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Numerical computation of sign-indefinite linear quadratic differential games for weakly coupled large-scale systems

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In this paper, N-player linear quadratic differential games that are sign-indefinite for infinite horizon weakly coupled large-scale systems are discussed. After establishing the asymptotic structure and local uniqueness of the solution for cross-coupled sign-indefinite algebraic Riccati equations (CSARE), a new algorithm for solving CSARE is provided. It is shown that the proposed algorithm attains linear convergence. Moreover, in order to reduce the computational workspace, the recursive algorithm is combined. Finally, a high-order approximation strategy based on the proposed iterative solutions is described. As a result, it was recently proved that the numerical strategy achieves a high-order approximation of the equilibrium value. As another important feature, when the small parameters are unknown, a parameter-independent strategy is developed.

1. Introduction

Linear quadratic Nash games and their applications have been widely studied in many literatures, see e.g., Starr and Ho (1969) and Broek *et al*. (2003). In particular, the robust equilibria in indefinite linear quadratic differential games under the disturbance input affecting the systems have been discussed in Broek *et al*. (2003). It is well known that in order to obtain the Nash equilibrium strategy, the cross-coupled algebraic Riccati equations (CARE) must be solved. The Newton-type algorithm for solving the CARE has been applied (Krikelis and Rekasius 1971). However, this research has focused on determining the feedback gain matrices for the 2-player Nash games. It should be noted that for general N-player Nash games, it is difficult to solve the N-coupled CARE because the required workspace is needed N times the dimension of the full-systems.

Recently, in order to avoid such drawbacks, an algorithm referred to as the Lyapunov iterations for solving the CARE has been introduced (Li and Gajić 1994). However, the convergence rate of the Lyapunov iterations for solving the CARE is unclear.

The control problems of large-scale systems have been investigated extensively, see e.g. Siljak (1978). In particular, the control problems of weakly coupled large-scale systems have been studied by several researchers (Delacour *et al*. 1978, Srikant and Basar 1992, Gajić and Borno 2000, Mukaidani 2005 and references therein). A new iterative approach to obtain the solution of a class of two-agent dynamic stochastic teams for weakly coupled systems has been derived (Srikant and Basar 1992). On the other hand, the N-player Nash games for such systems have been investigated via the Lyapunov iterations (Mukaidani 2006). However, since the connection between each control input and the input of each performance index has not been considered, the Lyapunov iterations are not applicable to a wider class of the Nash games.

This paper investigates the numerical computation for solving N-player sign-indefinite linear quadratic differential games of infinite horizon weakly coupled large-scale systems. The existence and local uniqueness of the solutions related to the CSARE are newly discussed. It should be noted that the CSARE has a sign-indefinite quadratic term. The main contribution

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is to propose a new algorithm for solving the CSARE. It is shown that this new algorithm has a linear convergence property even if the CSARE is different from the existing CARE that has a positive semidefinite quadratic term. In particular, it is noteworthy that even though the control input coupling of the performance indices are considered, the convergence rate of the proposed algorithm and its exact proof are derived first. Furthermore, although the proposed algorithm is based on the Lyapunov iterations, it is possible to use this algorithm for the CSARE because the convergence proof is given. Additionally, in order to reduce the computational workspace, the recursive algorithm is combined. As another important feature, a high-order approximation strategy based on the iterative solutions is provided. As a result, it is proved that the proposed strategy achieves a high-order approximation of the equilibrium value. It should be noted that the proof used in this paper is quite different from the existing result (Mukaidani 2006). Moreover, when the small parameters are unknown, the proposed parameter-independent strategy is used. Finally, in order to demonstrate the efficiency of the algorithm, a numerical example is included.

Notation: The notations used in this paper are fairly standard. The superscript T denotes the matrix transpose. Trace denotes the matrix trace. I_n denotes the $n \times n$ identity matrix. **block diag** denotes the block diagonal matrix. $\|\cdot\|$ denotes its Euclidean norm for a matrix. $\det M$ denotes the determinant of M. $\otimes$ denotes the Kronecker product. δ_{ij} denotes the Kronecker delta. The space of $\mathbf{R}^k$-valued functions that are quadratically integrable on $(0, \infty)$ are denoted by $L_2^k(0, \infty)$.

2. Problem formulation

Consider the weakly coupled large-scale linear systems with N-players

$$\left.\begin{aligned} \dot{x}_i(t) &= A_{ii}x_i(t) + B_{ii}u_i(t) + \varepsilon \sum_{j=1,\ j\neq i}^{N} A_{ij}x_j(t) \\ &\quad + \varepsilon \sum_{j=1,\ j\neq i}^{N} B_{ij}u_j(t) + E_{ii}w_i(t) + \varepsilon \sum_{j=1,\ j\neq i}^{N} E_{ij}w_j(t), \\ x_i(0) &= x_i^0, \quad i = 1, \ldots, N, \end{aligned}\right\} \tag{1}$$

where $x_i \in \mathbf{R}^{n_i}$, $i = 1, \ldots, N$ represent ith state vectors. $u_i \in \mathbf{R}^{m_i}$, $i = 1, \ldots, N$ represent ith control inputs. $w_i \in \mathbf{R}^{k_i}$, $i = 1, \ldots, N$ represent ith disturbance vectors. ε denotes a small positive weak coupling parameter which connect the other subsystems.

Let us introduce the partitioned matrices

$$A_\varepsilon := \begin{bmatrix} A_{11} & \varepsilon A_{12} & \cdots & \varepsilon A_{1N} \\ \varepsilon A_{21} & A_{22} & \cdots & \varepsilon A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon A_{N1} & \varepsilon A_{N2} & \cdots & A_{NN} \end{bmatrix}, \quad B_{i\varepsilon} := \begin{bmatrix} \varepsilon^{1-\delta_{1i}} B_{1i} \\ \varepsilon^{1-\delta_{2i}} B_{2i} \\ \vdots \\ \varepsilon^{1-\delta_{Ni}} B_{Ni} \end{bmatrix},$$

$$E_\varepsilon := \begin{bmatrix} E_{11} & \varepsilon E_{12} & \cdots & \varepsilon E_{1N} \\ \varepsilon E_{21} & E_{22} & \cdots & \varepsilon E_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon E_{N1} & \varepsilon E_{N2} & \cdots & E_{NN} \end{bmatrix}.$$

By using above relations, the system (1) can be changed as

$$\dot{x}(t) = A_\varepsilon x(t) + \sum_{i=1}^{N} B_{i\varepsilon}u_i(t) + E_\varepsilon w(t), \tag{2}$$

where

$$x(t) := \begin{bmatrix} x_1(t)^T & \cdots & x_N(t)^T \end{bmatrix}^T \in \mathbf{R}^{\bar{n}}, \quad \bar{n} := \sum_{i=1}^{N} n_i,$$
$$w(t) := \begin{bmatrix} w_1(t)^T & \cdots & w_N(t)^T \end{bmatrix}^T \in \mathbf{R}^{\bar{k}}, \quad \bar{k} := \sum_{i=1}^{N} k_i.$$

The cost performance for each strategy subset is defined by

$$\begin{aligned} J_i(u_1, \ldots, u_N, w, x(0)) &= \int_0^\infty \Bigg[x^T(t)Q_{i\varepsilon}x(t) + u_i^T(t)R_{ii}u_i(t) \\ &\quad + \mu \sum_{j=1,\ j\neq i}^{N} u_j^T(t)R_{ij}u_j(t) - w^T(t)V_{i\mu}w(t) \Bigg] \mathrm{d}t, \end{aligned} \tag{3}$$

where

$$Q_{i\varepsilon} = \begin{bmatrix} \varepsilon^{1-\delta_{i1}} Q_{i1} & \varepsilon Q_{i12} & \cdots & \varepsilon Q_{i1N} \\ \varepsilon Q_{i12}^T & \varepsilon^{1-\delta_{i2}} Q_{i2} & \cdots & \varepsilon Q_{i2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon Q_{i1N}^T & \varepsilon Q_{i2N}^T & \cdots & \varepsilon^{1-\delta_{iN}} Q_{iN} \end{bmatrix} \in \mathbf{R}^{\bar{n}\times\bar{n}},$$
$$R_{ii} = R_{ii}^T > 0 \in \mathbf{R}^{m_i\times m_i}, \quad R_{ij} = R_{ij}^T \geq 0 \in \mathbf{R}^{m_j\times m_j},$$
$$V_{i\mu} = \textbf{block diag}\big(\mu^{-(1-\delta_{i1})}V_{i1} \quad \mu^{-(1-\delta_{i2})}V_{i2} \quad \cdots \quad \mu^{-(1-\delta_{iN})}V_{iN}\big) \geq 0 \in \mathbf{R}^{\bar{k}\times\bar{k}}, \quad i, j = 1, \ldots, N.$$

The state weight matrices $Q_{i\varepsilon}$ is symmetric and assumed to be sign-indefinite (Broek *et al.* 2003). Furthermore, it should be noted that μ denotes a small positive parameter which is the same order for the parameter ε. That is, the following assumption is made.

