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ROBUST STABILIZATION OF NON-STANDARD SINGULARLY PERTURBED SYSTEMS WITH UNCERTAINTIES

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EXTENDED ABSTRACT

This paper considers the robust stabilization of singularly perturbed systems with time-varying unknown-but-bounded uncertainties. The implicit function theorem is used to prove the sufficient condition for stability of the closed-loop system. The construction of the stabilizing controller involves solving slow and fast algebraic Riccati equations. It is shown that if the reduced-order slow and fast algebraic Riccati equations have positive definite stabilizing solution then the obtained uncertain closed-loop system with the proposed ε-independent controller is quadratically stable. The main contribution of this paper is that the sufficient condition for stability derived here is independent of the parameter ε. Furthermore, our new results apply to the both standard and non-standard singularly perturbed systems.

KEYWORDS

· **Robust stability** · **Uncertain linear systems** · **Singular perturbations** · **Algebraic Riccati equations** · **Jacobian matrices**

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Abstract: This paper considers the robust stabilization of singularly perturbed systems with time-varying unknown-but-bounded uncertainties. The implicit function theorem is used to prove the sufficient condition for stability of the closed-loop system. The construction of the stabilizing controller involves solving slow and fast algebraic Riccati equations. It is shown that if the reduced-order slow and fast algebraic Riccati equations have positive definite stabilizing solution then the obtained uncertain closed-loop system with the proposed ε-independent controller is quadratically stable. The main contribution of this paper is that the sufficient condition for stability derived here is independent of the parameter ε. Furthermore, our new results apply to the both standard and non-standard singularly perturbed systems. *Copyright © 1999 IFAC*

Keywords: Robust stability, Uncertain linear systems, Singular perturbations, Algebraic Riccati equations, Jacobian matrices

1. INTRODUCTION

In recent papers, some authors have concerned with the problem of stabilizing the singularly perturbed systems containing uncertain parameters (Shao and Sawan 1993, Corless et al. 1993). Shao and Sawan (1993) showed that the robust stability conditions of singularly perturbed systems can be obtained by using a singularly perturbation method (Kokotovic et al. 1986). It is obvious that the basic assumption in Theorem 1 of Shao and Sawan (1993), that is, the uncertain matrix $A_{22} + \Delta_{22}(t)$ is Hurwitz, play an important role in the study of the problem. Corless et al. (1993) propose a class of nonlinear composite controllers which assure global uniform ultimate boundedness of the trajectories of closed loop singular perturbed systems. It is also obvious that Assumption 3 of Corless et al. (1993), that is, function $A_{22}(q)$ is invertible, is needed to construct the stabilizing controller. However, these assumptions are restricted because they contains uncertainties.

In this paper, based on Khargonekar (1990), the robust stabilization for singularly perturbed systems with uncertainties is studied in a different point of view. In order to prove the existence of stabilizing controller, the implicit function theorem is used. In general, in order to obtain the stabilizing controller we must solve a certain full-order algebraic Riccati equation with small parameter $\varepsilon > 0$. The aim of the present paper is to propose a method which, instead of solving the full-order algebraic Riccati equation, solve the reduced-order slow and fast algebraic Riccati equations without small parameter ε. It is also proposed that an ε-independent stabilizing controller can be obtained by making use of the solutions for reduced-order slow and fast algebraic Riccati equations. As the result, it is shown that if the reduced-order slow and fast Riccati equations have positive definite stabilizing solution, then the obtained closed-loop system with the proposed controller is quadratically stable.

The main feature of this paper is that the sufficient condition for stability margin is derived from existence of solutions for the ε-independent slow and fast algebraic Riccati equations. Furthermore, although the uncertain $A_{22} + \Delta_{22}(t)$ has unstable mode, there exists the stabilizing controller for singularly perturbed systems with uncertainties. Thus, our new results apply to both standard and non–standard singularly perturbed systems.

2. PROBLEM FORMULATION

Consider a linear time-invariant non standard singularly perturbed system

$$\dot{x}(t) = (A_\varepsilon + F_\varepsilon \Delta(t) E) x(t) + B_\varepsilon u(t), \qquad (1)$$

where

$$A_\varepsilon = \begin{bmatrix} A_{11} & A_{12} \\ \varepsilon^{-1} A_{21} & \varepsilon^{-1} A_{22} \end{bmatrix},$$

$$B_\varepsilon = \begin{bmatrix} B_1 \\ \varepsilon^{-1} B_2 \end{bmatrix}, \quad F_\varepsilon = \begin{bmatrix} F_1 \\ \varepsilon^{-1} F_2 \end{bmatrix},$$

$$E = \begin{bmatrix} E_1 & E_2 \end{bmatrix}, \quad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

where ε is a small positive parameter, $x^T := (x_1^T, x_2^T)$ is the n–dimensional state vector, with x_1 of dimension n_1 and x_2 of dimension $n_2 := n - n_1$, u is the m–dimensional control, $\Delta(t)$ is a Lebesgue measurable matrix of uncertain parameters and satisfies norm conditions $\|\Delta(t)\| \le 1$. All matrices above are of appropriate dimensions. The system (1) is said to be in the standard form if the matrix A_{22} is nonsingular. Otherwise, it is called the non standard singularly perturbed system (Khalil 1989).

