Infinite Horizon Differential Games for Multiparameter Singularly Perturbed Systems

Hua Xu[1] and Hiroaki Mukaidani[2]
[1]Graduate School of Business Sciences, The University of Tsukuba
3-29-1, Otsuka, Bunkyo-ku, Tokyo 112-0012, Japan
E-mail:xuhua@gssm.otsuka.tsukuba.ac.jp
[2] Graduate School of Education, Hiroshima University
1-1-1 Kagamiyama, Higashi-Hiroshima, 739-8524 Japan
E-mail:mukaida@hiroshima-u.ac.jp

Abstract - In this paper, we study the infinite horizon zero-sum differential games for both standard and nonstandard multiparameter singularly perturbed systems. A composite approximation of the full-order linear feedback saddle-point solution is obtained by decomposing the full-order game problem into a slow game and two fast control problems. It is proven that such a composite approximation forms an $O(\|\mu\|)$ (near) saddle-point equilibrium of the full-order game, and the resulting value is $O(\|\mu\|^2)$ over or below the exact value of the full-order game depending on the given game parameters.

Keywords— **Multiparameter, singular perturbation, differential games, saddle-point equilibrium.**

I. INTRODUCTION

In this paper, we study the infinite horizon zero-sum differential games of multiparameter singularly perturbed systems. A composite approximation of the full-order linear feedback saddle-point solution is obtained by decomposing the full-order game into a slow game and two fast control problems. It is anticipated that such a composite approximation can achieve an $O(\|\mu\|)$ (near) saddle-point equilibrium of the full-order game, and the resulting value is $O(\|\mu\|^2)$ over or below the exact value of the full-order game which depends on the parameters of the system. In order to prove these anticipatations, we first investigate the properties of the multiparameter algebraic Riccati equation(MARE) and establish its asymptotic structure. Then, we compare the structures of the composite approximation of the full-order linear feedback saddle-point solution and the main terms of the asymptotic expansions of the MARE. We find that the main terms, which are independent of the small positive singular perturbation parameters, in the asymptotic expansions of the MARE determine the composite approximation. As the result, we prove that the anticipations stated above are correct.

Since we treat the slow game as the game for a special kind of descriptor systems, the results obtained are valid for both standard and nonstandard multiparameter singularly perturbed systems.

II. PROBLEM FORMULATION

We consider the linear time-invariant multiparameter singularly perturbed systems(MSPS)

$$\begin{aligned}
\dot{x}_0(t) &= A_{00}x_0(t) + A_{01}x_1(t) + A_{02}x_2(t) \\
&\quad + B_{01}u_1(t) + B_{02}u_2(t),\ x_0(0) = x_0^0, && \text{(1a)} \\
\varepsilon_1\dot{x}_1(t) &= A_{10}x_0(t) + A_{11}x_1(t) + B_{11}u_1(t), \\
&\quad x_1(0) = x_1^0, && \text{(1b)} \\
\varepsilon_2\dot{x}_2(t) &= A_{20}x_0(t) + A_{22}x_2(t) + B_{22}u_2(t), \\
&\quad x_2(0) = x_2^0, && \text{(1c)}
\end{aligned}$$

where $x_j \in \mathbf{R}^{n_j},\ j = 0,\ 1,\ 2$ are the state vectors, $u_1 \in \mathbf{R}^{m_1}$ is the control vector of Player 1, and $u_2 \in \mathbf{R}^{m_2}$ is the control vector of Player 2. All the matrices above are constant matrices of appropriate dimensions.
ε_1 and ε_2 are two unknown small positive singular perturbation parameters of the same order of magnitude such that

$$0 < k_1 \le \alpha \equiv \frac{\varepsilon_1}{\varepsilon_2} \le k_2 < \infty. \tag{2}$$

That is, we assume that the ratio of ε_1 and ε_2 is bounded by some positive constants $k_j,\ j = 1,\ 2$. Since we do not know the values of ε_1 and ε_2, we can not reduce them to a single singular perturbation parameter. Note that the fast state matrices $A_{jj}, j = 1,\ 2$ may be singular. The system (1) is called a standard multiparameter singularly perturbed system if the matrix A_{jj} is nonsingular, otherwise it is called a nonstandard MSPS.
In a differential game problem of the MSPS, a quadratic cost functional is given by

$$\begin{aligned}
J &= \frac{1}{2}\int_0^\infty [z^T(t)z(t) + u_1^T(t)R_1u_1(t) \\
&\quad - u_2^T(t)R_2u_2(t)]dt, && \text{(3)}
\end{aligned}$$

where $z^T(t) = [z_0^T(t)\ z_1^T(t)\ z_2^T(t)]$, $z_0(t) = C_{00}x_0(t) \in \mathbf{R}^{r_0}$, $z_j(t) = C_{j0}x_0(t) + C_{jj}x_j(t) \in \mathbf{R}^{r_j},\ j = 1,\ 2$. The goal of Player 1 is to minimize the cost function J while Player 2 would like to maximize it.

Definition 1. An admissible feedback strategy pair $(u_1^*, u_2^*) \in \Gamma_{u_1} \times \Gamma_{u_2}$ is in a saddle-point equilibrium if

$$J(u_1^*, u_2) \le J(u_1^*, u_2^*) \le J(u_1, u_2^*), \tag{4}$$

for all $(u_1^*, u_2) \in \Gamma_{u_1} \times \Gamma_{u_2}$ and $(u_1, u_2^*) \in \Gamma_{u_1} \times \Gamma_{u_2}$.

In Definition 1, the admissible strategy pairs set $\Gamma_{u_1} \times \Gamma_{u_2}$ is composed of the feedback strategy pair (u_1, u_2) such that the obtained closed-loop system is strictly feedback stabilizing for all initial conditions $x_0(0) = x_0^0, x_1(0) = x_1^0$ and $x_2(0) = x_2^0$. The existence of a saddle-point equilibrium is a strong condition in general. A weaker version of saddle-point equilibrium, called μ (near) saddle-point equilibrium, is defined below (Başar and Olsder, 1995).

Definition 2. For a given $\mu \ge 0$, an admissible feedback strategy pair $(u_1^*, u_2^*) \in \Gamma_{u_1} \times \Gamma_{u_2}$ is in a μ (near) saddle-point equilibrium if

$$J(u_1^*, u_2) - \mu \le J(u_1^*, u_2^*) \le J(u_1, u_2^*) + \mu \tag{5}$$

for all $(u_1^*, u_2) \in \Gamma_{u_1} \times \Gamma_{u_2}$ and $(u_1, u_2^*) \in \Gamma_{u_1} \times \Gamma_{u_2}$.

