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AN ORDER REDUCTION PROCEDURE TO COMPOSITE NASH SOLUTION OF SINGULARLY PERTURBED SYSTEMS

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EXTENDED ABSTRACT

In this paper, a Nash game of singularly perturbed systems is studied in a viewpoint of descriptor system theory. A new method based on generalized coupled algebraic Riccati equations arising in descriptor systems is presented to find the composite Nash equilibrium solution of singularly perturbed systems. In this method, the full-order system is decomposed into a slow subsystem and a fast subsystem, and the slow subsystem is viewed as a special kind of descriptor systems. Through the solution of the slow Nash game, which is a Nash game for descriptor system, the composite Nash equilibrium solution is obtained. It is proven that the composite Nash equilibrium solution achieves a performance which is $O(\varepsilon)$ close to the full-order performance. Two numerical examples are given to show that the proposed method is valid for both standard and nonstandard singularly perturbed systems.

KEYWORDS

· **Singular perturbation method** · **Descriptor systems** · **Nash games** · **Differential game** · **Reduced-order models**

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Abstract: In this paper, a Nash game of singularly perturbed systems is studied in a viewpoint of descriptor system theory. A new method based on generalized coupled algebraic Riccati equations arising in descriptor systems is presented to find the composite Nash equilibrium solution of singularly perturbed systems. In this method, the full-order system is decomposed into a slow subsystem and a fast subsystem, and the slow subsystem is viewed as a special kind of descriptor systems. Through the solution of the slow Nash game, which is a Nash game for descriptor system, the composite Nash equilibrium solution is obtained. It is proven that the composite Nash equilibrium solution achieves a performance which is $O(\varepsilon)$ close to the full-order performance. Two numerical examples are given to show that the proposed method is valid for both standard and nonstandard singularly perturbed systems. *Copyright© 1999 IFAC*

Keywords: Singular perturbation method, descriptor systems, Nash games, differential game, reduced-order models

1. INTRODUCTION

The descriptor equation is a natural representation of dynamical systems which describe static and dynamic relations of physical variables explicitly. The descriptor systems has found many applications in large-scale systems, singularly perturbed systems, circuit theory, economics and other areas. As one of applications in singularly perturbed systems, the descriptor system is used in the study of various control problems and dynamic games for both standard and nonstandard singularly perburbed systems (Wang *et al.*, 1988, 1994; Xu *et al.*, 1997; Xu and Mizukami, 1997).

In this paper, a Nash game for singularly perturbed systems is studied in a viewpoint of descriptor system theory. A new method, based on generalized coupled algebraic Riccati equation arising in descriptor systems, is proposed to find the composite Nash equilibrium solution. In this method, the full-order system is decomposed into a slow subsystem and a fast subsystem and the slow subsystem is viewed as a special kind of descriptor systems. Through the solution of the slow Nash game, which is a Nash game for descriptor system, the composite Nash equilibrium solution is obtained. It is shown that the composite Nash equilibrium solution can be obtained simply by only revising the solution of the slow

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Nash game. Therefore, the method is a reduced-order method and is independent of the small parameter ε. It is proven that the composite Nash equilibrium solution achieves a performance which is $O(\varepsilon)$ close to the full-order performance. Two numerical examples are given to show that the proposed method is valid for both standard and nonstandard singularly perturbed systems.

This paper is organized as follows. In next section, the closed-loop Nash equilibrium equation of Player 1 and 2 for the full-order Nash game is given. In Section 3, the full-order problem is decomposed into a slow subsystem Nash game and a fast subsystem Nash game. Their solutions are investigated. In Section 4, the composite Nash equilibrium solution is obtained, and its near-optimality property is studied. Section 5 discusses some conclusions.

