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Robust H_∞ Control Problem for Nonstandard Singularly Perturbed Systems and Application

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Abstract

This paper considers the robust H_∞ control problem for nonstandard singularly perturbed uncertain systems under imperfect state measurements. We propose a new algorithm which is based on the Kleinman algorithm with specific initial conditions. Furthermore, we also present a new algorithm for solving generalized Lyapunov equation on the basis of the generalized Schur method. Finally, in order to show the effectiveness of the proposed algorithms, a practical example is included.

1 Introduction

Robust control problems for singularly perturbed uncertain systems have been extensively studied in the past decade (see e.g., [1, 2] and the reference therein). Shi *et al.*[1] has studied the robust H_∞ control problem for standard singularly perturbed uncertain systems by making use of a singular perturbation method. However, there partially exist the uncertainties in the state and the output matrices. Recently, Shi *et al.*[2] has also studied the asymptotic H_∞ control of singularly perturbed systems such that there completely exist the uncertainties in the state and the output matrices. However, the practical design procedure for the output dynamic controller has not been reported. Moreover, it is found that the nonsingularity of the state matrix A_{22} for the fast subsystem in [1, 2] plays an important role in the study of the problem.

In this paper, we consider the robust H_∞ control problem for nonstandard singularly perturbed uncertain systems under imperfect state measurements where the uncertainties are time–varying norm–bounded perturbation parameters. The crucial difference from the existing results Shi *et al.* [1] are that there exist the uncertainties in both the state and output matrices and the fast state matrix A_{22} may be singular. In order to solve the Riccati equation, we propose a new algorithm which has quadratic convergence property. The resulting algorithm is quite different from the existing method [3]. Moreover, we present a new algorithm for solving the generalized Lyapunov equation which is based on a variant of the numerical approach to the generalized Sylvester equations [6]. In order to show the effectiveness of our algorithms, we apply the new algorithm to the manufacturing assembly process and show the validity of the full–order controller proposed in this paper.

2 Problem Formulation

Consider a class of uncertain singularly perturbed systems [2]

$$\dot{x}_1 = [A_{11} + \Delta A_{11}]x_1 + [A_{12} + \Delta A_{12}]x_2 + B_{11}w + [B_{12} + \Delta B_{12}]u, \quad x_1(0) = 0, \tag{1a}$$

$$\varepsilon\dot{x}_2 = [A_{21} + \Delta A_{21}]x_1 + [A_{22} + \Delta A_{22}]x_2 + B_{21}w + [B_{22} + \Delta B_{22}]u, \quad x_2(0) = 0, \tag{1b}$$

$$z = C_{11}x_1 + C_{12}x_2 + D_{12}u, \tag{1c}$$

$$y = [C_{21} + \Delta C_{21}]x_1 + [C_{22} + \Delta C_{22}]x_2 + D_{21}w, \tag{1d}$$

where ε is a small positive parameter, $x_1 \in \mathbf{R}^{n_1}$ and $x_2 \in \mathbf{R}^{n_2}$ are state vectors, $u \in \mathbf{R}^{m_1}$ is the control input, $w \in \mathbf{R}^{l_1}$ is the disturbance, $z \in \mathbf{R}^{l_2}$ is the controlled output, $y \in \mathbf{R}^{m_2}$ is the measurement. All matrices above are of appropriate dimensions. The system (1) is said to be in the standard form if the matrix A_{22} is nonsingular. Otherwise, it is called the nonstandard singularly perturbed systems.

The admissible parameter uncertainties are the following form

$$\begin{bmatrix} \Delta A_{11} & \Delta A_{12} & \Delta B_{12} \\ \Delta A_{21} & \Delta A_{22} & \Delta B_{22} \end{bmatrix} = \begin{bmatrix} H_{a1} \\ H_{a2} \end{bmatrix} F(t) \begin{bmatrix} E_{a1} & E_{a2} & E_b \end{bmatrix}, \tag{2a}$$

$$\begin{bmatrix} \Delta C_{21} & \Delta C_{22} \end{bmatrix} = H_c F(t) \begin{bmatrix} E_{a1} & E_{a2} \end{bmatrix}, \tag{2b}$$

$$F^T(t)F(t) \le I_s, \quad F(t) \in \mathbf{R}^{p\times s}, \tag{2c}$$

where $F(t) \in \mathbf{R}^{p\times s}$ is a Lebesgue measurable matrix of uncertain parameters. Note that there completely exist the uncertainties in all of the state and the output matrices compared with Shi *et al.*[1]. Let us introduce the partitioned matrices

$$
\begin{aligned}
&x = \begin{bmatrix} x_1^T & x_2^T \end{bmatrix}^T \in \mathbf{R}^n, \ n = n_1 + n_2,\\
&A_\varepsilon = \begin{bmatrix} A_{11} & A_{12} \\ \varepsilon^{-1}A_{21} & \varepsilon^{-1}A_{22} \end{bmatrix}, \ A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\\
&B_{1\varepsilon} = \begin{bmatrix} B_{11} \\ \varepsilon^{-1}B_{21} \end{bmatrix}, \ B_1 = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix},\\
&B_{2\varepsilon} = \begin{bmatrix} B_{12} \\ \varepsilon^{-1}B_{22} \end{bmatrix}, \ B_2 = \begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix},\\
&C_1 = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix}, \ C_2 = \begin{bmatrix} C_{21} & C_{22} \end{bmatrix},\\
&\Delta A_\varepsilon = \begin{bmatrix} \Delta A_{11} & \Delta A_{12} \\ \varepsilon^{-1}\Delta A_{21} & \varepsilon^{-1}\Delta A_{22} \end{bmatrix},\\
&\Delta B_{2\varepsilon} = \begin{bmatrix} \Delta B_{21} \\ \varepsilon^{-1}\Delta B_{22} \end{bmatrix}, \ \Delta C_2 = \begin{bmatrix} \Delta C_{21} & \Delta C_{22} \end{bmatrix},\\
&H_{a\varepsilon} = \begin{bmatrix} H_{a1} \\ \varepsilon^{-1}H_{a2} \end{bmatrix}, \ H_a = \begin{bmatrix} H_{a1} \\ H_{a2} \end{bmatrix} \in \mathbf{R}^{n\times p},\\
&E_a = \begin{bmatrix} E_{a1} & E_{a2} \end{bmatrix} \in \mathbf{R}^{s\times n}.
\end{aligned}
$$