Assumption 1: The ratio of the small positive parameters ε and μ is bounded by some positive constants $\tilde{k}$

$$0 < \tilde{k} := \frac{\mu}{\varepsilon} < \infty. \tag{4}$$

It is now assumed that the parameters ε and μ are of the same order of magnitude, that is, their ratio is bounded by some positive constants. The reason for this is given as follows. First, it is preferable that the connection between each control input and the input of the performance indices has the same order of the coupling parameter ε because the coupling parameter μ strongly depends on the connection of the systems. Moreover, even though these parameters have the same order of magnitude, the coupling parameter μ should be different from the coupling parameter ε such that the order of the connection can be changed by the control designer.

For the matrices A_ε, $B_{i\varepsilon}$, $i = 1, \ldots, \mathrm{N}$, the set $\mathcal{F}_N$ is defined by

$$\mathcal{F}_N := \left\{ (F_{1\varepsilon}, \ldots, F_{N\varepsilon}) \mid A_\varepsilon + \sum_{j=1}^{N} B_{j\varepsilon} F_{j\varepsilon} \text{ is stable} \right\}.$$

The soft-constrained Nash equilibrium strategy $(F_{1\varepsilon}^*, \ldots, F_{N\varepsilon}^*)$ is defined as satisfying the following conditions (Broek *et al.* 2003)

$$\begin{aligned}&\bar{J}_i\big(F_{1\varepsilon}^*, \ldots, F_{N\varepsilon}^*,\ x(0)\big)\\ &\quad \le \bar{J}_i\big(F_{1\varepsilon}^*, \ldots, F_{i-1\varepsilon}^*,\ F_{i\varepsilon},\ F_{i+1\varepsilon}^*, \ldots, F_{N\varepsilon}^*, x(0)\big),\\ &\quad i = 1, \ldots, N,\end{aligned} \tag{5}$$

where

$$\bar{J}_i(F_{1\varepsilon}, \ldots, F_{N\varepsilon},\ x(0)) := \sup_{w \in L_2^{\tilde{k}}(0,\infty)} J_i(F_{1\varepsilon}, \ldots, F_{N\varepsilon}, w, x(0)),$$

$$\begin{aligned}&J_i(F_{1\varepsilon}, \ldots, F_{N\varepsilon}, w, x(0))\\ &= \int_0^\infty \Bigg[x^T(t) \Bigg[Q_{i\varepsilon} + F_{i\varepsilon}^T R_{ii} F_{i\varepsilon} + \mu \sum_{j=1,\ j\ne i}^{N} F_{j\varepsilon}^T R_{ij} F_{j\varepsilon} \Bigg] x(t)\\ &\quad - w^T(t) V_{i\mu} w(t) \Bigg] \mathrm{d}t,\end{aligned}$$

for all $x(0)$ and for all $(F_{1\varepsilon}, \ldots, F_{N\varepsilon})$ that satisfy $(F_{1\varepsilon}^*, \ldots, F_{i-1\varepsilon}^*,\ F_{i\varepsilon},\ F_{i+1\varepsilon}^*, \ldots,\ F_{N\varepsilon}^*) \in \mathcal{F}_N$.

It should be noted that the following assumption guarantees the existence of the admissible strategy.

Assumption 2: Each player uses the linear feedback strategy $u_i(t) = K_{i\varepsilon} x(t)$, $i = 1, \ldots, N$ such that the closed-loop system is asymptotically stable for sufficiently small parameters ε and μ.

Obviously, this assumption is made in order to obtain a stable system. Using the fact studied by Broek *et al.* (2003), the soft-constrained feedback Nash equilibrium is given below.

Lemma 1: *Assume that there exist N real symmetric matrices $P_{i\varepsilon}$ and $W_{i\varepsilon}$, such that*

$$\begin{aligned}&\mathcal{G}_i(P_{1\varepsilon}, \ldots, P_{N\varepsilon})\\ &= P_{i\varepsilon}\left(A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon} P_{j\varepsilon}\right) + \left(A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon} P_{j\varepsilon}\right)^T P_{i\varepsilon}\\ &\quad + P_{i\varepsilon} S_{i\varepsilon} P_{i\varepsilon} + \mu \sum_{j=1,\ j\ne i}^{N} P_{j\varepsilon} S_{ij\varepsilon} P_{j\varepsilon}\\ &\quad + P_{i\varepsilon} M_{i\mu} P_{i\varepsilon} + Q_{i\varepsilon} = 0,\end{aligned} \tag{6}$$

where $S_{i\varepsilon} := B_{i\varepsilon} R_{ii}^{-1} B_{i\varepsilon}^T$, $S_{ij\varepsilon} := B_{j\varepsilon} R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_{j\varepsilon}^T$, $M_{i\mu} := E_\varepsilon V_{i\mu}^{-1} E_\varepsilon^T$.
$A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon} P_{j\varepsilon} + M_{i\mu} P_{i\varepsilon}$ *is stable for* $i = 1, \ldots, N$,
$A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon} P_{j\varepsilon}$ *is stable,*

$$\begin{aligned}&W_{i\varepsilon}\left(A_\varepsilon - \sum_{j=1,\ j\ne i}^{N} S_{j\varepsilon} P_{j\varepsilon}\right) + \left(A_\varepsilon - \sum_{j=1,\ j\ne i}^{N} S_{j\varepsilon} P_{j\varepsilon}\right)^T\\ &\quad \times W_{i\varepsilon} - W_{i\varepsilon} S_{i\varepsilon} W_{i\varepsilon} + \mu \sum_{j=1,\ j\ne i}^{N} P_{j\varepsilon} S_{ij\varepsilon} P_{j\varepsilon} + Q_{i\varepsilon} \ge 0.\end{aligned} \tag{7}$$

Define the N-tuple $(F_{1\varepsilon}^*, \ldots, F_{N\varepsilon}^*)$ by

$$u_i^*(t) := F_{i\varepsilon}^* x(t) = -R_{ii}^{-1} B_{i\varepsilon}^T P_{i\varepsilon} x(t),\ i = 1, \ldots, N. \tag{8}$$

Then, $(F_{1\varepsilon}^*, \ldots, F_{N\varepsilon}^*) \in \mathcal{F}_N$ and this N-tuple is a soft-constrained Nash equilibrium. Furthermore, $\bar{J}_i(F_{1\varepsilon}^*, \ldots,\ F_{N\varepsilon}^*,\ x(0)) = x(0)^T P_{i\varepsilon} x(0)$.

It should be noted that if $Q_{i\varepsilon} \ge 0$ and $S_{ij\varepsilon} \ge 0$ for all $i = 1, \ldots, N$, the matrix inequality (7) is trivially satisfied with $W_{i\varepsilon} = 0$ (Broek *et al.* 2003).

In the following analysis, the basic assumption is needed.

Assumption 3: The triples $(A_{ii},\ B_{ii},\ \sqrt{Q_{ii}})$, $i = 1, \ldots, N$ are stabilizable and detectable.

3. Asymptotic structure of the CSARE

Firstly, in order to obtain the strategy, the asymptotic structure of the CSARE (6) is established. Since A_ε, $S_{i\varepsilon}$, $S_{ij\varepsilon}$ and $M_{i\mu}$ include the term of the small parameters ε and μ, the solution $P_{i\varepsilon}$ of the CSARE (6), if it exists, must contain these parameters. Moreover, it should be noted that two parameters ε and μ are the same magnitude such that Assumption 1 holds. Taking these facts into account, the solution $P_{i\varepsilon}$ of the CSARE (6) with the following structure is considered (Shen *et al.* 1994, Mukaidani 2006)

$$P_{i\varepsilon} := \begin{bmatrix} \varepsilon^{1-\delta_{i1}} P_{i1} & \varepsilon P_{i12} & \cdots & \varepsilon P_{i1N} \\ \varepsilon P_{i12}^T & \varepsilon^{1-\delta_{i2}} P_{i2} & \cdots & \varepsilon P_{i2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon P_{i1N}^T & \varepsilon P_{i2N}^T & \cdots & \varepsilon^{1-\delta_{iN}} P_{iN} \end{bmatrix} \in \mathbf{R}^{\bar{n}\times\bar{n}}.$$

Substituting the matrices A_ε, $S_{i\varepsilon}$, $S_{ij\varepsilon}$, $M_{i\mu}$, $Q_{i\varepsilon}$ and $P_{i\varepsilon}$ into the CSARE (6), letting $\varepsilon=0$ and $\mu=0$, and partitioning the CSARE (6), the following reduced-order algebraic Riccati equations (AREs) are obtained, where $\bar{P}_{ii}$, $i=1,\ldots,N$ be the 0-order solutions of the CSARE (6) as $\varepsilon=\mu=0$.

$$\bar{P}_{ii}A_{ii} + A_{ii}^T\bar{P}_{ii} - \bar{P}_{ii}(S_{ii} - M_{ii})\bar{P}_{ii} + Q_{ii} = 0, \tag{9}$$

where $S_{ii} := B_{ii}R_{ii}^{-1}B_{ii}^T$ and $M_{ii} := E_{ii}V_{ii}^{-1}E_{ii}^T$.

