_{20}^T)\\
&-\frac{\varepsilon_1\varepsilon_2}{\mathcal{E}}Y_{10}V_{00}Y_{20}^T - \mathcal{E}(\sqrt{\alpha}F_{21}^T V_{22}F_{22}\\
&+\frac{1}{\sqrt{\alpha}}F_{11}^T V_{11}^T F_{21}^T) = 0, \qquad (17\text{e})\\
&H_{22}F_{22} + F_{22}H_{22}^T + \frac{\varepsilon_2}{\mathcal{E}}(H_{20}\bar{Y}_{20}^T + \bar{Y}_{20}H_{20}^T)\\
&+\varepsilon_2(H_{20}F_{20}^T + F_{20}H_{20}^T)\\
&-\frac{\varepsilon_2^2}{\mathcal{E}}Y_{20}V_{00}Y_{20}^T - \varepsilon_2(F_{22}V_{02}^T Y_{20}^T + Y_{20}V_{02}F_{22})\\
&-\frac{\varepsilon_2}{\sqrt{\alpha}}(F_{21}V_{01}^T Y_{20}^T + Y_{20}V_{01}F_{21}^T)\\
&-\mathcal{E}(F_{22}V_{22}F_{22} + \frac{1}{\alpha}F_{21}V_{11}F_{21}^T) = 0. \qquad (17\text{f})
\end{aligned}
$$

Hence, we propose the following iterative algorithm (18) which is based on the recursive algorithm.