From the existing theory on linear quadratic differential game, we know that for $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, the zero-sum differential game described by (1) and (3) has equal upper and lower values if the MARE

$$A_{\mathcal{E}}^T P_{\mathcal{E}} + P_{\mathcal{E}} A_{\mathcal{E}} - P_{\mathcal{E}} S_{\mathcal{E}} P_{\mathcal{E}} + Q = 0, \tag{6}$$

admits a minimal positive definite symmetric solution $P_{\mathcal{E}}^+$, where

$$P_{\mathcal{E}} = \begin{bmatrix} P_{00} & \varepsilon_1 P_{10}^T & \varepsilon_2 P_{20}^T \\ \varepsilon_1 P_{10} & \varepsilon_1 P_{11} & \sqrt{\varepsilon_1 \varepsilon_2} P_{21}^T \\ \varepsilon_2 P_{20} & \sqrt{\varepsilon_1 \varepsilon_2} P_{21} & \varepsilon_2 P_{22} \end{bmatrix}$$

$$P_{00} = P_{00}^T, \;\; P_{11} = P_{11}^T, \;\; P_{22} = P_{22}^T,$$

$$A_{\mathcal{E}} = \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ \varepsilon_1^{-1} A_{10} & \varepsilon_1^{-1} A_{11} & 0 \\ \varepsilon_2^{-1} A_{20} & 0 & \varepsilon_2^{-1} A_{22} \end{bmatrix},$$

$$S_{\mathcal{E}} = S_{\mathcal{E}}^T = \begin{bmatrix} S_{00} & \varepsilon_1^{-1} S_{01} & \varepsilon_2^{-1} S_{02} \\ \varepsilon_1^{-1} S_{01}^T & \varepsilon_1^{-2} S_{11} & 0 \\ \varepsilon_2^{-1} S_{02}^T & 0 & \varepsilon_2^{-2} S_{22} \end{bmatrix}$$

$$= B_{1\mathcal{E}} R_1^{-1} B_{1\mathcal{E}}^T - B_{2\mathcal{E}} R_2^{-1} B_{2\mathcal{E}}^T,$$

$$B_{1\mathcal{E}} = \begin{bmatrix} B_{01} \\ \varepsilon_1^{-1} B_{11} \\ 0 \end{bmatrix}, \;\; B_{2\mathcal{E}} = \begin{bmatrix} B_{02} \\ 0 \\ \varepsilon_2^{-1} B_{22} \end{bmatrix},$$

$$Q = Q^T = \begin{bmatrix} Q_{00} & Q_{01} & Q_{02} \\ Q_{01}^T & Q_{11} & 0 \\ Q_{02}^T & 0 & Q_{22} \end{bmatrix},$$

$$Q_{00} = C_{00}^T C_{00} + C_{10}^T C_{10} + C_{20}^T C_{20},$$
$$Q_{11} = C_{11}^T C_{11}, \;\; Q_{22} = C_{22}^T C_{22},$$
$$Q_{01} = C_{10}^T C_{11}, \;\; Q_{02} = C_{20}^T C_{22},$$

and the value of the game is given by

$$J^* = \frac{1}{2} x_0^T P_{\mathcal{E}}^+ x_0, \tag{7}$$

where $x_0^T = [x_0^{0T} \; x_1^{0T} \; x_2^{0T}]$. Moreover, if we further assume that $P_{\mathcal{E}}^+$ is the strictly feedback stabilizing solution to the MARE (6), then the linear feedback strategy pair

$$u_1^*(t) = -R_1^{-1} B_{1\mathcal{E}}^T P_{\mathcal{E}}^+ x(t), \tag{8a}$$
$$u_2^*(t) = R_2^{-1} B_{2\mathcal{E}}^T P_{\mathcal{E}}^+ x(t), \tag{8b}$$

is in saddle-point equilibrium in the restricted class of feedback strategies that are strictly feedback stabilizing and under which $x(t) \to 0$ as $t \to \infty$ for all x_0.

For the convenience of the comparison between the full-order saddle-point equilibrium and the near saddle-point equilibrium later, we introduce the following useful lemma.

Lemma 1: The MARE (6) is equivalent to the following generalized multiparameter algebraic Riccati equation (GMARE) (9a)

$$A^T P + P^T A - P^T S P + Q = 0, \tag{9a}$$
$$P_{\mathcal{E}} = \Phi_{\mathcal{E}} P = P^T \Phi_{\mathcal{E}}, \tag{9b}$$

where

$$\Phi_{\mathcal{E}} = \begin{bmatrix} I_{n_0} & 0 & 0 \\ 0 & \varepsilon_1 I_{n_1} & 0 \\ 0 & 0 & \varepsilon_2 I_{n_2} \end{bmatrix},$$

$$A = \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & 0 \\ A_{20} & 0 & A_{22} \end{bmatrix},$$

$$S = S^T = \begin{bmatrix} S_{00} & S_{01} & S_{02} \\ S_{01}^T & S_{11} & 0 \\ S_{02}^T & 0 & S_{22} \end{bmatrix}$$

$$= B_1 R_1^{-1} B_1^T - B_2 R_2^{-1} B_2^T,$$

$$B_1 = \begin{bmatrix} B_{01} \\ B_{11} \\ 0 \end{bmatrix}, \;\; B_2 = \begin{bmatrix} B_{02} \\ 0 \\ B_{22} \end{bmatrix},$$

$$P = \begin{bmatrix} P_{00} & \varepsilon_1 P_{10}^T & \varepsilon_2 P_{20}^T \\ P_{10} & P_{11} & \frac{1}{\sqrt{\alpha}} P_{21}^T \\ P_{20} & \sqrt{\alpha} P_{21} & P_{22} \end{bmatrix}.$$

Combining the existing results recalled above and Lemma 1, we arrive at the following conclusions readily.
(i) The value of the full-order game, whenever it exists, can be expressed as

$$J^* = \frac{1}{2} x_0^T \Phi_{\mathcal{E}} P^+ x_0. \tag{10}$$

(ii) The linear feedback saddle-point strategy pair, if it exists, can be expressed as

$$u_1^*(t) = -R_1^{-1} B_1^T P^+ x(t), \tag{11a}$$
$$u_2^*(t) = R_2^{-1} B_2^T P^+ x(t), \tag{11b}$$

where P^+ denotes the minimal positive definite solution in the sense $P_{\mathcal{E}}^+ = \Phi_{\mathcal{E}} P^+ > 0$.

III. DECOMPOSITION OF THE FULL-ORDER PROBLEM

In this section, we first decompose the full-order game into a slow game and two fast control problems. Then, we discuss the solutions of the slow game and two fast control problems respectively.

Slow game: The slow subsystem is formed by neglecting the fast modes, which is equivalent to letting $\varepsilon_{1,2} = 0$ in (1),

$$\begin{aligned} E\dot{x}_s(t) &= Ax_s(t) + B_1 u_{1s}(t) + B_2 u_{2s}(t), \\ Ex_s(0) &= Ex_0, \end{aligned} \tag{12}$$

where $x_s(t) = [x_{0s}^T(t) \ \ x_{1s}^T(t) \ \ x_{2s}^T(t)]^T$, $E = \Phi_{\mathcal{E}}|_{\varepsilon_{1,2}=0}$ and A, B_1, B_2 are defined in Lemma 1. The corresponding reduced-order (slow) cost function is

$$\begin{aligned} J_s &= \frac{1}{2} \int_0^\infty [z_s^T(t) z_s(t) + u_{1s}^T(t) R_1 u_{1s}(t) \\ &\quad - u_{2s}^T(t) R_2 u_{2s}(t)] dt, \end{aligned} \tag{13}$$

where $z_s^T(t) = [z_{0s}^T(t) \ \ z_{1s}^T(t) \ \ z_{2s}^T(t)]$, $z_{0s}(t) = C_{00} x_{0s}(t) \in \mathbf{R}^{r_0}$, $z_{js}(t) = C_{j0} x_{0s}(t) + C_{jj} x_{js}(t) \in \mathbf{R}^{r_j}$, $j = 1, \ 2$.