Then equations (1) can be rewritten as

$$
\begin{aligned}
&\dot{x} = [A_\varepsilon + \Delta A_\varepsilon]x + B_{1\varepsilon}w + [B_{2\varepsilon} + \Delta B_{2\varepsilon}]u, \quad (3a)\\
&x(0) = 0,\\
&z = C_1x + D_{12}u, \quad (3b)\\
&y = [C_2 + \Delta C_2]x + D_{21}w, \quad (3c)
\end{aligned}
$$

where

$$
\Delta A_\varepsilon = H_{a\varepsilon}F(t)E_a, \ \ \Delta B_{2\varepsilon} = H_{a\varepsilon}F(t)E_b, \quad \Delta C_2 = H_cF(t)E_a.
$$

If an uncertain singularly perturbed system (1) or (3) is QS–H_∞–γ, then the following conditions are satisfied [7].

i) The closed–loop system is uniformly asymptotically stable.

ii) For any square integrable signal w, the uncertain singularly perturbed system (1) has an H_∞ norm bound $\gamma > 0$ in the sense

$$
\|z\|_2 \le \gamma \|w\|_2. \quad (4)
$$

The following lemma was shown by Gu [7].

Lemma 1 *The singularly perturbed uncertain system (1) or (3) can be made QS–H_∞–γ by a strictly proper linear output feedback control if and only if there exists a $\lambda > 0$ such that the following auxiliary system without uncertainties*

$$
\begin{aligned}
&\dot{x} = A_\varepsilon x + \begin{bmatrix} B_{1\varepsilon} & \gamma\lambda H_{a\varepsilon} \end{bmatrix}\begin{bmatrix} w \\ \hat{w} \end{bmatrix} + B_{2\varepsilon}u, \quad (5a)\\
&x(0) = 0,\\
&\begin{bmatrix} z \\ \hat{z} \end{bmatrix} = \begin{bmatrix} C_1 \\ \dfrac{1}{\lambda}E_a \end{bmatrix} x + \begin{bmatrix} D_{12} \\ \dfrac{1}{\lambda}E_b \end{bmatrix} u, \quad (5b)\\
&y = C_2x + \begin{bmatrix} D_{21} & \gamma\lambda H_c \end{bmatrix}\begin{bmatrix} w \\ \hat{w} \end{bmatrix} \quad (5c)
\end{aligned}
$$

where $\hat{w} \in \mathbf{R}^p$ is the disturbance, $\hat{z} \in \mathbf{R}^s$ is the controlled output, can be stabilized with its H_∞ norm less than γ by an output feedback control. That is, for any square integrable auxiliary signal w_e, the uncertain singularly perturbed system (5) has an H_∞ norm bound $\gamma > 0$ in the sense

$$
\|z_e\|_2 < \gamma \|w_e\|_2 \quad (6)
$$

where

$$
z_e = \begin{bmatrix} z \\ \hat{z} \end{bmatrix} \in \mathbf{R}^{l_2+s}, \ w_e = \begin{bmatrix} w \\ \hat{w} \end{bmatrix} \in \mathbf{R}^{l_1+p}.
$$

Define

$$
\begin{aligned}
&B_{1\gamma\lambda\varepsilon} = \begin{bmatrix} B_{1\varepsilon} & \gamma\lambda H_{a\varepsilon} \end{bmatrix}, \ C_{1\lambda} = \begin{bmatrix} C_1 \\ \dfrac{1}{\lambda}E_a \end{bmatrix},\\
&D_{12\lambda} = \begin{bmatrix} D_{12}\dfrac{1}{\lambda}E_b \end{bmatrix}, \ D_{21\gamma\lambda} = \begin{bmatrix} D_{21} & \gamma\lambda H_c \end{bmatrix},\\
&\tilde{D}_1 = (D_{12\lambda}^T D_{12\lambda})^{-1}, \ \tilde{D}_2 = (D_{21\gamma\lambda}D_{21\gamma\lambda}^T)^{-1}.
\end{aligned}
$$

The equations (5) can be also rewritten as

$$
\begin{aligned}
\dot{x} &= A_\varepsilon x + B_{1\gamma\lambda\varepsilon}w_e + B_{2\varepsilon}u, \ x(0) = 0, \quad (7a)\\
z_e &= C_{1\lambda}x + D_{12\lambda}u, \quad (7b)\\
y &= C_2x + D_{21\gamma\lambda}w_e. \quad (7c)
\end{aligned}
$$

We shall make the following basic structure assumptions for the full–order systems (7), which are typical in the H_∞ control problem.

