

$c(s; \mathbf{q}^0) = (s - q_1^0)(s - q_2^0) = (s - 1)(s - 2)$ . Hence, the following controller results (note that  $m = 2$ ):

$$C(s; \alpha) = \alpha \frac{(s^2 + 1.2s + 1)(s + 2)}{sd(s)}. \quad (33)$$

We selected  $d(s) = s^2 + 35$ , so that all the poles of  $C(s; \alpha)$  are purely imaginary. In order to have  $r(t; \alpha, \tau) \in C^0$ , a third-order transition polynomial ( $v = 1$ , note that the plant relative order is  $\rho = 1$ ) has been chosen as output function, so that we have

$$y(t; \tau) = -\frac{2}{\tau^3}t^3 + \frac{3}{\tau^2}t^2 \quad t \in [0, \tau].$$

The optimal values of  $\alpha$  and  $\tau$  have been determined by means of a genetic algorithm [9]. The resulting values are  $\alpha^* = 7.57$  and  $\tau^* = 6.99$  s, which results in an optimal worst-case settling time  $t_{wcs}^*$  equal to 16.44 s, with preaction time  $t_0^* = -6.2$  s ( $\varepsilon_0 = 10^{-3}$ ). The command input  $r_a(t; \alpha^*, \tau^*)$ , the worst-case output (which occurs when  $\mathbf{q} = [0.8, 1.6, 0.5]$ ) and the corresponding control variable are reported in Fig. 2. For technical convenience, in all the plots the zero time has been shifted to  $t_0^*$ .

To better evaluate how the new proposed approach works, in Fig. 3 we report the envelope of the Bode plots over the uncertain parameters box  $\mathcal{Q}$ , both of the plant (open-loop) and of the closed-loop system. Then, in Fig. 4 the normalized power spectrum of the command signal is shown. It can be noted that for the range of frequencies of the command signal, the presence of the feedback controller almost completely cancels the effects of the plant uncertainties, therefore greatly increasing the effectiveness of the use of the dynamic inversion.

## VI. CONCLUSION

In this note, we have proposed a new inversion-based control architecture for the set-point constrained regulation of nonminimum-phase scalar systems subject to parametric uncertainties. This new control scheme, which can be regarded as a radical generalization of the classic two-degrees-of-freedom configuration (the filter is indeed substituted by the command signal generator, see Fig. 1), permits obtaining high performances as exemplified in Section V. An extension to multivariable plants is possible and is currently investigated by the authors.

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## New Iterative Algorithm for Algebraic Riccati Equation Related to $H_\infty$ Control Problem of Singularly Perturbed Systems

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**Abstract**—In this note, we present the solution to the algebraic Riccati equation (ARE) with indefinite sign quadratic term related to the  $H_\infty$  control problem for singularly perturbed system by means of a Kleinman's type algorithm. The resulting algorithm is very efficient from the numerical point of view because the ARE is solvable even if the quadratic term has an indefinite sign. Moreover, the resulting iterative algorithm is quadratically convergent. We also present a new algorithm for solving the generalized algebraic Lyapunov equation (GALE) on the basis of the fixed point algorithm.

**Index Terms**—Fixed point algorithm,  $H_\infty$  control, Kleinman algorithm, singularly perturbed systems.

## I. INTRODUCTION

$H_\infty$  control problems for linear singularly perturbed systems were considered in many papers [1]–[9]. In particular, a great deal of studies on the composite controller design for singularly perturbed systems in  $H_\infty$  sense have been made [2], [3], [6], [8].

In order to obtain the optimal controller, we must solve the algebraic Riccati equation (ARE). Various reliable approaches to the theory of the ARE have been well documented in many literatures (see e.g., [11]–[14]). These methods consist of the invariant subspace approach which is based on the Hamiltonian matrix [11], [12] and the general matrix pencil technique which is based on the extended Hamiltonian pencil [13], [14] (in particular, the reference [14] is the most complete reference to date dealing with ARE by means of the matrix pencils). However, such approaches are not adequate to the singularly perturbed systems because of high dimension and numerical stiffness [10].

The recursive algorithm for the solution of ARE of singularly perturbed systems have been developed in many literatures (see, e.g., [15]). From a practical point of view, it has been shown that the recursive algorithm is very effective to solve the ARE when the system matrices are functions of a small perturbation parameter  $\varepsilon$ . However, the recursive algorithm converge only to the approximation solution. Moreover, such an algorithm is the linear convergence. On the other hand, the exact slow-fast decomposition method for solving the singularly perturbed systems has been proposed (see, for example, [7] and the references therein). However, in order to obtain the exact solution, ones need the same workspace compared with with the full-order ARE for calculating the inverse matrix.

In this paper, we study the numerical solution to the ARE with indefinite sign quadratic term related to the  $H_\infty$  control problem of singularly perturbed systems. The objective of this paper is to extend the convergence result of [17] to the ARE with indefinite sign quadratic term. Our new idea is to set the initial condition to the solutions of the reduced-order ARE. Because of such a choice, we can prove that our iterative algorithm converges to a unique solution of the ARE

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with indefinite sign quadratic term by using Newton–Kantorovich theorem [16]. Also, while the classical recursive algorithm is of the linear convergence property, the new iterative algorithm achieves the quadratic convergence property since the resulting algorithm is based on the Kleinman algorithm [17]. Furthermore, we also present a new algorithm for solving the generalized algebraic Lyapunov equation (GALE). The proof of the algorithm is based on the fixed point theorem. As another important feature, it is easy to construct an  $O(\varepsilon^{2^i})$  high-order approximation controller compared with the existing methods (e.g., [2], [3], [7], [20]).

## II. PROBLEM STATEMENT AND PRELIMINARIES

The AREs of singularly perturbed system corresponding to  $H_\infty$  control problem [4], [6] have the following form:

$$A_\varepsilon^T P_\varepsilon + P_\varepsilon A_\varepsilon + \gamma^{-2} P_\varepsilon G_\varepsilon G_\varepsilon^T P_\varepsilon - (P_\varepsilon B_\varepsilon + C_1^T D_{12}) \cdot (D_{12}^T D_{12})^{-1} (B_\varepsilon^T P_\varepsilon + D_{12}^T C_1) + C_1^T C_1 = 0 \quad (1)$$

$$A_\varepsilon W_\varepsilon + W_\varepsilon A_\varepsilon^T + \gamma^{-2} W_\varepsilon C_1^T C_1 W_\varepsilon - (W_\varepsilon C_2^T + G_\varepsilon D_{21}^T) \cdot (D_{21} D_{21}^T)^{-1} (C_2 W_\varepsilon + G_{21} B_\varepsilon^T) + G_\varepsilon G_\varepsilon^T = 0 \quad (2)$$

where  $\varepsilon$  is a small positive parameter. Let us introduce the following matrices:

$$\begin{aligned} P_\varepsilon &= P_\varepsilon(\gamma) = \begin{bmatrix} P_{11}(\varepsilon, \gamma) & \varepsilon P_{21}(\varepsilon, \gamma)^T \\ \varepsilon P_{21}(\varepsilon, \gamma) & \varepsilon P_{22}(\varepsilon, \gamma) \end{bmatrix} \\ W_\varepsilon &= W_\varepsilon(\gamma) = \begin{bmatrix} W_{11}(\varepsilon, \gamma) & W_{12}(\varepsilon, \gamma) \\ W_{12}(\varepsilon, \gamma)^T & \varepsilon^{-1} W_{22}(\varepsilon, \gamma) \end{bmatrix} \\ A_\varepsilon &= \begin{bmatrix} A_{11} & A_{12} \\ \varepsilon^{-1} A_{21} & \varepsilon^{-1} A_{22} \end{bmatrix} & A &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \\ B_\varepsilon &= \begin{bmatrix} B_1 \\ \varepsilon^{-1} B_2 \end{bmatrix} & B &= \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ G_\varepsilon &= \begin{bmatrix} G_1 \\ \varepsilon^{-1} G_2 \end{bmatrix} & G &= \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \\ C_1 &= [C_{11} \quad C_{12}] & C_2 &= [C_{21} \quad C_{22}]. \end{aligned}$$

In addition, dimensions of block matrices are as follows:

$$\begin{aligned} P_{11} &= P_{11}^T & W_{11} &= W_{11}^T & A_{11} &\in \mathbf{R}^{n_1 \times n_1} \\ P_{22} &= P_{22}^T & W_{22} &= W_{22}^T \\ A_{22} &\in \mathbf{R}^{n_2 \times n_2} & n_1 + n_2 &= n \\ B_1 &\in \mathbf{R}^{n_1 \times p} & B_2 &\in \mathbf{R}^{n_2 \times p} \\ G_1 &\in \mathbf{R}^{n_1 \times q} & G_2 &\in \mathbf{R}^{n_2 \times q} \\ C_{11} &\in \mathbf{R}^{r \times n_1} & C_{12} &\in \mathbf{R}^{r \times n_2} \\ C_{21} &\in \mathbf{R}^{s \times n_1} & C_{22} &\in \mathbf{R}^{s \times n_2}. \end{aligned}$$

The remaining matrices are constant matrices of appropriate dimensions. For technical simplification, we shall make the following basic assumptions.

