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Asymptotic Expansions of Solutions of Cross–Coupled Algebraic Riccati Equations of Multimodeling Systems Related to Nash Games

Hiroaki Mukaidani
Graduate School of Education
Hiroshima University
1–1–1, Kagamiyama,
Higashi–Hiroshima,
Hiroshima, 739–8524 Japan
mukaida@hiroshima-u.ac.jp

Tetsu Shimomura
Graduate School of Education
Hiroshima University
1–1–1, Kagamiyama,
Higashi–Hiroshima,
Hiroshima, 739–8524 Japan
tshimo@hiroshima-u.ac.jp

Hua Xu
Graduate School of Business Sciences
The University of Tsukuba
3–29–1, Otsuka,
Bunkyou–ku,
Tokyo, 112–0012 Japan
xuhua@gssm.otsuka.tsukuba.ac.jp

Abstract— The linear quadratic Nash games for infinite horizon multiparameter singularly perturbed systems (MSPS) are considered. The existence of bounded solutions and asymptotic expansions of solutions for the generalized cross–coupled multiparameter algebraic Riccati equations (GCMARE) are established without non–singularity assumptions of the fast subsystems.

I. Introduction

The linear quadratic Nash games and their applications have been studied intensively in many papers (see e.g., [1, 2]). In particular, Starr and Ho [1] derived the closed–loop perfect–state linear Nash equilibrium strategies for a class of analytic differential games. In [3, 4], linear quadratic Nash games for the multiparameter singularly perturbed systems (MSPS) have been studied by using the two–time–scale design method. However, to obtain the reduced–order systems, the non–singularity assumptions of the fast subsystems are needed.

It is well–known that in order to obtain the Nash equilibrium strategies for the MSPS, we must solve the generalized cross–coupled algebraic Riccati equations (GCARE). The existence of their solutions plays a crucial role in the theory of the Nash games for the MSPS. This important problem has been studied in [6] under the conservative conditions. However, the results for the asymptotic expansions of solutions for the GCARE have not been investigated so far. Moreover, in [6], the relation between the GCARE and the reduced–order equations and the formulation to calculate the reduced–order equations have not been studied.

In this paper the linear quadratic Nash games for the infinite horizon MSPS are considered without non–singularity assumptions of the fast subsystems. After defining the GCMARE, the boundedness of the solution to the GCMARE and its asymptotic structure are newly derived under less conservative condition compared with the previous result [6]. The proof of the existence of the solution to the GCMARE with asymptotic expansion is obtained by an implicit function theorem [5] under assumptions imposed on the reduced–order subsystems. As another important feature of this paper, since the non–singularity assumptions of the state matrices of the fast subsystems are not needed, the obtaining theoretical results can be extended over the existing results [3].

Notation: The notations used in this paper are fairly standard. The superscript T denotes matrix transpose. I_n denotes the $n \times n$ identity matrix. $\det M$ denotes the determinant of M. $\mathrm{Re}(\lambda)$ denotes a real part of $\lambda \in \mathbf{C}$.

II. Problem Formulation

Consider a linear time–invariant MSPS [3]

$$\begin{aligned} \dot{x}_0(t) &= \sum_{i=0}^{2} A_{0i}x_i(t) + \sum_{i=1}^{2} B_{0i}u_i(t), && \text{(1a)} \\ \varepsilon_1 \dot{x}_1(t) &= A_{10}x_0(t) + A_{11}x_1(t) + B_{11}u_1(t), && \text{(1b)} \\ \varepsilon_2 \dot{x}_2(t) &= A_{20}x_0(t) + A_{22}x_2(t) + B_{22}u_2(t), && \text{(1c)} \\ x_j(0) &= x_j^0, \ j = 0, \ 1, \ 2, \end{aligned}$$

with quadratic cost functions

$$\begin{aligned} J_i &= \frac{1}{2}\int_0^{\infty} [y_i^T(t)y_i(t) + u_i^T(t)R_{ii}u_i(t)]dt, && \text{(2a)} \\ & \quad R_{ii} > 0, \ i = 1, \ 2, \\ y_i(t) &= C_{i0}x_0(t) + C_{ii}x_i(t) = C_i x(t), && \text{(2b)} \\ x(t) &= \begin{bmatrix} x_0^T(t) & x_1^T(t) & x_2^T(t) \end{bmatrix}^T, \end{aligned}$$

where $x_i \in \mathbf{R}^{n_i}, \ i = 0, \ 1, \ 2$ are the state vector, $u_i \in \mathbf{R}^{m_i}, \ i = 1, \ 2$ are the control input. All the matrices are constant matrices of appropriate dimensions.

ε_1 and ε_2 are two small positive singular parameters of the same order of magnitude [3] such that

$$0 < k_1 \le \alpha \equiv \frac{\varepsilon_1}{\varepsilon_2} \le k_2 < \infty, \qquad \text{(3a)}$$

$$\bar{\alpha} = \lim_{\substack{\varepsilon_1 \to +0 \\ \varepsilon_2 \to +0}} \alpha. \qquad \text{(3b)}$$