It should be noted that since the CSARE (6) is continuous and differentiable in $\varepsilon=\mu=0$, there exist $\bar{P}_{ii}$, $i=1,\ldots,N$ at $\varepsilon=\mu=0$. It should also be noted that the assumption that $S_{ii}-M_{ii}$ is positive semidefinite because of H_∞ control problem setting is not needed.

In order to guarantee the existence of a positive semidefinite stabilizing solution of the ARE (9), the following condition is assumed (Mukaidani 2004 and 2006).

Assumption 4: The ARE (9) has a positive semidefinite stabilizing solution such that $A_{ii} - S_{ii}\bar{P}_{ii}$ is stable.

The asymptotic expansion of the CSARE (6) at $\varepsilon=\mu=0$ is described by the following lemma.

Lemma 2: *Under Assumptions* 1–4, *there exist the small constants* σ^* *and* ρ^* *such that for all* $\varepsilon\in(0,\ \sigma^*)$ and $\mu\in(0,\ \rho^*)$, *the CSARE* (6) *admits a unique positive semidefinite solution* $P_{i\varepsilon}^*$ *that can be written as*

$$P_{i\varepsilon} := P_{i\varepsilon}^* = \bar{P}_i + O(\varepsilon) = \textbf{block diag}\big(0\cdots\bar{P}_{ii}\cdots 0\big) + O(\varepsilon). \tag{10}$$

Proof: The proof can be derived by using the implicit function theorem (Gajić *et al.* 1990) for the CSARE (6). Using the implicit function theorem, it can be shown that there exists a neighbourhood of $\varepsilon=\mu=0$ and a unique function $P_{i\varepsilon} := \bar{P}_i + O(\varepsilon)$. It should be noted that under Assumption 4, since the solution of the reduced-order ARE (9) is unique (see e.g. Theorem 13.5 of Zhou *et al.* (1996), $\bar{P}_i$ is a unique solution. Therefore, the CSARE (6) has a unique positive semidefinite solution $P_{i\varepsilon}^*$ under the sufficiently small parameters ε and μ. □

4. Iterative algorithm for solving CSARE

In order to obtain the strategy, the following useful algorithm is given.

Consider the following iterative algorithm that is called Lyapunov iterations

$$\begin{aligned} &P_{i\varepsilon}^{(k+1)}\left(A_\varepsilon - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(k)} + M_{i\mu}P_{i\varepsilon}^{(k)}\right) \\ &\quad + \left(A_\varepsilon - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(k)} + M_{i\mu}P_{i\varepsilon}^{(k)}\right)^T P_{i\varepsilon}^{(k+1)} \\ &\quad + P_{i\varepsilon}^{(k)}S_{i\varepsilon}P_{i\varepsilon}^{(k)} - P_{i\varepsilon}^{(k)}M_{i\mu}P_{i\varepsilon}^{(k)} + \mu\sum_{j=1,\ j\neq i}^N P_{j\varepsilon}^{(k)}S_{ij\varepsilon}P_{j\varepsilon}^{(k)} \\ &\quad + Q_{i\varepsilon} = 0,\quad k=0,1,\ \ldots, \end{aligned} \tag{11a}$$

$$P_{i\varepsilon}^{(k)} := \begin{bmatrix} \varepsilon^{1-\delta_{i1}} P_{i1}^{(k)} & \varepsilon P_{i12}^{(k)} & \cdots & \varepsilon P_{i1N}^{(k)} \\ \varepsilon P_{i12}^{(k)T} & \varepsilon^{1-\delta_{i2}} P_{i2}^{(k)} & \cdots & \varepsilon P_{i2N}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon P_{i1N}^{(k)T} & \varepsilon P_{i2N}^{(k)T} & \cdots & \varepsilon^{1-\delta_{iN}} P_{iN}^{(k)} \end{bmatrix} \tag{11b}$$

with the initial conditions

$$P_{i\varepsilon}^{(0)} = \bar{P}_i = \textbf{block diag}\big(0\cdots\bar{P}_{ii}\cdots 0\big). \tag{12}$$

It has been shown that Lyapunov iterations yield the positive semidefinite stabilizing solution for the positive sign-definite CARE (Mukaidani 2006). However, so far, there are no results for the convergence property for the CSARE (6). The following theorem indicates that the proposed algorithm which is based on Lyapunov iterations attain the linear convergence.

Theorem 1: *Under Assumptions* 1–4, *there exist the small constants* $\bar{\sigma}$ *and* $\bar{\rho}$ *such that for all* $\varepsilon\in(0,\ \bar{\sigma})$, $\bar{\sigma}\le\sigma^*$ *and* $\mu\in(0,\ \bar{\rho})$, $\rho\le\rho^*$, *the iterative algorithm* (11) *converges to the exact solution of* $P_{i\varepsilon}^*$ *with the rate of the linear convergence, where* $P_{i\varepsilon}^{(k)}$ *is positive semidefinite matrix and*

$A_\varepsilon - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(k)} + M_{i\mu}P_{i\varepsilon}^{(k)}$ is stable. That is, the following conditions are satisfied

$$\|P_{i\varepsilon}^{(k)} - P_{i\varepsilon}^*\| = O(\varepsilon^{k+1}), \tag{13a}$$

$$\mathrm{Re}\lambda\left[A_\varepsilon - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(k)} + M_{i\mu}P_{i\varepsilon}^{(k)}\right] < 0, \quad k = 0, 1, \ldots. \tag{13b}$$

Proof: The proof of this theorem can be derived by using the mathematical induction. When $k=0$, taking (10) into account, it is easy to verify that the first order approximations $P_{i\varepsilon}^*$ corresponding to the small parameters ε and μ are the same as $P_{i\varepsilon}^{(0)}$. Moreover, since

$$A_\varepsilon - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(0)} + M_{i\mu}P_{i\varepsilon}^{(0)} = \textbf{block diag}\left(D_{11} \ \cdots \ D_{NN}\right) + O(\varepsilon) := \mathcal{D}_A + O(\varepsilon),$$

where $D_{ii} := A_{ii} - S_{ii}\bar{P}_{ii} + M_{ii}\bar{P}_{ii}$, $D_{jj} = A_{jj} - S_{jj}\bar{P}_{jj}$, $j \neq i,\ j = 1, \ldots, N$, there exists the small perturbation parameter σ_0 such that

$$A_\varepsilon - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(0)} + M_{i\mu}P_{i\varepsilon}^{(0)}$$

is stable because $\mathcal{D}_A$ is stable for sufficiently small ε. When $k = h,\ h \geq 1$, it is assumed that

$$\|P_{i\varepsilon}^{(h)} - P_{i\varepsilon}^*\| = O(\varepsilon^{h+1}), \tag{14a}$$

$$\mathrm{Re}\lambda\left[A_\varepsilon - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(h)} + M_{i\mu}P_{i\varepsilon}^{(h)}\right] < 0. \tag{14b}$$

Subtracting (6) from (11a) and setting $k=h$, the following equations are satisfied

$$\begin{aligned}
&\left(P_{i\varepsilon}^{(h+1)} - P_{i\varepsilon}^*\right)\left(A_\varepsilon - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(h)} + M_{i\mu}P_{i\varepsilon}^{(h)}\right) \\
&\quad + \left(A_\varepsilon - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(h)} + M_{i\mu}P_{i\varepsilon}^{(h)}\right)^T\left(P_{i\varepsilon}^{(h+1)} - P_{i\varepsilon}^*\right) \\
&\quad + \sum_{j=1,\ j\neq i}^N P_{i\varepsilon}^*S_{j\varepsilon}\left(P_{j\varepsilon}^* - P_{j\varepsilon}^{(h)}\right) + \sum_{j=1,\ j\neq i}^N \left(P_{j\varepsilon}^* - P_{j\varepsilon}^{(h)}\right)S_{j\varepsilon}P_{i\varepsilon}^* \\
&\quad + \left(P_{i\varepsilon}^{(h)} - P_{i\varepsilon}^*\right)S_{i\varepsilon}\left(P_{i\varepsilon}^{(h)} - P_{i\varepsilon}^*\right) \\
&\quad - \left(P_{i\varepsilon}^{(h)} - P_{i\varepsilon}^*\right)M_{i\mu}\left(P_{i\varepsilon}^{(h)} - P_{i\varepsilon}^*\right) \\
&\quad + \mu\left[\sum_{j=1,\ j\neq i}^N P_{j\varepsilon}^{(h)}S_{ij\varepsilon}P_{j\varepsilon}^{(h)} - \sum_{j=1,\ j\neq i}^N P_{j\varepsilon}^*S_{ij\varepsilon}P_{j\varepsilon}^*\right] = 0.
\end{aligned} \tag{15}$$