Fast control problems: Two fast subsystems are derived by assuming that the slow variables are constant during fast transients, that is, $\dot{x}_{js} = 0, j = 1, 2$ and $x_{0s} =$ a constant. The fast subsystems are defined, respectively, as

$$\begin{aligned} \varepsilon_j \dot{x}_{jf}(t) &= A_{jj} x_{jf}(t) + B_{jj} u_{jf}(t), \\ x_{jf}(0) &= x_j^0 - x_{js}(0), \ j = 1, \ 2 \end{aligned} \tag{14}$$

where $x_{jf} = x_j - x_{js}$, $u_{1f} = u_1 - u_{1s}$, and $u_{2f} = u_2 - u_{2s}$. The corresponding reduced-order (fast) cost functions become

$$J_{1f} = \frac{1}{2} \int_0^\infty (x_{1f}^T C_{11}^T C_{11} x_{1f} + u_{1f}^T R_1 u_{1f}) dt \tag{15}$$

and

$$J_{2f} = \frac{1}{2} \int_0^\infty (x_{2f}^T C_{22}^T C_{22} x_{2f} - u_{2f}^T R_2 u_{2f}) dt \tag{16}$$

respectively. The purpose of Player 1 is to minimize J_{1f}, while Player 2 will maximize J_{2f}.
We now give some results concerning the solutions of the slow game and two fast optimal control problems in a reverse order.
Two fast control problems are formulated as the linear quadratic optimal control problems for state space systems when stretched time scales $\tau_j = t/\varepsilon_j, j = 1, 2$ are introduced respectively. Therefore, we have
Proposition 1. Consider the fast control problems described by (14), (15) and (14), (16) respectively, where (A_{jj}, C_{jj}), $j = 1, \ 2$, are observable respectively. If the algebraic Riccati equation

$$A_{jj}^T P_{jjf} + P_{jjf} A_{jj} - P_{jjf} S_{jj} P_{jjf} + C_{jj}^T C_{jj} = 0, \ j = 1, \ 2 \tag{17}$$

admits a unique stabilizing positive semidefinite solution P_{jjf}^+, $j = 1, \ 2$, then the linear feedback strategies

$$u_{1f}^*(t) = -R_1^{-1} B_{11}^T P_{11f}^+ x_{1f}(t), \tag{18a}$$

$$u_{2f}^*(t) = R_2^{-1} B_{22}^T P_{22f}^+ x_{2f}(t), \tag{18b}$$

constitute the optimal controls to the optimal control problems respectively. Moreover, the obtained optimal values are

$$J_{1f}^* = \frac{1}{2} x_{1f}^T(0) P_{11f}^+ x_{1f}(0)$$

and

$$J_{2f}^* = \frac{1}{2} x_{2f}^T(0) P_{22f}^+ x_{2f}(0),$$

respectively.
Proposition 2. Consider the slow game described by (12), (13), and suppose that the generalized algebraic Riccati equation (GARE)

$$(i) \qquad A^T P_s + P_s^T A - P_s^T S P_s + Q = 0, \tag{19a}$$
$$(ii) \qquad EP_s = P_s^T E, \tag{19b}$$

admits a minimal positive-definite strictly stabilizing solution P_s^+ in the sense of $EP_s^+ > 0$, where

$$P_s^+ = \begin{bmatrix} P_{00s}^+ & 0 & 0 \\ P_{10s}^+ & P_{11f}^+ & 0 \\ P_{20s}^+ & 0 & P_{22f}^+ \end{bmatrix}.$$

Then, a linear feedback strategy pair

$$u_{1s}^* = -R_1^{-1} B_1^T P_s^+ x_s, \quad (20a)$$
$$u_{2s}^* = R_2^{-1} B_2^T P_s^+ x_s, \quad (20b)$$

is in a saddle-point equilibrium in the class of feedback strategies that are strictly feedback stabilizing and under which $x_s(t)$ is impulse-free and $x_s(t) \to 0$, as $t \to \infty$ for all Ex_0. and the value of the slow game is given by

$$J_s^* = \frac{1}{2} x_0^{0T} P_{00s}^+ x_0^0.$$

Remark 1: It can be found that the block decomposition of the GARE (19) is the same as the block decomposition of the GMARE (9a) when $\varepsilon_1 \to +0$ and $\varepsilon_2 \to +0$. The details are omitted (Xu and Mizukami, 1994)

IV. $O(\|\mu\|)$ NEAR SADDLE-POINT EQUILIBRIUM

In this section, we will construct a composite strategy pair and prove that the obtained composite strategy pair is in fact a $O(\|\mu\|)$ (near) saddle-point solution to the full-order game.

Let us construct the composite strategy pair as follows.

$$u_{1c}^*(t) = u_{1s}^{+*} + u_{1f}^* + u_{2f}^* = -R_1^{-1} B_1^T P_s^+ x(t), \quad (21a)$$

$$u_{2c}^*(t) = u_{2s}^{+*} + u_{1f}^* + u_{2f}^* = R_2^{-1} B_2^T P_s^+ x(t), \quad (21b)$$

where $x_0(t) \approx x_{0s}(t)$, $x_1(t) \approx x_{1s}(t) + x_{1f}(t)$ and $x_2(t) \approx x_{2s}(t) + x_{2f}(t)$.

$$P_s^+ = \begin{bmatrix} P_{00s}^+ & 0 & 0 \\ P_{10s}^+ & P_{11f}^+ & 0 \\ P_{20s}^+ & 0 & P_{22f}^+ \end{bmatrix}$$

will be explained later.

We now want to apply the composite strategy pair (u_{1c}^*, u_{2c}^*) to the full-order game and compare it with the exact linear feedback strategy pair (11). In order to do that, we first study the asymptotic expansions of the MARE (6) or the GMARE (9).

It is assumed that the limit of α exists as ε_1 and ε_2 tend to zero, that is

$$\bar{\alpha} = \lim_{\substack{\varepsilon_1 \to +0 \\ \varepsilon_2 \to +0}} \alpha.$$