Let us introduce the partitioned matrices

$$A_e = \Phi_e^{-1}A, \ B_{1e} = \Phi_e^{-1}B_1, \ B_{2e} = \Phi_e^{-1}B_2,$$

$$S_{ie} = B_{ie}R_{ii}^{-1}B_{ie}^T = \Phi_e^{-1}S_i\Phi_e^{-1},\ i = 1,\ 2,$$

$$\Phi_e = \begin{bmatrix} I_{n_0} & 0 & 0 \\ 0 & \varepsilon_1 I_{n_1} & 0 \\ 0 & 0 & \varepsilon_2 I_{n_2} \end{bmatrix},$$

$$A = \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & 0 \\ A_{20} & 0 & A_{22} \end{bmatrix},$$

$$B_1 = \begin{bmatrix} B_{01} \\ B_{11} \\ 0 \end{bmatrix},\ B_2 = \begin{bmatrix} B_{02} \\ 0 \\ B_{22} \end{bmatrix},$$

$$S_1 = B_1R_{11}^{-1}B_1^T = \begin{bmatrix} S_{001} & S_{011} & 0 \\ S_{011}^T & S_{111} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$S_2 = B_1R_{22}^{-1}B_2^T = \begin{bmatrix} S_{002} & 0 & S_{022} \\ 0 & 0 & 0 \\ S_{022}^T & 0 & S_{222} \end{bmatrix},$$

$$Q_1 = C_1^TC_1 = \begin{bmatrix} Q_{001} & Q_{011} & 0 \\ Q_{011}^T & Q_{111} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$Q_2 = C_2^TC_2 = \begin{bmatrix} Q_{002} & 0 & Q_{022} \\ 0 & 0 & 0 \\ Q_{022}^T & 0 & Q_{222} \end{bmatrix}.$$

We now consider the linear quadratic Nash games for infinite horizon MSPS (1) under the following basic assumptions [3].

Assumption 1 *There exists an* $\mu^* > 0$ *such that the triplet* $(A_e,\ B_{ie},\ C_i)$, $i = 1,\ 2$ *are stabilizable and detectable for all* $\mu \in (0, \mu^*]$, *where* $\mu := \sqrt{\varepsilon_1\varepsilon_2}$.

Assumption 2 *The triplet* $(A_{ii},\ B_{ii},\ C_{ii})$, $i = 1,\ 2$ *are stabilizable and detectable.*

These conditions are quite natural since at least one control agent has to be able to control and observe unstable modes. The purpose is to find a linear feedback controller $(u_1^*,\ u_2^*)$ such that

$$J_i(u_i^*,\ u_j^*) \le J_i(u_i,\ u_j^*),\quad i,\ j = 1, 2,\quad i \ne j. \tag{4}$$

The Nash inequality shows that u_i^* regulates the state to zero with minimum output energy. The following lemma is already known [1].

Lemma 1 *Under Assumption 1, there exists an admissible controller such that* (4) *hold iff the following full–order CMARE*

$$A_e^TX_e + X_eA_e + Q_1 - X_eS_{1e}X_e$$
$$-X_eS_{2e}Y_e - Y_eS_{2e}X_e = 0, \tag{5a}$$
$$A_e^TY_e + Y_eA_e + Q_2 - Y_eS_{2e}Y_e$$
$$-Y_eS_{1e}X_e - X_eS_{1e}Y_e = 0, \tag{5b}$$

have stabilizing solutions $X_e \ge 0$ *and* $Y_e \ge 0$ *where*

$$X_e = \begin{bmatrix} X_{00} & \varepsilon_1 X_{10}^T & \varepsilon_2 X_{20}^T \\ \varepsilon_1 X_{10} & \varepsilon_1 X_{11} & \sqrt{\varepsilon_1\varepsilon_2}X_{21}^T \\ \varepsilon_2 X_{20} & \sqrt{\varepsilon_1\varepsilon_2}X_{21} & \varepsilon_2 X_{22} \end{bmatrix},$$

$$Y_e = \begin{bmatrix} Y_{00} & \varepsilon_1 Y_{10}^T & \varepsilon_2 Y_{20}^T \\ \varepsilon_1 Y_{10} & \varepsilon_1 Y_{11} & \sqrt{\varepsilon_1\varepsilon_2}Y_{21}^T \\ \varepsilon_2 Y_{20} & \sqrt{\varepsilon_1\varepsilon_2}Y_{21} & \varepsilon_2 Y_{22} \end{bmatrix}.$$

Then, the closed–loop linear Nash equilibrium solutions to the full–order problem are given by

$$u_1^*(t) = -R_{11}^{-1}B_{1e}^TX_ex(t), \tag{6a}$$
$$u_2^*(t) = -R_{22}^{-1}B_{2e}^TY_ex(t). \tag{6b}$$

III. Asymptotic Expansion

To study the property of solutions of the CMARE (5), we introduce the following useful lemma [6].

Lemma 2 *The CMARE* (5) *is equivalent to the following GCMARE* (7), *respectively.*

$$A^TX + X^TA + Q_1 - X^TS_1X$$
$$-X^TS_2Y - Y^TS_2X = 0, \tag{7a}$$
$$A^TY + Y^TA + Q_2 - Y^TS_2Y$$
$$-Y^TS_1X - X^TS_1Y = 0, \tag{7b}$$

where

$$X_e = \Phi_eX = X^T\Phi_e,\ X_{ii} = X_{ii}^T,\ i = 0,\ 1,\ 2,$$
$$X = \begin{bmatrix} X_{00} & \varepsilon_1 X_{10}^T & \varepsilon_2 X_{20}^T \\ X_{10} & X_{11} & \sqrt{\alpha}^{-1}X_{21}^T \\ X_{20} & \sqrt{\alpha}X_{21} & X_{22} \end{bmatrix},$$
$$Y_e = \Phi_eY = Y^T\Phi_e,\ Y_{ii} = Y_{ii}^T,\ i = 0,\ 1,\ 2,$$
$$Y = \begin{bmatrix} Y_{00} & \varepsilon_1 Y_{10}^T & \varepsilon_2 Y_{20}^T \\ Y_{10} & Y_{11} & \sqrt{\alpha}^{-1}Y_{21}^T \\ Y_{20} & \sqrt{\alpha}Y_{21} & Y_{22} \end{bmatrix}.$$