*Assumption 1:*

- 1)  $D_{12}^T D_{12}$  and  $D_{21} D_{21}^T$  are nonsingular.
- 2)

$$\begin{aligned} \text{rank} \begin{bmatrix} A - sI_n & B \\ C_1 & D_{12} \end{bmatrix} &= n + p, \forall s \in \mathbf{C}^+ \\ \text{rank} \begin{bmatrix} A - sI_n & G \\ C_2 & D_{21} \end{bmatrix} &= n + q, \forall s \in \mathbf{C}^+. \end{aligned}$$

*Assumption 2:*

- 1) The pair  $(A_{22}, B_2)$  is stabilizable and  $(C_{12}, A_{22})$  is observable.

2)

$$\begin{aligned} \text{rank} \begin{bmatrix} sI_{n_1} - A_{11} & -A_{12} & B_1 \\ -A_{21} & -A_{22} & B_2 \end{bmatrix} &= n, \forall s \in \mathbf{C}^+ \\ \text{rank} \begin{bmatrix} sI_{n_1} - A_{11}^T & -A_{12}^T & C_{11}^T \\ -A_{12}^T & -A_{22}^T & C_{12}^T \end{bmatrix} &= n, \forall s \in \mathbf{C}^+. \end{aligned}$$

*Assumption 3:*

- 1) The pair  $(A_{22}^T, C_{22}^T)$  is stabilizable and  $(G_2^T, A_{22}^T)$  is observable.
- 2)

$$\begin{aligned} \text{rank} \begin{bmatrix} sI_{n_1} - A_{11}^T & -A_{12}^T & C_{21}^T \\ -A_{12}^T & -A_{22}^T & C_{22}^T \end{bmatrix} &= n, \forall s \in \mathbf{C}^+ \\ \text{rank} \begin{bmatrix} sI_{n_1} - A_{11} & -A_{12} & G_1 \\ -A_{21} & -A_{22} & G_2 \end{bmatrix} &= n, \forall s \in \mathbf{C}^+. \end{aligned}$$

It is well known in [3], [4] that a controller which stabilizes the singularly perturbed system with disturbance attenuation level measured by  $\gamma$  does exist if and only if (1) and (2) admit the positive-semidefinite solutions  $P_\varepsilon(\gamma)$  and  $W_\varepsilon(\gamma)$ , respectively, such that

- a)  $A_\varepsilon + G_\varepsilon F_1 + B_\varepsilon F_2$  with  $F_1 = F_1(\varepsilon) = \gamma^{-2} G_\varepsilon^T P_\varepsilon$ ,  $F_2 = F_2(\varepsilon) = (D_{12}^T D_{12})^{-1} (B_\varepsilon^T P_\varepsilon + D_{12}^T C_1)$  is stable;
- b)  $A_\varepsilon + H_{1\varepsilon} C_1 + H_{2\varepsilon} C_2$  with  $H_{1\varepsilon} = \gamma^{-2} W_\varepsilon C_1^T$ ,  $H_{2\varepsilon} = (W_\varepsilon C_2^T + G_\varepsilon D_{21}^T) (D_{21} D_{21}^T)^{-1}$  is stable;
- c)  $\rho(P_\varepsilon W_\varepsilon) < \gamma^2$ , where  $\rho(\cdot)$  is the spectral radius.

In order to solve the AREs (1) and (2) with indefinite sign quadratic term, we introduce the following useful lemma for the generalized algebraic Riccati equation (GARE) [20].

*Lemma 1:* The AREs (1) and (2) are equivalent to the following GAREs (3) and (4), respectively

$$\begin{aligned} \mathcal{F}_1(P) &= A^T P + P^T A + \gamma^{-2} P^T G G^T P \\ &\quad - (P^T B + C_1^T D_{12}) \tilde{D}_1 (B^T P + D_{12}^T C_1) \\ &\quad + C_1^T C_1 = 0 \end{aligned} \quad (3a)$$

$$P_\varepsilon = \Pi_\varepsilon P = P^T \Pi_\varepsilon \quad (3b)$$

$$\begin{aligned} \mathcal{F}_2(W) &= A W^T + W A^T + \gamma^{-2} W C_1^T C_1 W^T \\ &\quad - (W C_2^T + G D_{21}^T) \tilde{D}_2 (C_2 W^T + D_{21} G^T) \\ &\quad + G G^T = 0 \end{aligned} \quad (4a)$$

$$W_\varepsilon = \Pi_\varepsilon^{-1} W = W^T \Pi_\varepsilon^{-1} \quad (4b)$$

where

$$\Pi_\varepsilon = \text{diag}(I_{n_1} \quad \varepsilon I_{n_2}) \quad P = \begin{bmatrix} P_{11} & \varepsilon P_{21}^T \\ P_{21} & P_{22} \end{bmatrix}$$

$$W = \begin{bmatrix} W_{11} & W_{12} \\ \varepsilon W_{12}^T & W_{22} \end{bmatrix}$$

$$\tilde{D}_1 = (D_{12}^T D_{12})^{-1} \quad \tilde{D}_2 = (D_{21} D_{21}^T)^{-1}$$

$$P_{11} = P_{11}^T \quad P_{22} = P_{22}^T \quad W_{11} = W_{11}^T \quad W_{22} = W_{22}^T$$

$$A = \Pi_\varepsilon A_\varepsilon \quad B = \Pi_\varepsilon B_\varepsilon \quad G = \Pi_\varepsilon G_\varepsilon.$$

Partitioning for the ARE (3a) and letting  $\varepsilon = 0$ , we obtain the following equations:

$$\begin{aligned} \bar{A}_{11}^T \bar{P}_{11} + \bar{P}_{11} \bar{A}_{11} + \bar{A}_{21}^T \bar{P}_{21} + \bar{P}_{21}^T \bar{A}_{21} - \bar{P}_{11} S_{11}^T \bar{P}_{11} \\ - \bar{P}_{21}^T S_{22}^T \bar{P}_{21} - \bar{P}_{11} S_{12}^T \bar{P}_{21} - \bar{P}_{21}^T S_{12}^T \bar{P}_{11} + Q_{11} = 0 \end{aligned} \quad (5a)$$

$$\begin{aligned} \bar{P}_{22}^T \bar{A}_{21} + \bar{A}_{12}^T \bar{P}_{11} + \bar{A}_{22}^T \bar{P}_{21} \\ - \bar{P}_{22} S_{12}^T \bar{P}_{11} - \bar{P}_{22} S_{22}^T \bar{P}_{21} + Q_{12}^T = 0 \end{aligned} \quad (5b)$$

$$\bar{A}_{22}^T \bar{P}_{22} + \bar{P}_{22} \bar{A}_{22} - \bar{P}_{22} S_{22}^T \bar{P}_{22} + Q_{22} = 0 \quad (5c)$$

where

$$\begin{aligned}\bar{A}^\gamma &= A - B\bar{D}_1 D_{12}^T C_1 = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \\ S^\gamma &= B\bar{D}_1 B^T - \gamma^{-2} G G^T = \begin{bmatrix} S_{11}^\gamma & S_{12}^\gamma \\ S_{12}^{\gamma T} & S_{22}^\gamma \end{bmatrix} \\ Q &= C_1^T (I_r - D_{12} \bar{D}_1 D_{12}^T) C_1 = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}\end{aligned}$$

and  $\bar{P}_{11}$ ,  $\bar{P}_{21}$  and  $\bar{P}_{22}$  are 0-order solutions of the ARE (3a). The ARE (5c) will produce the unique positive-semidefinite stabilizing solution under assumption 2 if  $\gamma$  is large enough.

Let  $\gamma_{1f} = \inf\{\gamma > 0 \mid \text{the ARE (5c) has a positive-semidefinite stabilizing solution}\}$ .