The GMARE (9a) can be partitioned into

$$\begin{aligned} f_1 &= A_{00}^T P_{00} + P_{00} A_{00} + A_{10}^T P_{10} + P_{10}^T A_{10} \\ &\quad + A_{20}^T P_{20} + P_{20}^T A_{20} - P_{20}^T S_{22} P_{20} + Q_{00} \\ &\quad - P_{00} S_{00} P_{00} - P_{10}^T S_{01}^T P_{00} - P_{00} S_{01} P_{10} \\ &\quad - P_{20}^T S_{02}^T P_{00} - P_{00} S_{02} P_{20} - P_{10}^T S_{11} P_{10} = 0, \quad (22a) \\ f_2 &= P_{00} A_{01} + P_{10}^T A_{11} + \varepsilon_1 A_{00}^T P_{10}^T + A_{10}^T P_{11} \\ &\quad - \varepsilon_1 (P_{00} S_{00} P_{10}^T + P_{10}^T S_{01}^T P_{10}^T + P_{20}^T S_{02}^T P_{10}^T) \\ &\quad - P_{00} S_{01} P_{11} - P_{10}^T S_{11} P_{11} + \sqrt{\alpha} A_{20}^T P_{21} \\ &\quad - \sqrt{\alpha} (P_{00} S_{02} P_{21} + P_{20}^T S_{22} P_{21}) + Q_{01} = 0, \quad (22b) \\ f_3 &= P_{00} A_{02} + P_{20}^T A_{22} + \varepsilon_2 A_{00}^T P_{20}^T + A_{20}^T P_{22} \\ &\quad - \varepsilon_2 (P_{00} S_{00} P_{20}^T + P_{10}^T S_{01}^T P_{20}^T + P_{20}^T S_{02}^T P_{20}^T) \\ &\quad - P_{00} S_{02} P_{22} - P_{20}^T S_{22} P_{22} + \sqrt{\alpha}^{-1} A_{10}^T P_{21}^T \\ &\quad - \sqrt{\alpha}^{-1} (P_{00} S_{01} P_{21}^T + P_{10}^T S_{11} P_{21}^T) + Q_{02} = 0, \quad (22c) \\ f_4 &= A_{11}^T P_{11} + P_{11} A_{11} + \varepsilon_1 (A_{01}^T P_{10}^T + P_{10} A_{01}) \\ &\quad - \varepsilon_1 (\varepsilon_1 P_{10} S_{00} P_{10}^T + P_{11} S_{01}^T P_{10}^T + \sqrt{\alpha} P_{21}^T S_{02}^T P_{10}^T) \\ &\quad - \varepsilon_1 (P_{10} S_{01} P_{11} + \sqrt{\alpha} P_{10} S_{02} P_{21}) \\ &\quad - P_{11} S_{11} P_{11} - \alpha P_{21}^T S_{22} P_{21} + Q_{11} = 0, \quad (22d) \\ f_5 &= \varepsilon_1 P_{10} A_{02} + \varepsilon_2 A_{01}^T P_{20}^T - \varepsilon_1 \varepsilon_2 P_{10} S_{00} P_{20}^T \\ &\quad - \varepsilon_2 (P_{11} S_{01}^T P_{20}^T + \sqrt{\alpha} P_{21}^T S_{02}^T P_{20}^T) \\ &\quad - \varepsilon_1 (P_{10} S_{02} P_{22} + \sqrt{\alpha}^{-1} P_{10} S_{01} P_{21}^T) \\ &\quad + \sqrt{\alpha}^{-1} (A_{11} - S_{11} P_{11})^T P_{21}^T \\ &\quad + \sqrt{\alpha} P_{21}^T (A_{22} - S_{22} P_{22}) = 0, \quad (22e) \\ f_6 &= A_{22}^T P_{22} + P_{22} A_{22} + \varepsilon_2 (A_{02}^T P_{20}^T + P_{20} A_{02}) \\ &\quad - \varepsilon_2 (\varepsilon_2 P_{20} S_{00} P_{20}^T + P_{22} S_{02}^T P_{20}^T + \sqrt{\alpha}^{-1} P_{21} S_{01}^T P_{20}^T) \\ &\quad - \varepsilon_2 (P_{20} S_{02} P_{22} + \sqrt{\alpha}^{-1} P_{20} S_{01} P_{21}^T) \\ &\quad - P_{22} S_{22} P_{22} - \alpha^{-1} P_{21} S_{11} P_{21}^T + Q_{22} = 0. \quad (22f) \end{aligned}$$

Taking the limitation of the GMARE (9a) or (22) when $\varepsilon_1 \to +0$ and $\varepsilon_2 \to +0$, we obtain the following equations

$$\begin{aligned} & A_{00}^T \bar{P}_{00} + \bar{P}_{00} A_{00} + A_{10}^T \bar{P}_{10} + \bar{P}_{10}^T A_{10} + A_{20}^T \bar{P}_{20} \\ &\quad + \bar{P}_{20}^T A_{20} - \bar{P}_{00} S_{00} \bar{P}_{00} - \bar{P}_{00} S_{02} \bar{P}_{20} \\ &\quad - \bar{P}_{10}^T S_{01}^T \bar{P}_{00} - \bar{P}_{00} S_{01} \bar{P}_{10} - \bar{P}_{20}^T S_{02}^T \bar{P}_{00} \\ &\quad - \bar{P}_{10}^T S_{11} \bar{P}_{10} - \bar{P}_{20}^T S_{22} \bar{P}_{20} + Q_{00} = 0, \quad (23a) \\ & \bar{P}_{00} A_{01} + \bar{P}_{10}^T A_{11} + A_{10}^T \bar{P}_{11} + \sqrt{\bar{\alpha}} A_{20}^T \bar{P}_{21} \\ &\quad - \bar{P}_{10}^T S_{11} \bar{P}_{11} - \bar{P}_{00} S_{01} \bar{P}_{11} + Q_{01} \\ &\quad - \sqrt{\bar{\alpha}} (\bar{P}_{00} S_{02} \bar{P}_{21} + \bar{P}_{20}^T S_{22} \bar{P}_{21}) = 0, \quad (23b) \\ & \bar{P}_{00} A_{02} + \bar{P}_{20}^T A_{22} + A_{20}^T \bar{P}_{22} + \sqrt{\bar{\alpha}}^{-1} A_{10}^T \bar{P}_{21}^T \\ &\quad - \bar{P}_{00} S_{02} \bar{P}_{22} - \bar{P}_{20}^T S_{22} \bar{P}_{22} + Q_{02} \\ &\quad - \sqrt{\bar{\alpha}}^{-1} (\bar{P}_{00} S_{01} \bar{P}_{21}^T + \bar{P}_{10}^T S_{11} \bar{P}_{21}^T) = 0, \quad (23c) \\ & A_{11}^T \bar{P}_{11} + \bar{P}_{11} A_{11} - \bar{P}_{11} S_{11} \bar{P}_{11} \\ &\quad - \bar{\alpha} \bar{P}_{21}^T S_{22} \bar{P}_{21} + Q_{11} = 0, \quad (23d) \\ & \sqrt{\bar{\alpha}} \bar{P}_{21}^T (A_{22} - S_{22} \bar{P}_{22}) \\ &\quad + \sqrt{\bar{\alpha}}^{-1} (A_{11} - S_{11} \bar{P}_{11})^T \bar{P}_{21}^T = 0, \quad (23e) \\ & A_{22}^T \bar{P}_{22} + \bar{P}_{22} A_{22} - \bar{P}_{22} S_{22} \bar{P}_{22} \\ &\quad - \bar{\alpha}^{-1} \bar{P}_{21} S_{11} \bar{P}_{21}^T + Q_{22} = 0, \quad (23f) \end{aligned}$$

where $\bar{P}_{00}$, $\bar{P}_{10}$, $\bar{P}_{20}$, $\bar{P}_{11}$, $\bar{P}_{21}$ and $\bar{P}_{22}$ are the 0-order solutions of the GMARE (9a).

To solve the problem, we make the following basic condition without loss of generality (Dragan,1993).

(H1) The AREs $A_{jj}^T\tilde{P}_{jj} + \tilde{P}_{jj}A_{jj} - \tilde{P}_{jj}S_{jj}\tilde{P}_{jj} + Q_{jj} = 0,\ j = 1, 2$ admit the unique positive semidefinite stabilizing solutions respectively.