Let us introduce the partitioned matrices

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \; B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \; F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}.$$

Now, let us consider the stabilization of such non–standard singularly perturbed systems by using linear state feedback under the following basic assumption.

Assumption 1 *The pair $(A_\varepsilon,\ B_\varepsilon)$ is stabilizable for $\varepsilon \in (0, \varepsilon^*]$ $(\varepsilon^* > 0)$.*

Clearly, this assumption is necessary for even the nominal system to be considered. The following lemma is already known (see Khargonekar et al. 1990).

Lemma 1 *Under Assumption 1, if there exists $\mu > 0$ such that the following algebraic Riccati equation*

$$A_\varepsilon^T P_\varepsilon + P_\varepsilon A_\varepsilon - P_\varepsilon(\mu^{-1} B_\varepsilon B_\varepsilon^T - F_\varepsilon F_\varepsilon^T) P_\varepsilon + E^T E + \mu I = 0 \qquad (2)$$

has the unique positive definite symmetric solution, then the non–standard singularly perturbed system (1) is quadratically stable via linear control. In this case a stabilizing linear state feedback control law is given by

$$u(t) = -\frac{1}{2\mu} B_\varepsilon^T P_\varepsilon x(t). \qquad (3)$$

Conversely, if the non–standard singularly perturbed systems (1) is quadratically stable via linear control (3), then there exists $\mu^* > 0$ such that for all $\mu \in (0,\ \mu^*)$, the algebraic Riccati equation (2) admits a unique positeve definite stabilizing solution.

However, it is difficult to solve the algebraic Riccati equation (2) because of the different magnitudes of their coefficients caused by the small perturbation parameter ε and high dimension. In this paper, in order to overcome the computation difficulties caused by numerical stiffness, we propose a method which, instead of solving the full-order algebraic Riccati equation (2) with ε, solves the reduced–order slow and fast Riccati equations without small perturbation parameter ε.

3. GENERALIZED ALGEBRAIC RICCATI EQUATIONS

In order to solve the algebraic Riccati equation (2), we introduce the following useful lemma (Mukaidani et al.).

Lemma 2 *The algebraic Riccati equation (2) is equivalent to the following generalized algebraic Riccati equation (4)*

$$P^T A + A^T P - P^T(\mu^{-1} B B^T - F F^T) P + E^T E + \mu I = 0, \qquad (4a)$$

$$P_\varepsilon = \Pi_\varepsilon^T P = P^T \Pi_\varepsilon, \qquad (4b)$$

where

$$P_\varepsilon = \begin{bmatrix} P_{11} & \varepsilon P_{21}^T \\ \varepsilon P_{21} & \varepsilon P_{22} \end{bmatrix}, \; \Pi_\varepsilon = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \varepsilon I_{n_2} \end{bmatrix}.$$

Proof: Firstly, from (4b), P has the following partitioned form

$$P = \begin{bmatrix} P_{11} & \varepsilon P_{21}^T \\ P_{21} & P_{22} \end{bmatrix}, \; P_{11} = P_{11}^T, \; P_{22} = P_{22}^T.$$

It is worth to note that P is not symmetric, but $P_\varepsilon = \Pi_\varepsilon^T P = P^T \Pi_\varepsilon$ is. Secondly, we can observe the following useful relationships between A_ε, B_ε, F_ε, Π_ε, A, B and F.

$$A_\varepsilon = \Pi_\varepsilon^{-1} A, \; B_{1\varepsilon} = \Pi_\varepsilon^{-1} B_1, \; B_{2\varepsilon} = \Pi_\varepsilon^{-1} B_2.$$

Substituting the above relations and $P_\varepsilon = \Pi_\varepsilon^T P = P^T \Pi_\varepsilon$ into the Riccati equation (4a). Then, (2) can be rewritten as (4a). Thus, to solve the algebraic Riccati equation (4a) is equivalent to solving the generalized algebraic Riccati equation (2). ■

By making use of the relation (4b), we can change the form of the controller (3).

$$u(t) = -\frac{1}{2\mu} \begin{bmatrix} B_1^T & B_2^T \end{bmatrix} P x(t) \qquad (5)$$

4. THE LINEAR STATE FEEDBACK CONTROLLER

In this section, the linear state feedback controller for singularly perturbed systems with structured uncertainties are presented.