Then, the matrix  $\bar{A}_{22} - S_{22}^\gamma \bar{P}_{22}$  is nonsingular if we choose  $\gamma > \gamma_{1f}$ . Therefore, we obtain the following 0-order equations:

$$A_p^T \bar{P}_{11} + \bar{P}_{11} A_p^\gamma - \bar{P}_{11} S_p^\gamma \bar{P}_{11} + Q_p^\gamma = 0 \quad (6a)$$

$$\bar{P}_{21} = -N_2^T + N_1^T \bar{P}_{11} \quad (6b)$$

$$\bar{A}_{22}^T \bar{P}_{22} + \bar{P}_{22} \bar{A}_{22} - \bar{P}_{22} S_{22}^\gamma \bar{P}_{22} + Q_{22} = 0 \quad (6c)$$

where

$$T_0 = T_1 - T_2 T_4^{-1} T_3 = \begin{bmatrix} A_p^\gamma & -S_p^\gamma \\ -Q_p^\gamma & -A_p^{\gamma T} \end{bmatrix}$$

$$T_1 = \begin{bmatrix} \bar{A}_{11} & -S_{11}^\gamma \\ -Q_{11} & -\bar{A}_{11}^T \end{bmatrix}$$

$$T_2 = \begin{bmatrix} \bar{A}_{12} & -S_{12}^\gamma \\ -Q_{12} & -\bar{A}_{21}^T \end{bmatrix}$$

$$T_3 = \begin{bmatrix} \bar{A}_{21} & -S_{12}^{\gamma T} \\ -Q_{12}^T & -\bar{A}_{12}^T \end{bmatrix}$$

$$T_4 = \begin{bmatrix} \bar{A}_{22} & -S_{22}^\gamma \\ -Q_{22} & -\bar{A}_{22}^T \end{bmatrix}$$

$$N_2^T = D_4^{-T} \hat{Q}_{12}^T \quad N_1^T = -D_4^{-T} D_2^T$$

$$D_1 = \bar{A}_{11} - S_{11}^\gamma \bar{P}_{11} - S_{12}^\gamma \bar{P}_{21}$$

$$D_3 = \bar{A}_{21} - S_{12}^{\gamma T} \bar{P}_{11} - S_{22}^\gamma \bar{P}_{21}$$

$$D_2 = \bar{A}_{12} - S_{12}^\gamma \bar{P}_{22} \quad D_4 = \bar{A}_{22} - S_{22}^\gamma \bar{P}_{22}$$

$$D_0 = D_1 - D_2 D_4^{-1} D_3 \quad \hat{Q}_{12} = Q_{12} + \bar{A}_{21}^T \bar{P}_{22}.$$

*Remark 1:* The matrices  $A_p^\gamma$ ,  $S_p^\gamma$  and  $Q_p^\gamma$  do not depend on  $\bar{P}_{22}$  because their matrices can be computed by using  $T_m$ ,  $m = 1, \dots, 4$  which is independent of  $\bar{P}_{22}$  [8], [9], [18].

Let us define  $\gamma_{1s} = \inf\{\gamma > 0 \mid \text{the ARE (6a) has a positive semidefinite stabilizing solution}\}$ .

By following the similar steps, we obtain the following equations:

$$A_w^\gamma \bar{W}_{11} + \bar{W}_{11} A_w^{\gamma T} - \bar{W}_{11} R_w^\gamma \bar{W}_{11} + M_w^\gamma = 0 \quad (7a)$$

$$\bar{W}_{12} = -L_2 + \bar{W}_{11} L_1 \quad (7b)$$

$$\hat{A}_{22} \bar{W}_{22} + \bar{W}_{22} \hat{A}_{22}^T - \bar{W}_{22} R_{22}^\gamma \bar{W}_{22} + M_{22} = 0 \quad (7c)$$

where

$$\hat{A}^\gamma = A - G D_{21}^T \bar{D}_2 C_2 = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}$$

$$R^\gamma = C_2^T \bar{D}_2 C_2 - \gamma^{-2} C_1^T C_1 = \begin{bmatrix} R_{11}^\gamma & R_{12}^\gamma \\ R_{12}^{\gamma T} & R_{22}^\gamma \end{bmatrix}$$

$$M = G (I_q - D_{21}^T \bar{D}_2 D_{21}) G^T = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}$$

$$H_0 = H_1 - H_2 H_4^{-1} H_3 = \begin{bmatrix} A_w^{\gamma T} & -R_w^\gamma \\ -M_w^\gamma & -A_w^\gamma \end{bmatrix}$$

$$H_1 = \begin{bmatrix} \hat{A}_{11}^T & -R_{11}^\gamma \\ -M_{11} & -\hat{A}_{11} \end{bmatrix}$$

$$H_2 = \begin{bmatrix} \hat{A}_{21}^T & -R_{12}^\gamma \\ -M_{12} & -\hat{A}_{12} \end{bmatrix}$$

$$H_3 = \begin{bmatrix} \hat{A}_{12}^T & -R_{12}^{\gamma T} \\ -M_{12}^T & -\hat{A}_{21} \end{bmatrix}$$

$$H_4 = \begin{bmatrix} \hat{A}_{22}^T & -R_{22}^\gamma \\ -M_{22} & -\hat{A}_{22} \end{bmatrix}$$

$$L_2^T = U_4^{-1} \hat{V}_{12} \quad L_1^T = -U_4^{-1} U_3$$

$$U_1 = \hat{A}_{11} - \bar{W}_{12} R_{12}^{\gamma T} - \bar{W}_{11} R_{11}^{\gamma T}$$

$$U_2 = \hat{A}_{12} - \bar{W}_{12} R_{22}^{\gamma T} - \bar{W}_{11} R_{12}^{\gamma T}$$

$$U_3 = \hat{A}_{21} - \bar{W}_{22} R_{12}^{\gamma T}$$

$$U_4 = \hat{A}_{22} - \bar{W}_{22} R_{22}^{\gamma T}$$

$$U_0 = U_1 - U_2 U_4^{-1} U_3$$

$$\hat{V}_{12} = \bar{W}_{22} \hat{A}_{12}^T + M_{12}^T.$$

The AREs (7c) and (7a) will produce the unique positive semidefinite stabilizing solution under assumption 3 if  $\gamma$  is large enough.

Let  $\gamma_{2f} = \inf\{\gamma > \gamma_{1f} \mid \text{the ARE (7c) has a positive-semidefinite stabilizing solution, and } \rho(\bar{P}_{22} \bar{W}_{22}) < \gamma^2\}$ . Moreover, let us define  $\gamma_{2s} = \inf\{\gamma > \gamma_{1s} \mid \text{the ARE (7a) has a positive-semidefinite stabilizing solution, and } \rho(\bar{P}_{11} \bar{W}_{11}) < \gamma^2\}$ .

As the results, for every  $\gamma > \bar{\gamma} = \max\{\gamma_{1s}, \gamma_{1f}, \gamma_{2s}, \gamma_{2f}\}$ , the ARE's (6) and (7) have the positive semidefinite stabilizing solutions if  $\varepsilon > 0$  is small enough. Thus, we have the following result.

*Lemma 2:* Under the Assumptions 1–3, if we select a parameter  $\gamma > \bar{\gamma} = \max\{\gamma_{1s}, \gamma_{1f}, \gamma_{2s}, \gamma_{2f}\}$ , then there exists a small  $\varepsilon > 0$  such that for all  $\varepsilon \in (0, \varepsilon)$ , the ARE's (1) and (2) admits a positive-semidefinite solution, which can be written as

$$P_\varepsilon = \begin{bmatrix} \bar{P}_{11} + O(\varepsilon) & \varepsilon \bar{P}_{21}^T + O(\varepsilon^2) \\ \varepsilon \bar{P}_{21} + O(\varepsilon^2) & \varepsilon \bar{P}_{22} + O(\varepsilon^2) \end{bmatrix} \quad (8)$$

$$W_\varepsilon = \begin{bmatrix} \bar{W}_{11} + O(\varepsilon) & \bar{W}_{12} + O(\varepsilon) \\ \bar{W}_{12}^T + O(\varepsilon) & \varepsilon^{-1}(\bar{W}_{22} + O(\varepsilon)) \end{bmatrix}. \quad (9)$$

*Proof:* By using the implicit function theorem, Lemma 2 can be proved. The proof is omitted since it is similar to that of the references [4] and [20]. ■

*Remark 2:* We can prove Lemma 2 by using a method similar to that given in the proof of [4, Theorems 2.1, 2.2]. Note that the proof given in [4] is made on the invertible assumption of  $A_{22}$ . However, this note improves the proof of Lemma 2 in the sense that the invertible assumption is not needed.

### III. THE NEW ITERATIVE ALGORITHM

In this section, we establish an elegant and simple algorithm which converges globally to the positive-semidefinite symmetric solution of AREs (1) and (2). The algorithm is given in term of the standard GALE, which have to be solved iteratively. We present the new iterative algorithm based on the Kleinman algorithm. Here, we note that the Kleinman algorithm is based on the Newton type algorithm. In general, the stabilizable-detectable conditions will guarantee the convergence of the Kleinman algorithm for the standard linear-quadratic regulator type ARE to the positive semidefinite solutions. However, it is difficult to apply the Kleinman algorithm to the GAREs (3) and (4) because the matrix  $S^\gamma = \gamma^{-2} G G^T - B \bar{D}_1 B^T$  and/or  $R^\gamma = \gamma^{-2} C_1^T C_1 - C_2 \bar{D}_2 C_2^T$  are in general indefinite.