If the condition (H1) holds, there exist the solutions P_{jj}^+ such that the matrices $A_{jj} - S_{jj}P_{jj}^+$ are stable. Therefore, we chose the solutions $\bar{P}_{jj}$ as P_{jj}^+, where $j = 1,\ 2$. Then, $P_{21}^+ = 0$ in (23e) because the matrices $A_{jj} - S_{jj}\bar{P}_{jj} = A_{jj} - S_{jj}P_{jj}^+$ are stable. As a result, the parameter $\bar{\alpha}$ disappears from (23) automatically, that is, it does not affect the equation (23) in the limit when ε_1 and ε_2 tend to zero. Thus, the AREs (23) have the same structure with the block decomposition of the GARM (19).

After some computations, we now obtain the following 0-order equations.

$$A_s^T\bar{P}_{00} + \bar{P}_{00}A_s - \bar{P}_{00}S_s\bar{P}_{00} + Q_s = 0, \quad (24a)$$
$$\bar{P}_{j0}^T = \bar{P}_{00}N_{0j} - M_{0j},\ j = 1,\ 2, \quad (24b)$$
$$A_{jj}^T\bar{P}_{jj} + \bar{P}_{jj}A_{jj} - \bar{P}_{jj}S_{jj}\bar{P}_{jj} + Q_{jj} = 0, \quad (24c)$$

where

$$\begin{aligned}
A_s &= A_{00} + N_{01}A_{10} + N_{02}A_{20} + S_{01}M_{01}^T \\
&\quad + S_{02}M_{02}^T + N_{01}S_{11}M_{01}^T + N_{02}S_{22}M_{02}^T, \\
S_s &= S_{00} + N_{01}S_{01}^T + S_{01}N_{01}^T + N_{02}S_{02}^T \\
&\quad + S_{02}N_{02}^T + N_{01}S_{11}N_{01}^T + N_{02}S_{22}N_{02}^T, \\
Q_s &= Q_{00} - M_{01}A_{10} - A_{10}^TM_{01}^T - M_{02}A_{20} \\
&\quad - A_{20}^TM_{02}^T - M_{01}S_{11}M_{01}^T - M_{02}S_{22}M_{02}^T, \\
N_{0j} &= -D_{0j}D_{jj}^{-1},\ M_{0j} = \bar{Q}_{0j}D_{jj}^{-1}, \\
\bar{Q}_{0j} &= A_{j0}^T\bar{P}_{jj} + Q_{0j}, \\
D_{00} &= A_{00} - S_{00}\bar{P}_{00} - S_{01}\bar{P}_{10} - S_{02}\bar{P}_{20}, \\
D_{0j} &= A_{0j} - S_{0j}\bar{P}_{jj}, \\
D_{j0} &= A_{j0} - S_{0j}^T\bar{P}_{00} - S_{jj}\bar{P}_{j0}, \\
D_{jj} &= A_{jj} - S_{jj}\bar{P}_{jj},\ j = 1,\ 2.
\end{aligned}$$

The matrices $A_s,\ S_s$ and Q_s do not depend on $\bar{P}_{jj},\ j = 1,\ 2$ because their matrices can be computed by using $T_{pq},\ p,\ q = 0,\ 1,\ 2$ which are independent of $\bar{P}_{jj},\ j = 1,\ 2$, that is,

$$T_s = T_{00} - T_{01}T_{11}^{-1}T_{10} - T_{02}T_{22}^{-1}T_{20}$$

$$= \begin{bmatrix} A_s & -S_s \\ -Q_s & -A_s^T \end{bmatrix},$$

$$T_{00} = \begin{bmatrix} A_{00} & -S_{00} \\ -Q_{00} & -A_{00}^T \end{bmatrix},\ T_{0j} = \begin{bmatrix} A_{0j} & -S_{0j} \\ -Q_{0j} & -A_{j0}^T \end{bmatrix},$$

$$T_{j0} = \begin{bmatrix} A_{j0} & -S_{0j}^T \\ -Q_{0j}^T & -A_{0j}^T \end{bmatrix},\ T_{jj} = \begin{bmatrix} A_{jj} & -S_{jj} \\ -Q_{jj} & -A_{jj}^T \end{bmatrix}.$$

Note that the Hamiltonian matrices

$$T_{jj} := \begin{bmatrix} A_{jj} & -S_{jj} \\ -Q_{jj} & -A_{jj}^T \end{bmatrix},\ j = 1,\ 2$$

are nonsingular under the condition (H1) because of

$$T_{jj} = \begin{bmatrix} I_{n_j} & 0 \\ \bar{P}_{jj}^T & I_{n_j} \end{bmatrix}\begin{bmatrix} D_{jj} & -S_{jj} \\ 0 & -D_{jj}^T \end{bmatrix}\begin{bmatrix} I_{n_j} & 0 \\ -\bar{P}_{jj} & I_{n_j} \end{bmatrix}$$

$$\Leftrightarrow T_{jj}^{-1} = \begin{bmatrix} I_{n_j} & 0 \\ \bar{P}_{jj} & I_{n_j} \end{bmatrix}\begin{bmatrix} D_{jj}^{-1} & -D_{jj}^{-1}S_{jj}D_{jj}^{-T} \\ 0 & -D_{jj}^{-T} \end{bmatrix}$$

$$\times \begin{bmatrix} I_{n_j} & 0 \\ -\bar{P}_{jj}^T & I_{n_j} \end{bmatrix}.$$

We now assume,

(H2) The ARE (24a) has the minimal positive definite stabilizing solution.

It should be remarked that the solution $P_{\mathcal{E}}$ of (6) is a function of the parameters ε_1 and ε_2. But, the solutions $\bar{P}_{00}$ and $\bar{P}_{jj},\ j = 1,\ 2$ of (24a) and (24c) are independent of the parameters ε_1 and ε_2, respectively. The following theorem will establish the relation between $P_{\mathcal{E}}$ and the reduced-order solutions (23).

Theorem 1: Under the conditions (H1) and (H2), there exist small ε_1^* and ε_2^* such that for all $\varepsilon_1 \in (0,\ \varepsilon_1^*)$ and $\varepsilon_2 \in (0,\ \varepsilon_2^*)$, the MARE (6) admits a symmetric positive semidefinite stabilizing solution $P_{\mathcal{E}}$ which can be written as

$$P_{\mathcal{E}}$$
$$= \begin{bmatrix} \bar{P}_{00} + \mathcal{F}_{00} & \varepsilon_1(\bar{P}_{10} + \mathcal{F}_{10})^T & \varepsilon_2(\bar{P}_{20} + \mathcal{F}_{20})^T \\ \varepsilon_1(\bar{P}_{10} + \mathcal{F}_{10}) & \varepsilon_1(\bar{P}_{11} + \mathcal{F}_{11}) & \sqrt{\varepsilon_1\varepsilon_2}\mathcal{F}_{21}^T \\ \varepsilon_2(\bar{P}_{20} + \mathcal{F}_{20}) & \sqrt{\varepsilon_1\varepsilon_2}\mathcal{F}_{21} & \varepsilon_2(\bar{P}_{22} + \mathcal{F}_{22}) \end{bmatrix}, \quad (25)$$

where

$$\mathcal{F}_{pq} = O(\|\mu\|),\ \mu = [\varepsilon_1, \varepsilon_2],\ \|\mathcal{F}_{pq}\| = c_{pq} < \infty,$$
$$pq = 00,\ 10,\ 20,\ 11,\ 21,\ 22.$$

In order to prove Theorem 1, we need the following lemma (Khalil, 1978).