$$
\begin{aligned}
&H_{11}F_{11}^{(i+1)} + F_{11}^{(i+1)}H_{11}^T\\
&= -\frac{\varepsilon_1}{\mathcal{E}}(H_{10}\bar{Y}_{10}^T + \bar{Y}_{10}H_{10}^T)\\
&\quad -\varepsilon_1(H_{10}F_{10}^{(i)T} + F_{10}^{(i)}H_{10}^T) + \frac{\varepsilon_1^2}{\mathcal{E}}Y_{10}^{(i)}V_{00}Y_{10}^{(i)T}\\
&\quad +\varepsilon_1(F_{11}^{(i)}V_{01}^T Y_{10}^{(i)T} + Y_{10}^{(i)}V_{01}F_{11}^{(i)})\\
&\quad +\varepsilon_1\sqrt{\alpha}(F_{21}^{(i)T}V_{02}^T Y_{10}^{(i)T} + Y_{10}^{(i)}V_{02}F_{21}^{(i)})\\
&\quad +\mathcal{E}(F_{11}^{(i)}V_{11}F_{11}^{(i)} + \alpha F_{21}^{(i)T}V_{22}F_{21}^{(i)}), \qquad (18\text{a})\\
&H_{22}F_{22}^{(i+1)} + F_{22}^{(i+1)}H_{22}^T\\
&= -\frac{\varepsilon_2}{\mathcal{E}}(H_{20}\bar{Y}_{20}^T + \bar{Y}_{20}H_{20}^T)\\
&\quad -\varepsilon_2(H_{20}F_{20}^{(i)T} + F_{20}^{(i)}H_{20}^T) + \frac{\varepsilon_2^2}{\mathcal{E}}Y_{20}^{(i)}V_{00}Y_{20}^{(i)T}\\
&\quad +\varepsilon_2(F_{22}^{(i)}V_{02}^T Y_{20}^{(i)T} + Y_{20}^{(i)}V_{02}F_{22}^{(i)})\\
&\quad +\frac{\varepsilon_2}{\sqrt{\alpha}}(F_{21}^{(i)}V_{01}^T Y_{20}^{(i)T} + Y_{20}^{(i)}V_{01}F_{21}^{(i)T})\\
&\quad +\mathcal{E}(F_{22}^{(i)}V_{22}F_{22}^{(i)} + \frac{1}{\alpha}F_{21}^{(i)}V_{11}F_{21}^{(i)T}), \qquad (18\text{b})\\
&\sqrt{\alpha}F_{21}^{(i+1)T}H_{22} + \frac{1}{\sqrt{\alpha}}H_{11}F_{21}^{(i+1)T}\\
&= -\frac{\varepsilon_1}{\mathcal{E}}\bar{Y}_{10}H_{20}^T - \frac{\varepsilon_2}{\mathcal{E}}H_{10}\bar{Y}_{20}^T\\
&\quad -\varepsilon_1 F_{10}^{(i)}H_{20}^T - \varepsilon_2 H_{10}F_{20}^{(i)T} + \frac{\varepsilon_1\varepsilon_2}{\mathcal{E}}Y_{10}^{(i)}V_{00}Y_{20}^{(i)T}\\
&\quad +\varepsilon_1(Y_{10}^{(i)}V_{02}F_{22}^{(i)} + \frac{1}{\sqrt{\alpha}}Y_{10}^{(i)}V_{01}F_{21}^{(i)T})\\
&\quad +\varepsilon_2(F_{11}^{(i)}V_{01}^T Y_{20}^{(i)T} + \sqrt{\alpha}F_{21}^{(i)T}V_{02}^T Y_{20}^{(i)T})\\
&\quad +\mathcal{E}(\sqrt{\alpha}F_{21}^{(i)T}V_{22}F_{22}^{(i)} + \frac{1}{\sqrt{\alpha}}F_{11}^{(i)}V_{11}^T F_{21}^{(i)T}), \qquad (18\text{c})\\
&H_0 F_{00}^{(i+1)} + F_{00}^{(i+1)}H_0^T\\
&= -H_{01}H_{11}^{-1}L_{01}^{(i)T} - L_{01}^{(i)}H_{11}^{-T}H_{01}^T\\
&\quad -H_{02}H_{22}^{-1}L_{02}^{(i)T} - L_{02}^{(i)}H_{22}^{-T}H_{02}^T\\
&\quad +\mathcal{E}(F_{00}^{(i)}V_{00}F_{00}^{(i)} + F_{10}^{(i)T}V_{01}^T F_{00}^{(i)} + F_{00}^{(i)}V_{01}F_{10}^{(i)}\\
&\quad +F_{20}^{(i)T}V_{02}^T F_{00}^{(i)} + F_{00}^{(i)}V_{02}F_{20}^{(i)}\\
&\quad +F_{10}^{(i)T}V_{11}^T F_{10}^{(i)} + F_{20}^{(i)T}V_{22}F_{20}^{(i)}), \qquad (18\text{d})\\
&F_{j0}^{(i+1)T} = (L_{0j}^{(i)} - F_{00}^{(i+1)}H_{j0}^T)H_{jj}^{-T}, \qquad (18\text{e})
\end{aligned}
$$

where $j = 1,\ 2,$

$$
\begin{aligned}
L_{01}^{(i)} &= -H_{01}F_{11}^{(i+1)} - \sqrt{\alpha}H_{02}^T F_{21}^{(i+1)} - \frac{\varepsilon_1}{\mathcal{E}}H_{00}^T Y_{10}^T\\
&\quad +\mathcal{E}(F_{00}^{(i)}V_{01}F_{11}^{(i)} + F_{10}^{(i)T}V_{11}F_{11}^{(i)})\\
&\quad +\mathcal{E}\sqrt{\alpha}(F_{00}^{(i)}V_{02}F_{21}^{(i)} + F_{20}^{(i)T}V_{22}^T F_{21}^{(i)})\\
&\quad +\varepsilon_1(F_{00}^{(i)}V_{00} + F_{10}^{(i)T}V_{01}^T + F_{20}^{(i)T}V_{02}^T)Y_{10}^{(i)T},\\
L_{02}^{(i)} &= -H_{02}^T F_{22}^{(i+1)} - \frac{1}{\sqrt{\alpha}}H_{01}^T F_{21}^{(i+1)T} - \frac{\varepsilon_2}{\mathcal{E}}H_{00}^T Y_{20}^T\\
&\quad +\mathcal{E}(F_{00}^{(i)}V_{02}F_{22}^{(i)} + F_{20}^{(i)T}V_{22}F_{22}^{(i)})\\
&\quad +\frac{\mathcal{E}}{\sqrt{\alpha}}(F_{00}^{(i)}V_{01}F_{21}^{(i)T} + F_{10}^{(i)T}V_{11}^T F_{21}^{(i)T})\\
&\quad +\varepsilon_2(F_{00}^{(i)}V_{00} + F_{10}^{(i)T}V_{01}^T + F_{20}^{(i)T}V_{02}^T)Y_{20}^{(i)T},\\
Y_{j0}^{(i)} &= \bar{Y}_{j0} + \mathcal{E}F_{j0}^{(i)}, \quad i = 0,\ 1,\ \cdots.\\
F_{00}^{(0)} &= F_{10}^{(0)} = F_{20}^{(0)} = F_{11}^{(0)} = F_{21}^{(0)} = F_{22}^{(0)} = 0.
\end{aligned}
$$