We propose the following algorithm for solving the GAREs (3) and (4), respectively

$$\begin{aligned} & \left( \bar{A}^\gamma - S^\gamma P^{(i)} \right)^T P^{(i+1)} + P^{(i+1)T} \left( \bar{A}^\gamma - S^\gamma P^{(i)} \right) \\ & + P^{(i)T} S^\gamma P^{(i)} + Q = 0 \end{aligned} \quad (10)$$

$$\begin{aligned}
\Pi_\varepsilon P^{(i+1)} &= P^{(i+1)\top} \Pi_\varepsilon \\
&\left( \hat{A}^\gamma - W^{(i)} R^\gamma \right) W^{(i+1)\top} + W^{(i+1)} (\hat{A}^\gamma - W^{(i)} R^\gamma)^\top \\
&\quad + W^{(i)} R^\gamma W^{(i)\top} + M = 0 \\
W^{(i+1)} \Pi_\varepsilon &= \Pi_\varepsilon W^{(i+1)\top} \\
i &= 0, 1, 2, 3, \dots
\end{aligned} \tag{11}$$

with  $P^{(0)} = \begin{bmatrix} \bar{P}_{11} & \varepsilon \bar{P}_{21}^\top \\ \bar{P}_{21} & \bar{P}_{22} \end{bmatrix}$ ,  $W^{(0)} = \begin{bmatrix} \bar{W}_{11} & \bar{W}_{12} \\ \varepsilon \bar{W}_{12}^\top & \bar{W}_{22} \end{bmatrix}$ . The Kleinman algorithm (10) can be constructed by setting  $P^{(i+1)} = P^{(i)} + \Delta P^{(i)}$  and neglecting  $\Delta P^{(i)\top} S^\gamma \Delta P^{(i)}$  term. By following the similar steps, we obtain the Kleinman algorithm (11). Kleinman algorithm is well-known and is widely used to find a solution of ARE, and its local convergence properties are well understood. We are concerned with good choices of the starting points which guarantee to find a required solution of a given GARE. Our new idea is to set the initial conditions to the matrices  $P^{(0)}$  and  $W^{(0)}$ . Using Lemma 2, the fundamental idea is based on  $\|P - P^{(0)}\| = O(\varepsilon)$  and  $\|W - W^{(0)}\| = O(\varepsilon)$ . Although the matrices  $S^\gamma$  and/or  $R^\gamma$  are in general indefinite, we can get the required solution with the rate of quadratic convergence by using the Kleinman algorithm. By using Newton–Kantorovich theorem, we now prove the existence of the unique solution for the GARE (3a). The main result of this section is as follows.

**Theorem 1:** Under the assumptions 1–3, if we select a parameter  $\gamma > \bar{\gamma} = \max\{\gamma_{1s}, \gamma_{1f}, \gamma_{2s}, \gamma_{2f}\}$ , then the new iterative algorithm (10) converges to the exact solution  $P^*$  of the GARE (3a) with the rate of quadratic convergence. The unique bounded solution  $P^{(i)}$  of the GARE (3a) is in the neighborhood of the exact solution  $P^*$ . Furthermore,  $P_\varepsilon^{(i)} = \Pi_\varepsilon^\top P^{(i)} = P^{(i)\top} \Pi_\varepsilon$  is positive semidefinite and  $\bar{A}_\varepsilon^\gamma - S_\varepsilon^\gamma P_\varepsilon^{(i)}$  is stable for all  $i, i = 0, 1, 2, \dots$ . That is, the following conditions are satisfied:

$$\|P^{(i)} - P^*\| = O(\varepsilon^{2^i}) \tag{12a}$$

$$\begin{aligned}
\|P^{(i)}\| &\leq c < \infty \\
P_\varepsilon^{(i)} &= \Pi_\varepsilon P^{(i)} = P^{(i)\top} \Pi_\varepsilon \geq 0 \\
\text{Re} \lambda \left[ \bar{A}_\varepsilon^\gamma - S_\varepsilon^\gamma P_\varepsilon^{(i)} \right] &< 0
\end{aligned} \tag{12b}$$

where

$$\begin{aligned}
P &= P^* = \begin{bmatrix} P_{11} & \varepsilon P_{21}^\top \\ P_{21} & P_{22} \end{bmatrix} \\
P^{(i)} &= \begin{bmatrix} P_{11}^{(i)} & \varepsilon P_{21}^{(i)\top} \\ P_{21}^{(i)} & P_{22}^{(i)} \end{bmatrix}.
\end{aligned}$$

Before proving this theorem, we will first establish a following useful lemma.

**Lemma 3:** Let us consider the GALE (13)

$$\mathcal{A}^\top \Xi + \Xi^\top \mathcal{A} + \varepsilon^j Q = 0 \tag{13}$$

where  $\Xi$  is the solution of the GALE (13) and  $\mathcal{A}$  and  $Q$  known matrices defined by

$$\begin{aligned}
\Xi &= \begin{bmatrix} \Xi_{11}(\varepsilon) & \varepsilon \Xi_{21}^\top(\varepsilon) \\ \Xi_{21}(\varepsilon) & \Xi_{22}(\varepsilon) \end{bmatrix} \\
\mathcal{A} &= \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix} \quad Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^\top & Q_{22} \end{bmatrix} \\
\Xi_{11} &= \Xi_{11}^\top \quad \mathcal{A}_{11} \quad Q_{11} = Q_{11}^\top \in \mathbf{R}^{n_1 \times n_1} \\
\Xi_{22} &= \Xi_{22}^\top \quad \mathcal{A}_{22} \quad Q_{22} = Q_{22}^\top \in \mathbf{R}^{n_2 \times n_2}.
\end{aligned}$$

Assume that the matrix  $\mathcal{A}_{22}$  is nonsingular and  $\mathcal{A}_0 \equiv \mathcal{A}_{11} - \mathcal{A}_{12} \mathcal{A}_{22}^{-1} \mathcal{A}_{21}$  and  $\mathcal{A}_{22}$  are stable. Then

$$\Xi = \sum_{k=j}^{\infty} \frac{\varepsilon^k}{k!} \begin{bmatrix} \Xi_{11}^{(k)}(0) & \varepsilon \Xi_{21}^{(k)\top}(0) \\ \Xi_{21}^{(k)}(0) & \Xi_{22}^{(k)}(0) \end{bmatrix} = O(\varepsilon^j) \tag{14}$$

where

$$\Xi_{lm}^{(k)}(0) = \frac{d^k}{d\varepsilon^k} \Xi_{lm}(\varepsilon)|_{\varepsilon=0} \quad lm = 11, 21, 22.$$

**Proof:** Partitioning for the GALE (13), we obtain the following equations:

$$\mathcal{A}_{11}^\top \Xi_{11} + \Xi_{11} \mathcal{A}_{11} + \mathcal{A}_{21}^\top \Xi_{21} + \Xi_{21}^\top \mathcal{A}_{21} + \varepsilon^j Q_{11} = 0 \tag{15a}$$

$$\Xi_{11} \mathcal{A}_{12} + \Xi_{21}^\top \mathcal{A}_{22} + \mathcal{A}_{21}^\top \Xi_{22} + \varepsilon \mathcal{A}_{11}^\top \Xi_{21}^\top + \varepsilon^j Q_{12} = 0 \tag{15b}$$

$$\mathcal{A}_{22}^\top \Xi_{22} + \Xi_{22} \mathcal{A}_{22} + \varepsilon \left( \mathcal{A}_{12}^\top \Xi_{21}^\top + \Xi_{21} \mathcal{A}_{12} \right) + \varepsilon^j Q_{22} = 0. \tag{15c}$$