4.1 The Linear State Feedback full-order Controller

The generalized algebraic Riccati equation (4a) can be partitioned into

$$\begin{aligned}&A_{11}^T P_{11} + P_{11}^T A_{11} + A_{21}^T P_{21} + P_{21}^T A_{21}\\&\quad - P_{11}^T S_{11}^\mu P_{11} - P_{21}^T S_{22}^\mu P_{21} - P_{11}^T S_{12}^\mu P_{21}\\&\quad - P_{21}^T S_{12}^{\mu T} P_{11} + Q_{11}^\mu = 0, \end{aligned} \tag{6a}$$

$$\begin{aligned}&\varepsilon P_{21} A_{11} + P_{22}^T A_{21} + A_{12}^T P_{11} + A_{22}^T P_{21}\\&\quad - \varepsilon P_{21} S_{11}^\mu P_{11} - \varepsilon P_{21} S_{12}^\mu P_{21} - P_{22}^T S_{12}^{\mu T} P_{11}\\&\quad - P_{22}^T S_{22}^\mu P_{21} + Q_{12}^T = 0, \end{aligned} \tag{6b}$$

$$\begin{aligned}&A_{22}^T P_{22} + P_{22}^T A_{22} + \varepsilon A_{12}^T P_{21}^T + \varepsilon P_{21} A_{12}\\&\quad - P_{22}^T S_{22}^\mu P_{22} - \varepsilon P_{22}^T S_{12}^{\mu T} P_{21}^T - \varepsilon P_{21} S_{12}^\mu P_{22}\\&\quad - \varepsilon^2 P_{21} S_{11}^\mu P_{21}^T + Q_{22}^\mu = 0, \end{aligned} \tag{6c}$$

where

$$\begin{aligned} Q_{11}^\mu &= E_1^T E_1 + \mu I,\ Q_{12} = E_1^T E_2,\\ Q_{22}^\mu &= E_2^T E_2 + \mu I,\\ S_{11}^\mu &= \mu^{-1} B_1 B_1^T - F_1 F_1^T,\\ S_{12}^\mu &= \mu^{-1} B_1 B_2^T - F_1 F_2^T,\\ S_{22}^\mu &= \mu^{-1} B_2 B_2^T - F_2 F_2^T. \end{aligned}$$

For the previous equations (6), setting $\varepsilon = 0$, we obtain the following equations

$$\begin{aligned}&A_{11}^T \bar{P}_{11} + \bar{P}_{11}^T A_{11} + A_{21}^T \bar{P}_{21} + \bar{P}_{21}^T A_{21}\\&\quad - \bar{P}_{11}^T S_{11}^\mu \bar{P}_{11} - \bar{P}_{21}^T S_{22}^\mu \bar{P}_{21}\\&\quad - \bar{P}_{11}^T S_{12}^\mu \bar{P}_{21} - \bar{P}_{21}^T S_{12}^{\mu T} \bar{P}_{11} + Q_{11}^\mu = 0, \end{aligned} \tag{7a}$$

$$\begin{aligned}&\bar{P}_{22}^T A_{21} + A_{12}^T \bar{P}_{11} + A_{22}^T \bar{P}_{21} - \bar{P}_{22}^T S_{12}^{\mu T} \bar{P}_{11}\\&\quad - \bar{P}_{22}^T S_{22}^\mu \bar{P}_{21} + Q_{12}^T = 0. \end{aligned} \tag{7b}$$

$$A_{22}^T \bar{P}_{22} + \bar{P}_{22}^T A_{22} - \bar{P}_{22}^T S_{22}^\mu \bar{P}_{22} + Q_{22}^\mu = 0. \tag{7c}$$

The Riccati equation (7c) will produce the unique positive definite stabilizing solution under the following assumption and condition.

Assumption 2 *The pair $(A_{22},\ B_{22})$ is stabilizable.*

Let
$\Gamma_f := \{\mu > 0 |$the Riccati equation (7c) has a positive definite stabilizing solution$\}$,
$\mu_f := \sup\{\mu | \mu \in \Gamma_f\}$.

Then, the matrix $A_{22} - S_{22}^\mu \bar{P}_{22}$ is non–singular if we choose μ such that $0 < \mu < \mu_f$. Therefore, we obtain the following 0–order equations

$$\bar{P}_{11}^T A_0^\mu + A_0^{T\mu} \bar{P}_{11} - \bar{P}_{11}^T S_0^\mu \bar{P}_{11} + Q_0^\mu = 0, \tag{8a}$$

$$\bar{P}_{21} = -N_2^T + N_1^T \bar{P}_{11}, \tag{8b}$$

$$A_{22}^T \bar{P}_{22} + \bar{P}_{22}^T A_{22} - \bar{P}_{22}^T S_{22}^\mu \bar{P}_{22} + Q_{22}^\mu = 0, \tag{8c}$$

