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# On the Near-Optimality of Composite Optimal Control for Nonstandard Singularly Perturbed Systems

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**Abstract**

In this paper, a new method based on a generalized algebraic Riccati equation arising in descriptor systems is presented to solve the composite optimal control problem of nonstandard singularly perturbed systems. It is shown that the composite optimal control can be obtained very simply by only revising the solution of the slow regulator problem. It is proven that the composite optimal control can achieve a performance which is $O(\varepsilon^2)$ close to the optimal performance. Although this result is well-known for the standard singularly perturbed systems, it is new in the nonstandard case.

## 1 Full-Order Regulator Problem

Consider the linear time-invariant singularly perturbed system

$$\dot{x} = A_{11}x + A_{12}z + B_1 u,\ x(0) = x_0, \tag{1a}$$

$$\varepsilon\dot{z} = A_{21}x + A_{22}z + B_2 u,\ z(0) = z_0, \tag{1b}$$

with a performance index

$$J = \frac{1}{2}\int_0^\infty (\begin{bmatrix} x \\ z \end{bmatrix}^T Q \begin{bmatrix} x \\ z \end{bmatrix} + u^T Ru)dt, \tag{2}$$

which has to be minimized, where

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} = \begin{bmatrix} C_1^T C_1 & C_1^T C_2 \\ C_2^T C_1 & C_2^T C_2 \end{bmatrix},\ R > 0, \tag{3}$$

and $\varepsilon$ is a small positive parameter, $x(t) \in R^n$ and $z(t) \in R^m$ are states, and $u(t) \in R^r$ is the control, and all matrices are of appropriate dimensions. The system (1) is called the nonstandard singularly perturbed system if the matrix $A_{22}$ is singular.

Let us introduce a generalized algebraic Riccati equation.

$$(i)\ A^T P + P^T A - P^T BR^{-1}B^T P + Q = 0, \tag{4a}$$

$$(ii)\ E_\varepsilon P = P^T E_\varepsilon, \tag{4b}$$

where $E_\varepsilon = diag[I_n, \varepsilon I_m]$. Corresponding to the parameter matrices of (4), $P$ has the following partitioned form

$$P = \begin{bmatrix} P_{11} & \varepsilon P_{21}^T \\ P_{21} & P_{22} \end{bmatrix},\ P_{11} = P_{11}^T,\ P_{22} = P_{22}^T, \tag{5}$$

since it satisfies (4b). It is worthy to note that $P$ is not symmetric, but $E_\varepsilon P$ is.

**Theorem 1.** *Suppose that there exists a small positive parameter $\varepsilon^*$ such that, for all $\varepsilon \in (0, \varepsilon^*)$, the generalized algebraic Riccati equation (4) admits a unique solution $P$ for which $E_\varepsilon P \geq 0$. Then,*

$$u^*(t) = -R^{-1}B^T Py(t), \tag{6}$$

*constitutes the optimal feedback control for the full-order regulator problem, and the optimal performance is*

$$J^* = \frac{1}{2}y^T(0)E_\varepsilon Py(0). \tag{7}$$

## 2 Decomposition of Slow and Fast Regulator Problems

Similar to the standard singularly perturbed systems[1], we decompose the full-order regulator problem into two subsystem regulator problems.

**Slow regulator problem:** Find $u_s$ to minimize

$$J_s = \frac{1}{2}\int_0^\infty (\begin{bmatrix} x_s \\ z_s \end{bmatrix}^T Q \begin{bmatrix} x_s \\ z_s \end{bmatrix} + u_s^T Ru_s)dt, \tag{8}$$

for the slow subsystem

$$E\dot{y}_s = Ay_s + Bu_s,\quad Ey_s(0) = Ey_0, \tag{9}$$

where $y_s(t) = [x_s^T(t)\ \ z_s^T(t)]^T$, $E = E_\varepsilon|_{\varepsilon=0}$, $A$, $B$ are defined in (4), and $Q$ in (3).

**Fast regulator problem:** Find $u_f$ to minimize

$$J_f = \frac{1}{2}\int_0^\infty (z_f^T C_2^T C_2 z_f + u_f^T Ru_f)dt, \tag{10}$$

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for the fast subsystem

$$\varepsilon \dot{z}_f = A_{22} z_f + B_2 u_f, \quad z_f(0) = z_0 - z_s(0), \qquad (11)$$

where $z_f = z - z_s$, $u_f = u - u_s$.

We now consider the solution of the slow and fast regulator problems under appropriate assumptions.