Setting  $\varepsilon = 0$  for the above (15), since  $\mathcal{A}_0$  and  $\mathcal{A}_{22}$  are stable we deduce that  $\Xi_{lm}(0) = \Xi_{lm}^{(0)}(0) = 0$ ,  $lm = 11, 21, 22$ . We now obtain the derivative of (15) at parameter  $\varepsilon$  as follows:

$$\begin{aligned}
&\mathcal{A}_{11}^\top \Xi_{11}^{(1)}(\varepsilon) + \Xi_{11}^{(1)}(\varepsilon) \mathcal{A}_{11} + \mathcal{A}_{21}^\top \Xi_{21}^{(1)}(\varepsilon) \\
&\quad + \Xi_{21}^{(1)\top}(\varepsilon) \mathcal{A}_{21} + j \varepsilon^{j-1} Q_{11} = 0 \\
&\Xi_{11}^{(1)}(\varepsilon) \mathcal{A}_{12} + \Xi_{21}^{(1)\top}(\varepsilon) \mathcal{A}_{22} + \mathcal{A}_{21}^\top \Xi_{22}^{(1)}(\varepsilon) \\
&\quad + \mathcal{A}_{11}^\top \Xi_{21}^{(1)}(\varepsilon) + \varepsilon \mathcal{A}_{11}^\top \Xi_{21}^{(1)\top}(\varepsilon) + j \varepsilon^{j-1} Q_{12} = 0 \\
&\mathcal{A}_{22}^\top \Xi_{22}^{(1)}(\varepsilon) + \Xi_{22}^{(1)}(\varepsilon) \mathcal{A}_{22} + \varepsilon \left( \mathcal{A}_{12}^\top \Xi_{21}^{(1)\top}(\varepsilon) + \Xi_{21}^{(1)}(\varepsilon) \mathcal{A}_{12} \right) \\
&\quad + \mathcal{A}_{12}^\top \Xi_{21}^{(1)}(\varepsilon) + \Xi_{21}(\varepsilon) \mathcal{A}_{12} + j \varepsilon^{j-1} Q_{22} = 0.
\end{aligned}$$

Using  $\Xi_{lm}^{(0)}(0) = 0$  and the fact that  $\mathcal{A}_0$  and  $\mathcal{A}_{22}$  are stable, we get  $\Xi_{lm}^{(1)}(0) = 0$ . By following the similar steps, we have  $\Xi_{lm}^{(k)}(0) = 0$ ,  $k = 0, 1, 2, j-1$ . Note that the exact proof is done by using mathematical induction. On the other hand, it is well known that the matrix  $\Xi$  possess a power series expansion at  $\varepsilon = 0$  as follows:

$$\Xi = \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} \begin{bmatrix} \Xi_{11}^{(k)}(0) & \varepsilon \Xi_{21}^{(k)\top}(0) \\ \Xi_{21}^{(k)}(0) & \Xi_{22}^{(k)}(0) \end{bmatrix}.$$

Substituting  $\Xi_{lm}^{(k)}(0) = 0$ ,  $k = 0, 1, 2, j-1$  into  $\Xi$ , it is straightforward to verify that (14). This is the required result. ■

Now, let us prove Theorem 1.

**Proof:** The proof of (12b) has been given in [21]. Thus, we will prove the quadratic convergence property corresponding to (12a) and the existence of the unique solution of the ARE (1). We first prove that under the assumptions in Theorem 1 the algorithm (10) converges to the desired solution of (3a) with (12a). The proof is done by using mathematical induction and Lemma 3. Subtracting (3a) from (10), it is easy to derive

$$\begin{aligned}
&\left( \bar{A}^\gamma - S^\gamma P^{(i)} \right)^\top \left( P^{(i+1)} - P \right) + \left( P^{(i+1)} - P \right)^\top \left( \bar{A}^\gamma - S^\gamma P^{(i)} \right) \\
&\quad = - \left( P^{(i)} - P \right)^\top S^\gamma \left( P^{(i)} - P \right).
\end{aligned}$$

When  $i = 0$  for the above equations, by noting that  $\|P^{(0)} - P\| = O(\varepsilon)$  based on the Lemma 2, we have  $\left( \bar{A}^\gamma - S^\gamma P^{(0)} \right)^\top (P^{(1)} - P) + (P^{(1)} - P)^\top (\bar{A}^\gamma - S^\gamma P^{(0)}) = O(\varepsilon^2)$ . Using

$$\bar{A}^\gamma - S^\gamma P^{(0)} = \begin{bmatrix} \bar{D}_1 & \bar{D}_2 + O(\varepsilon) \\ \bar{D}_3 & \bar{D}_4 + O(\varepsilon) \end{bmatrix}$$

and the known results that  $\bar{D}_4$  and  $\bar{D}_0 = \bar{D}_1 - \bar{D}_2 \bar{D}_4^{-1} \bar{D}_3$  are stable, it follows from Lemma 3 that  $\|P^{(1)} - P\| = O(\varepsilon^2)$ . We now assume that  $\|P^{(N)} - P\| = O(\varepsilon^{2N})$ . Using this assumption, we conclude that

$$\begin{aligned} & (\bar{A}^\gamma - S^\gamma P^{(N)})^\top (P^{(N+1)} - P) \\ & + (P^{(N+1)} - P)^\top (\bar{A}^\gamma - S^\gamma P^{(N)}) = -O(\varepsilon^{2N+1}). \end{aligned}$$

Thus, using Lemma 3, we have  $\|P^{(N+1)} - P\| = O(\varepsilon^{2N+1})$ . Consequently, the (12a) holds for all  $i \in \mathbb{N}$ . Secondly, we show that there exists the unique solution for the ARE (1). This proof is equivalent to the proof of existence of the unique solution for the GARE (3a). Thus, the proof follows directly by applying Newton–Kantorovich theorem (see [16, p. 155]) for the GARE (3a). We now observe that function  $\mathcal{F}_1(P)$  is differentiable on a convex set  $\mathcal{D}$ . Using the fact that

$$\nabla \mathcal{F}_1(P) = (\bar{A}^\gamma - S^\gamma P)^\top \otimes I_n + I_n \otimes (\bar{A}^\gamma - S^\gamma P)^\top$$

where  $\otimes$  denotes Kronecker product [19] and  $\mathcal{F}_1(P) = \bar{A}^{\gamma\top} P + P^\top \bar{A}^\gamma - P^\top S^\gamma P + Q$ ,  $\nabla \mathcal{F}_1(P) = \partial \text{vec} \mathcal{F}_1(P) / \partial (\text{vec} P)^\top$ , where  $\text{vec}$  denotes an ordered stack of the columns of its matrix [19], we have

$$\|\nabla \mathcal{F}_1(P_1) - \nabla \mathcal{F}_1(P_2)\| \leq \bar{\gamma} \|P_1 - P_2\|$$

where  $\bar{\gamma} = 2\|S^\gamma\|$ . Moreover, using the fact that

$$\nabla \mathcal{F}_1(P^{(0)}) = \begin{bmatrix} \bar{D}_1 & \bar{D}_2 + O(\varepsilon) \\ \bar{D}_3 & \bar{D}_4 + O(\varepsilon) \end{bmatrix}^\top \otimes I_n + I_n \otimes \begin{bmatrix} \bar{D}_1 & \bar{D}_2 + O(\varepsilon) \\ \bar{D}_3 & \bar{D}_4 + O(\varepsilon) \end{bmatrix}^\top$$

it follows that  $\nabla \mathcal{F}_1(P^{(0)})$  is nonsingular because  $\bar{D}_4$  and  $\bar{D}_0$  are stable. Therefore, there exists  $\bar{\beta}$  such that  $\left\| [\nabla \mathcal{F}_1(P^{(0)})]^{-1} \right\| \equiv \bar{\beta}$ .

On the other hand, since  $\mathcal{F}_1(P^{(0)}) = O(\varepsilon)$ , there exists  $\bar{\eta}$  such that  $\left\| [\nabla \mathcal{F}_1(P^{(0)})]^{-1} \right\| \cdot \left\| \mathcal{F}_1(P^{(0)}) \right\| = O(\varepsilon) \equiv \bar{\eta}$ . Thus, there exists  $\bar{\alpha}$  such that  $\bar{\alpha} \equiv \bar{\beta} \bar{\eta} < 2^{-1}$  because of  $\bar{\eta} = O(\varepsilon)$ . Now, let us define

$$\begin{aligned} t^* & \equiv \frac{1}{\bar{\gamma} \bar{\beta}} [1 - \sqrt{1 - 2\bar{\alpha}}] \\ & = \frac{1}{2\|S^\gamma\| \cdot \left\| [\nabla \mathcal{F}_1(P^{(0)})]^{-1} \right\|} [1 - \sqrt{1 - 2\bar{\alpha}}]. \end{aligned}$$

Clearly,  $\mathcal{S} \equiv \{P : \|P - P^{(0)}\| \leq t^*\}$  is in the convex set  $\mathcal{D}$ . In the sequel, since  $\|P^* - P^{(0)}\| = O(\varepsilon)$  holds for small  $\varepsilon$ , we have shown that  $P^*$  is the unique solution in  $\mathcal{S}$ . Therefore, the proof is completed. ■

**Remark 3:** The algorithm (10) which is based on the Kleinman algorithm might facilitate new approach to the singularly perturbed ARE with indefinite sign quadratic term, that is, conceptually simpler and numerically more efficient than those previously used in [7] and [15]. Moreover, by applying the results of this paper, we can get rather easily the solution for various singularly perturbed ARE with indefinite sign quadratic term.