The following theorem indicates the convergence of the algorithm (18).

Theorem 2: If we select a parameter $\gamma > \bar{\gamma}$, there exist the unique and bounded solutions F_{pq} of the error equation in a neighborhood of $\|\mu\| = 0$. Moreover, the algorithm (18) converges to the exact solution F_{pq} with the rate of convergence of $O(\|\mu\|^i)$, that is

$$\|F_{pq} - F_{pq}^{(i)}\| = O(\|\mu\|^i), \quad i = 1,\ 2,\ \cdots. \qquad (19)$$

Proof: As a starting point we need to show the existence of a unique and bounded solution of F_{pq} in neighborhood of $\|\mu\| = 0$. To prove that by the implicit function theorem, it is enough to show that the corresponding Jacobian J_F of (17) is nonsingular at $\|\mu\| = 0$. The Jacobian is given by

$$J_F = J_Y. \qquad (20)$$

Taking into consideration the fact that J_Y is nonsingular at $\|\mu\| = 0$, J_F is also nonsingular. Therefore, there exists a unique and bounded solution of the error equations (17). Secondly, the proof of (18) uses mathematical induction. However, in order to respect pages limitations, the proof is omitted since it is similar to that of the reference Mukaidani et al,. (2001). □

Table 1.

$$Y = \begin{bmatrix} 7.7443e-1 & -1.5934e-1 & 1.1995e-2 & 3.0145e-2 & 3.0483e-1 & -1.7552e-2 & -5.2303e-2 & -1.2068e-1 & -1.1517e-1 \\ -1.5934e-1 & 7.7443e-1 & 3.0145e-2 & 1.1995e-2 & -3.0483e-1 & -1.2068e-1 & -1.1517e-1 & -1.7552e-2 & -5.2303e-2 \\ 1.1995e-2 & 3.0145e-2 & 6.4581e-3 & 2.9175e-3 & 1.2093e-2 & 2.1547e-2 & -8.7334e-3 & -7.5655e-3 & -1.1394e-2 \\ 3.0145e-2 & 1.1995e-2 & 2.9175e-3 & 6.4581e-3 & -1.2093e-2 & -7.5655e-3 & -1.1394e-2 & 2.1547e-2 & -8.7334e-3 \\ 3.0483e-1 & -3.0483e-1 & 1.2093e-2 & -1.2093e-2 & 1.1827e+0 & 1.1885e-1 & -3.7194e-2 & -1.1885e-1 & 3.7194e-2 \\ -1.7552e-4 & -1.2068e-3 & 2.1547e-4 & -7.5655e-5 & 1.1885e-3 & 1.6204e-2 & 1.6222e-2 & 3.5482e-4 & 3.4417e-4 \\ -5.2303e-4 & -1.1517e-3 & -8.7334e-5 & -1.1394e-4 & -3.7194e-4 & 1.6222e-2 & 5.0369e-2 & 3.4417e-4 & 4.5092e-4 \\ -1.2068e-3 & -1.7552e-4 & -7.5655e-5 & 2.1547e-4 & -1.1885e-3 & 3.5482e-4 & 3.4417e-4 & 1.6204e-2 & 1.6222e-2 \\ -1.1517e-3 & -5.2303e-4 & -1.1394e-4 & -8.7334e-5 & 3.7194e-4 & 3.4417e-4 & 4.5092e-4 & 1.6222e-2 & 5.0369e-2 \end{bmatrix}.$$