2. FULL-ORDER NASH GAME

Consider a linear time-invariant singularly perturbed system

$$\dot{x} = A_{11}x + A_{12}z + B_{11}u_1 + B_{12}u_2, \quad (1a)$$

$$\varepsilon\dot{z} = A_{21}x + A_{22}z + B_{21}u_1 + B_{22}u_2, \quad (1b)$$

with $x(0) = x_0, z(0) = z_0$ and performance criteria

$$J_i = \frac{1}{2}\int_0^\infty (\begin{bmatrix} x \\ z \end{bmatrix}^T Q_i \begin{bmatrix} x \\ z \end{bmatrix} + u_i^T R_{ii} u_i$$

$$+ u_j^T R_{ij} u_j)dt, \; i,j = 1,2, \; i \neq j \quad (2)$$

where $R_{ii} > 0, R_{ij} \geq 0$,

$$Q_i = \begin{bmatrix} Q_{i11} & Q_{i12} \\ Q_{i12}^T & Q_{i22} \end{bmatrix} = \begin{bmatrix} C_{i1}^T C_{i1} & C_{i1}^T C_{i2} \\ C_{i2}^T C_{i1} & C_{i2}^T C_{i2} \end{bmatrix}, \quad (3)$$

and ε is a small positive parameter, $x(t) \in R^n$ and $z(t) \in R^m$ are states, $u_1(t) \in R^{r_1}$, $u_2(t) \in R^{r_2}$ are the controls of Player 1 and Player 2, and all matrices are of appropriate dimensions. The system (1) is called the nonstandard singularly perturbed system if the matrix A_{22} is singular.

A Nash equilibrium solution is a pair (u_1^*, u_2^*) such that

$$J_i(u_i^*, u_j^*) \leq J_i(u_i, u_j^*), \quad i = 1,2, \quad i \neq j. \quad (4)$$

for all admissible u_i.

Let us consider the closed-loop Nash equilibrium solution to the full-order problem. Before doing that, let us define the following matrices.

$$A_\varepsilon = \begin{bmatrix} A_{11} & A_{12} \\ A_{21}/\varepsilon & A_{22}/\varepsilon \end{bmatrix}; \; A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, (5a)$$

$$B_{i\varepsilon} = \begin{bmatrix} B_{1i} \\ B_{2i}/\varepsilon \end{bmatrix}; \; B_i = \begin{bmatrix} B_{1i} \\ B_{2i} \end{bmatrix}, \quad (5b)$$

$$S_{i\varepsilon} = B_{i\varepsilon} R_{ii}^{-1} B_{i\varepsilon}^T; \; S_i = B_i R_{ii}^{-1} B_i^T, \quad (5c)$$

$$G_{j\varepsilon} = B_{j\varepsilon} R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_{j\varepsilon}^T; \quad (5d)$$

$$G_j = B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j^T, \quad (5e)$$

with $i,j = 1,2, i \neq j$. Furthermore, the klth block of matrices S_i, G_j are denoted henceforth by $S_{kli}, G_{klj}, k,l = 1,2$, and $S_{kli}^T = S_{lki}, G_{klj}^T = G_{lkj}, k \neq l$.

The closed-loop Nash equilibrium solution to the full-order problem is given by

$$u_i^* = -R_{ii}^{-1} B_{i\varepsilon} \hat{K}_i y, \quad i = 1,2, \quad i \neq j, \quad (6)$$

where $y := [x^T, z^T]^T$ and $\hat{K}_i$ is a unique stabilizing solution of the coupled algebraic Riccati equation

$$A_\varepsilon^T \hat{K}_i + \hat{K}_i A_\varepsilon + Q_i - \hat{K}_i S_{i\varepsilon} \hat{K}_i -$$

$$\hat{K}_i S_{j\varepsilon} \hat{K}_j - \hat{K}_j S_{j\varepsilon} \hat{K}_i + \hat{K}_j G_{j\varepsilon} \hat{K}_j = 0, \quad (7)$$

with $i,j = 1,2, i \neq j$. The above Nash equilibrium solution can also be expressed as, equivalently,

$$u_i^* = -R_{ii}^{-1} B_i K_i y, \quad (8)$$

where K_i is a unique stabilizing solution of the generalized coupled algebraic Riccati equation

$$(i) \; A^T K_i + K_i^T A + Q_i - K_i^T S_i K_i -$$

$$K_i^T S_j K_j - K_j^T S_j K_i + K_j^T G_j K_j = 0, \quad (9a)$$

$$(ii) \; E_\varepsilon K_i = K_i^T E_\varepsilon, \quad (9b)$$

where

$$E_\varepsilon = \begin{bmatrix} I_n & 0 \\ 0 & \varepsilon I_m \end{bmatrix}. \quad (10)$$