Using the fact that the assumption (14a) holds, it is easy to derive that

$$\begin{aligned}
\sum_{j=1,\ j\neq i}^N P_{i\varepsilon}^*S_{j\varepsilon}\left(P_{j\varepsilon}^* - P_{j\varepsilon}^{(h)}\right) &= O(\varepsilon^{h+2}), \\
\left(P_{i\varepsilon}^{(h)} - P_{i\varepsilon}^*\right)S_{i\varepsilon}\left(P_{i\varepsilon}^{(h)} - P_{i\varepsilon}^*\right) &= O(\varepsilon^{2h+2}), \\
\left(P_{i\varepsilon}^{(h)} - P_{i\varepsilon}^*\right)M_{i\mu}\left(P_{i\varepsilon}^{(h)} - P_{i\varepsilon}^*\right) &= O(\varepsilon^{2h+2}),
\end{aligned}$$

$$\begin{aligned}
&\mu\left[\sum_{j=1, j\neq i}^N P_{j\varepsilon}^{(h)}S_{ij\varepsilon}P_{j\varepsilon}^{(h)} - \sum_{j=1,\ j\neq i}^N P_{j\varepsilon}^*S_{ij\varepsilon}P_{j\varepsilon}^*\right] \\
&= \mu\left[\sum_{j=1, j\neq i}^N \left(P_{j\varepsilon}^* + O(\varepsilon^{h+1})\right)S_{ij\varepsilon}\left(P_{j\varepsilon}^* + O(\varepsilon^{h+1})\right) - \sum_{j=1,\ j\neq i}^N P_{j\varepsilon}^*S_{ij\varepsilon}P_{j\varepsilon}^*\right] \\
&= \mu\left[\sum_{j=1, j\neq i}^N \left(P_{j\varepsilon}^*S_{ij\varepsilon}O(\varepsilon^{h+1}) + O(\varepsilon^{h+1})S_{ij\varepsilon}P_{j\varepsilon}^* + O(\varepsilon^{2h+2})\right)\right] \\
&= O(\varepsilon^{h+2}).
\end{aligned}$$

It should be noted that if $i \neq j,\ P_{i\varepsilon}^*S_{j\varepsilon} = O(\varepsilon)$ holds. Thus, the following relation is satisfied

$$\begin{aligned}
&\left(P_{i\varepsilon}^{(h+1)} - P_{i\varepsilon}^*\right)\left(A_\varepsilon - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(h)} + M_{i\mu}P_{i\varepsilon}^{(h)}\right) \\
&\quad + \left(A_\varepsilon - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(h)} + M_{i\mu}P_{i\varepsilon}^{(h)}\right)^T\left(P_{i\varepsilon}^{(h+1)} - P_{i\varepsilon}^*\right) \\
&\quad + O(\varepsilon^{h+2}) = 0.
\end{aligned} \tag{16}$$

Taking into account the fact that the stability assumption (14b) holds and using relation (14a), the following result is satisfied.

$$\begin{aligned}
(16) \Leftrightarrow P_{i\varepsilon}^{(h+1)} - P_{i\varepsilon}^* &= \int_0^\infty \exp\left[\Psi_\varepsilon^T t\right]O(\varepsilon^{h+2})\exp[\Psi_\varepsilon t]\,\mathrm{d}t \\
&= \int_0^\infty \exp[O(\varepsilon)t]\exp\left[\mathcal{D}_A^T t\right]O(\varepsilon^{h+2}) \\
&\quad \times \exp[\mathcal{D}_A t]\exp[O(\varepsilon)t]\mathrm{d}t
\end{aligned}$$

where $\Psi_\varepsilon = (A_\varepsilon - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(h)} + M_{i\mu}P_{i\varepsilon}^{(h)}) = \mathcal{D}_A + O(\varepsilon)$.

Since there exist the ε-independent scalar parameters α and β such that

$$\|\exp[O(\varepsilon)t]\| \leq \beta e^{\alpha\varepsilon t},$$

it is easy to verify that

$$\|P_{i\varepsilon}^{(h+1)} - P_{i\varepsilon}^{*}\| \le \int_0^{\infty} \beta^2 e^{2\alpha\varepsilon t} \left\| \exp\left[\mathcal{D}_A^T t\right] O(\varepsilon^{h+2}) \times \exp[\mathcal{D}_A t] \right\| \mathrm{d}t = O(\varepsilon^{h+2}). \tag{17}$$

Furthermore, using the relation (17), it is shown that there exists the small positive perturbation parameter σ_{h+1} such that

$$\begin{aligned} A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon} P_{j\varepsilon}^{(h+1)} + M_{i\mu} P_{i\varepsilon}^{(h+1)} &= A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon} P_{j\varepsilon}^{*} \\ &\quad + M_{i\mu} P_{i\varepsilon}^{*} + O(\varepsilon^{h+2}) \\ &= \mathcal{D}_A + O(\varepsilon) \end{aligned}$$

is stable. Consequently, choosing $\bar{\sigma} := \min\{\sigma_0, \ldots, \sigma_{h+1}\}$, the relation (13b) holds for all $k \in \mathbf{N}$. This completes the proof of Theorem 1 concerned with the Lyapunov iterations. □

Using the asymptotic structure of the solutions (10), the local uniqueness of the convergence solutions is studied.

Theorem 2: *Under Assumptions* 1–4, *there exist the sufficiently small constants* $\hat{\sigma}$ *and* $\hat{\rho}$ *such that for all* $\varepsilon \in (0,\ \hat{\sigma})$, $\hat{\sigma} \le \bar{\sigma} \le \sigma^*$ *and* $\mu \in (0,\ \hat{\rho})$, $\hat{\rho} \le \bar{\rho} \le \rho^*$, *the convergence solution* $P_{i\varepsilon}^{*}$ *of the iterative solution* $P_{i\varepsilon}^{(k)}$ *is unique in the neighbourhood of* $\varepsilon = \mu = 0$.

Proof: First, under Assumptions 1–4, there exists the neighbourhood of $\varepsilon = \mu = 0$ such that the CSARE (6) admits a unique positive semidefinite solution $P_{i\varepsilon}^{*}$ by means of the implicit function theorem (see the proof of Lemma 2). That is, for sufficiently small ε and μ, the CSARE (6) has a unique positive semidefinite solution $P_{i\varepsilon}^{*}$. Taking into account the fact that the solutions $P_{i\varepsilon}^{*}$ of (10) and (13a) are equivalent, the iterative solution $P_{i\varepsilon}^{(k)}$ converges to $P_{i\varepsilon}^{*}$ and it is a unique solution for sufficiently small ε and μ. □

Although the Lyapunov iterations (11) yield the positive semidefinite solution of the CSARE (6) in general, the local uniqueness of the convergence solution is guaranteed in the neighbourhood of $\varepsilon = 0$ under the weakly coupled large-scale system.

On the other hand, the convergence solution may not be maximum solution. However, since the solution is unique in the neighbourhood of $\varepsilon = 0$, other solution cannot be used to the weakly coupled Nash games as long as the sufficiently small parameter ε is considered. As a result, it is worth pointing out that the convergence solutions satisfy the local uniqueness and the positive semidefiniteness in the neighbourhood of $\varepsilon = 0$.

It is well known that the CSARE (6) could have several positive definite solutions and even some indefinite solutions (Abou-Kandil *et al.* 2003). However, as long as the sufficiently small parameters ε and μ are chosen, the obtained iterative solutions guarantee the positive semidefiniteness and admissibility. Furthermore, the positive semidefinite stabilizing properties of the solutions obtained by means of the proposed algorithm are guaranteed (Li and Gajić 1994).