5. NUMERICAL EXAMPLE

In order to demonstrate the efficiency of our proposed algorithm, we have run a numerical example. The system matrix is given by

$$A_{00} = \begin{bmatrix} 0 & 0 & 4.5 & 0 & 1 \\ 0 & 0 & 0 & 4.5 & -1 \\ 0 & 0 & -0.05 & 0 & -0.1 \\ 0 & 0 & 0 & -0.05 & 0.1 \\ 0 & 0 & 32.7 & -32.7 & 0 \end{bmatrix},$$

$$A_{jj} = \begin{bmatrix} -0.05 & 0.05 \\ 0 & -0.1 \end{bmatrix}$$

$$A_{01} = \begin{bmatrix} 0_{2\times 2} \\ A_p \\ 0_{2\times 2} \end{bmatrix}, \ A_{02} = \begin{bmatrix} 0_{3\times 2} \\ A_p \\ 0_{1\times 2} \end{bmatrix}, \ A_p = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}^T$$

$$A_{10} = \begin{bmatrix} 0_{2\times 2} & -4A_q & 0_{2\times 2} \end{bmatrix}, \ A_q = \begin{bmatrix} 0 & 0.1 \end{bmatrix}^T$$

$$A_{20} = \begin{bmatrix} 0_{2\times 3} & -4A_q & 0_{2\times 1} \end{bmatrix}, \ D_{0j} = 0_{5\times 1}, \ D_{jj} = A_q$$

$$C^T C = \mathrm{diag}(1,\ 1,\ 1,\ 1,\ 1,\ 0,\ 0,\ 0,\ 0),\ R = 20$$

$$G^T R G = \mathrm{diag}(1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1), \gamma = 5.$$

The small parameters are chosen as $\varepsilon_1 = \varepsilon_2 = 0.01$. Note that we can not apply the technique proposed in Coumarbatch and Gajić, (2000) to the MSPS since the Hamiltonian matrices $Z_{jj},\ j = 1,\ 2$ have eigenvalues in common. We give a solution of the GMARE (10a) in Table 1. We find that the solution of the GMARE (10a) converge to the exact solution with accuracy of $\|\mathcal{F}(Y^{(i)})\| < e - 10$ after 22 iterative iterations, where "$e - x$" stands for "$\times 10^{-x}$. For different values of ε_1 and ε_2, in order to verify the exactitude of the solution, the errors (i.e. $\|\mathcal{F}(Y^{(i)})\|$) and the necessary iteration numbers of the algorithm (18) are given by Table 2. From Table 2, since for sufficiently small perturbation parameters the convergence speed is quite good, the resulting algorithm of this paper is very useful.

Table 2. Error $\|\mathcal{F}(Y)\|$

ε_1	ε_2	Iterations	Errors
$1e-2$	$1e-2$	22	$3.5484e-10$
$1e-2$	$5e-3$	24	$7.3850e-10$
$1e-3$	$1e-3$	5	$5.6755e-10$
$1e-3$	$5e-4$	5	$3.5433e-10$
$1e-4$	$1e-4$	3	$2.1241e-10$
$1e-4$	$5e-5$	3	$1.2052e-10$
$1e-5$	$1e-5$	2	$2.1017e-10$

6. CONCLUSION

In this paper, we have proposed a new recursive algorithm for solving the MARE which has the indefinite sign quadratic term. We have proven that the solution of the MARE converges to a positive semi–definite stabilizing solution with the rate of convergence of $O(\|\mu\|^{i+1})$. As another important feature, since we do not assume that the Hamiltonian matrices $Z_{jj},\ j = 1,\ 2$ for the fast fast subsystems have no eigenvalues in common compared with Coumarbatch and Gajić, (2000), our new results are applicable to more realistic MSPS.

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