Lemma 2: Consider the system

$$\begin{aligned}
\dot{x}_0(t) &= A_{00}x_0(t) + A_{01}x_1(t) + A_{02}x_2(t),\ x_0^0, \\
\varepsilon_1\dot{x}_1(t) &= A_{10}x_0(t) + A_{11}x_1(t) + \varepsilon_3A_{12}x_2(t),\ x_1^0, \\
\varepsilon_2\dot{x}_2(t) &= A_{20}x_0(t) + \varepsilon_4A_{21}x_1(t) + A_{22}x_2(t),\ x_2^0,
\end{aligned}$$

where $x_0 \in \mathrm{R}^{n_0}$, $x_1 \in \mathrm{R}^{n_1}$ and $x_2 \in \mathrm{R}^{n_2}$ are the state vector. ε_3, ε_4 are small weak coupling parameters, ε_1 and

ε_2 are small positive singular perturbation parameters of the same order of magnitude with (3). If $A_{jj}^{-1},\ j = 1,\ 2$ exist, and if $A_0 \equiv A_{00} - A_{01}A_{11}^{-1}A_{10} - A_{02}A_{22}^{-1}A_{20}$, $A_{jj},\ j = 1,\ 2$ are stable matrices, then there exist small $\hat{\varepsilon}_1$ and $\hat{\varepsilon}_2$ such that for all $\varepsilon_1 \in (0,\ \hat{\varepsilon}_1)$ and $\varepsilon_2 \in (0,\ \hat{\varepsilon}_2)$, the system is asymptotically stable.

Now, let us prove Theorem 1.

Proof: Since the MARE (6) is equivalent to the GMARE (9a) from Lemma 1, we apply the implicit function theorem (Gajic, 1988) to (9a). To do so, it is sufficient to show that the corresponding Jacobian is nonsingular at $\varepsilon_1 = 0$ and $\varepsilon_2 = 0$. It can be shown, after some computations, that the Jacobian of (9a) in the limit is given by

$$\begin{aligned} J &= \nabla \mathbf{F}|_{(\mu,\ \mathcal{P})=(\mu_0,\ \mathcal{P}_0)} \\ &= \frac{\partial \mathrm{vec}(f_1,\ f_2,\ f_3,\ f_4,\ f_5,\ f_6)}{\partial \mathrm{vec}(P_{00},\ P_{10},\ P_{20},\ P_{11},\ P_{21},\ P_{22})^T} \\ &= \begin{bmatrix} J_{00} & J_{01} & J_{02} & 0 & 0 & 0 \\ J_{10} & J_{11} & 0 & J_{13} & J_{14} & 0 \\ J_{20} & 0 & J_{22} & 0 & J_{24} & J_{25} \\ 0 & 0 & 0 & J_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & J_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & J_{55} \end{bmatrix}, \end{aligned} \quad (26)$$

where vec denotes an ordered stack of the columns of its matrix and

$$\begin{aligned} &\mu = (\varepsilon_1,\ \varepsilon_2),\ \mu_0 = (0,\ 0), \\ &\mathcal{P} = (P_{00},\ P_{10},\ P_{20},\ P_{11},\ P_{21},\ P_{22}), \\ &\mathcal{P}_0 = (\bar{P}_{00},\ \bar{P}_{10},\ \bar{P}_{20},\ \bar{P}_{11},\ 0,\ \bar{P}_{22}), \\ &J_{00} = I_{n_0} \otimes D_{00}^T + D_{00}^T \otimes I_{n_0}, \\ &J_{0j} = (I_{n_0} \otimes D_{j0}^T)U_{n_0 n_j} + D_{j0}^T \otimes I_{n_0}, \\ &J_{j0} = D_{0j}^T \otimes I_{n_0},\ J_{jj} = D_{jj}^T \otimes I_{n_0},\ j = 1,\ 2, \\ &J_{13} = I_{n_1} \otimes D_{10},\ J_{14} = \sqrt{\bar{\alpha}}(I_{n_1} \otimes D_{20})U_{n_1 n_2}, \\ &J_{24} = \sqrt{\bar{\alpha}}^{-1} I_{n_2} \otimes D_{10},\ J_{25} = I_{n_2} \otimes D_{20}, \\ &J_{33} = (I_{n_1} \otimes D_{11}^T)U_{n_1 n_1} + D_{11}^T \otimes I_{n_1}, \\ &J_{44} = \sqrt{\bar{\alpha}} D_{22}^T \otimes I_{n_1} + \sqrt{\bar{\alpha}}^{-1} I_{n_2} \otimes D_{11}^T, \\ &J_{55} = (I_{n_2} \otimes D_{22}^T)U_{n_2 n_2} + D_{22}^T \otimes I_{n_2}, \end{aligned}$$

where $\otimes$ denotes Kronecker products and $U_{n_j n_j},\ j = 0,\ 1,\ 2$ is the permutation matrix in Kronecker matrix sense.

The Jacobian (26) can be expressed as

$$\begin{aligned} \det J &= \det J_{33} \cdot \det J_{44} \cdot \det J_{55} \\ &\quad \cdot \det \begin{bmatrix} J_{00} & J_{01} & J_{02} \\ J_{10} & J_{11} & 0 \\ J_{20} & 0 & J_{22} \end{bmatrix} \\ &= \det J_{33} \cdot \det J_{44} \cdot \det J_{55} \cdot \det J_{11} \cdot \det J_{22} \\ &\quad \cdot \det(J_{00} - J_{01}J_{11}^{-1}J_{10} - J_{02}J_{22}^{-1}J_{20}) \\ &= \det J_{11} \cdot \det J_{22} \cdot \det J_{33} \cdot \det J_{44} \cdot \det J_{55} \\ &\quad \cdot \det[I_{n_0} \otimes D_0^T U_{n_0 n_0} + D_0^T \otimes I_{n_0}], \end{aligned} \quad (27)$$

where $D_0 \equiv D_{00} - D_{01}D_{11}^{-1}D_{10} - D_{02}D_{22}^{-1}D_{20}$. Obviously, $J_{jj},\ j = 1, \cdots, 5$ are nonsingular because the matrices $D_{jj} = A_{jj} - S_{jj}\bar{P}_{jj},\ j = 1,\ 2$ are nonsingular under the condition (H1). After some straightforward algebra but tedious, we see that the $A_s - S_s\bar{P}_{00} = D_{00} - D_{01}D_{11}^{-1}D_{10} - D_{02}D_{22}^{-1}D_{20} = D_0$. Therefore, the matrix D_0 is nonsingular if the condition (H2) holds. Thus, $\det J \neq 0$, i.e., J is nonsingular at $(\mu,\ \mathcal{P}) = (\mu_0,\ \mathcal{P}_0)$. The conclusion of the first part of Theorem 1 is obtained directly by using the implicit function theorem. The second part of the proof of Theorem 1 is performed by direct calculation. By using (25), we obtain

$$\begin{aligned} &\Phi_{\mathcal{E}}^{-1}(A - SP) \\ &= \Phi_{\mathcal{E}}^{-1}\left(\begin{bmatrix} D_{00} & D_{01} & D_{02} \\ D_{10} & D_{11} & 0 \\ D_{20} & 0 & D_{22} \end{bmatrix} + O(\|\mu\|) \right). \end{aligned}$$