**Proposition 1.** *The fast regulator problem admits a unique optimal feedback control*

$$u_f^* = -R^{-1} B_{22} P_{22f}^{+} z_f, \qquad (12)$$

*where $P_{22f}^{+}$ is a unique stabilizing positive semidefinite symmetric solution of the algebraic Riccati equation*

$$P_{22f} A_{22} + A_{22}^T P_{22f} - P_{22f} B_2 R^{-1} B_2^T P_{22f} + Q_{22} = 0. \qquad (13)$$

In the following, we will consider the solution of the slow regulator problem. Before doing that, we first introduce another generalized algebraic Riccati equation [2],

$$(i)\ A^T P_s + P_s^T A - P_s^T B R^{-1} B^T P_s + Q = 0, \qquad (14a)$$

$$(ii)\ E^T P_s = P_s^T E. \qquad (14b)$$

where $Q$ is the same as that in (4). The solution $P_s$ of (14) has a lower-triangular block form

$$P_s = \begin{bmatrix} P_{11s} & 0 \\ P_{21s} & P_{22s} \end{bmatrix}, \quad P_{11s}^T = P_{11s}, \qquad (15)$$

because of (14b). It is worthy to note that $P_{22s}$ may not be symmetric.

**Proposition 2.** *The slow regulator problem admits a unique optimal open-loop control, which can be implemented by a class of linear feedback controls given by*

$$u_s^* = -R^{-1} B^T P_s y_s, \qquad (16)$$

*where*

$$P_s = \begin{bmatrix} P_{11s}^{+} & 0 \\ P_{21s} & P_{22s} \end{bmatrix}, \qquad (17)$$

*is the solution of the generalized algebraic Riccati equation (14).*

## 3 Near-Optimality of Composite Optimal Control

The composite optimal control is constructed as follows.

$$u_c^* = u_s^{*+} + u_f^* = -R^{-1} [B_1^T \ \ B_2^T] \begin{bmatrix} P_{11s}^{+} & 0 \\ P_{21s}^{+} & P_{22s}^{+} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}, \qquad (18)$$

where $x(t) \approx x_s(t)$ and $z(t) \approx z_s(t) + z_f(t)$.

We now apply the composite optimal control $u_c^*$ to the full-order system (1) and compare it with the exact optimal control (6). In order to do that, we first study the existence conditions of the unique solution $P$ of the generalized algebraic Riccati equation (4).

**Theorem 2.** *There exists a small positive parameter $\varepsilon^*$ such that, for all $\varepsilon \in [0, \varepsilon^*)$, the generalized algebraic Riccati equation (4) admits a unique stabilizing solution $P$ for which $E_\varepsilon P \geq 0$. Moreover, the solution $P$ possesses a power series expansion at $\varepsilon = 0$, that is,*

$$P = \begin{bmatrix} P_{11}^{(0)} & \varepsilon P_{21}^{(0)T} \\ P_{21}^{(0)} & P_{22}^{(0)} \end{bmatrix} + \sum_{i=1}^{\infty} \frac{\varepsilon^i}{i!} \begin{bmatrix} P_{11}^{(i)} & \varepsilon P_{21}^{(i)T} \\ P_{21}^{(i)} & P_{22}^{(i)} \end{bmatrix}. \qquad (19)$$

Now, we can compare the composite optimal control $u_c^*$ with the exact optimal control $u^*$ and show the $O(\varepsilon^2)$ approximation of $J^*$. Applying the composite optimal control $u_c^*$ to the full-order system (1), we have

$$J^c = \frac{1}{2} y^T(0) E_\varepsilon P_c y(0), \qquad (20)$$

where $P_c$ is the solution of the generalized Lyapunov equation

$$(i)\ (A - SP_s^{+})^T P_c + P_c^T (A - SP_s^{+}) = -P_s^{+T} S P_s^{+} - Q, \qquad (21a)$$

$$(ii)\ E_\varepsilon P_c = P_c^T E_\varepsilon, \qquad (21b)$$

with $S = BR^{-1}B^T$.

**Theorem 3.** *The first two terms of the power series of $J^c$ and $J^*$ at $\varepsilon = 0$ are the same, that is,*

$$J^c = J^* + O(\varepsilon^2), \qquad (22)$$

*and hence the composite optimal control (18) is an $O(\varepsilon^2)$ near-optimal solution to the full-order regulator problem (1),(2).*

We have therefore provided a complete theoretic analysis of the near-optimality of the composite optimal control for both standard and nonstandard singularly perturbed systems. It is further proven that the new composite optimal controller is equivalent to the existing one in the case of the standard singularly perturbed systems. The detail is omitted for the limit of the paper space. Therefore, we claim that the new composite optimal controller includes the existing composite optimal controller [1] as a special case.

## References


[1] P. V. Kokotovic, H. K. Khalil, and J. O'Reilly, Singular Perturbation Methods in Control: Analysis and Design. New York: Academic Press, 1986.

[2] H. Xu and K. Mizukami, The Linear-Quadratic Optimal Regulator for Continuous-Time Descriptor Systems : A Dynamic Programming Approach. *International Journal of Systems Sciences*, 25, pp.1889-1898, 1994.