where

$$\begin{aligned}& A_0^\mu = A_{11} + N_1 A_{21} + S_{12}^\mu N_2^T + N_1 S_{22}^\mu N_2^T,\\& S_0^\mu = S_{11}^\mu + N_1 S_{12}^{T\mu} + S_{12}^\mu N_1^T + N_1 S_{22}^\mu N_1^T,\\& Q_0^\mu = Q_{11}^\mu - N_2 A_{21} - A_{21}^T N_2^T - N_2 S_{22}^\mu N_2^T,\\& N_2^T = D_4^{-T} \hat{Q}_{12}^T,\ N_1^T = -D_4^{-T} D_2^T,\\& D_2 = A_{12} - S_{12}^\mu \bar{P}_{22},\ D_4 = A_{22} - S_{22}^\mu \bar{P}_{22},\\& \hat{Q}_{12} = Q_{12} + A_{21}^T \bar{P}_{22}. \end{aligned}$$

Remark 1 *Although the expressions of the matrix A_0^μ , S_0^μ and Q_0^μ contain the matrix $\bar{P}_{22}$, they do not depend on it (Tan et al. 1998, Xu and Mizukami 1996). In fact, the coefficient matrices of the equation (8a) is obtained from the formula*

$$T_0 = T_1 - T_2 T_4^{-1} T_3 = \begin{bmatrix} A_0^\mu & -S_0^\mu \\ -Q_0^\mu & -A_0^{\mu T} \end{bmatrix}, \tag{9}$$

where

$$\begin{aligned} T_1 &= \begin{bmatrix} A_{11} & -S_{11}^\mu \\ -Q_{11}^\mu & -A_{11}^T \end{bmatrix},\\ T_2 &= \begin{bmatrix} A_{12} & -S_{12}^\mu \\ -Q_{12} & -A_{21}^T \end{bmatrix},\\ T_3 &= \begin{bmatrix} A_{21} & -S_{12}^{\mu T} \\ -Q_{12}^T & -A_{12}^T \end{bmatrix},\\ T_4 &= \begin{bmatrix} A_{22} & -S_{22}^\mu \\ -Q_{22}^\mu & -A_{22}^T \end{bmatrix}. \end{aligned}$$

Let us define
$\Gamma_s := \{\mu > 0 |$the Riccati equation (8a) has a positive definite stabilizing solution$\}$,
$\mu_s := \sup\{\mu | \mu \in \Gamma_s\}$.
As the results, for every $0 < \mu < \bar{\mu} = \min\{\mu_s,\ \mu_f\}$, the Riccati equations (8a) and (8c) have the positive definite stabilizing solutions.
Now, let us introduce

$$\begin{aligned}& P_{11} = \bar{P}_{11} + \varepsilon E_{11},\ P_{21} = \bar{P}_{21} + \varepsilon E_{21},\\& P_{22} = \bar{P}_{22} + \varepsilon E_{22}. \end{aligned} \tag{10}$$

Substituting (10) into (6) and subtracting (6) from (7), we arrive at the error equations. Hence, we propose the following recursive algorithm (11) (Gajic *et al.* 1990).

$$\begin{aligned}& E_{11}^{T(i+1)} D_0 + D_0^T E_{11}^{(i+1)} = -V^T H_1^{T(i)}\\&\quad - H_1^{(i)} V + V^T H_3^{(i)} V + \varepsilon H_2^{(i)}, \end{aligned} \tag{11a}$$

$$\begin{aligned}& E_{11}^{T(i+1)} D_2 + E_{21}^{T(i+1)} D_4 + D_3^T E_{22}^{(i+1)}\\&\quad = H_1^{(i)}, \end{aligned} \tag{11b}$$

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$$E_{22}^{T(i+1)}D_4 + D_4^T E_{22}^{(i+1)} = H_3^{(i)}, \tag{11c}$$