Moreover, we can change the form of the strategies (6) *as follows.*

$$u_1^*(t) = -R_{11}^{-1}B_1^TXx(t), \tag{8a}$$
$$u_2^*(t) = -R_{22}^{-1}B_2^TYx(t). \tag{8b}$$

Proof: Since the proof is similar to the proof of Lemma 3 in [6], it is omitted. ■

After partitioning the GCARE (7), we obtain the reduced–order equations (10) as $\varepsilon_i \to +0$, $i = 1,\ 2$, where $\bar{X}_{lm}$, $\bar{Y}_{lm}$, $lm = 00,\ 10,\ 20,\ 11,\ 21,\ 22$ are the zeroth order solutions.

If Assumption 2 holds, there exist the matrices $\tilde{X}_{11} \ge 0$ and $\tilde{Y}_{22} \ge 0$ such that the matrices $A_{11} - S_{111}\tilde{X}_{11}$ and $A_{22} - S_{222}\tilde{Y}_{22}$ are stable, where $A_{11}^T\tilde{X}_{11} + \tilde{X}_{11}A_{11} - \tilde{X}_{11}S_{111}\tilde{X}_{11} + Q_{11} = 0$ and $A_{22}^T\tilde{X}_{22} + \tilde{X}_{22}A_{22} - \tilde{X}_{22}S_{222}\tilde{X}_{22} + Q_{22} = 0$. Now we chose $\bar{X}_{11}$ and $\bar{Y}_{22}$ to be $\tilde{X}_{11}$ and $\tilde{Y}_{22}$, respectively. Then there exist λ_x and λ_y such that

$$(A_{11} - S_{111}\tilde{X}_{11})v_x = \lambda_xv_x,\ \mathrm{Re}(\lambda_x) < 0, \tag{9a}$$
$$(A_{22} - S_{222}\tilde{Y}_{22})v_y = \lambda_yv_y,\ \mathrm{Re}(\lambda_y) < 0, \tag{9b}$$

where $v_x \in \mathbf{C}^{n_1}$ and $v_y \in \mathbf{C}^{n_2}$ are any vectors.