We have the relation $\hat{K}_i = E_\varepsilon K_i$.

3. DECOMPOSITION OF SLOW AND FAST GAMES

Similar to the standard singularly perturbed systems, we decompose the full-order game into two subsystem games.

Slow Nash game: Find a Nash equilibrium solution (u_{1s}^*, u_{2s}^*) of

$$J_{is} = \frac{1}{2}\int_0^\infty (\begin{bmatrix} x_s \\ z_s \end{bmatrix}^T Q_i \begin{bmatrix} x_s \\ z_s \end{bmatrix} + u_{is}^T R_{ii} u_{is}$$

$$+ u_{js}^T R_{ij} u_{js})dt, \quad i,j = 1,2, \; i \neq j \quad (11)$$

for the slow subsystem

$$E\dot{y}_s = Ay_s + B_1u_{1s} + B_2u_{2s},$$
$$Ey_s(0) = Ey_0, \quad (12)$$

where $y_s := [x_s^T, z_s^T]^T$, $E = E_\varepsilon|_{\varepsilon=0}$, A, B_i are defined in (5), and Q_i in (3).

Fast Nash game: Find a Nash equilibrium solution (u_{1f}^*, u_{2f}^*) of

$$J_{if} = \frac{1}{2}\int_0^\infty (z_f^T C_{i2}^T C_{i2} z_f + u_{if}^T R_{ii} u_{if}$$
$$+ u_{jf}^T R_{ij} u_{jf})dt, \quad i,j = 1,2, \ i \neq j \quad (13)$$

for the fast subsystem

$$\varepsilon \dot{z}_f = A_{22}z_f + B_{21}u_{1f} + B_{22}u_{2f},$$
$$z_f(0) = z_0 - z_s(0), \quad (14)$$

where $z_f = z - z_s$, $u_{1f} = u_1 - u_{1s}$ and $u_{2f} = u_2 - u_{2s}$.

The fast subsystem (14) is derived by assuming that the slow variables are constant during fast transients, that is, $\dot{z}_s = 0$ and $x_s =$ a constant.

Let us first consider the solution of the fast Nash game.

Proposition 1. The fast Nash game admits a Nash equilibrium solution

$$u_{if}^* = -R_{ii}^{-1}B_{2i}^T K_{if} z_f, \quad i = 1,2, \quad (15)$$

if there exists a unique stabilizing solution to the coupled algebraic Riccati equations

$$A_{22}^T K_{if} + K_{if}A_{22} + C_{i2}^T C_{i2}^T - K_{if}S_{22i}K_{if}$$
$$-K_{if}S_{22j}K_{jf} - K_{jf}S_{22j}K_{if}$$
$$+K_{jf}G_{22j}K_{jf} = 0, \quad (16)$$

with $i, j = 1, 2, \ i \neq j$.

Next let us consider the solution of the slow Nash game. Before doing that, let us first introduce the other generalized coupled algebraic Riccati equations.