where

$$\begin{aligned} H_1^{(i)} &= -A_{11}^T P_{21}^{T(i)} + P_{11}^{T(i)} S_{11}^{\mu} P_{21}^{T(i)} \\ &\quad + P_{21}^{T(i)} S_{12}^{\mu T} P_{21}^{T(i)} \\ &\quad + \varepsilon (E_{11}^{T(i)} S_{12}^{\mu} E_{22}^{(i)} + E_{21}^{T(i)} S_{22}^{\mu} E_{22}^{(i)}), \\ H_2^{(i)} &= E_{11}^{T(i)} S_{11}^{\mu} E_{11}^{(i)} + E_{21}^{T(i)} S_{22}^{\mu} E_{21}^{(i)} \\ &\quad + E_{11}^{T(i)} S_{12}^{\mu} E_{21}^{(i)} + E_{21}^{T(i)} S_{12}^{\mu T} E_{11}^{(i)}, \\ H_3^{(i)} &= -A_{12}^T P_{21}^{T(i)} - P_{21}^{(i)} A_{12} + \varepsilon P_{21}^{(i)} S_{11}^{\mu} P_{21}^{T(i)} \\ &\quad + \varepsilon E_{22}^{T(i)} S_{22}^{\mu} E_{22}^{(i)} + P_{21}^{(i)} S_{12}^{\mu} P_{22}^{(i)} \\ &\quad + P_{22}^{T(i)} S_{12}^{\mu T} P_{21}^{T(i)}, \\ D_1 &= A_{11} - S_{11}^{\mu} \bar{P}_{11} - S_{12}^{\mu} \bar{P}_{21}, \\ D_3 &= A_{21} - S_{12}^{\mu T} \bar{P}_{11} - S_{22}^{\mu} \bar{P}_{21}, \\ D_0 &= D_1 - D_2 D_4^{-1} D_3, \ V = D_4^{-1} D_3, \\ P_{11}^{(i)} &= \bar{P}_{11} + \varepsilon E_{11}^{(i)}, \ P_{21}^{(i)} = \bar{P}_{21} + \varepsilon E_{21}^{(i)}, \\ P_{22}^{(i)} &= \bar{P}_{22} + \varepsilon E_{22}^{(i)}, \ E_{11}^{(0)} = E_{21}^{(0)} = E_{22}^{(0)} = 0. \end{aligned}$$

Let $P_{11}^{(\infty)}$, $P_{21}^{(\infty)}$ and $P_{22}^{(\infty)}$ be the limit points of the recursive algorithm (11).
Our first observation is as follows.

Theorem 1 *Under the Assumption 1 and 2, if we can select a parameter μ such that $0 < \mu < \bar{\mu} = \min\{\mu_s, \mu_f\}$, then from the recursive algorithm (11) we have*

$$\begin{aligned} &P^{T(\infty)} A + A^T P^{(\infty)} - P^{T(\infty)} (\mu^{-1} B B^T \\ &\quad - F F^T) P^{(\infty)} + E^T E + \mu I = 0, \end{aligned} \tag{12}$$

where

$$P^{(\infty)} = \begin{bmatrix} P_{11}^{(\infty)} & \varepsilon P_{21}^{T(\infty)} \\ P_{21}^{(\infty)} & P_{22}^{(\infty)} \end{bmatrix}.$$

Furthermore, by using the linear state feedback full-order controller

$$u_{exa}(t) = -\frac{1}{2\mu} \begin{bmatrix} B_1^T & B_2^T \end{bmatrix} P^{(\infty)} x(t) \tag{13}$$

the uncertain linear singularly perturbed system (1) is quadratically stable.

Proof: By using the implicit function theorem, the theorem can be proved (Mukaidani et al.). ■

4.2 The Linear State Feedback ε-Independent Controller

Our attention in this section is focused on linear state feedback ε-independent controller design for the non-standard singularly perturbed systems. In Gajic *et al.* 1990 and Mukaidani *et al.*, by making use of the recursive algorithm, a linear state feedback controller for the standard or non-standard singularly perturbed systems are given. However, controller design need to repeat the recursive algorithm. Therefore, it will take a lot of time to get the required positive definite solution. We proposed that an ε-independent stabilizing controller can be obtained by making use of the solutions for reduced-order slow and fast algebraic Riccati equations, that is, 0-order solutions. By analogy with the linear state feedback controller (5), the linear state feedback ε-independent controller are obtained by neglecting $O(\varepsilon)$ for the linear state feedback controller (5). It is obvious that the linear state feedback controller (5) given in Lemma 2 can be approximated as

$$\begin{aligned} u(t) &= u_{exa}(t) \approx u_{app}(t) \\ &= -\frac{1}{2\mu} \begin{bmatrix} B_1^T \bar{P}_{11} + B_2^T \bar{P}_{21} & B_2^T \bar{P}_{22} \end{bmatrix} x(t). \end{aligned} \tag{14}$$

The main result of this section is as follows.

Theorem 2 *Under the Assumption 1 and 2, for small ε, if we can select a parameter μ such that $0 < \mu < \bar{\mu} = \min\{\mu_s, \mu_f\}$, then by using the linear state feedback ε-independent controller (14) the uncertain linear singularly perturbed system (1) is quadratically stable.*

Proof: Applying the proposed controller (14) to the system (1) yields a closed-loop system as follows:

$$\dot{x}(t) = (\bar{A}_\varepsilon + F_\varepsilon \Delta(t) E) x(t), \tag{15}$$

where

$$\begin{aligned} \bar{A}_\varepsilon &= \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \varepsilon^{-1} \bar{A}_{21} & \varepsilon^{-1} \bar{A}_{22} \end{bmatrix}, \\ \bar{A}_{i1} &= A_{i1} - \frac{1}{2\mu} (B_i B_1^T \bar{P}_{11} + B_i B_2^T \bar{P}_{21}), \\ \bar{A}_{i2} &= A_{i2} - \frac{1}{2\mu} B_i B_2^T \bar{P}_{22}, \ (i = 1, 2). \end{aligned}$$