Assumption 1 A1. *The pair (A_{22}, B_{22}) is stabilizable and (C_{22}, A_{22}) is observable.*

A2. $\text{rank}\begin{bmatrix} A_{22} - sI_{n_2} & B_{22} \\ C_{12} & D_{12} \\ E_{a2} & E_b \end{bmatrix} = n_2 + m_1,$

$\text{rank}\begin{bmatrix} A_{22} - sI_{n_2} & B_{12} & H_{a2} \\ C_{22} & D_{21} & H_c \end{bmatrix} = n_2 + m_2,$

A3. $\text{rank}\begin{bmatrix} \mathcal{A}(s) & B_2 \\ C_1 & D_{12} \\ E_a & E_b \end{bmatrix} = n + m_1,$

$\text{rank}\begin{bmatrix} \mathcal{A}(s) & B_1 & H_a \\ C_2 & D_{21} & H_c \end{bmatrix} = n + m_2,$

where $\mathcal{A}(s) = \begin{bmatrix} A_{11} - sI_{n_1} & A_{12} \\ A_{22} & A_{22} \end{bmatrix}.$

The QS–H_∞–γ condition can be written in a more convenient form (Shi *et al.* [1]).

Lemma 2 *A system (1) is QS–H_∞–γ if and only if there exists a positive scalar λ such that the following* *algebraic Riccati equations* (8a) *and* (8b) *have symmetric positive semidefinite stabilizing solutions* X_ε *and* Y_ε, *respectively, which satisfy* $\rho(X_\varepsilon Y_\varepsilon) < \gamma^2$.

$$\begin{aligned} &A_\varepsilon^T X_\varepsilon + X_\varepsilon A_\varepsilon + \gamma^{-2} X_\varepsilon B_{1\gamma\lambda\varepsilon} B_{1\gamma\lambda\varepsilon}^T X_\varepsilon \\ &\quad -(X_\varepsilon B_{2\varepsilon} + C_{1\lambda}^T D_{12\lambda})\tilde{D}_1 (B_{2\varepsilon}^T X_\varepsilon + D_{12\lambda}^T C_{1\lambda}) \\ &\quad + C_{1\lambda}^T C_{1\lambda} = 0, \qquad (8a) \\ &A_\varepsilon Y_\varepsilon + Y_\varepsilon A_\varepsilon^T + \gamma^{-2} Y_\varepsilon C_{1\lambda}^T C_{1\lambda} Y_\varepsilon \\ &\quad -(Y_\varepsilon C_2^T + B_{1\gamma\lambda\varepsilon} D_{21\gamma\lambda}^T)\tilde{D}_2 (C_2 Y_\varepsilon + D_{21\gamma\lambda} B_{1\gamma\lambda\varepsilon}^T) \\ &\quad + B_{1\gamma\lambda\varepsilon} B_{1\gamma\lambda\varepsilon}^T = 0, \qquad (8b) \end{aligned}$$

where

$$X_\varepsilon = \begin{bmatrix} X_{11} & \varepsilon X_{21}^T \\ \varepsilon X_{21} & \varepsilon X_{22} \end{bmatrix}, \; Y_\varepsilon = \begin{bmatrix} Y_{11} & Y_{12}^T \\ Y_{12} & \varepsilon^{-1} Y_{22} \end{bmatrix}.$$

Moreover, a suitable dynamic output feedback controller is given by

$$\begin{aligned} \dot{\xi} &= A_{c\varepsilon}\xi + L_\varepsilon y \qquad (9a) \\ u &= F_{2\varepsilon}\xi \qquad (9b) \end{aligned}$$

where

$$\begin{aligned} &F_{2\varepsilon} = -\tilde{D}_1 (B_{2\varepsilon}^T X_\varepsilon + D_{12\lambda}^T C_{1\lambda}), \\ &K_{2\varepsilon} = -(Y_\varepsilon C_2^T + B_{1\gamma\lambda\varepsilon} D_{21\gamma\lambda}^T)\tilde{D}_2, \; L_\varepsilon = -Z_\varepsilon K_{2\varepsilon}, \\ &A_{c\varepsilon} = A_\varepsilon + B_{2\varepsilon} F_{2\varepsilon} + \gamma^{-2} B_{1\gamma\lambda\varepsilon} B_{1\gamma\lambda\varepsilon}^T X_\varepsilon \\ &\quad -L_\varepsilon (C_2 + \gamma^{-2} D_{21\gamma\lambda} B_{1\gamma\lambda\varepsilon}^T X_\varepsilon), \\ &Z_\varepsilon = (I_n - \gamma^{-2} Y_\varepsilon X_\varepsilon)^{-1}, \; \xi \in \mathbf{R}^n. \end{aligned}$$

Remark 1 *Recall that the solution* X_ε *for* (8a) *is called stabilizing if* $A_\varepsilon + B_{1\gamma\lambda\varepsilon} F_{1\varepsilon} + B_{2\varepsilon} F_{2\varepsilon}$ *is stable matrix with* $F_{1\varepsilon} = \gamma^{-2} B_{1\gamma\lambda\varepsilon}^T X_\varepsilon$. *Similarly, the solution* Y_ε *of* (8b) *is stabilizing if* $A_\varepsilon + K_{1\varepsilon} C_{1\lambda} + K_{2\varepsilon} C_2$ *is stable matrix with* $K_{1\varepsilon} = \gamma^{-2} Y_\varepsilon C_{1\lambda}^T$.