$$(i) \ A^T P_{is} + P_{is}^T A + Q_i - P_{is}^T S_i P_{is} -$$
$$P_{is}^T S_j P_{js} - P_{js}^T S_j P_{is} + P_{js}^T G_j P_{js} = 0, \quad (17a)$$
$$(ii) \ EP_{is} = P_{is}^T E, \quad (17b)$$

with $i, j = 1, 2, \ i \neq j$. The solution P_{is} of (17) takes a lower-triangular block form

$$P_{is} = \begin{bmatrix} P_{11is} & 0 \\ P_{21is} & P_{22is} \end{bmatrix}, \quad P_{11is}^T = P_{11is}, \quad (18)$$

because of (17b). It is worthy to note that P_{22is} may not be symmetric. The coupled algebraic Riccati equations (17) can be partitioned into

$$P_{11is}A_{11} + P_{21is}^T A_{21} + A_{11}^T P_{11is} + A_{21}^T P_{21is}$$
$$-P_{11is}S_{11i}P_{11is} - P_{21is}^T S_{12i}^T P_{11is}$$
$$-P_{11is}S_{12i}P_{21is} - P_{21is}^T S_{22i}P_{21is}$$
$$-P_{11js}S_{11j}P_{11is} - P_{21js}^T S_{12j}^T P_{11is}$$
$$-P_{11js}S_{12j}P_{21is} - P_{21js}^T S_{22j}P_{21is}$$
$$-P_{11is}S_{11j}P_{11js} - P_{21is}^T S_{12j}^T P_{11js}$$
$$-P_{11is}S_{12j}P_{21js} - P_{21is}^T S_{22j}P_{21js}$$
$$+P_{11js}G_{11j}P_{11js} + P_{21js}^T G_{12j}^T P_{11js}$$
$$+P_{11js}G_{12j}P_{21js} + P_{21js}^T G_{22j}P_{21js}$$
$$+Q_{i11} = 0, \quad (19a)$$

$$P_{22is}^T A_{21} + A_{12}^T P_{11is} + A_{22}^T P_{21is}$$
$$-P_{22is}^T S_{12i}^T P_{11is} - P_{22is}^T S_{22i}P_{21is}$$
$$-P_{22js}^T S_{12j}^T P_{11is} - P_{22js}^T S_{22j}P_{21is}$$
$$-P_{22is}^T S_{12j}^T P_{11js} - P_{22is}^T S_{22j}P_{21js}$$
$$+P_{22js}^T G_{12j}^T P_{11js} + P_{22js}^T G_{22j}P_{21js}$$
$$+Q_{i12}^T = 0, \quad (19b)$$

$$P_{22is}^T A_{22} + A_{22}^T P_{22is} - P_{22is}^T S_{22i}P_{22is}$$
$$-P_{22js}^T S_{22j}P_{22is} - P_{22is}^T S_{22j}P_{22js}$$
$$+P_{22js}^T G_{22j}P_{22js} + Q_{i22} = 0. \quad (19c)$$

Suppose that (19c) admit a real solution pair (P_{221s}, P_{222s}). Then, (19b) can be rewritten as

$$\hat{A}_{22}^T P_{21is} - W_{22j}^T P_{21js} + \hat{A}_{12}^T P_{11is} + W_{12j}^T P_{11js}$$
$$+\hat{Q}_{i12}^T = 0, \quad i,j = 1,2, \ i \neq j, \quad (20)$$

where

$$\hat{A}_{12} = A_{12} - S_{121}P_{221s} - S_{122}P_{222s}, \quad (21a)$$
$$\hat{A}_{22} = A_{22} - S_{221}P_{221s} - S_{222}P_{222s}, \quad (21b)$$
$$W_{12i} = G_{12i}P_{22is} - S_{12i}P_{22js}, \quad (21c)$$
$$W_{22i} = S_{22i}P_{22js} - G_{22i}P_{22is}, \quad (21d)$$
$$\hat{Q}_{i12} = Q_{i12} + A_{21}^T P_{22is}. \quad (21e)$$

From (20),

$$\begin{bmatrix} \hat{A}_{22}^T & -W_{222}^T \\ -W_{221}^T & \hat{A}_{22}^T \end{bmatrix} \begin{bmatrix} P_{211s} \\ P_{212s} \end{bmatrix} =$$
$$-\begin{bmatrix} \hat{A}_{12}^T & -W_{122}^T \\ -W_{121}^T & \hat{A}_{12}^T \end{bmatrix} \begin{bmatrix} P_{111s} \\ P_{112s} \end{bmatrix}$$