By using Theorem (3.9)(see Khargonekar *et al.* 1990), we will consider the following algebraic Riccati equation

$$\begin{aligned} &\bar{A}_\varepsilon^T X_\varepsilon + X_\varepsilon \bar{A}_\varepsilon + X_\varepsilon F_\varepsilon F_\varepsilon^T X_\varepsilon \\ &\quad + E^T E + kI = 0, \ \forall k > 0, \end{aligned} \tag{16}$$

where

$$X_\varepsilon = \begin{bmatrix} X_{11} & \varepsilon X_{21}^T \\ \varepsilon X_{21} & \varepsilon X_{22} \end{bmatrix}.$$

In order to verify the existence of solution for the algebraic Riccati equation (16), substituting X_ε into (16) yields

$$\begin{aligned} &\bar{A}_{11}^T X_{11} + X_{11}^T \bar{A}_{11} + \bar{A}_{21}^T X_{21} + X_{21}^T \bar{A}_{21} \\ &\quad + X_{11}^T F_1 F_1^T X_{11} + X_{21}^T F_2 F_2^T X_{21} \\ &\quad + X_{11}^T F_1 F_2^T X_{21} + X_{21}^T F_2 F_1^T X_{11} \\ &\quad + R_{11} = 0, \end{aligned} \tag{17a}$$

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$$\begin{aligned}&\varepsilon X_{21}\tilde{A}_{11} + X_{22}^T\tilde{A}_{21} + \tilde{A}_{12}^T X_{11} + \tilde{A}_{22}^T X_{21}\\ &+\varepsilon X_{21}F_1F_1^T X_{11} + \varepsilon X_{21}F_1F_2^T X_{21}\\ &+ X_{22}^T F_2F_1^T X_{11} + X_{22}^T F_2F_2^T X_{21}\\ &+Q_{12}^T = 0, \qquad (17b)\end{aligned}$$

$$\begin{aligned}&\tilde{A}_{22}^T X_{22} + X_{22}^T\tilde{A}_{22} + \varepsilon\tilde{A}_{12}^T X_{21}^T + \varepsilon X_{21}\tilde{A}_{12}\\ &+X_{22}^T F_2F_2^T X_{22} + \varepsilon X_{22}^T F_2F_1^T X_{21}^T\\ &+\varepsilon X_{21}F_1F_2^T X_{22} + \varepsilon^2 X_{21}F_1F_1^T X_{21}^T\\ &+R_{22} = 0, \qquad (17c)\end{aligned}$$

where

$$R_{11} = E_1^T E_1 + kI,\ R_{22} = E_2^T E_2 + kI.$$

For the previous equations (17), setting $\varepsilon = 0$, we obtain the following equations

$$\begin{aligned}&\tilde{A}_{11}^T\bar{X}_{11} + \bar{X}_{11}^T\tilde{A}_{11} + \tilde{A}_{21}^T\bar{X}_{21} + \bar{X}_{21}^T\tilde{A}_{21}\\ &+\bar{X}_{11}^T F_1F_1^T\bar{X}_{11} + \bar{X}_{21}^T F_2F_2^T\bar{X}_{21}\\ &+\bar{X}_{11}^T F_1F_2^T\bar{X}_{21} + \bar{X}_{21}^T F_2F_1^T\bar{X}_{11}\\ &+R_{11} = 0, \qquad (18a)\end{aligned}$$

$$\begin{aligned}&\bar{X}_{22}^T\tilde{A}_{21} + \tilde{A}_{12}^T\bar{X}_{11} + \tilde{A}_{22}^T\bar{X}_{21} + \bar{X}_{22}^T F_2F_1^T\bar{X}_{11}\\ &+\bar{X}_{22}^T F_2F_2^T\bar{X}_{21} + Q_{12}^T = 0, \qquad (18b)\end{aligned}$$

$$\begin{aligned}&\tilde{A}_{22}^T\bar{X}_{22} + \bar{X}_{22}^T\tilde{A}_{22} + \bar{X}_{22}^T F_2F_2^T\bar{X}_{22}\\ &+R_{22} = 0. \qquad (18c)\end{aligned}$$