In order to solve the algebraic Riccati equation (8a) and (8b), we introduce the following generalized algebraic Riccati equation (10a) and (11a), respectively.

$$\begin{aligned} &\mathcal{F}_1(X) = A^T X + X^T A + \gamma^{-2} X^T B_{1\gamma\lambda} B_{1\gamma\lambda}^T X \\ &\quad -(X^T B_2 + C_{1\lambda}^T D_{12\lambda})\tilde{D}_1 (B_2^T X \\ &\quad + D_{12\lambda}^T C_{1\lambda}) + C_{1\lambda}^T C_{1\lambda} = 0, \qquad (10a) \\ &\Pi_\varepsilon X = X^T \Pi_\varepsilon = X_\varepsilon, \qquad (10b) \\ &\mathcal{F}_2(Y) = AY^T + YA^T + \gamma^{-2} Y C_{1\lambda}^T C_{1\lambda} Y^T \\ &\quad -(YC_2^T + B_{1\gamma\lambda} D_{21\gamma\lambda}^T)\tilde{D}_2 (C_2 Y^T \\ &\quad + D_{21\gamma\lambda} B_{1\gamma\lambda}^T) + B_{1\gamma\lambda} B_{1\gamma\lambda}^T = 0, \qquad (11a) \\ &\Pi_\varepsilon^{-1} Y = Y^T \Pi_\varepsilon^{-1} = Y_\varepsilon, \qquad (11b) \end{aligned}$$

where

$$\begin{aligned} &\Pi_\varepsilon = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \varepsilon I_{n_2} \end{bmatrix}, \; B_{1\gamma\lambda} = \begin{bmatrix} B_1 & \gamma\lambda H_a \end{bmatrix}, \\ &X = \begin{bmatrix} X_{11} & \varepsilon X_{21}^T \\ X_{21} & X_{22} \end{bmatrix}, \; Y = \begin{bmatrix} Y_{11} & Y_{12} \\ \varepsilon Y_{12}^T & Y_{22} \end{bmatrix}, \\ &X_{11} = X_{11}^T, \; X_{22} = X_{22}^T, \; Y_{11} = Y_{11}^T, \; Y_{22} = Y_{22}^T, \\ &A_\varepsilon = \Pi_\varepsilon^{-1} A, \; B_{1\gamma\lambda\varepsilon} = \Pi_\varepsilon^{-1} B_{1\gamma\lambda}, \; B_{2\varepsilon} = \Pi_\varepsilon^{-1} B_2. \end{aligned}$$

3 Primary Results

In this section, we study the robust H_∞ control problem by using the dynamic output feedback control law for the linear time–invariant singularly perturbed system (7).

Let us define the following partition matrices

$$\begin{aligned} &\bar{A} = A - B_2 \tilde{D}_1 D_{12\lambda}^T C_{1\lambda}, \\ &\bar{R} = B_2 \tilde{D}_1 B_2^T - \gamma^{-2} B_{1\gamma\lambda} B_{1\gamma\lambda}^T, \\ &\bar{Q} = C_{1\lambda}^T [I_{l_2+s} - D_{12\lambda}\tilde{D}_1 D_{12\lambda}^T] C_{1\lambda}, \\ &\hat{A} = A - B_{1\gamma\lambda} D_{21\gamma\lambda}^T \tilde{D}_2 C_2, \\ &\hat{R} = C_2^T \tilde{D}_2 C_2 - \gamma^{-2} C_{1\lambda}^T C_{1\lambda}, \\ &\hat{Q} = B_{1\gamma\lambda} [I_{l_1+p} - D_{21\gamma\lambda}^T \tilde{D}_2 D_{21\gamma\lambda}] B_{1\gamma\lambda}^T, \end{aligned}$$

where

$$\begin{aligned} &\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \; \bar{R} = \begin{bmatrix} \bar{R}_{11} & \bar{R}_{12} \\ \bar{R}_{12}^T & \bar{R}_{22} \end{bmatrix}, \\ &\bar{Q} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{12}^T & \bar{Q}_{22} \end{bmatrix}, \; \hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \\ &\hat{R} = \begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} \\ \hat{R}_{12}^T & \hat{R}_{22} \end{bmatrix}, \; \hat{Q} = \begin{bmatrix} \hat{Q}_{11} & \hat{Q}_{12} \\ \hat{Q}_{12}^T & \hat{Q}_{22} \end{bmatrix}. \end{aligned}$$

Then, it follows that

$$\begin{aligned} &X^T \bar{A} + \bar{A}^T X - X^T \bar{R} X + \bar{Q} = 0, \qquad (12a) \\ &\hat{A} Y^T + Y \hat{A}^T - Y \hat{R} Y^T + \hat{Q} = 0. \qquad (12b) \end{aligned}$$

The generalized algebraic Riccati equations (12a) and (12a) will produce the unique positive semidefinite stabilizing solution under the assumption 1.