$$
\begin{aligned}
&A_{00}^T\bar{X}_{00}+\bar{X}_{00}A_{00}+A_{10}^T\bar{X}_{10}+\bar{X}_{10}^TA_{10}+A_{20}^T\bar{X}_{20}+\bar{X}_{20}^TA_{20}-\bar{X}_{00}S_{001}\bar{X}_{00}-\bar{X}_{00}S_{002}\bar{Y}_{00}\\
&\quad-\bar{X}_{10}^TS_{011}^T\bar{X}_{00}-\bar{X}_{00}S_{011}\bar{X}_{10}-\bar{X}_{10}^TS_{111}\bar{X}_{10}-\bar{X}_{20}^TS_{022}^T\bar{Y}_{00}-\bar{X}_{00}S_{022}\bar{Y}_{20}-\bar{X}_{20}^TS_{222}\bar{Y}_{20}\\
&\quad-\bar{Y}_{00}S_{002}\bar{X}_{00}-\bar{Y}_{20}^TS_{022}^T\bar{X}_{00}-\bar{Y}_{00}S_{022}\bar{X}_{20}-\bar{Y}_{20}^TS_{222}\bar{X}_{20}+Q_{001}=0, &\text{(10a)}\\
&\bar{X}_{00}A_{01}+\bar{X}_{10}^TA_{11}+A_{10}^T\bar{X}_{11}+\sqrt{\bar{\alpha}}A_{20}^T\bar{X}_{21}-(\bar{X}_{00}S_{011}\bar{X}_{11}+\bar{X}_{10}^TS_{111}\bar{X}_{11})-\sqrt{\bar{\alpha}}(\bar{X}_{00}S_{022}\bar{Y}_{21}+\bar{X}_{20}^TS_{222}\bar{Y}_{21})\\
&\quad-\sqrt{\bar{\alpha}}(\bar{Y}_{00}S_{022}\bar{X}_{21}+\bar{Y}_{20}^TS_{222}\bar{X}_{21})+Q_{011}=0, &\text{(10b)}\\
&\bar{X}_{00}A_{02}+\bar{X}_{20}^TA_{22}+A_{20}^T\bar{X}_{22}+\frac{1}{\sqrt{\bar{\alpha}}}A_{10}^T\bar{X}_{21}^T-\frac{1}{\sqrt{\bar{\alpha}}}(\bar{X}_{00}S_{011}\bar{X}_{21}^T+\bar{X}_{10}^TS_{111}\bar{X}_{21}^T)-(\bar{X}_{00}S_{022}\bar{Y}_{22}+\bar{X}_{20}^TS_{222}\bar{Y}_{22})\\
&\quad-(\bar{Y}_{00}S_{022}\bar{X}_{22}+\bar{Y}_{20}^TS_{222}\bar{X}_{22})=0, &\text{(10c)}\\
&A_{11}^T\bar{X}_{11}+\bar{X}_{11}A_{11}-\bar{X}_{11}S_{111}\bar{X}_{11}-\bar{\alpha}\bar{X}_{21}^TS_{222}\bar{Y}_{21}-\bar{\alpha}\bar{Y}_{21}^TS_{222}\bar{X}_{21}+Q_{111}=0, &\text{(10d)}\\
&\sqrt{\bar{\alpha}}\bar{X}_{21}^TA_{22}+\frac{1}{\sqrt{\bar{\alpha}}}A_{11}^T\bar{X}_{21}^T-\frac{1}{\sqrt{\bar{\alpha}}}\bar{X}_{11}S_{111}\bar{X}_{21}^T-\sqrt{\bar{\alpha}}\bar{X}_{21}^TS_{222}\bar{Y}_{22}-\sqrt{\bar{\alpha}}\bar{Y}_{21}^TS_{222}\bar{X}_{22}=0, &\text{(10e)}\\
&A_{22}^T\bar{X}_{22}+\bar{X}_{22}A_{22}-\frac{1}{\bar{\alpha}}\bar{X}_{21}S_{111}\bar{X}_{21}^T-\bar{X}_{22}S_{222}\bar{Y}_{22}-\bar{Y}_{22}S_{222}\bar{X}_{22}=0, &\text{(10f)}\\
&A_{00}^T\bar{Y}_{00}+\bar{Y}_{00}A_{00}+A_{10}^T\bar{Y}_{10}+\bar{Y}_{10}^TA_{10}+A_{20}^T\bar{Y}_{20}+\bar{Y}_{20}^TA_{20}-\bar{Y}_{00}S_{002}\bar{Y}_{00}-\bar{Y}_{00}S_{001}\bar{X}_{00}\\
&\quad-\bar{Y}_{20}^TS_{022}^T\bar{Y}_{00}-\bar{Y}_{00}S_{022}\bar{Y}_{20}-\bar{Y}_{20}^TS_{222}\bar{Y}_{20}-\bar{Y}_{10}^TS_{011}^T\bar{X}_{00}-\bar{Y}_{00}S_{011}\bar{X}_{10}-\bar{Y}_{10}^TS_{111}\bar{X}_{10}\\
&\quad-\bar{X}_{00}S_{001}\bar{Y}_{00}-\bar{X}_{10}^TS_{011}^T\bar{Y}_{00}-\bar{X}_{00}S_{011}\bar{Y}_{10}-\bar{X}_{10}^TS_{111}\bar{Y}_{10}+Q_{002}=0, &\text{(10g)}\\
&\bar{Y}_{00}A_{01}+\bar{Y}_{10}^TA_{11}+A_{10}^T\bar{Y}_{11}+\sqrt{\bar{\alpha}}A_{20}^T\bar{Y}_{21}-\sqrt{\bar{\alpha}}(\bar{Y}_{00}S_{022}\bar{Y}_{21}+\bar{Y}_{20}^TS_{222}\bar{Y}_{21})-(\bar{Y}_{00}S_{011}\bar{X}_{11}+\bar{Y}_{10}^TS_{111}\bar{X}_{11})\\
&\quad-(\bar{X}_{00}S_{011}\bar{Y}_{11}+\bar{X}_{10}^TS_{111}\bar{Y}_{11})=0, &\text{(10h)}\\
&\bar{Y}_{00}A_{02}+\bar{Y}_{20}^TA_{22}+A_{20}^T\bar{Y}_{22}+\frac{1}{\sqrt{\bar{\alpha}}}A_{10}^T\bar{Y}_{21}^T-(\bar{Y}_{00}S_{022}\bar{Y}_{22}+\bar{Y}_{20}^TS_{222}\bar{Y}_{22})-\frac{1}{\sqrt{\bar{\alpha}}}(\bar{Y}_{00}S_{011}\bar{X}_{21}^T+\bar{Y}_{10}^TS_{111}\bar{X}_{21}^T)\\
&\quad-\frac{1}{\sqrt{\bar{\alpha}}}(\bar{X}_{00}S_{011}\bar{Y}_{21}^T+\bar{X}_{10}^TS_{111}\bar{Y}_{21}^T)+Q_{022}=0, &\text{(10i)}\\
&A_{11}^T\bar{Y}_{11}+\bar{Y}_{11}A_{11}-\bar{\alpha}\bar{Y}_{21}^TS_{222}\bar{Y}_{21}-\bar{Y}_{11}S_{111}\bar{X}_{11}-\bar{X}_{11}S_{111}\bar{Y}_{11}=0, &\text{(10j)}\\
&\sqrt{\bar{\alpha}}\bar{Y}_{21}^TA_{22}+\frac{1}{\sqrt{\bar{\alpha}}}A_{11}^T\bar{Y}_{21}^T-\sqrt{\bar{\alpha}}\bar{Y}_{21}^TS_{222}\bar{Y}_{22}-\frac{1}{\sqrt{\bar{\alpha}}}\bar{Y}_{11}S_{111}\bar{X}_{21}^T-\frac{1}{\sqrt{\bar{\alpha}}}\bar{X}_{11}S_{111}\bar{Y}_{21}^T=0, &\text{(10k)}\\
&A_{22}^T\bar{Y}_{22}+\bar{Y}_{22}A_{22}-\bar{Y}_{22}S_{222}\bar{Y}_{22}-\frac{1}{\bar{\alpha}}\bar{Y}_{21}S_{111}\bar{X}_{21}^T-\frac{1}{\bar{\alpha}}\bar{X}_{21}S_{111}\bar{Y}_{21}^T+Q_{222}=0. &\text{(10l)}
\end{aligned}
$$