Thus, setting $k = \mu$ for equation (18) and comparing (18) with (7) yields

$$\bar{X}_{11} = \bar{P}_{11},\ \bar{X}_{21} = \bar{P}_{21},\ \bar{X}_{22} = \bar{P}_{22} \qquad (19)$$

directly. Now, let us introduce error term

$$\begin{aligned}&X_{11} = \bar{X}_{11} + \varepsilon M_{11},\ X_{21} = \bar{X}_{21} + \varepsilon M_{21},\\ &X_{22} = \bar{X}_{22} + \varepsilon M_{22}. \qquad (20)\end{aligned}$$

Substituting (20) into (17) and subtracting (17) from (18), we arrive at the error equations

$$\begin{aligned}&M_{11}^T\hat{D}_0 + \hat{D}_0^T M_{11} = \hat{V}^T\hat{H}_1^T\\ &+\hat{H}_1\hat{V} - \hat{V}^T\hat{H}_3\hat{V} - \varepsilon\hat{H}_2, \qquad (21a)\end{aligned}$$

$$M_{11}^T\hat{D}_2 + M_{21}^T\hat{D}_4 + \hat{D}_3^T M_{22} + \hat{H}_1 = 0, \quad (21b)$$

$$M_{22}^T\hat{D}_4 + \hat{D}_4^T M_{22} + \hat{H}_3 = 0, \qquad (21c)$$

where

$$\begin{aligned}\hat{H}_1 &= A_{11}^T X_{21}^T + X_{11}^T F_1F_1^T X_{21}^T\\ &+X_{21}^T F_2F_1^T X_{21}^T\\ &+\varepsilon(M_{11}^T F_1F_2^T M_{22} + M_{21}^T F_2F_2^T M_{22}),\\ \hat{H}_2 &= M_{11}^T F_1F_1^T M_{11} + M_{21}^T F_2F_2^T M_{21}\\ &+M_{11}^T F_1F_2^T M_{21} + M_{21}^T F_2F_1^T M_{11},\\ \hat{H}_3 &= \tilde{A}_{12}^T X_{21}^T + X_{21}\tilde{A}_{12} + \varepsilon X_{21}F_1F_1^T X_{21}^T\\ &+\varepsilon M_{22}^T F_2F_2^T M_{22} + X_{21}F_1F_2^T X_{22}\\ &+X_{22}^T F_2F_1^T X_{21}^T,\\ \hat{D}_1 &= \tilde{A}_{11} + F_1F_1^T\bar{X}_{11} + F_1F_2^T\bar{X}_{21},\\ \hat{D}_2 &= \tilde{A}_{12} + F_1F_2^T\bar{X}_{22},\\ \hat{D}_3 &= \tilde{A}_{21} + F_2F_1^T\bar{X}_{11} + F_2F_2^T\bar{X}_{21},\\ \hat{D}_4 &= \tilde{A}_{22} + F_2F_2^T\bar{X}_{22},\\ \hat{D}_0 &= \hat{D}_1 - \hat{D}_2\hat{D}_4^{-1}\hat{D}_3,\ \hat{V} = \hat{D}_4^{-1}\hat{D}_3.\end{aligned}$$

We need to show the existence of a bounded solution of M_{ij} $(i,\ j = 1,\ 2)$ in neighborhood of $\varepsilon = 0$. To prove that by the implicit function theorem (Gajic *et al.* 1990), it is enough to show that the corresponding Jacobian is non-singular at $\varepsilon = 0$. The Jacobian is given by

$$J|_{\varepsilon=0} = \begin{bmatrix} J_{11} & 0 & 0\\ J_{21} & J_{22} & J_{23}\\ 0 & 0 & J_{33}\end{bmatrix} \qquad (22)$$

where

$$\begin{aligned}J_{11} &= I\otimes\hat{D}_0 + \hat{D}_0^T\otimes I,\ J_{22} = I\otimes\hat{D}_4,\\ J_{33} &= I\otimes\hat{D}_4 + \hat{D}_4^T\otimes I,\\ J_{j1} &= \frac{\partial L_j}{\partial M_{11}}|_{\varepsilon=0},\ J_{j2} = \frac{\partial L_j}{\partial M_{21}}|_{\varepsilon=0},\\ J_{j3} &= \frac{\partial L_j}{\partial M_{22}}|_{\varepsilon=0},\ (j = 1,2,3),\\ L_1 &= M_{11}^T\hat{D}_0 + \hat{D}_0^T M_{11} - \hat{V}^T\hat{H}_1^T - \hat{H}_1\hat{V}\\ &+\hat{V}^T\hat{H}_3\hat{V} + \varepsilon\hat{H}_2,\\ L_2 &= M_{11}^T\hat{D}_2 + M_{21}^T\hat{D}_4 + \hat{D}_3^T M_{22} + \hat{H}_1,\\ L_3 &= M_{22}^T\hat{D}_4 + \hat{D}_4^T M_{22} + \hat{H}_3.\end{aligned}$$