Firstly, we define the 0–order solutions to the Riccati equation (10a). Substitutes X of (10b) into (12a) and sets $\varepsilon = 0$. If the Riccati equation (13b) has a unique positive semidefinite stabilizing solution, then we obtain the following 0–order equations

$$\begin{aligned} &\bar{A}_0^T \bar{X}_{11} + \bar{X}_{11}^T \bar{A}_0 - \bar{X}_{11}^T \bar{R}_0 \bar{X}_{11} + \bar{Q}_0 = 0, \qquad (13a) \\ &\bar{A}_{22}^T \bar{X}_{22} + \bar{X}_{22}^T \bar{A}_{22} - \bar{X}_{22}^T \bar{R}_{22} \bar{X}_{22} + \bar{Q}_{22} = 0, \qquad (13b) \\ &\bar{X}_{21} = -N_2^T + N_1^T \bar{X}_{11}, \qquad (13c) \end{aligned}$$

where

$$\begin{aligned} &T_0 = T_1 - T_2 T_4^{-1} T_3 = \begin{bmatrix} \bar{A}_0 & -\bar{R}_0 \\ -\bar{Q}_0 & -\bar{A}_0^T \end{bmatrix}, \\ &T_1 = \begin{bmatrix} \bar{A}_{11} & -\bar{R}_{11} \\ -\bar{Q}_{11} & -\bar{A}_{11}^T \end{bmatrix}, \; T_2 = \begin{bmatrix} \bar{A}_{12} & -\bar{R}_{12} \\ -\bar{Q}_{12} & -\bar{A}_{21}^T \end{bmatrix}, \\ &T_3 = \begin{bmatrix} \bar{A}_{21} & -\bar{R}_{12}^T \\ -\bar{Q}_{12}^T & -\bar{A}_{12}^T \end{bmatrix}, \; T_4 = \begin{bmatrix} \bar{A}_{22} & -\bar{R}_{22} \\ -\bar{Q}_{22} & -\bar{A}_{22}^T \end{bmatrix}, \\ &N_2^T = \Lambda_4^{-T} \bar{L}_{12}^T, \; N_1^T = -\Lambda_4^{-T} \Lambda_2^T, \\ &\Lambda_1 = \bar{A}_{11} - \bar{R}_{11} \bar{X}_{11} - \bar{R}_{12} \bar{X}_{21}, \\ &\Lambda_3 = \bar{A}_{21} - \bar{R}_{12}^T \bar{X}_{11} - \bar{R}_{22} \bar{X}_{21}, \\ &\Lambda_2 = \bar{A}_{12} - \bar{R}_{12} \bar{X}_{22}, \; \Lambda_4 = \bar{A}_{22} - \bar{R}_{22} \bar{X}_{22}, \\ &\bar{L}_{12} = \bar{Q}_{12} + \bar{A}_{21}^T \bar{X}_{22}, \; \Lambda_0 = \Lambda_1 - \Lambda_2 \Lambda_4^{-1} \Lambda_3. \end{aligned}$$

Remark 2 *The matrices $\bar{A}_0$, $\bar{R}_0$ and $\bar{Q}_0$ do not depend on $\bar{X}_{22}$ because their matrices can be computed by using T_i, $i = 1, \cdots, 4$ which is independent of $\bar{X}_{22}$* [8].

By following the similar steps in equation (12a), if the Riccati equation (14b) has a unique positive semidefinite stabilizing solution, then we obtain the following 0–order equations for the equation (12b)

$$\hat{A}_0\bar{Y}_{11}^T + \bar{Y}_{11}\hat{A}_0^T - \bar{Y}_{11}\hat{R}_0\bar{Y}_{11}^T + \hat{Q}_0 = 0, \quad (14a)$$
$$\hat{A}_{22}\bar{Y}_{22}^T + \bar{Y}_{22}\hat{A}_{22}^T - \bar{Y}_{22}\hat{R}_{22}\bar{Y}_{22}^T + \hat{Q}_{22} = 0, \quad (14b)$$
$$\bar{Y}_{12} = -M_2 + \bar{Y}_{11}M_1, \quad (14c)$$

where

$$H_0 = H_1 - H_2H_4^{-1}H_3 = \begin{bmatrix} \hat{A}_0^T & -\hat{R}_0 \\ -\hat{Q}_0 & -\hat{A}_0 \end{bmatrix},$$
$$H_1 = \begin{bmatrix} \hat{A}_{11}^T & -\hat{R}_{11} \\ -\hat{Q}_{11} & -\hat{A}_{11} \end{bmatrix}, \ H_2 = \begin{bmatrix} \hat{A}_{21}^T & -\hat{R}_{12} \\ -\hat{Q}_{12} & -\hat{A}_{12} \end{bmatrix},$$
$$H_3 = \begin{bmatrix} \hat{A}_{12}^T & -\hat{R}_{12}^T \\ -\hat{Q}_{12}^T & -\hat{A}_{21} \end{bmatrix}, \ H_4 = \begin{bmatrix} \hat{A}_{22}^T & -\hat{R}_{22} \\ -\hat{Q}_{22} & -\hat{A}_{22} \end{bmatrix},$$
$$M_2^T = \Gamma_4^{-1}\hat{L}_{21}, \ M_1^T = -\Gamma_4^{-1}\Gamma_3,$$
$$\Gamma_1 = \hat{A}_{11} - \bar{Y}_{12}\hat{R}_{12}^T - \bar{Y}_{11}\hat{R}_{11}^T,$$
$$\Gamma_2 = \hat{A}_{12} - \bar{Y}_{12}\hat{R}_{22}^T - \bar{Y}_{11}\hat{R}_{12},$$
$$\Gamma_3 = \hat{A}_{21} - \bar{Y}_{22}\hat{R}_{12}^T, \ \Gamma_4 = \hat{A}_{22} - \bar{Y}_{22}\hat{R}_{22}^T,$$
$$\hat{L}_{21} = \bar{Y}_{22}\hat{A}_{12}^T + \hat{Q}_{12}^T, \ \Gamma_0 = \Gamma_1 - \Gamma_2\Gamma_4^{-1}\Gamma_3.$$

Now we are in a position to establish our primary result in this paper.