When the ALE (11a) is solved, the dimension of the workspace as $\bar{n} := \sum_{i=1}^{N} n_i$ larger than the dimensions n_i, $i = 1, \ldots, N$ is needed. Thus, in order to reduce the dimension of the workspace, a new algorithm for solving the ALE (11a) which is based on the recursive algorithm is established. Let us consider the following ALE (18), in a general form of the ALE (11a)

$$X_\varepsilon \Lambda_\varepsilon + \Lambda_\varepsilon^T X_\varepsilon + U_\varepsilon = 0. \tag{18}$$

In particular, the following special matrices X_ε, Λ_ε and U_ε which are related to the ALE (18) are considered because the other case $i = 2, \ldots, N$ can be changed into the similar form by using the similarity transformation $\mathcal{T}_i$, where

$$X_\varepsilon := \mathcal{T}_i^{-1} P_{i\varepsilon}^{(k+1)} \mathcal{T}_i,$$

$$\Lambda_\varepsilon := \mathcal{T}_i^{-1} \Bigg(A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon} P_{j\varepsilon}^{(k)} + M_{i\mu} P_{i\varepsilon}^{(k)} \Bigg) \mathcal{T}_i,$$

$$\begin{aligned} U_\varepsilon := \mathcal{T}_i^{-1} \Bigg(& P_{i\varepsilon}^{(k)} S_{i\varepsilon} P_{i\varepsilon}^{(k)} - P_{i\varepsilon}^{(k)} M_{i\mu} P_{i\varepsilon}^{(k)} \\ & + \mu \sum_{j=1,\, j\neq i}^{N} P_{j\varepsilon}^{(k)} S_{ij\varepsilon} P_{j\varepsilon}^{(k)} + Q_{i\varepsilon} \Bigg) \mathcal{T}_i, \end{aligned}$$

$$\mathcal{T}_i := \begin{bmatrix} 0 & \cdots & I_{n_i} & \cdots & 0 \\ \vdots & \textbf{block diag}(1 \ldots 1) & \vdots & \ddots & \vdots \\ I_{n_i} & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \textbf{block diag}(1 \ldots 1) & \vdots \\ 0 & \cdots & 0 & \cdots & I_{n_N} \end{bmatrix},$$

$$X_\varepsilon := \begin{bmatrix} X_{11} & \varepsilon X_{12} & \cdots & \varepsilon X_{1N} \\ \varepsilon X_{12}^T & \varepsilon X_{22} & \cdots & \varepsilon X_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon X_{1N}^T & \varepsilon X_{2N}^T & \cdots & \varepsilon X_{NN} \end{bmatrix},$$

$$\Lambda_\varepsilon := \begin{bmatrix} \Lambda_{11} & \varepsilon \Lambda_{12} & \cdots & \varepsilon \Lambda_{1N} \\ \varepsilon \Lambda_{21} & \Lambda_{22} & \cdots & \varepsilon \Lambda_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon \Lambda_{N1} & \varepsilon \Lambda_{N2} & \cdots & \Lambda_{NN} \end{bmatrix},$$

$$U_\varepsilon := \begin{bmatrix} U_{11} & \varepsilon U_{12} & \cdots & \varepsilon U_{1N} \\ \varepsilon U_{12}^T & \varepsilon U_{22} & \cdots & \varepsilon U_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon U_{1N}^T & \varepsilon U_{2N}^T & \cdots & \varepsilon U_{NN} \end{bmatrix}.$$

In order to guarantee the existence of the solution and the convergence of the algorithm, another assumption is needed.

Assumption 5: $\Lambda_{11}, \ldots, \Lambda_{NN}$ are stable.

Without loss of generality, it should be noted that the above assumption is satisfied automatically under the condition of Theorem 1.

In order to calculate briefly, the following partitioned matrices are introduced.

$$X_\varepsilon := \begin{bmatrix} X_{11} & \varepsilon X_{1f} \\ \varepsilon X_{1f}^T & \varepsilon X_f \end{bmatrix}, \quad \Lambda_\varepsilon := \begin{bmatrix} \Lambda_{11} & \varepsilon \Lambda_{1f} \\ \varepsilon \Lambda_{f1} & \Lambda_f \end{bmatrix},$$
$$U_\varepsilon := \begin{bmatrix} U_{11} & \varepsilon U_{1f} \\ \varepsilon U_{1f}^T & \varepsilon U_f \end{bmatrix}.$$

As a result, the ALE (18) can be changed as follows by partitioning.

$$X_{11}\Lambda_{11} + \Lambda_{11}^T X_{11} + \varepsilon^2(X_{1f}\Lambda_{f1} + \Lambda_{f1}^T X_{1f}^T) + U_{11} = 0, \tag{19a}$$
$$X_f\Lambda_f + \Lambda_f^T X_f + \varepsilon(X_{1f}^T\Lambda_{1f} + \Lambda_{1f}^T X_{1f}) + U_f = 0, \tag{19b}$$
$$X_{11}\Lambda_{1f} + X_{1f}\Lambda_f + \Lambda_{11}^T X_{1f} + \varepsilon\Lambda_{f1}^T X_f + U_{1f} = 0. \tag{19c}$$

It should be noted that the ALE (19) is quite different from the existing one (Gajić *et al.* 1990). Moreover, Λ_f is stable because Assumption 5 holds.

Defining approximation errors as

$$X_{11} = \bar{X}_{11} + \varepsilon H_{11}, \quad X_{1f} = \bar{X}_{1f} + \varepsilon H_{1f},$$
$$X_f = \bar{X}_f + \varepsilon H_f. \tag{20}$$

where

$$\bar{X}_{11}\Lambda_{11} + \Lambda_{11}^T \bar{X}_{11} + U_{11} = 0,$$
$$\bar{X}_f\Lambda_f + \Lambda_f^T \bar{X}_f + U_f = 0,$$
$$\bar{X}_{11}\Lambda_{1f} + \bar{X}_{1f}\Lambda_f + \Lambda_{11}^T \bar{X}_{1f} + U_{1f} = 0.$$

Substituting (20) into (19), the following ALEs for the matrices H_{11}, H_{1f} and H_f are derived

$$H_{11}\Lambda_{11} + \Lambda_{11}^T H_{11} + \varepsilon(X_{1f}\Lambda_{f1} + \Lambda_{f1}^T X_{1f}^T) = 0, \tag{21a}$$
$$H_f\Lambda_f + \Lambda_f^T H_f + X_{1f}\Lambda_{1f} + \Lambda_{1f}^T X_{1f} = 0, \tag{21b}$$
$$H_{11}\Lambda_{1f} + H_{1f}\Lambda_f + \Lambda_{11}^T H_{1f} + \Lambda_{f1}^T X_f = 0. \tag{21c}$$

Taking the form of (20) into account, the algorithm to solve the ALE (21) is given by (22).

$$H_{11}^{(k+1)}\Lambda_{11} + \Lambda_{11}^T H_{11}^{(k+1)} + \varepsilon\left(X_{1f}^{(k)}\Lambda_{f1} + \Lambda_{f1}^T X_{1f}^{(k)T}\right) = 0, \tag{22a}$$
$$H_f^{(k+1)}\Lambda_f + \Lambda_f^T H_f^{(k+1)} + X_{1f}^{(k)T}\Lambda_{1f} + \Lambda_{1f}^T X_{1f}^{(k)} = 0, \tag{22b}$$
$$H_{1f}^{(k+1)}\Lambda_f + \Lambda_{11}^T H_{1f}^{(k+1)} + H_{11}^{(k+1)}\Lambda_{1f} + \Lambda_{f1}^T X_f^{(k+1)} = 0. \tag{22c}$$

where $k = 0, 1, \ldots,$

$$X_{11}^{(k)} = \bar{X}_{11} + \varepsilon H_{11}^{(k)}, \quad X_{1f}^{(k)} = \bar{X}_{1f} + \varepsilon H_{1f}^{(k)},$$
$$X_f^{(k)} = \bar{X}_f + \varepsilon H_f^{(k)}, \quad H_{11}^{(0)} = 0, \ H_{1f}^{(0)} = 0, \ H_f = 0.$$

It should be noted that $H_{1f}^{(k+1)}$ can be computed as the Sylvester equation by using the solutions $H_{11}^{(k+1)}$ and $H_f^{(k+1)}$ that are obtained from ALEs (22a) and (22b) because $X_f^{(k+1)} = \bar{X}_f + \varepsilon H_f^{(k+1)}$.

The following theorem indicates the convergence of the algorithm (22).