When $0 < \mu < \bar{\mu} = \min\{\mu_s,\ \mu_f\}$, since the algebraic Riccati equation (7c) has a positive definite stabilizing solution, the matrix $\hat{D}_4 = \tilde{A}_{22} + F_2F_2^T\bar{P}_{22} = \tilde{A}_{22} + F_2F_2^T\bar{X}_{22}$ is non-singular. Similarly, the matrix $\hat{A} + \hat{S}\bar{P}_{11} = \hat{A} + \hat{S}\bar{X}_{11}$ is non-singular. Then, we obtain the following relation.

$$\hat{A} + \hat{S}\bar{P}_{11} = \hat{A} + \hat{S}\bar{X}_{11} = \hat{D}_0$$

Thus, the matrix $\hat{D}_0$ is stable also. Therefore, for a sufficiently small parameter ε, the Jacobian is non-singular. ■

5. DESIGN PROCEDURE AND EXAMPLE

In light of the above Theorem 2, the ε-independent controller can be obtained by using the solutions of the reduced-order slow and fast algebraic Riccati equations. Therefore, the design procedure of the proposed controller is simple. The basic steps are as follows.

Step 1. Calculate A_0^μ, S_0^μ, Q_0^μ by using relation (9).

Step 2. By making use of method of bisection, find the $\bar{\mu} = \min\{\mu_s,\ \mu_f\}$. This time, if a positive-definite symmetric solutions exists for the algebraic Riccati equations (8a) and (8c), then proceed to Step 3. If not or $\bar{\mu}$ is less than some computational accuracy μ, then stop and declare that system (1) is not quadratically stable.

Step 3. Calculate $\bar{P}_{21}$ in (8b) by using $\bar{P}_{11}$, $\bar{P}_{22}$.

Step 4. Substituting $\bar{P}_{11}$, $\bar{P}_{22}$ and $\bar{P}_{21}$ into (14), we obtain the ε-independent controller (14).

Example: Consider a non–standard singularly perturbed system

$$\begin{bmatrix} \dot{x}_1 \\ \varepsilon \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & \Delta(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad \|\Delta(t)\| \le 1. \quad (23)$$

It is obvious that the existing method (Shao and Sawan 1993) to find the stabilizing controller is not availed for this example. By making use of relation (9), we obtain A_0^μ, S_0^μ, Q_0^μ as follows.

$$A_0^\mu = 0, \; S_0^\mu = \frac{1}{\mu+1}, \; Q_0^\mu = \frac{\mu(2-\mu)}{1-\mu}.$$

From the algebraic Riccati equations (8a) and (8c), we obtain two quantities $\mu_f = \mu_s = 1$ analytically. Hence, $0 < \mu < \bar{\mu} = \min\{\mu_f, \; \mu_s\} = 1$. Then, we get

$$\begin{aligned} \bar{P}_{11} &= \sqrt{\frac{\mu(1+\mu)(2-\mu)}{1-\mu}}, \\ \bar{P}_{21} &= \frac{\mu}{1-\mu}(1+\sqrt{2-\mu}), \\ \bar{P}_{22} &= \sqrt{\frac{\mu(1+\mu)}{1-\mu}}. \end{aligned}$$

Therefore, we obtain an ε–independent controller as follows:

$$u = -\frac{1+\sqrt{2-\mu}}{2(1-\mu)}x_1 - \frac{1}{2}\sqrt{\frac{1+\mu}{\mu(1-\mu)}}x_2. \quad (24)$$

On the other hand, substituting the controller (24) into the system (23), the solution of the algebraic Riccati equation (16) is given by

$$X_\varepsilon = \begin{bmatrix} \alpha\beta & 0 \\ 0 & \varepsilon\beta \end{bmatrix}, \quad (25)$$

where

$$\alpha = \frac{2\mu - 1 + \sqrt{2-\mu}}{2(1-\mu)} > 0,$$

$$\beta = \frac{1}{2}\left(\sqrt{\frac{\mu+1}{\mu(1-\mu)}} - \sqrt{\frac{\mu+1}{\mu(1-\mu)} - 4}\right) > 0.$$

We observe that a matrix X_ε is positive definite symmetric solution because of $\alpha > 0$ and $\beta > 0$. Hence, the closed–loop uncertain system (23) with proposed ε–independent controller (24) is quadratically stable.

6. CONCLUSION

In this paper, the robust stabilization of singularly perturbed systems with uncertainties was investigated. ε–independent sufficient conditions for the existence of a controller were derived in a different way. We proposed a ε–independent stabilizing controller such that the closed–loop uncertain linear singularly perturbed systems is quadratically stable. This structure is achieved by solving slow and fast algebraic Riccati equations. The implicit function theorem is used to prove the sufficient condition for stability of the closed–loop system. It is pointed out that our results are applicable to both standard and non–standard singularly perturbed systems.

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