Note that we can change the form (10f) and (10j) as follows

$$
\begin{aligned}
&v_y^T(A_{22}-S_{222}\tilde{Y}_{22})^T\bar{X}_{22}v_y+v_y^T\bar{X}_{22}(A_{22}-S_{222}\tilde{Y}_{22})v_y\\
&\quad-\frac{1}{\bar{\alpha}}v_y^T\bar{X}_{21}S_{111}\bar{X}_{21}^Tv_y=0,\\
&\Leftrightarrow 2\lambda_y v_y^T\bar{X}_{22}v_y-\frac{1}{\bar{\alpha}}v_y^T\bar{X}_{21}S_{111}\bar{X}_{21}^Tv_y=0, &\text{(11a)}\\
&v_x^T(A_{11}-S_{111}\tilde{X}_{11})^T\bar{Y}_{11}v_x+v_x^T\bar{Y}_{11}(A_{11}-S_{111}\tilde{X}_{11})v_x\\
&\quad-\bar{\alpha}v_x^T\bar{Y}_{21}S_{222}\bar{Y}_{21}^Tv_x=0,\\
&\Leftrightarrow 2\lambda_x v_x^T\bar{Y}_{11}v_x-\bar{\alpha}v_x^T\bar{Y}_{21}S_{222}\bar{Y}_{21}^Tv_x=0. &\text{(11b)}
\end{aligned}
$$

Taking $\mathrm{Re}(\lambda_x)<0$ and $\mathrm{Re}(\lambda_y)<0$ into account, we have $\bar{X}_{22}=\bar{Y}_{11}=0$. Then, from (10e) and (10k), the following equalities (12) hold.

$$
\begin{aligned}
&\bar{\alpha}\bar{X}_{21}^T(A_{22}-S_{222}\bar{Y}_{22})+(A_{11}-S_{111}\bar{X}_{11})^T\bar{X}_{21}^T=0, &\text{(12a)}\\
&\bar{\alpha}\bar{Y}_{21}^T(A_{22}-S_{222}\bar{Y}_{22})+(A_{11}-S_{111}\bar{X}_{11})^T\bar{Y}_{21}^T=0. &\text{(12b)}
\end{aligned}
$$

Hence, the unique solutions of (10f) and (10j) are given by $\bar{X}_{21}=\bar{Y}_{21}=0$ because of the stability $A_{11}-S_{111}\bar{X}_{11}$ and $A_{22}-S_{222}\bar{Y}_{22}$. Thus the parameter $\bar{\alpha}$ does not appear in (10), that is, it does not affect the equation (10) in the limit when ε_1 and ε_2 tend to zero. Therefore, we obtain the zeroth order equations (13). Then, the Nash equilibrium strategies for the MSPS will be studied. It is noted that we need the following basic assumption, so that ones can apply the proposed method to the nonstandard MSPS.

Assumption 3 *The Hamiltonian matrices T_{iii}, $i=1,\ 2$ are nonsingular, where*

$$
T_{iii}:=\begin{bmatrix} A_{ii} & -S_{iii}\\ -Q_{iii} & -A_{ii}^T\end{bmatrix}. \tag{14}
$$

Under Assumptions 2 and 3, we obtain the following zeroth order equations

$$
\begin{aligned}
&A_s^T\bar{X}_{00}+\bar{X}_{00}A_s+Q_{s1}\\
&\quad-\bar{X}_{00}S_{s1}\bar{X}_{00}-\bar{X}_{00}S_{s2}\bar{Y}_{00}-\bar{Y}_{00}S_{s2}\bar{X}_{00}=0, &\text{(15a)}\\
&A_s^T\bar{Y}_{00}+\bar{Y}_{00}A_s+Q_{s2}\\
&\quad-\bar{Y}_{00}S_{s2}\bar{Y}_{00}-\bar{Y}_{00}S_{s1}\bar{X}_{00}-\bar{X}_{00}S_{s1}\bar{Y}_{00}=0, &\text{(15b)}\\
&A_{11}^T\bar{X}_{11}+\bar{X}_{11}A_{11}-\bar{X}_{11}S_{111}\bar{X}_{11}+Q_{111}=0, &\text{(15c)}\\
&A_{22}^T\bar{Y}_{22}+\bar{Y}_{22}A_{22}-\bar{Y}_{22}S_{222}\bar{Y}_{22}+Q_{222}=0, &\text{(15d)}\\
&\bar{X}_{10}=-D_{x11}^{-T}D_{x01}^T\bar{X}_{00}-D_{x11}^{-T}N_{x01}^T, &\text{(15e)}\\
&\bar{Y}_{10}=-D_{x11}^{-T}D_{x01}^T\bar{Y}_{00}, &\text{(15f)}\\
&\bar{X}_{20}=-D_{y22}^{-T}D_{y02}^T\bar{X}_{00}, &\text{(15g)}\\
&\bar{Y}_{20}=-D_{y22}^{-T}D_{y02}^T\bar{Y}_{00}-D_{y22}^{-T}N_{y02}^T, &\text{(15h)}
\end{aligned}
$$