**Remark 4:** Note that our proposed method is not a straightforward extension to the continuous-time case of the methods given in [22],

[23]. First, it is most different from [22], [23] that our convergence proof of the iterative algorithm (10) is based on Newton–Kantorovich theorem. Moreover, how to select the initial condition is quite different. That is, we choose the initial condition as solutions of the reduced-order AREs close to the exact solutions using the property of singularly perturbed system. As a result, while the recursion in [22] has exponential convergence property, the new iterative algorithm achieves the quadratic convergence property stronger than exponential convergence property. Secondly, we have newly proved the existence of the unique solution for the GARE with indefinite sign quadratic term.

We now summarize a perturbation analysis of the GALE (10). Setting  $\varepsilon$  to zero and using Kronecker products, the GALE (10) can be written as

$$\begin{aligned} \mathcal{V} \begin{bmatrix} \text{vec} \bar{P}_{11}^{(i+1)} \\ \text{vec} \bar{P}_{21}^{(i+1)} \\ \text{vec} \bar{P}_{22}^{(i+1)} \end{bmatrix} &= \begin{bmatrix} \text{vec} \bar{Q}_{11} \\ \text{vec} \bar{Q}_{12} \\ \text{vec} \bar{Q}_{22} \end{bmatrix} \\ \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} &= \bar{A}^\gamma - S^\gamma \begin{bmatrix} \bar{P}_{11}^{(i)} & 0 \\ \bar{P}_{21}^{(i)} & \bar{P}_{22}^{(i)} \end{bmatrix} \\ \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{12}^\top & \bar{Q}_{22} \end{bmatrix} &= \begin{bmatrix} \bar{P}_{11}^{(i)} & \bar{P}_{21}^{(i)\top} \\ 0 & \bar{P}_{22}^{(i)} \end{bmatrix} S^\gamma \begin{bmatrix} \bar{P}_{11}^{(i)} & 0 \\ \bar{P}_{21}^{(i)} & \bar{P}_{22}^{(i)} \end{bmatrix} + Q \end{aligned}$$

where  $\mathcal{V}$  is given at the bottom of the page.  $U_{n_1 n_1}$  denotes a permutation matrix in Kronecker matrix sense [19] and  $\text{vec} \bar{P}_{lm}^{(i+1)}$  denotes an ordered stack of the columns of  $P_{lm}^{(i+1)}$  when  $\varepsilon = 0$ . It can be shown, after some algebra, that the determinant of  $\mathcal{V}$  is expressed as  $\det \mathcal{V} = \det [(I_{n_2} \otimes \bar{A}_{22}^\top) U_{n_2 n_2} + \bar{A}_{22}^\top \otimes I_{n_2}] \cdot \det [(I_{n_1} \otimes \bar{A}_{22}^\top) U_{n_1 n_1} + \bar{A}_{22}^\top \otimes I_{n_1}]$ , where  $\bar{A}_0 = \bar{A}_{11} - \bar{A}_{12} \bar{A}_{22}^{-1} \bar{A}_{21}$ . Obviously,  $\bar{A}_{22}$  and  $\bar{A}_0$  are nonsingular matrices. Thus, there exists  $\mathcal{V}^{-1}$ . Therefore, the condition number [16] of  $\mathcal{V}$ , that is,  $K(\mathcal{V}) = \|\mathcal{V}\| \cdot \|\mathcal{V}^{-1}\|$  is given by  $K(\mathcal{V}) = O(1)$ . Since  $K(\mathcal{V})$  is not large, the matrix  $\mathcal{V} + O(\varepsilon)$  is well-conditioned for small  $\varepsilon$ .

We next give the convergence theorem of the algorithm (11) by similar argument corresponding to the algorithm (10).

**Theorem 2:** Under the Assumptions 1–3, if we select a parameter  $\gamma > \bar{\gamma} = \max\{\gamma_{1s}, \gamma_{1f}, \gamma_{2s}, \gamma_{2f}\}$ , then the new iterative algorithm (11) converges to the exact solution  $W^*$  of the GARE (4a) with the rate of quadratic convergence. The unique bounded solution  $W^{(i)}$  of the GARE (4a) is in the neighborhood of the exact solution  $W^*$ . Furthermore,  $W_\varepsilon^{(i)} = W^{(i)\top} \Pi_\varepsilon^{-\top} = \Pi_\varepsilon^{-1} W^{(i)}$  is positive semidefinite and  $\bar{A}_\varepsilon^\gamma - W_\varepsilon^{(i)} R_\varepsilon^\gamma$  is stable for all  $i, i = 0, 1, 2, \dots$ .

**Proof:** Since the proof of Theorem 2 is performed by a dual argument of Theorem 1, it is omitted. ■

As a result of applying the idea of the Kleinman algorithm, we have managed to replace the computation of the GAREs (3) and (4) which contain the small parameter  $\varepsilon$  with a sequence of the GALEs (10) and (11).

Now, we consider a method for solving the GALEs (10) and (11). Since the algorithm for solving the GALE (10) is virtually identical to the GALE (11), we give only the algorithm of the GALE (10). In order to reduce the dimension of the workspace, a new algorithm for solving the GALE which is based on the fixed point algorithm is established. Let us consider the following GALE (16), in a general form:

$$\Lambda^\top X + X^\top \Lambda + V = 0 \quad (16)$$

$$\mathcal{V} = \begin{bmatrix} (I_{n_1} \otimes \bar{A}_{11}^\top) U_{n_1 n_1} + \bar{A}_{11}^\top \otimes I_{n_1} & (I_{n_1} \otimes \bar{A}_{21}^\top) U_{n_1 n_2} + \bar{A}_{21}^\top \otimes I_{n_1} & 0 \\ (I_{n_1} \otimes \bar{A}_{12}^\top) U_{n_1 n_1} & (I_{n_1} \otimes \bar{A}_{22}^\top) U_{n_1 n_2} & \bar{A}_{21}^\top \otimes I_{n_2} \\ 0 & 0 & (I_{n_2} \otimes \bar{A}_{22}^\top) U_{n_2 n_2} + \bar{A}_{22}^\top \otimes I_{n_2} \end{bmatrix}.$$

where  $X$  is the solution of the GALE (16) and  $\Lambda$  and  $V$  are known matrices defined by

$$\begin{aligned} X &= \begin{bmatrix} X_{11}(\varepsilon) & \varepsilon X_{21}(\varepsilon)^T \\ X_{21}(\varepsilon) & X_{22}(\varepsilon) \end{bmatrix} := P^{(i+1)} \\ \Lambda &= \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} := \bar{A}^\gamma - S^\gamma P^{(i)} \\ V &= \begin{bmatrix} V_{11} & V_{12} \\ V_{12}^T & V_{22} \end{bmatrix} := P^{(i)T} S^\gamma P^{(i)} + Q \\ X_{11} &= X_{11}^T \quad \Lambda_{11} \quad V_{11} = V_{11}^T \in \mathbf{R}^{n_1 \times n_1} \\ X_{22} &= X_{22}^T \quad \Lambda_{22} \quad V_{22} = V_{22}^T \in \mathbf{R}^{n_2 \times n_2}. \end{aligned}$$

In order to solve the GALE (16) in a iterative method, we need another assumption.

**Assumption 4:** The matrix  $\Lambda_{22}$  is nonsingular and  $\Lambda_0 \equiv \Lambda_{11} - \Lambda_{12}\Lambda_{22}^{-1}\Lambda_{21}$  and  $\Lambda_{22}$  are stable.