We know from Lemma 2 that for sufficiently small $\|\mu\|$ the matrix $\Phi_{\mathcal{E}}^{-1}(A - SP)$ will be stable. On the other hand, since $\bar{P}_{00} \geq 0$, $\bar{P}_{11} \geq 0$ and $\bar{P}_{22} \geq 0$, $P_{\mathcal{E}}$ is positive semidefinite as long as $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ by using the Schur complement. Therefore, the proof on Theorem 1 ends. ■

Since $P_{\mathcal{E}} = \Phi_{\mathcal{E}} P$, we have

$$\begin{aligned} &P \\ &= \begin{bmatrix} \bar{P}_{00} + \mathcal{F}_{00} & \varepsilon_1(\bar{P}_{10} + \mathcal{F}_{10})^T & \varepsilon_2(\bar{P}_{20} + \mathcal{F}_{20})^T \\ \bar{P}_{10} + \mathcal{F}_{10} & \bar{P}_{11} + \mathcal{F}_{11} & \sqrt{\alpha}^{-1}\mathcal{F}_{21}^T \\ \bar{P}_{20} + \mathcal{F}_{20} & \sqrt{\alpha}\mathcal{F}_{21} & \bar{P}_{22} + \mathcal{F}_{22} \end{bmatrix}. \end{aligned} \quad (28)$$

Under the conditions (H1) and (H2), we have $\bar{P}_{11} = P_{11f}^+$, $\bar{P}_{22} = P_{22f}^+$, $\bar{P}_{00} = P_{00s}^+$, $\bar{P}_{10} = P_{10s}^+$, $\bar{P}_{20} = P_{20s}^+$. Therefore, we readily have

$$u_1^*(t) = u_{1c}^*(t) + O(\|\mu\|), \quad (29a)$$

$$u_2^*(t) = u_{2c}^*(t) + O(\|\mu\|). \quad (29b)$$

Furthermore, we will show the $O(\|\mu\|^2)$ approximation between J^* and J_c^*.

Applying the composite strategy pair (u_{1c}^*, u_{2c}^*) to the full-order game described by (1),(3), we have

$$J_c^* = \frac{1}{2}x^T(0)P_{c\mathcal{E}}x(0), \quad (30)$$

where $P_{c\mathcal{E}}$ is the solution of the Lyapunov equation

$$\begin{aligned} &(A_{\mathcal{E}} - S_{\mathcal{E}}P_{s\mathcal{E}}^+)^T P_{c\mathcal{E}} + P_{c\mathcal{E}}(A_{\mathcal{E}} - S_{\mathcal{E}}P_{s\mathcal{E}}^+) \\ &= -P_{s\mathcal{E}}^+ S_{\mathcal{E}} P_{s\mathcal{E}}^+ - Q, \end{aligned} \quad (31)$$

where $P_{s\mathcal{E}}^{+} = \Phi_{\mathcal{E}} P_s^{+}$.

Theorem 2: Under the conditions of Theorem 1, we have

$$J(u_{1c}^*, u_{2c}^*) = J(u_1^*,\ u_2^*) + O(\|\mu\|^2), \qquad (32)$$

with $J_c^* > J^*$ for $S > 0$, and $J_c^* < J^*$ when $S < 0$, where $S := \Phi_{\mathcal{E}}^{-1} S_{\mathcal{E}} \Phi_{\mathcal{E}}^{-1}$.

Before proving this theorem, we introduce the following lemma (Mukaidani et al., 2001).

Lemma 3: Consider the iterative algorithm which is based on the Kleinman algorithm

$$\begin{aligned}(A - SP^{(i)})^T P^{(i+1)} + P^{(i+1)T}(A - SP^{(i)})\\ + P^{(i)T} S P^{(i)} + Q = 0,\ \ i = 0,\ 1,\ \ldots,\end{aligned} \qquad (33a)$$

$$P^{(i)} = \begin{bmatrix} P_{00}^{(i)} & \varepsilon_1 P_{10}^{(i)T} & \varepsilon_2 P_{20}^{(i)T} \\ P_{10}^{(i)} & P_{11}^{(i)} & \sqrt{\alpha}^{-1} P_{21}^{(i)T} \\ P_{20}^{(i)} & \sqrt{\alpha} P_{21}^{(i)} & P_{22}^{(i)} \end{bmatrix}, \qquad (33b)$$

with the initial condition obtained from

$$P^{(0)} = \begin{bmatrix} \bar{P}_{00} & 0 & 0 \\ \bar{P}_{10} & \bar{P}_{11} & \sqrt{\bar{\alpha}}^{-1} \bar{P}_{21} \\ \bar{P}_{20} & \sqrt{\bar{\alpha}} \bar{P}_{21} & \bar{P}_{22} \end{bmatrix}. \qquad (34)$$

Under the conditions (H1) and (H2), there exists a small $\bar{\sigma}$ such that for all $\|\mu\| \in (0,\ \bar{\sigma})$, $\bar{\sigma} \le \sigma^*$ Kleinman algorithm (33) converges to the exact solution of $P_{\mathcal{E}} = \Phi_{\mathcal{E}} P = P^T \Phi_{\mathcal{E}}$ with the rate of quadratic convergence, where $P_{\mathcal{E}}^{(i)} = \Phi_{\mathcal{E}} P^{(i)} = P^{(i)T} \Phi_{\mathcal{E}}$ is positive semidefinite.

$$\|P^{(i)} - P\| = O(\|\mu\|^{2^i}),\ i = 0,\ 1,\ \ldots, \qquad (35)$$

where

$$\gamma = 2\|S\| < \infty,\ \beta = \|[\nabla \mathcal{G}(P^{(0)})]^{-1}\|,$$

$$\eta = \beta \cdot \|\mathcal{G}(P^{(0)})\|,\ \theta = \beta\eta\gamma,\ \nabla \mathcal{G}(P) = \frac{\partial \text{vec}\mathcal{G}(P)}{\partial(\text{vec}P)^T}.$$

Proof: When u_{1c}^* and u_{2c}^* are used, the value of the performance index is given by (30). Subtracting (6) from (31) we find that $V_{\mathcal{E}} = P_{c\mathcal{E}} - P_{\mathcal{E}}$ satisfies the following multiparameter algebraic Lyapunov equation (MALE)

$$\begin{aligned}(A_{\mathcal{E}} - S_{\mathcal{E}} P_{s\mathcal{E}}^{+})^T V_{\mathcal{E}} + V_{\mathcal{E}}(A_{\mathcal{E}} - S_{\mathcal{E}} P_{s\mathcal{E}}^{+})\\ + (P_{\mathcal{E}} - P_{s\mathcal{E}}^{+}) S_{\mathcal{E}} (P_{\mathcal{E}} - P_{s\mathcal{E}}^{+}) = 0.\end{aligned} \qquad (36)$$