$$A_{00}^T\bar{X}_{00}+\bar{X}_{00}A_{00}+A_{10}^T\bar{X}_{10}+\bar{X}_{10}^TA_{10}+A_{20}^T\bar{X}_{20}+\bar{X}_{20}^TA_{20}-\bar{X}_{00}S_{001}\bar{X}_{00}-\bar{X}_{00}S_{002}\bar{Y}_{00}$$
$$-\bar{X}_{10}^TS_{011}^T\bar{X}_{00}-\bar{X}_{00}S_{011}\bar{X}_{10}-\bar{X}_{10}^TS_{111}\bar{X}_{10}-\bar{X}_{20}^TS_{022}^T\bar{Y}_{00}-\bar{X}_{00}S_{022}\bar{Y}_{20}-\bar{X}_{20}^TS_{222}\bar{Y}_{20}$$
$$-\bar{Y}_{00}S_{002}\bar{X}_{00}-\bar{Y}_{20}^TS_{022}^T\bar{X}_{00}-\bar{Y}_{00}S_{022}\bar{X}_{20}-\bar{Y}_{20}^TS_{222}\bar{X}_{20}+Q_{001}=0, \quad (13a)$$
$$\bar{X}_{00}A_{01}+\bar{X}_{10}^TA_{11}+A_{10}^T\bar{X}_{11}-(\bar{X}_{00}S_{011}\bar{X}_{11}+\bar{X}_{10}^TS_{111}\bar{X}_{11})+Q_{011}=0, \quad (13b)$$
$$\bar{X}_{00}A_{02}+\bar{X}_{20}^TA_{22}-(\bar{X}_{00}S_{022}\bar{Y}_{22}+\bar{X}_{20}^TS_{222}\bar{Y}_{22})=0, \quad (13c)$$
$$A_{11}^T\bar{X}_{11}+\bar{X}_{11}A_{11}-\bar{X}_{11}S_{111}\bar{X}_{11}+Q_{111}=0, \quad (13d)$$
$$A_{00}^T\bar{Y}_{00}+\bar{Y}_{00}A_{00}+A_{10}^T\bar{Y}_{10}+\bar{Y}_{10}^TA_{10}+A_{20}^T\bar{Y}_{20}+\bar{Y}_{20}^TA_{20}-\bar{Y}_{00}S_{002}\bar{Y}_{00}-\bar{Y}_{00}S_{001}\bar{X}_{00}$$
$$-\bar{Y}_{20}^TS_{022}^T\bar{Y}_{00}-\bar{Y}_{00}S_{022}\bar{Y}_{20}-\bar{Y}_{20}^TS_{222}\bar{Y}_{20}-\bar{Y}_{10}^TS_{011}^T\bar{X}_{00}-\bar{Y}_{00}S_{011}\bar{X}_{10}-\bar{Y}_{10}^TS_{111}\bar{X}_{10}$$
$$-\bar{X}_{00}S_{001}\bar{Y}_{00}-\bar{X}_{10}^TS_{011}^T\bar{Y}_{00}-\bar{X}_{00}S_{011}\bar{Y}_{10}-\bar{X}_{10}^TS_{111}\bar{Y}_{10}+Q_{002}=0, \quad (13e)$$
$$\bar{Y}_{00}A_{01}+\bar{Y}_{10}^TA_{11}-(\bar{Y}_{00}S_{011}\bar{X}_{11}+\bar{Y}_{10}^TS_{111}\bar{X}_{11})=0, \quad (13f)$$
$$\bar{Y}_{00}A_{02}+\bar{Y}_{20}^TA_{22}+A_{20}^T\bar{Y}_{22}-(\bar{Y}_{00}S_{022}\bar{Y}_{22}+\bar{Y}_{20}^TS_{222}\bar{Y}_{22}^T)+Q_{022}=0, \quad (13g)$$
$$A_{22}^T\bar{Y}_{22}+\bar{Y}_{22}A_{22}-\bar{Y}_{22}S_{222}\bar{Y}_{22}+Q_{222}=0. \quad (13h)$$

where

$$\begin{aligned}
A_s &= A_{00}-D_{x01}D_{x11}^{-1}A_{10}-D_{y02}D_{y22}^{-1}A_{20}\\
&\quad+(S_{011}-D_{x01}D_{x11}^{-1}S_{111})D_{x11}^{-T}N_{x01}^T\\
&\quad+(S_{022}-D_{y02}D_{y22}^{-1}S_{222})D_{y22}^{-T}N_{y02}^T,\\
S_{s1} &= S_{001}-D_{x01}D_{x11}^{-1}S_{011}^T-S_{011}D_{x11}^{-T}D_{x01}^T\\
&\quad+D_{x01}D_{x11}^{-1}S_{111}D_{x11}^{-T}D_{x01}^T,\\
S_{s2} &= S_{002}-D_{y02}D_{y22}^{-1}S_{022}^T-S_{022}D_{y22}^{-T}D_{y02}^T\\
&\quad+D_{y02}D_{y22}^{-1}S_{222}D_{y22}^{-T}D_{y02}^T,\\
Q_{s1} &= Q_{001}-A_{10}^TD_{x11}^{-T}N_{x01}^T-N_{x01}D_{x11}^{-1}A_{10}\\
&\quad-N_{x01}D_{x11}^{-1}S_{111}D_{x11}^{-T}N_{x01}^T,\\
Q_{s2} &= Q_{002}-A_{20}^TD_{y22}^{-T}N_{y02}^T-N_{y02}D_{y22}^{-1}A_{20}\\
&\quad-N_{y02}D_{y22}^{-1}S_{222}D_{y22}^{-T}N_{y02}^T,\\
D_{x01} &= A_{01}-S_{011}\bar{X}_{11},\ D_{x11}=A_{11}-S_{111}\bar{X}_{11},\\
D_{y02} &= A_{02}-S_{022}\bar{Y}_{22},\ D_{y22}=A_{22}-S_{222}\bar{Y}_{22},\\
N_{x01} &= A_{10}^T\bar{X}_{11}+Q_{011},\ N_{y02}=A_{20}^T\bar{Y}_{22}+Q_{022}.
\end{aligned}$$

The following theorem gives how to calculate the coefficient matrices of the reduced–order equations (15a) and (15b).