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$$= \begin{bmatrix} Q_{112}^T \\ Q_{212}^T \end{bmatrix}. \tag{22}$$

Therefore, solving the equations (22) gives (P_{211s}, P_{212s}) which are expressed in terms of (P_{111s}, P_{112s}).

$$P_{21is} = -Q_{si}^T - \hat{W}_{ii} P_{11is} - \hat{W}_{ij} P_{11js}, \quad i, j = 1, 2, \; i \neq j, \tag{23}$$

where, the inverse of the corresponding matrix is assumed to exist and the explicit expressions for Q_{si}, $\hat{W}_{ii}$, $\hat{W}_{ij}$ are omitted for brevity. Substituting (23) into (19a) arrive at nonstandard coupled algebaric Riccati equations of (P_{111s}, P_{112s}), that is,

$$\begin{aligned}
&P_{11is} A_{11} - (Q_{si}^T + \hat{W}_{ii} P_{11is} + \hat{W}_{ij} P_{11js})^T A_{21} \\
&+A_{11}^T P_{11is} - A_{21}^T (Q_{si}^T + \hat{W}_{ii} P_{11is} + \hat{W}_{ij} P_{11js}) \\
&-P_{11is} S_{11i} P_{11is} + (Q_{si}^T + \hat{W}_{ii} P_{11is} \\
&+\hat{W}_{ij} P_{11js})^T S_{12i}^T P_{11is} \\
&+P_{11is} S_{12i} (Q_{si}^T + \hat{W}_{ii} P_{11is} + \hat{W}_{ij} P_{11js}) \\
&-(Q_{si}^T + \hat{W}_{ii} P_{11is} + \hat{W}_{ij} P_{11js})^T S_{22i} (Q_{si}^T \\
&+\hat{W}_{ii} P_{11is} + \hat{W}_{ij} P_{11js}) \\
&-P_{11js} S_{11j} P_{11is} + (Q_{sj}^T + \hat{W}_{jj} P_{11js} \\
&+\hat{W}_{ji} P_{11is})^T S_{12j}^T P_{11is} \\
&+P_{11js} S_{12j} (Q_{si}^T + \hat{W}_{ii} P_{11is} + \hat{W}_{ij} P_{11js}) \\
&-(Q_{sj}^T + \hat{W}_{jj} P_{11js} + \hat{W}_{ji} P_{11is})^T S_{22j} (Q_{si}^T \\
&+\hat{W}_{ii} P_{11is} + \hat{W}_{ij} P_{11js}) \\
&-P_{11is} S_{11j} P_{11js} + (Q_{si}^T + \hat{W}_{ii} P_{11is} \\
&+\hat{W}_{ij} P_{11js})^T S_{12j}^T P_{11js} \\
&+P_{11is} S_{12j} (Q_{sj}^T + \hat{W}_{jj} P_{11js} + \hat{W}_{ji} P_{11is}) \\
&-(Q_{si}^T + \hat{W}_{ii} P_{11is} + \hat{W}_{ij} P_{11js})^T S_{22j} (Q_{sj}^T \\
&+\hat{W}_{jj} P_{11js} + \hat{W}_{ji} P_{11is}) \\
&+P_{11js} G_{11j} P_{11js} - (Q_{sj}^T + \hat{W}_{jj} P_{11js} \\
&+\hat{W}_{ji} P_{11is})^T G_{12j}^T P_{11js} \\
&-P_{11js} G_{12j} (Q_{sj}^T + \hat{W}_{jj} P_{11js} + \hat{W}_{ji} P_{11is}) \\
&+(Q_{sj}^T + \hat{W}_{jj} P_{11js} + \hat{W}_{ji} P_{11is})^T G_{22j} (Q_{sj}^T \\
&+\hat{W}_{jj} P_{11js} + \hat{W}_{ji} P_{11is}) \\
&+Q_{i11} = 0, \quad i, j = 1, 2, \; i \neq j.
\end{aligned} \tag{24}$$

Clearly, the existence of a real solution to the coupled algebraic Riccati equations (19c) and a real solution to the nonstandard coupled algebraic Riccati equations (24) is sufficient for the existence of a real solution to the generalized coupled algebraic Riccati equations (17) suppose that the related inverse matrix exists.