Theorem 1 *Under the assumption 1, if there exists a positive scalar λ such that the reduced–order Riccati equations* (13a), (13b), (14a) *and* (14b) *have positive semidefinite stabilizing solutions $\bar{X}_{11} \geq 0$, $\bar{X}_{22} \geq 0$, $\bar{Y}_{11} \geq 0$ and $\bar{Y}_{22} \geq 0$, respectively, then there exists sufficient small ε^* such that $\forall\varepsilon \in [0, \varepsilon^*)$, the algebraic Riccati equations* (8a) *and* (8b) *admit the positive semidefinite stabilizing solutions. Furthermore, the solutions of* (8a) *and* (8b) *can be approximated by*

$$X_\varepsilon = \begin{bmatrix} \bar{X}_{11} + \varepsilon E_{11} & \varepsilon\{\bar{X}_{21} + \varepsilon E_{21}\}^T \\ \varepsilon\{\bar{X}_{21} + \varepsilon E_{21}\} & \varepsilon\{\bar{X}_{22} + \varepsilon E_{22}\} \end{bmatrix}, \quad (15a)$$
$$Y_\varepsilon = \begin{bmatrix} \bar{Y}_{11} + \varepsilon F_{11} & \bar{Y}_{12} + \varepsilon F_{12} \\ \{\bar{Y}_{12} + \varepsilon F_{12}\}^T & \varepsilon^{-1}\{\bar{Y}_{22} + \varepsilon F_{22}\} \end{bmatrix}. \quad (15b)$$

If such conditions are met, a control is given by (9).

Proof: By using the implicit function theorem [3], the theorem can be proved. The proof is omitted since it is similar to that of the references [9]. ∎

Remark 3 *It can be easily shown that if there exists positive scalar λ such that the Riccati equations* (13) *and* (14) *have positive semidefinite stabilizing solutions, then the uncertain singularly perturbed system with controller* (9) *is robust stabilizable and has a robust H_∞ performance γ.*

4 New Algorithm

We propose the following new algorithm for solving (12a) and (12b) with parameter ε.

$$[\bar{A} - \bar{R}X^{(i)}]^T X^{(i+1)} + X^{(i+1)T}[\bar{A} - \bar{R}X^{(i)}] + X^{(i)T}\bar{R}X^{(i)} + \bar{Q} = 0, \quad (16a)$$
$$[\hat{A} - Y^{(i)}\hat{R}]Y^{(i+1)T} + Y^{(i+1)}[\hat{A} - Y^{(i)}\hat{R}]^T + Y^{(i)}\hat{R}Y^{(i)T} + \hat{Q} = 0, \quad (16b)$$

with the initial condition obtained from

$$X^{(0)} = \begin{bmatrix} \bar{X}_{11} & 0 \\ \bar{X}_{21} & \bar{X}_{22} \end{bmatrix}, \ Y^{(0)} = \begin{bmatrix} \bar{Y}_{11} & \bar{Y}_{12} \\ 0 & \bar{Y}_{22} \end{bmatrix} \quad (16c)$$

where

$$X^{(i)} = \begin{bmatrix} X_{11}^{(i)} & \varepsilon X_{21}^{(i)T} \\ X_{21}^{(i)} & X_{22}^{(i)} \end{bmatrix}, \ Y^{(i)} = \begin{bmatrix} Y_{11}^{(i)} & Y_{12}^{(i)} \\ \varepsilon Y_{12}^{(i)T} & Y_{22}^{(i)} \end{bmatrix},$$

and $\bar{X}_{ij}$ and $\bar{Y}_{ij}$ are defined by (13) and (14), respectively.

The main result of this section is as follows.

Theorem 2 *Under the stabilizability and detectability conditions, imposed in the assumption 1, we assume that the Riccati equations* (13a), (13b), (14a) *and* (14b) *have the positive semidefinite stabilizing solutions respectively. Then the proposed algorithm* (16a) *converges to the exact solution of the generalized algebraic Riccati equation* (10a) *or* (12a) *with the rate of quadratic convergence. Moreover, there exists a unique solution of the algebraic Riccati equation* (10a) *in neighborhood of the required solution.*

Proof: The proof of the quadratic convergence property has been given in [9]. Thus, we will prove existence of the unique solution for the Riccati equation (8a). We now observe that function $\mathcal{F}_1(P)$ is differentiable on a convex set $\mathcal{D}$. Using the fact that

$$\nabla\mathcal{F}_1(X) = (\bar{A} - \bar{R}X)^T \otimes I_n + I_n \otimes (\bar{A} - \bar{R}X)^T,$$

we have

$$\|\nabla\mathcal{F}_1(X_1) - \nabla\mathcal{F}_1(X_2)\| \leq \bar{\gamma}\|X_1 - X_2\|.$$