Theorem 1 *The matrices A_s, S_{s1}, S_{s2}, Q_{s1} and Q_{s2} do not depend on $\bar{X}_{11}$, $\bar{X}_{21}$, $\bar{Y}_{21}$ and $\bar{Y}_{22}$, that is, their matrices can be computed by using the following Hamiltonian matrices.*

$$\begin{bmatrix} A_s & * \\ * & -A_s^T \end{bmatrix}=\begin{bmatrix} A_{00} & * \\ * & -A_{00}^T \end{bmatrix}-T_{011}T_{111}^{-1}T_{101}-T_{022}T_{222}^{-1}T_{202}, \quad (16a)$$
$$\begin{bmatrix} * & -S_{s1} \\ -Q_{s1} & * \end{bmatrix}=T_{001}-T_{011}T_{111}^{-1}T_{101}, \quad (16b)$$
$$\begin{bmatrix} * & -S_{s2} \\ -Q_{s2} & * \end{bmatrix}=T_{002}-T_{022}T_{222}^{-1}T_{202}, \quad (16c)$$
$$T_{001}=\begin{bmatrix} A_{00} & -S_{001} \\ -Q_{001} & -A_{00}^T \end{bmatrix},\ T_{011}=\begin{bmatrix} A_{01} & -S_{011} \\ -Q_{011} & -A_{10}^T \end{bmatrix},$$
$$T_{101}=\begin{bmatrix} A_{10} & -S_{011}^T \\ -Q_{011}^T & -A_{01}^T \end{bmatrix},\ T_{111}=\begin{bmatrix} A_{11} & -S_{111} \\ -Q_{111} & -A_{11}^T \end{bmatrix},$$
$$T_{002}=\begin{bmatrix} A_{00} & -S_{002} \\ -Q_{002} & -A_{00}^T \end{bmatrix},\ T_{022}=\begin{bmatrix} A_{02} & -S_{022} \\ -Q_{022} & -A_{20}^T \end{bmatrix},$$
$$T_{202}=\begin{bmatrix} A_{20} & -S_{022}^T \\ -Q_{022}^T & -A_{02}^T \end{bmatrix},\ T_{222}=\begin{bmatrix} A_{22} & -S_{222} \\ -Q_{222} & -A_{22}^T \end{bmatrix}.$$

where $*$ stands for a appropriate matrix. Moreover, we can change the form of the solutions $\bar{X}_{10}$, $\bar{X}_{20}$, $\bar{Y}_{10}$ and $\bar{Y}_{20}$.

$$\begin{bmatrix} \bar{X}_{10} \\ \bar{Y}_{10} \end{bmatrix}^T=\begin{bmatrix} \bar{X}_{11} \\ -I_{n_1} \end{bmatrix}^T T_{111}^{-1}T_{101}\begin{bmatrix} I_{n_0} & 0 \\ \bar{X}_{00} & \bar{Y}_{00} \end{bmatrix}, \quad (17a)$$
$$\begin{bmatrix} \bar{X}_{20} \\ \bar{Y}_{20} \end{bmatrix}^T=\begin{bmatrix} \bar{Y}_{22} \\ -I_{n_2} \end{bmatrix}^T T_{222}^{-1}T_{202}\begin{bmatrix} 0 & I_{n_0} \\ \bar{X}_{00} & \bar{Y}_{00} \end{bmatrix}. \quad (17b)$$

Proof: Note the relation

$$T_{111}=\begin{bmatrix} I_{n_1} & 0 \\ \bar{X}_{11} & I_{n_1} \end{bmatrix}\begin{bmatrix} D_{x11} & -S_{111} \\ 0 & -D_{x11}^T \end{bmatrix}\begin{bmatrix} I_{n_1} & 0 \\ -\bar{X}_{11} & I_{n_1} \end{bmatrix}.$$

Since T_{111} is nonsingular under Assumption 3 and the algebraic Riccati equation (ARE) (15c) has a stabilizing solution under Assumption 2, D_{x11} is also nonsingular. This means that T_{111}^{-1} can be expressed explicitly in terms of D_{x11}^{-1}. Using the similar manner, we have the following relations.