Proposition 2. Suppose that the generalized coupled algebraic Riccati equation (17) admit an impulse-free and stabilizing solution P_{is} and EP_{is} is unique. Then, the slow Nash game admits a Nash equilibrium solution, given by

$$u_{is}^* = -R_{ii}^{-1} B_i^T P_{is} y_s, \quad i = 1, 2. \tag{25}$$

Remark 3. Similar to descriptor systems, an important feature of the slow Nash game is that the feedback Nash equilibrium solution is not unique. This fact is clear if we note that any real solution of the coupled algebraic Riccati equation (19c) is allowed in the feedback gain of (25). But $EP_{is} = P_{is}^T E$ is required to be unique.

4. NEAR-OPTIMALITY OF COMPOSITE NASH EQUILIBRIUM

In this section, let us construct a composite Nash equilibrium solution $u_{ic}^* = u_{is}^* + u_{if}^*$ and analyze its near-optimality. It has been known that the Nash equilibrium solution for the slow Nash game is not unique. However, the corresponding Nash equilibrium solution for the fast Nash game is unique. Comparing (16) and (19c), it is found that they have the same parameter matrices. This means that a unique stabilizing solution of (16) is also a solution of (19c). Taking the solution $P_{22is} = K_{if}$, we get a special Nash equilibrium solution to the slow Nash game, denoted by

$$u_{is}^{*+} = -R_{ii}^{-1} B_i^T P_{is}^+ y_s, \tag{26}$$

where

$$P_{is}^+ = \begin{bmatrix} P_{11is}^+ & 0 \\ P_{21is}^+ & P_{22is}^+ \end{bmatrix}, \tag{27}$$

$P_{22is}^+ = K_{if}$ and P_{11is}^+ is the corresponding solution of (24) when $P_{22is} = P_{22is}^+$. P_{21is}^+ is obtained from (22).

From the analyses above, it is obtained readily

$$\begin{aligned}
u_{ic}^* &= u_{is}^{*+} + u_{if}^* = -R_{ii}^{-1} [B_{1i}^T \;\; B_{2i}^T] \times \\
&\begin{bmatrix} P_{11is}^+ & 0 \\ P_{21is}^+ & P_{22is}^+ \end{bmatrix} \begin{bmatrix} x_s \\ z_s \end{bmatrix} - R_{ii}^{-1} B_{2i}^T K_{if} z_f = \\
&-R_{ii}^{-1} [B_{1i}^T \;\; B_{2i}^T] \begin{bmatrix} P_{11is}^+ & 0 \\ P_{21is}^+ & P_{22is}^+ \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix},
\end{aligned} \tag{28}$$

where $x(t) \approx x_s(t)$ and $z(t) \approx z_s(t) + z_f(t)$.

Remark 4. Let us compare (25) with (28). it is found that u_{ic}^* is obtained by computing and

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revising the Nash equilibrium solution u_{is}^* to the slow Nash game.

Now, let us apply the composite Nash equilibrium solution u_c^* to the full-order system (1) and compare it with the exact Nash equilibrium solution (8). In order to do that, let us first make the following remarks on the unique stabilizing solution K_i of the generalized coupled algebraic Riccati equation (9).