Moreover, using the fact that

$$\nabla\mathcal{F}_1(X^{(0)}) = \begin{bmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_3 & \Lambda_4 \end{bmatrix} \otimes I_n + I_n \otimes \begin{bmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_3 & \Lambda_4 \end{bmatrix},$$

it follows that $\nabla\mathcal{F}_1(X^{(0)})$ is nonsingular because Λ_4 and $\Lambda_0 = \Lambda_1 - \Lambda_2\Lambda_4^{-1}\Lambda_3$ are stable. Therefore, there exists $\bar{\beta}$ such that $\|[\nabla\mathcal{F}_1(X^{(0)})]^{-1}\| \equiv \bar{\beta}$. On the other hand, since $\mathcal{F}_1(X^{(0)}) = 0$, there exists $\bar{\eta}$ such that $\|[\nabla\mathcal{F}_1(X^{(0)})]^{-1}\mathcal{F}_1(X^{(0)})\| \leq \bar{\eta}$. Thus, there exists $\bar{\alpha}$ such that $\bar{\alpha} \equiv \bar{\beta}\bar{\gamma}\bar{\eta} < 2^{-1}$. Now, let us define

$$t^* \equiv \frac{1}{\bar{\gamma}\bar{\beta}}[1 - \sqrt{1 - 2\bar{\alpha}}].$$

Clearly, $\mathcal{S} \equiv \{ X \ : \|X - X^{(0)}\| \le t^* \}$ is in the convex set $\mathcal{D}$. In the sequel, X^* is the unique solution in $\mathcal{S}$ by applying Newton–Kantorovich theorem (see [5], pp.155) because $\|X^* - X^{(0)}\| = O(\varepsilon)$ holds for small ε. This completes the proof of Theorem 2. ■

By comparing with Mukaidani et al. [9], since the proposed algorithm is quadratic convergence, the required solution can be easily obtained up to an arbitrary order of accuracy, that is, $O(\varepsilon^{2^i})$ where i is a iteration number.

We next give the convergence theorem for the algorithm (16b) by similar argument.

Theorem 3 *Under the stabilizability and detectability conditions, imposed in the assumption 1, the proposed algorithm* (16b) *converges to the exact solution of the generalized algebraic Riccati equation* (11a) *with the rate of quadratic convergence. Moreover, there exists a unique solution of the algebraic Riccati equation* (11a) *in neighborhood of the required solution* Y^*.

Proof: Since the proof of Theorem 3 is performed by a dual argument, it is omitted. ■

5 Algorithms for The Generalized Lyapunov Equation

In order to obtain the solution of the generalized Lyapunov equation (16a), we present the new algorithm by applying the generalized Schur (GS) method. The GS method is based upon the QZ–algorithm and the GS–algorithm [6]. Let us consider the simultaneous linear equations (17) by rearranging the generalized Lyapunov equation (16a)

$$W^T \mathcal{A} H H^T \mathcal{X} V + W^T \mathcal{Y} U U^T \mathcal{B} V + W^T \mathcal{C} V = 0, \quad (17a)$$
$$W^T \Pi_\varepsilon H H^T \mathcal{X} V - W^T \mathcal{Y} U U^T \Pi_\varepsilon V = 0, \quad (17b)$$

where the matrix W, H, U and V are an orthogonal matrix, e.g., $W^T = W^{-1}$, and

$$\mathcal{X} = X^{(i+1)},\ \mathcal{Y} = X^{(i+1)T},\ \mathcal{A} = (\bar{A} - \bar{R}X^{(i)})^T,$$
$$\mathcal{B} = \bar{A} - \bar{R}X^{(i)},\ \mathcal{C} = X^{(i)T}\bar{R}X^{(i)} + \bar{Q}.$$

Using the unitary equivalence transformations, such methods involve the following three steps.

Step 1: Transform $(\mathcal{A},\ \Pi_\varepsilon)$ and $(\mathcal{B},\ \Pi_\varepsilon)$ into simpler form $(\mathcal{A}_1,\ \Pi_{1\varepsilon}) := (W^T \mathcal{A} H,\ W^T \Pi_\varepsilon H)$, $(\mathcal{B}_1,\ \Pi_{2\varepsilon}) := (U^T \mathcal{B} V,\ U^T \Pi_\varepsilon V)$, respectively. Furthermore, modify the right–hand side $\mathcal{C}_1 := W^T \mathcal{C} V$.

Step 2: Solve the equations (18) for $\mathcal{X}_1 := H^T \mathcal{X} V$ and $\mathcal{Y}_1 := W^T \mathcal{Y}$.

$$\mathcal{A}_1 \mathcal{X}_1 + \mathcal{Y}_1 \mathcal{B}_1 + \mathcal{C}_1 = 0 \quad (18a)$$
$$\Pi_{1\varepsilon} \mathcal{X}_1 - \mathcal{Y}_1 \Pi_{2\varepsilon} = 0 \quad (18b)$$

Step 3: Transform the solution back to the original equation: $X^{(i+1)} = \mathcal{X} = H \mathcal{X}_1 V^T$, $X^{(i+1)T} = \mathcal{Y} = W \mathcal{Y}_1 U^T$.

The matrix pairs in Step 1 are transformed to generalized Schur form with upper quasi–triangular matrix pairs $\mathcal{A}_1 = W^T \mathcal{A} H$ and $\Pi_{1\varepsilon} = W^T \Pi_\varepsilon H$, and upper triangular matrix pairs $\mathcal{B}_1 = U^T \mathcal{B} V$ and $\Pi_{2\varepsilon} = U^T \Pi_\varepsilon V$. In step 2, using the GS–Algorithm and Kronecker products method, these solutions $\mathcal{X}_1$ and $\mathcal{Y}_1$ of (18) can be obtained [6]. It should be remarked that since the GS method is based upon the QZ–algorithm and the GS–algorithm, the method is apparently quite numerically stable and performs reliably on equations with dense matrices of high–order dimension.