Note that the assumption 4 is satisfied for all  $i \in \mathbf{N}$  because  $\bar{D}_4$  and  $\bar{D}_0$  are stable. The GALE (16) can be partitioned into

$$\begin{aligned} \Lambda_{11}^T X_{11} + X_{11} \Lambda_{11} + \Lambda_{21}^T X_{21} \\ + X_{21}^T \Lambda_{21} + V_{11} &= 0 \end{aligned} \quad (17a)$$

$$\begin{aligned} X_{11} \Lambda_{12} + X_{21}^T \Lambda_{22} + \Lambda_{21}^T X_{22} \\ + \varepsilon \Lambda_{11}^T X_{21}^T + V_{12} &= 0 \end{aligned} \quad (17b)$$

$$\begin{aligned} \Lambda_{22}^T X_{22} + X_{22} \Lambda_{22} + \varepsilon \left( \Lambda_{12}^T X_{21}^T + X_{21} \Lambda_{12} \right) \\ + V_{22} &= 0. \end{aligned} \quad (17c)$$

The fixed point algorithm for solving (17) is given by

$$\begin{aligned} \Lambda_{22}^T X_{22}^{(i+1)} + X_{22}^{(i+1)} \Lambda_{22} \\ + \varepsilon \left( \Lambda_{12}^T X_{21}^{(i)T} + X_{21}^{(i)} \Lambda_{12} \right) + V_{22} &= 0 \end{aligned} \quad (18a)$$

$$\begin{aligned} \Lambda_0^T X_{11}^{(i+1)} + X_{11}^{(i+1)} \Lambda_0 - \varepsilon \Lambda_{21}^T \Lambda_{22}^{-T} X_{21}^{(i)} \Lambda_0 \\ - \varepsilon \Lambda_0^T X_{21}^{(i)T} \Lambda_{22}^{-1} \Lambda_{21} + \Lambda_{21}^T \Lambda_{22}^{-T} V_{22} \Lambda_{22}^{-1} \Lambda_{21} \\ - V_{12} \Lambda_{22}^{-1} \Lambda_{21} - \Lambda_{21}^T \Lambda_{22}^{-T} V_{12}^T + V_{11} &= 0 \end{aligned} \quad (18b)$$

$$\begin{aligned} X_{21}^{(i+1)} &= -\Lambda_{22}^{-T} \left( \Lambda_{12}^T X_{11}^{(i+1)} + X_{22}^{(i+1)} \Lambda_{21} \right. \\ &\quad \left. + \varepsilon X_{21}^{(i)} \Lambda_{11} + V_{12}^T \right) \\ X_{21}^{(0)} &= 0 \quad i = 0, 1, 2, 3, \dots \end{aligned} \quad (18c)$$

The following theorem indicates the convergence of the algorithm (18).

**Theorem 3:** The algorithm (18) converges to the exact solution  $X_{lm}$  of (17) with the rate of convergence of  $O(\varepsilon^i)$ , that is

$$\begin{aligned} \|X_{lm}^{(i)} - X_{lm}\| &= O(\varepsilon^i) \\ i &= 1, 2, 3, \dots \quad lm = 11, 21, 22. \end{aligned} \quad (19)$$

**Proof:** The proof is done by using the mathematical induction. When  $i = 0$  for the (18), the solutions  $X_{lm}^{(1)}$  are equivalent to the first order approximations  $X_{lm}$  corresponding to the small parameters  $\varepsilon$  for the (17). It follows from these equations that  $\|X_{lm}^{(1)} - X_{lm}\| = O(\varepsilon)$ ,  $lm = 11, 21, 22$ . When  $i = N$  ( $N \geq 1$ ), we assume that  $\|X_{lm}^{(N)} - X_{lm}\| = O(\varepsilon^N)$ . Subtracting (18) from (17) and using the above assumptions, we arrive at the following equations:

$$\begin{aligned} \Lambda_{22}^T (X_{22}^{(N+1)} - X_{22}) + (X_{22}^{(N+1)} - X_{22}) \Lambda_{22} + O(\varepsilon^{N+1}) &= 0 \\ (X_{11}^{(N+1)} - X_{11}) \Lambda_{12} + (X_{21}^{(N+1)} - X_{21})^T \Lambda_{22} \\ + \Lambda_{21}^T (X_{22}^{(N+1)} - X_{22}) + O(\varepsilon^{N+1}) &= 0 \\ \Lambda_0^T (X_{11}^{(N+1)} - X_{11}) + (X_{11}^{(N+1)} - X_{11}) \Lambda_0 + O(\varepsilon^{N+1}) &= 0. \end{aligned}$$

Thus, using the standard properties of the algebraic Lyapunov equation (ALE) [24], we have  $\|X_{lm}^{(N+1)} - X_{lm}\| = O(\varepsilon^{N+1})$ . Consequently, the (19) holds for all  $i \in \mathbf{N}$ . This completes the proof of the theorem concerned with the fixed point algorithm. ■

The existing method [7] can obtain the solution by solving the ALEs of lower dimensions which are the same as the slow and fast subsystems. However, in order to obtain the exact solution of the sign indefinite ARE, ones need the same workspace for calculating the inverse matrix (see [7], equation (3.3)). On the other hand, the resulting algorithm is very useful because our proposed algorithm has only to solve the ALEs of lower dimensions. Moreover, note that the algorithm (18a) is quite different from the recursive algorithm [15]. As another important feature, since our proposed algorithm is the quadratic convergence, while the recursive algorithm is the linear convergence [15], the resulting algorithm is also efficient.

In the rest of this section, we will present an important implication. If the state information is available for feedback, then the following corollary is easily seen in view of Theorem 1.

**Corollary 1:** Assume that  $C_1^T D_{12} = 0$  and  $D_{12}^T D_{12} = I_p$ . Under the assumptions 1–3, the approximate feedback gain  $K^{(i)} = -[B_1^T B_2^T] P^{(i)}$  guarantees the performance level  $\left\| \left( C_1 + D_{12} K^{(i)} \right) \left( sI_n - A_\varepsilon - B_\varepsilon K^{(i)} \right)^{-1} G_\varepsilon \right\|_\infty < \gamma + O(\varepsilon^{2i})$ , where  $P^{(i)}$  is defined in the statement of Theorem 1.

**Proof:** It can be carried out via a similar technique used in [7] and [20]. ■

#### IV. NUMERICAL EXAMPLE

In order to demonstrate the efficiency of our proposed algorithm, we have run a simple numerical example. The system matrix is given by

$$\begin{aligned} A_{11} &= \begin{bmatrix} 0 & 0.4 \\ 0 & 0 \end{bmatrix} \quad A_{12} = \begin{bmatrix} 0 & 0 \\ 0.345 & 0 \end{bmatrix} \\ A_{21} &= \begin{bmatrix} 0 & -0.524 \\ 0 & 0 \end{bmatrix} \quad A_{22} = \begin{bmatrix} 0 & 0.262 \\ 0 & -1 \end{bmatrix} \\ B_1 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ G_1 &= \begin{bmatrix} 1.0 & 0 \\ 0 & 1.0 \end{bmatrix} \quad G_2 = \begin{bmatrix} 0.2 & 0.1 \\ 1.2 & 0.5 \end{bmatrix} \\ D_{12}^T &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad D_{21} = \begin{bmatrix} 1 & 0.5 \end{bmatrix} \\ C_{11}^T &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad C_{12}^T = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ C_{21} &= \begin{bmatrix} 1 & 0 \end{bmatrix} \quad C_{22} = \begin{bmatrix} 1 & 1 \end{bmatrix}. \end{aligned}$$

The numerical results are obtained for small parameter  $\varepsilon = 10^{-4}$ . Since  $\det A_{22} = 0$ , the system is nonstandard singularly perturbed systems. The four basic quantities for the system are  $\gamma_{1f} = \gamma_{2f} = 2.255156$ ,  $\gamma_{1s} = \gamma_{2s} = 7.359056$ . Thus, for every boundary value  $\gamma > \bar{\gamma} = \max \{\gamma_{1f}, \gamma_{1s}, \gamma_{2f}, \gamma_{2s}\} = 7.359056$ , the AREs (6a), (6c), (7a) and (7c) have the positive semidefinite stabilizing solutions. On the other hand, by using MATLAB, the minimum value of  $\hat{\gamma}$  such that there exists the dynamic feedback controller is  $\hat{\gamma} = 7.468750$ .

Now, we choose as  $\gamma = 8.0 > \bar{\gamma}$  to design the controller. We give the following solutions of the AREs (1) and (2) and in Table I. Table II shows the results of the errors  $\|\mathcal{F}_1(P^{(i)})\|$  and  $\|\mathcal{F}_2(W^{(i)})\|$  per iterations. We find that the solutions of the AREs (1) and (2) converge to the exact solution with accuracy of  $\|\mathcal{F}_1(P^{(i)})\| < 10^{-14}$  and  $\|\mathcal{F}_2(W^{(i)})\| < 10^{-14}$  after 2 iterative iterations, respectively. Moreover, it is interested in pointing out that the result of Table II shows that the algorithms (10) and (11) are quadratic convergence. Table III shows the results of the number of iterations required to the solution with the same accuracy of