Since $A_{\mathcal{E}} - S_{\mathcal{E}} P_{s\mathcal{E}}^{+}$ is stable, using the standard Lyapunov theorem (Zhou, 1998), we have $J(u_{1c}^*, u_{2c}^*) > J(u_1^*,\ u_2^*)$ for $S > 0$, and $J(u_{1c}^*, u_{2c}^*) < J(u_1^*,\ u_2^*)$ when $S < 0$. On the other hand, subtracting (6) from (33a) we also get the MALE

$$\begin{aligned}&(A_{\mathcal{E}} - S_{\mathcal{E}} P_{\mathcal{E}}^{(i)})^T (P_{\mathcal{E}}^{(i+1)} - P_{\mathcal{E}})\\ &\quad + (P_{\mathcal{E}}^{(i+1)} - P_{\mathcal{E}})(A_{\mathcal{E}} - S_{\mathcal{E}} P_{\mathcal{E}}^{(i)})\\ &\quad + (P_{\mathcal{E}} - P_{\mathcal{E}}^{(i)}) S_{\mathcal{E}} (P_{\mathcal{E}} - P_{\mathcal{E}}^{(i)}) = 0,\end{aligned} \qquad (37)$$

where $P_{\mathcal{E}}^{(i)} = \Phi_{\mathcal{E}} P^{(i)}$. When $i = 0$, we have

$$\begin{aligned}&(A_{\mathcal{E}} - S_{\mathcal{E}} P_{\mathcal{E}}^{(0)})^T (P_{\mathcal{E}}^{(1)} - P_{\mathcal{E}})\\ &\quad + (P_{\mathcal{E}}^{(1)} - P_{\mathcal{E}})(A_{\mathcal{E}} - S_{\mathcal{E}} P_{\mathcal{E}}^{(0)})\\ &\quad + (P_{\mathcal{E}} - P_{\mathcal{E}}^{(0)}) S_{\mathcal{E}} (P_{\mathcal{E}} - P_{\mathcal{E}}^{(0)})\\ &= (A_{\mathcal{E}} - S_{\mathcal{E}} P_{s\mathcal{E}}^{+})^T (P_{\mathcal{E}}^{(1)} - P_{\mathcal{E}})\\ &\quad + (P_{\mathcal{E}}^{(1)} - P_{\mathcal{E}})(A_{\mathcal{E}} - S_{\mathcal{E}} P_{s\mathcal{E}}^{+})\\ &\quad + (P_{\mathcal{E}} - P_{s\mathcal{E}}^{+}) S_{\mathcal{E}} (P_{\mathcal{E}} - P_{s\mathcal{E}}^{+}) = 0.\end{aligned}$$

Therefore, it is easy to verify that $V_{\mathcal{E}} = P_{\mathcal{E}}^{(1)} - P_{\mathcal{E}}$ because $A_{\mathcal{E}} - S_{\mathcal{E}} P_{s\mathcal{E}}^{+}$ is stable from Theorem 1 in Khalil and Kokotović (1979). Using Lemma 3 we obtain that

$$\begin{aligned}\|V_{\mathcal{E}}\| &= \|P_{c\mathcal{E}} - P_{\mathcal{E}}\| = \|P_{\mathcal{E}}^{(1)} - P_{\mathcal{E}}\|\\ &\le \|\Phi_{\mathcal{E}}\| \cdot \|P^{(1)} - P\|\\ &\le \|P^{(1)} - P\| = O(\|\mu\|^2).\end{aligned} \qquad (38)$$

Hence, we have $V_{\mathcal{E}} = P_{c\mathcal{E}} - P_{\mathcal{E}} = O(\|\mu\|^2)$, which implies (32). ■

Finally, by using the similar method(Xu and Mizukami, 1997), we show that the composite approximation (u_{1c}^*, u_{2c}^*) of the full-order linear feedback saddle-point solution constitutes the $O(\|\mu\|)$ near saddle-point equilibrium of the full-order game.

Theorem 3: Under the conditions of Theorem 1, the composite feedback strategy pair (u_{1c}^*, u_{2c}^*) constitutes the $O(\|\mu\|)$ near saddle-point equilibrium of the full-order game, that is,

$$\begin{aligned}&J(u_{1c}^*,\ u_2) - O(\|\mu\|)\\ \le\ &J(u_{1c}^*,\ u_{2c}^*)\\ \le\ &J(u_1,\ u_{2c}^*) + O(\|\mu\|).\end{aligned} \qquad (39)$$

V. CONCLUSION

In this paper, we have studied the infinite horizon zero-sum differential games for multiparameter singularly perturbed systems. We have shown that the composite approximation of the full-order linear feedback saddle-point solution (u_{1c}^*, u_{2c}^*) constitutes the $O(\|\mu\|)$ near saddle-point equilibrium of the full-order game, and the resulting value is $O(\|\mu\|^2)$ over or below the exact value of the full-order game which depends on the parameters of the system. The conclusions obtained in this paper are similar to those in the paper (Xu and Mizukami, 1997) where the same problem for singularly perturbed systems are considered. However, it is worth to note that the method used to prove the results in the paper (Xu and Mizukami, 1997) is not suitable to the differential games for the MSPS. In this paper, we have developed a different method to prove the results (Mukaidani, et al., 2001)

REFERENCES

Basar, T. and G. J. Olsder (1995).*Dynamic Noncooperative Game Theory.* Academic Press, New York.

Coumarbatch, C. and Z. Gajić (2000). Parallel optimal Kalman filtering for stochastic systems in multimodeling form. *Trans. ASME, J. Dynamic Systems, Measurement, and Control*, **122**, 542-550.

Gajić, Z. (1988). The existence of a unique and bounded solution of the algebraic Riccati equation of multimodel estimation and control problems. *Sys. & Con. Lett.*, **10**, 185-190.

Dragan, V. (1993) Asymptotic expansions for game-theoretic Riccati equations and stabilization with disturbance attenuation for singularly perturbed systems, *Systems and Control Lett.*, **20**, 455-463.

Khalil, H. K. and P. V. Kokotovič (1978). Control strategies for decision makers using different models of the same system. *IEEE Trans. A.C.*, **23**, 289-298.

Khalil, H. K. and P. V. Kokotovič (1979). Control of linear systems with multiparameter singular perturbations. *Automatica*, **15**, 197-207.

Laub, A. J. (1979). A Schur method for solving algebraic Riccati equations. *IEEE Trans. A.C.*, **24**, 913-921.

Mukaidani, H., H. Xu and K. Mizukami (2001). New iterative algorithm for algebraic Riccati equation related to H_∞ control problem of singularly perturbed systems. *IEEE Trans. A.C.*, **46**, 1659-1666.

Mukaidani, H., H. Xu and K. Mizukami (2003). New results for near-optimal control of linear multiparameter singularly perturbed systems. *Automatica*, **39**, 2157-2167.

Wang, Y-Y., M. Paul and N. E. Wu (1994). Near-optimal control of nonstandard singularly perturbed systems. *Automatica*, **30**, 277-292.

Xu, H. and K. Mizukami (1994). On the Isaacs equation of differential games for descriptor systems. *Journal of Optim. Theory Applic.*, **83**, 405-419.

Xu, H. and K. Mizukami (1997). Infinite-horizon differential games of singularly perturbed systems : A unified approach. *Automatica*, **33**, 273-276.

Xu, H., H. Mukaidani and K. Mizukami (1997). New method for composite optimal control of singularly perturbed systems. *Int. J. Systems Sciences*, **28**, 161-172.

Zhou, K., *Essentials of Robust Control*, Prentice Hall, New Jersey: 1998.