6 Numerical Example

In order to demonstrate the efficiency of the proposed algorithm, we consider a fourth order real world example, that is, manufacturing assembly process [1]. The system matrix is given by (see Shi *et al.* [1])

$$A_{11} = \begin{bmatrix} 0 & 1 & -1 \\ -2 & -0.2 & 0 \\ 2 & 0 & 0 \end{bmatrix},\ A_{12} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix},$$
$$A_{21} = \begin{bmatrix} 0 & -5.2 \times 10^{-2} & 0 \end{bmatrix},\ A_{22} = \begin{bmatrix} -2 \end{bmatrix},$$
$$B_{11} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ B_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.2 & 0 \end{bmatrix},$$
$$B_{21} = \begin{bmatrix} 0 & 0 & 0.1 \end{bmatrix},\ B_{22} = \begin{bmatrix} 0 & 1 \end{bmatrix},$$
$$C_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\ C_{12} = D_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$H_{a1} = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \end{bmatrix},\ H_{a2} = \begin{bmatrix} 0.2 \end{bmatrix},\ E_b = \begin{bmatrix} 1 & 1 \end{bmatrix},$$
$$E_{a1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix},\ E_{a2} = \begin{bmatrix} 0 \end{bmatrix},\ F(t) = \sin(10\pi t).$$

The numerical results are obtained for small parameter $\varepsilon = 0.1$. Now, choosing $\lambda = 1.0$ and $\gamma = 3.0$, then, the algebraic Riccati equations (13a) and (13b) have the positive semidefinite stabilizing solutions. Thus, we find that equation (8a) has solutions $X_\varepsilon \ge 0$ under the previous conditions. The solutions of the algebraic Riccati equation (13), that is, the 0–order solutions are shown in the following.

$$X^{(0)} = \begin{bmatrix} 4.1885 & 0.2495 & -0.2650 & 0.0 \\ 0.2495 & 1.7466 & -0.5352 & 0.0 \\ -0.2650 & -0.5352 & 1.5064 & 0.0 \\ 0.00505 & 0.11465 & -0.03556 & 0.00620 \end{bmatrix}.$$

On the basis of above 0–order solutions, by using the new algorithms (16), we get the exact solution (19). We find that the solutions of the algebraic Riccati equation

$$X^{(3)} = \begin{bmatrix} 4.2612453460 & 2.6135877367 \times 10^{-1} & -2.6952557244 \times 10^{-1} & -5.1479030477 \times 10^{-3} \\ 2.6135877367 \times 10^{-1} & 1.8137919882 \times 10^{-1} & -5.6897106792 \times 10^{-1} & 1.1627840749 \times 10^{-1} \\ -2.6952557244 \times 10^{-1} & -5.6897106792 \times 10^{-1} & 1.5285405402 & -3.6788833874 \times 10^{-2} \\ -5.1479030477 \times 10^{-2} & 1.1627840749 & -3.6788833874 \times 10^{-1} & 1.4664013731 \times 10^{-1} \end{bmatrix}. \quad (19)$$

$$u = \begin{bmatrix} -3.145562671 \times 10^{-1} & 1.301241674 \times 10^{-1} & -1.326434852 & 3.871185570 \times 10^{-1} \\ -3.169823512 \times 10^{-1} & -1.146454121 & 3.471615952 \times 10^{-1} & -7.668793471 \times 10^{-1} \end{bmatrix} x. \quad (20)$$

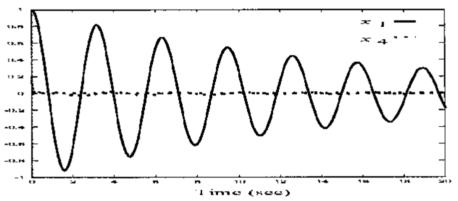

Fig.1 Response of the open loop system without any controller.

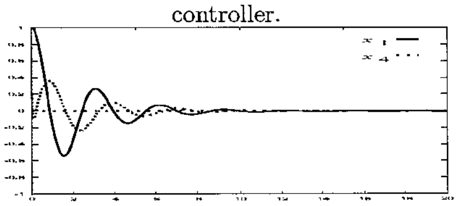

Fig.2 Response of the closed–loop system with the proposed controller.

(8a) converge to the exact solution with accuracy of $\|\mathcal{F}_1(X^{(i)})\| < 10^{-14}$ after 3 iterations. It can be seen that the initial guess (16c) for the proposed algorithm is quite good. It should be noted that if we set the initial condition to the above solutions with $O(10^{-4})$–order accuracy the classical recursive algorithm [3] does not converge to the exact solution with accuracy of $O(10^{-14})$, while the proposed algorithm converges to the exact solution with accuracy of $O(10^{-14})$ after 3 iterations.

Table 1.

i	$\|\mathcal{F}_1(X^{(i)})\|$
0	1.2215×10^{-2}
1	8.6001×10^{-6}
2	4.0334×10^{-13}
3	1.6372×10^{-15}

In this example, the QS–H_∞–γ controller is given by (20). The constructed controller (20) will be employed as the manufacturing system under bounded uncertain assembly goods. The results of the simulation of this example are depicted in Figures 1 and 2. The initial state is set as $x(0) = \begin{bmatrix} 1 & 0 & 0 & 0.2 \end{bmatrix}^T$. It is shown from Fig. 2 that the closed–loop system is asymptotically stable.

7 Conclusion

In this paper, we have considered the robust H_∞ control problem for nonstandard singularly perturbed systems under imperfect state measurements. We have proposed the new algorithm for solving the generalized algebraic Riccati equations and new method for solving the generalized Lyapunov equations. Comparing with [3], since the proposed algorithm is quadratic convergence, the required solution can be easily obtained up to an arbitrary order of accuracy. It should be emphasized that by applying the new algorithm to the algebraic Riccati equation, the obtained controller is very reliable to the manufacturing assembly process.

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