Theorem 3: *Under Assumption 5, the recursive algorithm* (22) *converges to the exact solution H_{11}, H_{1f} and H_f with the rate of*

$$\|H_{11}^{(k)} - H_{11}\| = O(\varepsilon^{2k}), \quad \|H_{1f}^{(k)} - H_{1f}\| = O(\varepsilon^{2k-1}),$$
$$\|H_f^{(k)} - H_f\| = O(\varepsilon^{2k}), \quad k = 1, \ldots. \tag{23}$$

Proof: The proof of Theorem 3 can be done by using the mathematical induction. Subtracting (21) from (22), the following equations hold

$$(H_{11}^{(k+1)} - H_{11})\Lambda_{11} + \Lambda_{11}^T(H_{11}^{(k+1)} - H_{11})$$
$$+ \varepsilon^2[(H_{1f}^{(k)} - H_{1f})\Lambda_{f1} + \Lambda_{f1}^T(H_{1f}^{(k)} - H_{1f})^T] = 0, \tag{24a}$$
$$(H_f^{(k+1)} - H_f)\Lambda_f + \Lambda_f^T(H_f^{(k+1)} - H_f)$$
$$+ \varepsilon[(H_{1f}^{(k)} - H_{1f})\Lambda_{1f} + \Lambda_{1f}^T(H_{1f}^{(k)} - H_{1f})] = 0, \tag{24b}$$
$$(H_{11}^{(k+1)} - H_{11})\Lambda_{1f} + (H_{1f}^{(k+1)} - H_{1f})\Lambda_f$$
$$+ \Lambda_{11}^T(H_{1f}^{(k+1)} - H_{1f}) + \varepsilon\Lambda_{f1}^T(H_f^{(k+1)} - H_f) = 0. \tag{24c}$$

First, $k = 0$ for the algorithms (24) is considered. Taking into account the fact that the stability assumption of Λ_{ii} holds and using the standard properties of the algebraic Lyapunov equation ALE (Zhou 1998), it is easy to verify that

$$\|H_{11}^{(1)} - H_{11}\| = O(\varepsilon^2), \quad \|H_f^{(1)} - H_f\| = O(\varepsilon),$$
$$\|H_{1f}^{(1)} - H_{1f}\| = O(\varepsilon^2). \tag{25}$$

Therefore, the equation (23) is true for $k=1$. When $k=h$, $h\geq 2$, it is assumed that

$$\|H_{11}^{(h)}-H_{11}\|=O(\varepsilon^{2h}),\quad \|H_{f}^{(h)}-H_{f}\|=O(\varepsilon^{2h-1}),$$
$$\|H_{1f}^{(h)}-H_{1f}\|=O(\varepsilon^{2h}). \tag{26}$$

Setting $k=h$ for the ALE (24) and using the above assumption, the following equations hold

$$(H_{11}^{(h+1)}-H_{11})\Lambda_{11}+\Lambda_{11}^{T}(H_{11}^{(h+1)}-H_{11})+\mathrm{O}(\varepsilon^{2h+2})=0, \tag{27a}$$
$$(H_{f}^{(h+1)}-H_{f})\Lambda_{f}+\Lambda_{f}^{T}(H_{f}^{(h+1)}-H_{f})+O(\varepsilon^{2h+1})=0, \tag{27b}$$

$$(H_{11}^{(h+1)}-H_{11})\Lambda_{1f}+(H_{1f}^{(h+1)}-H_{1f})\Lambda_{f}$$
$$+\Lambda_{11}^{T}(H_{1f}^{(h+1)}-H_{1f})+\varepsilon\Lambda_{f1}^{T}(H_{f}^{(h+1)}-H_{f})=0. \tag{27c}$$

After the cancellation takes place, since Λ_{ii}, $i=1,\ldots,N$ are stable from Assumption 5, the following relations hold

$$\|H_{11}^{(h+1)}-H_{11}\|=O(\varepsilon^{2h+2}),\quad \|H_{f}^{(h+1)}-H_{f}\|=O(\varepsilon^{2h+1}),$$
$$\|H_{1f}^{(h+1)}-H_{1f}\|=O(\varepsilon^{2h+2}). \tag{28}$$

Consequently, the error equations (23) are true for all $k\in\mathbf{N}$. This completes the proof of Theorem 3. □

5. High-order soft constrained Nash strategy

The required solution of the CSARE (6) exists under Assumptions 1–4. Moreover, it is very important to note that the iterative solutions $P_{i\varepsilon}^{(k)}$ by means of the Lyapunov iterations (11) satisfy the positive semidefiniteness, the local uniqueness in the neighbourhood of $\varepsilon=0$ and the admissibility from Li and Gajić (1994). That is, these convergence solutions will satisfy the soft constrained Nash equilibrium properties (5) for sufficiently small parameter ε.

The attention is focused on the design of the high-order Nash equilibrium strategy for the sign-indefinite linear quadratic games. Such strategy is obtained by using the iterative solution (11a).

$$u_i^{(k)*}(t)=-R_{ii}^{-1}B_{i\varepsilon}^{T}P_{i\varepsilon}^{(k)}x(t),\quad i=1,\ldots,N. \tag{29}$$

The degradation of the cost performance via the new high-order soft constrained Nash equilibrium strategy (29) is given as follows.

Theorem 4: *Under Assumptions* 1–4, *the use of the high-order soft constrained Nash equilibrium strategy* (29) *results in* (30)

$$\bar{J}_i\Big(F_{1\varepsilon}^{(k)*},\ldots,F_{N\varepsilon}^{(k)*},\ x(0)\Big)=\bar{J}_i\big(F_{1\varepsilon}^{*},\ldots,F_{N\varepsilon}^{*},x(0)\big)$$
$$+O(\varepsilon^{k+2}),\quad i=1,\ldots,N. \tag{30}$$

Proof: When $u_i^{(k)*}(t)=F_{i\varepsilon}^{(k)*}x(t)$ is used, the value of the cost performance is given by Broek *et al.* (2003)

$$\bar{J}_i\Big(F_{1\varepsilon}^{(k)*},\ldots,F_{N\varepsilon}^{(k)*},x(0)\Big)=x^T(0)Y_{i\varepsilon}x(0), \tag{31}$$

where $Y_{i\varepsilon}$ is a positive semidefinite solution of the following ARE

$$Y_{i\varepsilon}\left(A_\varepsilon-\sum_{j=1}^{N}S_{j\varepsilon}P_{j\varepsilon}^{(k)}\right)+\left(A_\varepsilon-\sum_{j=1}^{N}S_{j\varepsilon}P_{j\varepsilon}^{(k)}\right)^T$$
$$\times Y_{i\varepsilon}+Y_{i\varepsilon}M_{i\mu}Y_{i\varepsilon}+\mu\sum_{j=1,\ j\neq i}^{N}P_{j\varepsilon}^{(k)}S_{ij\varepsilon}P_{j\varepsilon}^{(k)}$$
$$+P_{i\varepsilon}^{(k)}S_{i\varepsilon}P_{i\varepsilon}^{(k)}+Q_{i\varepsilon}=0. \tag{32}$$

Subtracting the CSARE (6) from the ARE (32), $Z_\varepsilon=Y_{i\varepsilon}-P_{i\varepsilon}$ satisfies the following ARE

$$Z_{i\varepsilon}\left(A_\varepsilon-\sum_{j=1}^{N}S_{j\varepsilon}P_{j\varepsilon}^{(k)}+M_{i\mu}P_{i\varepsilon}^{(k)}+M_{i\mu}\Big(P_{i\varepsilon}-P_{i\varepsilon}^{(k)}\Big)\right)$$
$$+\left(A_\varepsilon-\sum_{j=1}^{N}S_{j\varepsilon}P_{j\varepsilon}^{(k)}+M_{i\mu}P_{i\varepsilon}^{(k)}+M_{i\mu}\Big(P_{i\varepsilon}-P_{i\varepsilon}^{(k)}\Big)\right)^T Z_{i\varepsilon}$$
$$+Z_{i\varepsilon}M_{i\mu}Z_{i\varepsilon}+\sum_{j=1,\ j\neq i}^{N}P_{i\varepsilon}S_{j\varepsilon}\Big(P_{j\varepsilon}-P_{j\varepsilon}^{(k)}\Big)$$
$$+\sum_{j=1,\ j\neq i}^{N}\Big(P_{j\varepsilon}-P_{j\varepsilon}^{(k)}\Big)S_{j\varepsilon}P_{i\varepsilon}$$
$$+\mu\left[\sum_{j=1,\ j\neq i}^{N}\Big(P_{j\varepsilon}^{(k)}S_{ij\varepsilon}P_{j\varepsilon}^{(k)}-P_{j\varepsilon}S_{ij\varepsilon}P_{j\varepsilon}\Big)\right]$$
$$+\Big(P_{i\varepsilon}-P_{i\varepsilon}^{(k)}\Big)S_{i\varepsilon}\Big(P_{i\varepsilon}-P_{i\varepsilon}^{(k)}\Big)=0. \tag{33}$$

Taking (13a) into account as $P_{i\varepsilon}=P_{i\varepsilon}^{*}$, it is easy to verify that

$$Z_{i\varepsilon}(\mathcal{D}_A+O(\varepsilon))+(\mathcal{D}_A+O(\varepsilon))^TZ_{i\varepsilon}$$
$$+Z_{i\varepsilon}M_{i\mu}Z_{i\varepsilon}+O(\varepsilon^{k+2})=0. \tag{34}$$