TABLE I

$P_\varepsilon^{(2)} =$	$\begin{bmatrix} 1.5886644664 \times 10^1 & 7.1692654025 & 1.4777259003 \times 10^{-3} & 3.2803557787 \times 10^{-4} \\ 7.1692654025 & 9.5689292890 & 7.3774420477 \times 10^{-4} & 1.2062616188 \times 10^{-4} \\ 1.4777259003 \times 10^{-3} & 7.3774420477 \times 10^{-4} & 4.8953376479 \times 10^{-4} & 1.0462699908 \times 10^{-4} \\ 3.2803557787 \times 10^{-4} & 1.2062616188 \times 10^{-4} & 1.0462699908 \times 10^{-4} & 2.4621105838 \times 10^{-5} \end{bmatrix}$
$W_\varepsilon^{(2)} =$	$\begin{bmatrix} 1.7234972390 & -9.6975066240 \times 10^{-2} & -1.5020487314 & -1.0113300267 \times 10^{-1} \\ -9.6975066240 \times 10^{-2} & 6.7468422739 \times 10^{-1} & -1.4273792265 & 7.8173878465 \times 10^{-1} \\ -1.5020487314 & -1.4273792265 & 4.8380921965 & -1.2998747057 \\ -1.0113300267 \times 10^{-1} & 7.8173878465 \times 10^{-1} & -1.2998747057 & 1.9265002490 \times 10^1 \end{bmatrix}$

TABLE II

$i$	$\ \mathcal{F}_1(P^{(i)})\ $	$\ \mathcal{F}_2(W^{(i)})\ $
0	$1.77984 \times 10^{-3}$	$3.11227 \times 10^{-4}$
1	$1.58061 \times 10^{-7}$	$8.84535 \times 10^{-8}$
2	$6.00958 \times 10^{-15}$	$6.83822 \times 10^{-15}$

TABLE III

Number of iterations such that $\ \mathcal{F}_1(P^{(i)})\  < 10^{-14}$ .		
$\varepsilon$	recursive algorithm	Kleinman algorithm
$10^{-1}$	17	5
$10^{-2}$	9	4
$10^{-3}$	5	3
$10^{-4}$	4	2
$10^{-5}$	3	2
$10^{-6}$	2	2

$\|\mathcal{F}_1(P^{(i)})\| < 10^{-14}$  for the classical recursive algorithm [15] versus the improved iterative algorithm. It can be seen that the convergence rate of the resulting algorithm is stable for all  $\varepsilon$  since the initial conditions  $P^{(0)}$  and  $W^{(0)}$  are quite good. On the other hand, the classical recursive algorithm converges very slowly for the  $\varepsilon$  that is not small. However, it is important to point out that the recursive algorithm is very useful when the parameter  $\varepsilon$  is sufficiently small.

## V. CONCLUSION

In this note, the  $H_\infty$  control problem for singularly perturbed systems has been investigated. We have shown that the Kleinman algorithm can be used well to solve the ARE under the appropriate initial condition. Comparing with [15], since our proposed algorithm is quadratic convergence, the required solution can be easily obtained up to an arbitrary order of accuracy, that is,  $O(\varepsilon^{2^i})$ . Moreover, we have also presented the method for solving the GALE by means of the fixed point algorithm. It avoids high dimension and numerical stiffness. Another important feature, if we use the state information, then the high-order  $O(\varepsilon^{2^i})$  accuracy controller achieves the performance  $\gamma + O(\varepsilon^{2^i})$  compared with the existing controller [7], [20].

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## Perturbation of Multivariable Linear Quadratic Systems With Jump Parameters

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**Abstract**—We consider the problem of the perturbation of a class of linear-quadratic systems where the change from one structure (for the dynamics and costs) to another is governed by a finite-state Markov process. The problem above leads to the analysis of some perturbed linearly coupled set of Riccati equations. We show that the matrix obtained as the solution of the equations, which determines the optimal value and control, has a Taylor expansion in the perturbation parameter. We compute explicitly the terms of this expansion.

**Index Terms**—Averaging and aggregation, coupled Riccati equations, linear quadratic control, Markov jump parameters, singular perturbation.

### I. INTRODUCTION

Many systems subject to frequent unpredictable structural changes can be modeled as piecewise deterministic systems, where the system dynamics takes on different forms depending on the value of an associated Markov process. In the linear case, these are also known as jump linear systems. Such a system model is useful particularly since it allows the decision maker to cope adequately with the discrete events that disrupt and/or change significantly the normal operation of a system by using the knowledge of their occurrence and the statistical information on the rate at which these events take place. Research in this class of systems and their applications into manufacturing management spans several decades. Some representative papers in this area are [4] and [5].

The solution of jump linear systems relies on solving a set of coupled Riccati equations involving the generator of the underlying Markov chain. In many applications, the state space of the Markov chain is large and it becomes difficult to obtain solutions to the Riccati equations. To overcome this difficulty, we use singular perturbation techniques in the modeling, control design, and optimization, which holds when the system displays certain two-time-scale behavior, a fast time scale and a slowly varying one. Presence of such a phenomenon is best expressed mathematically by introducing a small parameter  $\epsilon > 0$  and model the underlying system as one involving singularly perturbed Markov chain (see [8] for examples). Using averaging and aggregation techniques, we show that the set of coupled algebraic Riccati equations (CARE) possesses a unique solution which can be represented as a Taylor series in  $\epsilon$  and moreover we present an algorithm for computing the terms of the series.

An important paper related to ours is [6] which considers also linear dynamics with jump parameters. The transition probabilities are subject to the same type of perturbation that we consider. A more general setting is treated in that reference: that of an additional disturbance, which is handled using an  $H^\infty$  approach. The value of the limiting control problem is obtained. Our approach allows us to get more than the limiting control problem; it gives the solution in terms of a series expansion in the perturbation parameter. This note also extends the results in [3] which dealt with a one dimensional linear system.

The paper is organized as follows: Section II introduces the general model. Section III provides the Taylor expansion of the solution to the coupled algebraic Riccati equations. We study computational algorithms in Section IV. This note ends with the concluding remarks.

### II. GENERAL MODEL

The class of jump linear systems under consideration is described by

$$\frac{dx}{dt} = A(\theta(t))x(t) + B(\theta(t))u(t), \quad x(0) = x_0 \quad (1)$$

where

- $x$   $p$ -dimensional system state;
- $x_0$  fixed (known) initial state;
- $u$   $r$ -dimensional control, taking values in  $\mathbb{R}^r$ ;
- $\theta(t)$  finite state Markov chain defined on the state space  $S$ , of cardinality  $s$ , with the infinitesimal generator matrix  $\Lambda = (\lambda_{ij}), i, j \in S$ .

The  $\lambda_{ij}$ 's are real numbers such that for any  $i \neq j$ ,  $\lambda_{ij} \geq 0$ , and for all  $i \in S$ ,  $\lambda_{iai} = -\sum_{j \neq i} \lambda_{iaj}$ . With this system, the control  $u$  is generated by a control policy  $\gamma$  according to  $u(t) = \gamma(t, x_t, \theta_t)$ ,  $t \in [0, +\infty)$ , where  $\gamma$  is taken to be piecewise continuous in its first argument, and piecewise Lipschitz continuous in its second argument. Let us denote the class of all admissible controllers by  $\Gamma$ . Define the running (immediate) cost  $L : \mathbb{R}^p \times S \times \Gamma \rightarrow \mathbb{R}$  as:  $L(x, i, u) = |x|_{Q(i)}^2 + |u|_{R(i)}^2$ , where  $Q(\cdot) \geq 0$ ,  $R(\cdot) > 0$ , and where  $|x|_Q := x^T Q x$ , etc. The underlying probability space is the triple  $(\Omega, F, P)$ . Let  $E$  denote the expectation with respect to the underlying probability space. For each initial state  $(x_0, i_0)$  and strategy  $\gamma \in \Gamma$ , we introduce the discounted (expected) cost function

$$J_\beta(x_0, i_0, \gamma) = E_{x_0, i_0} \left( \int_0^{+\infty} e^{-\beta t} L(x(t), \theta(t), u(t)) dt \right) \quad (2)$$

where  $\beta \geq 0$  is a discount factor. For this infinite-horizon, we have to ensure that the cost is finite for at least one stationary policy. A sufficient condition for this is the following (see also Remark 2):

**Assumption 1:** The pair  $(A(\theta(t)), B(\theta(t)))$  is stochastically stabilizable and  $(A(i), Q(i))$  is observable for each  $i \in S$ .

The problem is the derivation of a solution to  $\hat{J}_\beta(x, i) = \inf_{\gamma \in \Gamma} J_\beta(x, i, \gamma)$ .

**The Perturbed Systems:** In many application, because of the various sources of uncertainties, the Markov chain involved is of large dimension. It is natural to group the large number of states into different collections of states, based on whether the interaction between any two states is weak or strong. This description using two scales of interactions is expressed mathematically by taking the probability transition rate matrix  $\Lambda$  in an appropriate singularly perturbed form, as already discussed in [2]:

$$\lambda_{ij}^\epsilon = \nu_{ij} + \frac{1}{\epsilon} \mu_{ij}$$

where  $(\nu_{ij})_{s \times s}$  and  $(\mu_{ij})_{s \times s}$  are the transition probability rate matrices corresponding to, respectively, weak interaction and strong interactions within the form process. The scalar  $\epsilon$  is a small positive number.

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