Under certain conditions, for example, the existence of the solutions to the coupled algebraic Riccati equations (19c),(24) and the existence of the related inverse matrices, the generalized coupled algebraic Riccati equations (9) admit a unique stabilizing solution K_i which possesses a power series expansion at $\varepsilon = 0$, that is,

$$K_i = \begin{bmatrix} K_{11i}^{(0)} & \varepsilon K_{21i}^{(0)T} \\ K_{21i}^{(0)} & K_{22i}^{(0)} \end{bmatrix} + \sum_{k=1}^{\infty} \frac{\varepsilon^k}{k!} \begin{bmatrix} K_{11i}^{(k)} & \varepsilon K_{21i}^{(k)T} \\ K_{21i}^{(k)} & K_{22i}^{(k)} \end{bmatrix}, \quad i = 1, 2. \quad (29)$$

Now, let us compare the composite Nash equilibrium solution u_{ic}^* with the exact Nash equilibrium solution u_i^* and show the $O(\varepsilon)$ approximation of J_i^*. Applying the composite Nash equilibrium solution u_{ic}^* to the full-order system (1), we have

$$J_{ic}^* = \frac{1}{2} y^T(0) E_\varepsilon P_{ic} y(0), \quad (30)$$

where P_{ic} is the solution of the generalized Lyapunov equations

$$\begin{aligned} (i)\ & (A - S_1 P_{1s}^+ - S_2 P_{2s}^+)^T P_{ic} \\ & + P_{ic}^T (A - S_1 P_{1s}^+ - S_2 P_{2s}^+) \\ = & -P_{is}^{+T} S_i P_{is}^+ - P_{js}^{+T} G_j P_{js}^+ - Q_i, \quad (31a) \end{aligned}$$

$$(ii)\ E_\varepsilon P_{ic} = P_{ic}^T E_\varepsilon. \quad (31b)$$

Similar to K_i, the generalized Lyapunov equations (31) also admit a unique stabilizing solution P_{ic} which possesses a power series expansion at $\varepsilon = 0$, that is,

$$P_{ic} = \begin{bmatrix} P_{11ic}^{(0)} & \varepsilon P_{21ic}^{(0)T} \\ P_{21ic}^{(0)} & P_{22ic}^{(0)} \end{bmatrix} + \sum_{k=1}^{\infty} \frac{\varepsilon^k}{k!} \begin{bmatrix} P_{11ic}^{(k)} & \varepsilon P_{21ic}^{(k)T} \\ P_{21ic}^{(k)} & P_{22ic}^{(k)} \end{bmatrix}, \quad i = 1, 2. \quad (32)$$

Theorem 1. The first term of the power series of J_{ic}^* and J_i^* at $\varepsilon = 0$ are the same, that is,

$$J_{ic}^* = J_i^* + O(\varepsilon), \quad i = 1, 2, \quad (33)$$

and hence the composite Nash equilibrium solution is an $O(\varepsilon)$ near-optimal solution to the full-order Nash game (1),(2).

To the end of this section, the design procedure to find the composite Nash equilibrium solution will be summarized and two numerical examples will be solved.

Step 1. Find the unique stabilizing solution K_{if} of the coupled algebraic Riccati equations (16).

Step 2. Based on K_{if}, find the impulse-free stabilizing solution P_{is}^+ of the generalized coupled algebraic Riccati equations (17).

Step 3. Substitute $y_s(t)$ of (26) by $y(t)$ to get the composite Nash equilibrium solution (28) directly.

Different from the optimal control problem and zero-sum differential game (Xu *et al.* 1997 and Xu and Mizukami, 1997), parallel computations for the solutions of (16),(17) are impossible.

Example 5. Consider a standard singularly perturbed system

$$\begin{bmatrix} \dot{x}(t) \\ \varepsilon \dot{z}(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u_1(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u_2(t), \quad x(0) = 1, \quad z(0) = 2. \quad (34)$$

The performance criteria are

$$J_1 = \frac{1}{2} \int_0^\infty \{ \begin{bmatrix} x \\ z \end{bmatrix}^T \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + u_1^2 + 2u_2^2 \} dt. \quad (35a)$$

$$J_2 = \frac{1}{2} \int_0^\infty \{ \begin{bmatrix} x \\ z \end{bmatrix}^T \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + 2u_1^2 + u_2^2 \} dt. \quad (35b)$$