Without loss of generality, using the similarity transformation $\mathcal{T}_i$, the following ARE is considered because the other case is similar,

$$\mathcal{Z}_\varepsilon \mathcal{A}_\varepsilon + \mathcal{A}_\varepsilon^T \mathcal{Z}_\varepsilon + \mathcal{Z}_\varepsilon M_\mu \mathcal{Z}_\varepsilon + O(\varepsilon^{k+2}) = 0, \tag{35}$$

where

$$\mathcal{Z}_\varepsilon := \mathcal{T}_i^{-1} Z_{i\varepsilon} \mathcal{T}_i = \begin{bmatrix} \bar{Z}_1 & \varepsilon \bar{Z}_{1f} \\ \varepsilon \bar{Z}_{1f}^T & \varepsilon \bar{Z}_f \end{bmatrix}, \ \bar{Z}_1 = Z_{ii},$$

$$\mathcal{A}_\varepsilon := \mathcal{T}_i^{-1}(\mathcal{D}_A + O(\varepsilon))\mathcal{T}_i = \mathbf{block\ diag}\big(\bar{D}_1\ \bar{D}_f\big) + O(\varepsilon),$$

$$\bar{D}_1 := D_{ii} = A_{ii} - S_{ii}\bar{P}_{ii} + M_{ii}\bar{P}_{ii},$$

$$\bar{D}_f := \mathbf{block\ diag}\big(D_{22} \cdots D_{i-1i-1}\ D_{i+1i+1} \cdots D_{NN}\big),$$

$$M_\mu := \mathcal{T}_i^{-1} M_{i\mu} \mathcal{T}_i = \begin{bmatrix} \bar{M}_1 & \varepsilon \bar{M}_{1f} \\ \varepsilon \bar{M}_{1f}^T & \varepsilon \bar{M}_f \end{bmatrix}, \ \bar{M}_1 = M_{ii}.$$

Letting $\varepsilon = \mu = 0$, the following reduced-order parameter independent algebraic Bernoulli equation (ABE) is given

$$\mathcal{Z}_0 \mathcal{A}_0 + \mathcal{A}_0^T \mathcal{Z}_0 + \mathcal{Z}_0 M_0 \mathcal{Z}_0 = 0, \tag{36}$$

where

$$\mathcal{Z}_0 := \begin{bmatrix} \bar{Z}_1^{(0)} & 0 \\ 0 & 0 \end{bmatrix},$$

$$\mathcal{A}_0 := \mathbf{block\ diag}\big(\bar{D}_1 \quad \bar{D}_f\big), \quad M_0 := \begin{bmatrix} \bar{M}_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The ABE (36) is equivalent to the following reduced-order ABE by partitioning

$$\bar{Z}_1^{(0)} \bar{D}_1 + \bar{D}_1^T \bar{Z}_1^{(0)} + \bar{Z}_1^{(0)} \bar{M}_1 \bar{Z}_1^{(0)} = 0. \tag{37}$$

Taking into account the fact that

$$\begin{bmatrix} \bar{D}_1 & \bar{M}_1 \\ 0 & -\bar{D}_1^T \end{bmatrix}$$

has no eigenvalues on the imaginary axis and $\bar{D}_1$ is stable under Assumption 4 and using the well-known result (see e.g. Theorem 13.5 of Zhou *et al.* (1996)), the reduced-order ABE (37) has a unique stabilizing solution $\bar{Z}_1^{(0)} = 0$. Thus, the following equation holds

$$\bar{Z}_1^{(0)} = 0 \Leftrightarrow \bar{Z}_1 = \bar{Z}_1^{(0)} + O(\varepsilon) = O(\varepsilon). \tag{38}$$

Hence, the solution (38) results in

$$\mathcal{Z}_\varepsilon = \begin{bmatrix} \bar{Z}_1 & \varepsilon \bar{Z}_{1f} \\ \varepsilon \bar{Z}_{1f}^T & \varepsilon \bar{Z}_f \end{bmatrix} = \varepsilon \mathcal{Z}_\varepsilon^{(1)}. \tag{39}$$

Substituting (39) into (35), it follows that

$$\mathcal{Z}_\varepsilon^{(1)} \mathcal{A}_\varepsilon + \mathcal{A}_\varepsilon^T \mathcal{Z}_\varepsilon^{(1)} + \varepsilon \mathcal{Z}_\varepsilon^{(1)} M_\mu \mathcal{Z}_\varepsilon^{(1)} + O(\varepsilon^{k+1}) = 0. \tag{40}$$

Using the similar technique, the following relation holds.

$$\mathcal{Z}_\varepsilon = \varepsilon^2 \mathcal{Z}_\varepsilon^{(2)}. \tag{41}$$

Finally, continuing the above steps results in (41)

$$\mathcal{Z}_\varepsilon = \varepsilon^{k+2} \mathcal{Z}_\varepsilon^{(k+2)} = O\big(\varepsilon^{k+2}\big). \tag{42}$$

Therefore, $Z_{i\varepsilon} = \mathcal{T}_i \mathcal{Z}_\varepsilon \mathcal{T}_i^{-1} = O(\varepsilon^{k+2})$ because of the stability condition (13b) and the standard properties of the ARE. Hence

$$\begin{aligned} x(0)^T Z_{i\varepsilon} x(0) &= x(0)^T Y_{i\varepsilon} x(0) - x^T(0) P_{i\varepsilon} x(0) \\ &= \bar{J}_i(F_{1\varepsilon}^{(k)*}, \ldots, F_{N\varepsilon}^{(k)*}, x(0)) \\ &\quad - \bar{J}_i\big(F_{1\varepsilon}^*, \ldots, F_{N\varepsilon}^*, x(0)\big) = O\big(\varepsilon^{k+2}\big) \end{aligned} \tag{43}$$

results in (30). □

It should be noted that the above results and its proof are novel compared with the existing results (Mukaidani 2006).

In the rest of this section, an important implication is given. If the parameters ε and μ are unknown, then the following corollary is easily seen in view of Theorem 4.

Corollary 1: *Consider the parameter-independent soft constrained Nash strategy*

$$\bar{u}_i^*(t) := u_i^{(0)*}(t) = -R_{ii}^{-1} B_i^T \bar{P}_i x(t), \ i = 1, \ldots, N, \tag{44}$$

where $B_i^T := \begin{bmatrix} 0 & \cdots & B_{ii}^T & \cdots & 0 \end{bmatrix}$.

Under Assumptions 1–4, *the use of the reduced-order strategy* (44) *results in* (45)

$$\begin{aligned} \bar{J}_i\big(\bar{F}_{1\varepsilon}^*, \ldots, \bar{F}_{N\varepsilon}^*, x(0)\big) &= \bar{J}_i\big(F_{1\varepsilon}^*, \ldots, F_{N\varepsilon}^*, x(0)\big) \\ &\quad + O(\varepsilon^2), \ i = 1, \ldots, N. \end{aligned} \tag{45}$$

Proof: Since the result of Corollary 1 can be proved by using the similar technique in Theorem 4 under the fact that $\|P_{i\varepsilon} - \bar{P}_i\| = O(\varepsilon)$, the proof is omitted. □

6. Numerical example

In order to demonstrate the efficiency of the proposed algorithm, an illustrative example is given. The system matrices are given as follows:

$$A_{11} = \begin{bmatrix} 0 & 1 & -0.266 & -0.009 \\ -2.75 & -2.78 & -1.36 & -0.037 \\ 0 & 0 & 0 & 1 \\ -4.95 & 0 & -55.5 & -0.039 \end{bmatrix},$$

$$\varepsilon A_{12} = \begin{bmatrix} 0.0024 & 0 & -0.087 & 0.002 \\ -0.185 & 0 & 1.11 & -0.011 \\ 0 & 0 & 0 & 0 \\ 0.222 & 0 & 8.17 & 0.004 \end{bmatrix},$$

$$\varepsilon A_{13} = \begin{bmatrix} 0.073 & 0 & -0.25 & 0.003 \\ -0.46 & 0 & 2.8 & -0.02 \\ 0 & 0 & 0 & 0 \\ 0.924 & 0 & 17.5 & 0.02 \end{bmatrix},$$

$$\varepsilon A_{21} = \begin{bmatrix} 0.021 & 0 & 0.121 & 0.003 \\ -1.1 & 0 & -1.62 & -0.015 \\ 0 & 0 & 0 & 0 \\ -2.43 & 0 & 1.37 & -0.034 \end{bmatrix},$$

$$A_{22} = \begin{bmatrix} -0.21 & 1 & -1.6 & -0.005 \\ -1.9 & -1.8 & 9.3 & -0.12 \\ 0 & 0 & 0 & 1 \\ -3.1 & 0 & -56 & 0.032 \end{bmatrix},$$

$$\varepsilon A_{23} = \begin{bmatrix} 0.06 & 0 & 0.46 & 0.002 \\ -1 & 0 & 1.49 & -0.04 \\ 0 & 0 & 0 & 0 \\ 0.12 & 0 & 29.8 & -0.028 \end{bmatrix},$$

$$\varepsilon A_{31} = \begin{bmatrix} -0.002 & 0 & 0.83 & 0 \\ -6.78 & 0 & -10.1 & 0.09 \\ 0 & 0 & 0 & 0 \\ -1.24 & 0 & 0.498 & -0.017 \end{bmatrix},$$

$$\varepsilon A_{32} = \begin{bmatrix} 0.011 & 0 & 0.22 & 0 \\ -2.1 & 0 & 1.7 & -0.123 \\ 0 & 0 & 0 & 0 \\ -0.07 & 0 & 6.38 & -0.011 \end{bmatrix},$$

$$A_{33} = \begin{bmatrix} -0.197 & 1 & -1.2 & -0.003 \\ -54.5 & -20 & 70.1 & -2.37 \\ 0 & 0 & 0 & 1 \\ -3.4 & 0 & -21.0 & -0.017 \end{bmatrix},$$

$$B_{11} = \begin{bmatrix} 0 \\ 36.1 \\ 0 \\ 0 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 0 \\ 78.9 \\ 0 \\ 0 \end{bmatrix},$$

$$B_{33} = \begin{bmatrix} 0 \\ 1000 \\ 0 \\ 0 \end{bmatrix}, \quad B_{ij} = 0, \quad i \neq j,$$

$$E_{11} = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0.1 \end{bmatrix},$$

$$E_{33} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0.1 \end{bmatrix},$$