Following the steps 1-3, the composite Nash equilibrium solution is obtained as follows.

$$u_{1c}^* = -[1 \ \ 2] \times \begin{bmatrix} \sqrt{2}/2 & 0 \\ (\sqrt{2}+1)(\sqrt{3}-1)/4 & (\sqrt{3}-1)/2 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}, \quad (36a)$$

$$u_{2c}^* = -[1 \ \ 2] \times \begin{bmatrix} \sqrt{2}/2 & 0 \\ (\sqrt{2}+1)(\sqrt{3}-1)/4 & (\sqrt{3}-1)/2 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}. \quad (36b)$$

The resulting performance values for several values of ε when the full-order Nash equilibrium

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solution and the composite Nash equilibrium solution are applied respectively are given in Table 1, where $J_1 = J_2 = J$ since symmetric property.

Table 1 Comparing the numerical results

ε	0.5	0.2	0.1	0.01	0.001
$J^*(\varepsilon)$	1.30	0.73	0.54	0.37	0.36
$J_c^*(\varepsilon)$	1.24	0.73	0.54	0.37	0.36

It is seen that $J^*(\varepsilon)$ and $J_c^*(\varepsilon)$ have the same limit as $\varepsilon \to 0$.

Example 6. Further consider a nonstandard singularly perturbed system

$$\begin{bmatrix} \dot{x}(t) \\ \varepsilon \dot{z}(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u_1(t)$$

$$+ \begin{bmatrix} 1 \\ 2 \end{bmatrix} u_2(t), \quad x(0) = 1, \quad z(0) = 2. \qquad (37)$$

The performance criteria are the same as (35). Following the steps 1-3, the composite Nash equilibrium solution is obtained as follows.

$$u_{1c}^* = -[1 \;\; 2]\times$$

$$\begin{bmatrix} (1+\sqrt{15})/4 & 0 \\ (3\sqrt{2}+\sqrt{30}-3-\sqrt{15})/8 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}, \quad (38a)$$

$$u_{2c}^* = -[1 \;\; 2]\times$$

$$\begin{bmatrix} (1+\sqrt{15})/4 & 0 \\ (3\sqrt{2}+\sqrt{30}-3-\sqrt{15})/8 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}, \quad (38b)$$

The resulting performance values for several values of ε when the full-order Nash equilibrium solution and the composite Nash equilibrium solution are applied respectively are given in Table 2, where $J_1 = J_2 = J$ since symmetric property.

Table 2 Comparing the numerical results

ε	0.5	0.1	0.01	0.001	0.0001
$J^*(\varepsilon)$	1.77	0.85	0.63	0.61	0.61
$J_c^*(\varepsilon)$	1.69	0.84	0.63	0.61	0.61

$J^*(\varepsilon)$ and $J_c^*(\varepsilon)$ have the same limit as $\varepsilon \to 0$.

5. CONCLUSIONS

In this paper, by using the generalized coupled algebraic Riccati equation arising in descriptor systems, the composite Nash game for singularly perturbed systems has been studied. The same problem has been treated by Gardner and Cruz (1978) in which the Hierarchical Reduction Scheme is proposed. An important observation in that paper is that the usual order reduction procedure for singularly perturbed optimal control systems does not lead to a well-posed problem when extended directly to the linear-quadratic nonzero-sum closed-loop Nash game. In this paper, the full-order system is decomposed into a slow subsystem and a fast subsystem, and the slow subsystem is viewed as a special kind of descriptor systems. It is shown that the proposed method really leads to a well-posed problem. Combining the results in this paper and those in authors's other papers (Xu *et al.* 1997 and Xu and Mizukami, 1997), it is claimed that the proposed order reduction procedure, that is, viewing the slow subsystem as a descriptor system, will lead to a well-posed solution no matter what a problem is, for example, an optimal control problem, a zero-sum differential game or a nonzero-sum differential game.

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