$E_{ij} = 0, \ i \neq j,$

$$\begin{aligned} V_{ii} &= \textbf{block diag}\begin{pmatrix} 1 & 2 & 2 & 1 \end{pmatrix}, \\ V_1 &= \textbf{block diag}\begin{pmatrix} V_{ii} & \mu^{-1} I_4 & \mu^{-1} I_4 \end{pmatrix}, \\ V_2 &= \textbf{block diag}\begin{pmatrix} \mu^{-1} I_4 & V_{ii} & \mu^{-1} I_4 \end{pmatrix}, \\ V_3 &= \textbf{block diag}\begin{pmatrix} \mu^{-1} I_4 & \mu^{-1} I_4 & V_{ii} \end{pmatrix}, \\ Q_1 &= \textbf{block diag}\begin{pmatrix} 0.5 I_4 & O_{8\times 8} \end{pmatrix}, \\ Q_2 &= \textbf{block diag}\begin{pmatrix} O_{4\times 4} & 0.5 I_4 & O_{4\times 4} \end{pmatrix}, \\ Q_3 &= \textbf{block diag}\begin{pmatrix} O_{8\times 8} & 0.5 I_4 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} R_{11} = R_{22} = R_{33} = 1, \ R_{12} = R_{13} = 0.2, \\ R_{23} = R_{21} = 0.3, \ R_{31} = R_{32} = 0.1. \end{aligned}$$

The small parameters are chosen as $\varepsilon = 0.01$ and $\mu = 0.005$. It should be noted that the algorithm (11a) converges to the exact solution with accuracy of $\|\mathcal{G}^{(k)}(\varepsilon)\| < 1.0e - 10$ after five iterations, where

$$\|\mathcal{G}^{(k)}(\varepsilon)\| := \sum_{i=1}^{3} \|\mathcal{G}_i(P_{1\varepsilon}^{(k)}, \ P_{2\varepsilon}^{(k)}, \ P_{3\varepsilon}^{(k)})\|. \tag{46}$$

In order to verify the exactitude of the solution, the remainder per iteration by substituting $P_{i\varepsilon}^{(k)}$ into the

Table 1.

k	$\|\mathcal{G}^{(k)}(1.0e-02)\|$	$\|\mathcal{G}^{(k)}(1.0e-03)\|$	$\|\mathcal{G}^{(k)}(1.0e-04)\|$
0	$3.5262e-01$	$3.5262e-02$	$3.5262e-03$
1	$4.8116e-03$	$4.8163e-05$	$4.8188e-07$
2	$1.6202e-05$	$1.5945e-08$	$1.6104e-11$
3	$1.1027e-07$	$1.1069e-11$	
4	$5.6047e-10$		
5	$2.7238e-12$		

Table 2.

k	$\|H_{11}^{(k)}-H_{11}\|$	$\|H_{1f}^{(k)}-H_{1f}\|$	$\|H_{f}^{(k)}-H_{f}\|$
0	$5.3991e-04$	$1.6457e-03$	$3.1859e-02$
1	$5.5514e-07$	$2.4153e-06$	$7.1316e-05$
2	$9.8057e-10$	$3.4647e-09$	$8.4322e-08$
3	$1.2965e-12$	$5.0174e-12$	$1.3434e-10$
4	$1.8468e-15$	$7.1821e-15$	$2.0040e-13$
5	$4.7322e-17$	$3.0454e-16$	$1.7938e-15$

CSARE (6) is computed. In table 1, the results of the error $\|\mathcal{G}^{(k)}(\varepsilon)\|$ per iterations are given for several values ε and $\mu=0.5\varepsilon$. As a result, it can be seen that the algorithm (11a) has the linear convergence. On the other hand, it should be noted that the existence of more than one soft-constrained Nash equilibrium is possible because it is not a sufficiently small parameter as $\varepsilon=0.01$.

Second, the norm condition (22) is evaluated. Choosing $\varepsilon=0.01$ and $\mu=0.005$, the errors between the exact solution and the iterative solution per iterations are given in table 2. It should be noted that the result for the first iteration of the algorithm (11a) is demonstrated. It can be found that the norm condition (23) for the numerical error of the algorithm (22) are true.

Finally, the costs using the near-optimal strategy (29) are computed. The initial conditions are chosen as $x(0)=[\,1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\,]^T$. The cost functional-to-perturbation ε^{k+2} ratio for various ε and μ are given in table 3, where

$$\phi_i=\frac{|J_{i\mathrm{app}}-J_{i\mathrm{opt}}|}{\varepsilon^{k+2}} =\frac{|x^T(0)Y_{i\varepsilon}x(0)-x^T(0)P_{i\varepsilon}x(0)|}{\varepsilon^{k+2}}, \tag{47}$$

with $\varepsilon=0.01$, $\mu=0.005$,

$$J_{i\mathrm{app}}:=\bar{J}_i\Big(F_{1\varepsilon}^{(k)*},\ldots,\ F_{N\varepsilon}^{(k)*},\ x(0)\Big)=x(0)^TY_{i\varepsilon}x(0),$$
$$J_{i\mathrm{opt}}:=\bar{J}_i\big(F_{1\varepsilon}^{*},\ldots,\ F_{N\varepsilon}^{*},\ x(0)\big)=x^T(0)P_{i\varepsilon}x(0).$$

Table 3.

k	ϕ_1	ϕ_2	ϕ_3
0	$1.8753e+01$	$1.0841e+01$	$2.3362e-01$
1	$4.0209e-02$	$2.0298e-01$	1.5967
2	$9.9697e-02$	$1.5860e-01$	$8.4013e-01$
3	$1.2945e-02$	$2.0882e-01$	$2.2566e-01$
4	$6.2172e-03$	$2.9310e-02$	$1.5787e-02$
5	3.3196	$8.6597e-01$	$2.3981e+01$

It is easy to verify that $\bar{J}_i(F_{1\varepsilon}^{(k)*},\ldots,\ F_{N\varepsilon}^{(k)*},\ x(0))-\bar{J}_i(F_{1\varepsilon}^{*},\ldots,\ F_{N\varepsilon}^{*},\ x(0))=O(\varepsilon^{k+2})$ because of $\phi_i<\infty$.

7. Conclusion

In this paper, N-player indefinite linear quadratic differential games for large-scale systems connected by a weak small positive coupling parameter have been studied. The main contribution of this study is to propose a new algorithm for solving the large-scale CSARE with a sign-indefinite quadratic term. Comparing with the existing result (Mukaidani 2006), the control input coupling of the performance indices has been newly considered. It is noteworthy that although the proposed design method is based on the Lyapunov iterations (Petrovic and Gajić1988, Gajić *et al.* 1990) for solving the sign-indefinite CARE, the convergence rate has been newly proved as a linear convergence. Moreover, to reduce the computational workspace, the recursive algorithm has been combined. Finally, both fast convergence and a reduced-order calculation are attained. As another important feature, the asymptotic structure and local uniqueness of the solution for the CSARE has been proved.

It is well known that the implicit function theorem (Gajić *et al.* 1990) guarantees the existence of the small parameters σ^* and ρ^* such that for all parameters $\varepsilon\in(0,\ \sigma^*)$ and $\mu\in(0,\ \rho^*)$, the CSARE admits a positive semidefinite stabilizing solution. However, there is no information about the magnitude of these coupling parameters which maintains the assertion. Furthermore, the proposed algorithm may not converge under the large parameters ε and μ. These problems will be addressed in future investigations via the Newton–Kantorovich theorem (Yamamoto 1986).

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