$$T_{111}^{-1}=\begin{bmatrix} I_{n_1} & 0 \\ \bar{X}_{11} & I_{n_1} \end{bmatrix}\begin{bmatrix} D_{x11}^{-1} & -D_{x11}^{-1}S_{111}D_{x11}^{-T} \\ 0 & -D_{x11}^{-T} \end{bmatrix}\cdot\begin{bmatrix} I_{n_1} & 0 \\ -\bar{X}_{11} & I_{n_1} \end{bmatrix}, \quad (18a)$$
$$T_{222}^{-1}=\begin{bmatrix} I_{n_2} & 0 \\ \bar{Y}_{22} & I_{n_2} \end{bmatrix}\begin{bmatrix} D_{y22}^{-1} & -D_{y22}^{-1}S_{222}D_{y22}^{-T} \\ 0 & -D_{y22}^{-T} \end{bmatrix}\cdot\begin{bmatrix} I_{n_2} & 0 \\ -\bar{Y}_{22} & I_{n_2} \end{bmatrix}. \quad (18b)$$

$$J = \frac{\partial \mathrm{vec}(f_{x1},\ f_{x2},\ f_{x3},\ f_{x4},\ f_{x5},\ f_{x6},\ f_{y1},\ f_{y2},\ f_{y3},\ f_{y4},\ f_{y5},\ f_{y6})}{\partial \mathrm{vec}(X_{00},\ X_{10},\ X_{20},\ X_{11},\ X_{21},\ X_{22},\ Y_{00},\ Y_{10},\ Y_{20},\ Y_{11},\ Y_{21},\ Y_{22})^T}\Big|_{\mu=\mu_0,\ \mathcal{P}=\mathcal{P}_0}$$

$$= \begin{bmatrix} J_{00} & J_{01} & J_{02} & 0 & 0 & 0 & J_{06} & 0 & J_{08} & 0 & 0 & 0 \\ J_{10} & J_{11} & 0 & J_{13} & J_{14} & 0 & 0 & 0 & 0 & 0 & J_{110} & 0 \\ J_{20} & 0 & J_{22} & 0 & J_{24} & J_{25} & 0 & 0 & 0 & 0 & 0 & J_{211} \\ 0 & 0 & 0 & J_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & J_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & J_{55} & 0 & 0 & 0 & 0 & 0 & 0 \\ J_{60} & J_{61} & 0 & 0 & 0 & 0 & J_{00} & J_{01} & J_{02} & 0 & 0 & 0 \\ 0 & 0 & 0 & J_{73} & 0 & 0 & J_{10} & J_{11} & 0 & J_{13} & J_{14} & 0 \\ 0 & 0 & 0 & 0 & J_{84} & 0 & J_{20} & 0 & J_{22} & 0 & J_{24} & J_{25} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_{55} \end{bmatrix}, \quad (21)$$

$$\mathcal{P} = (X_{00},\ X_{10},\ X_{20},\ X_{11},\ X_{21},\ X_{22},\ Y_{00},\ Y_{10},\ Y_{20},\ Y_{11},\ Y_{21},\ Y_{22}),$$

$$\mathcal{P}_0 = (\bar{X}_{00},\ \bar{X}_{10},\ \bar{X}_{20},\ \bar{X}_{11},\ 0,\ 0,\ \bar{Y}_{00},\ \bar{Y}_{10},\ \bar{Y}_{20},\ 0,\ 0,\ \bar{Y}_{22}),\ \mu = \begin{bmatrix} \varepsilon_1 & \varepsilon_2 \end{bmatrix}^T,\ \mu_0 = \begin{bmatrix} 0 & 0 \end{bmatrix}^T,$$

$$A^TX + X^TA + Q_1 - X^TS_1X - X^TS_2Y - Y^TS_2X = \begin{bmatrix} f_{x1} & f_{x2} & f_{x3} \\ f_{x2}^T & f_{x4} & f_{x5} \\ f_{x3}^T & f_{x5}^T & f_{x6} \end{bmatrix},$$

$$A^TY + Y^TA + Q_2 - Y^TS_2Y - Y^TS_1X - X^TS_1Y = \begin{bmatrix} f_{y1} & f_{y2} & f_{y3} \\ f_{y2}^T & f_{y4} & f_{y5} \\ f_{y3}^T & f_{y5}^T & f_{y6} \end{bmatrix}.$$

Therefore, it suffices the proof of Lemma 3 to show that the relations (16) hold. These formulations can be proved after direct algebraic manipulations, which are omitted here for brevity. ■

It should be noted that the results of Theorem 1 have never been obtained compared with the existing results [3, 6]. The following theorem will establish the relation between the solutions X and Y and the solutions $\bar{X}_{lm}$ and $\bar{Y}_{lm}$ for the reduced–order equations (15).

Theorem 2 *Suppose that*

$$\det \begin{bmatrix} \hat{A}_s^T \otimes I_{n_0} + I_{n_0} \otimes \hat{A}_s & -[(S_{s_s}\bar{X}_{00})^T \otimes I_{n_0} + I_{n_0} \otimes (S_{s_2}\bar{X}_{00})^T] \\ -[(S_{s_1}\bar{Y}_{00})^T \otimes I_{n_0} + I_{n_0} \otimes (S_{s_1}\bar{Y}_{00})^T] & \hat{A}_s^T \otimes I_{n_0} + I_{n_0} \otimes \hat{A}_s. \end{bmatrix} \neq 0, \quad (19)$$

where $\hat{A}_s := A_s - S_{s1}\bar{X}_{00} - S_{s2}\bar{Y}_{00}$.

Under Assumptions 1–3, the GCMARE (7) admits the stabilizing solutions X and Y such that these matrices possess a power series expansion at $\|\mu\| = 0$. That is,

$$X = \begin{bmatrix} \bar{X}_{00} & 0 & 0 \\ \bar{X}_{10} & \bar{X}_{11} & 0 \\ \bar{X}_{20} & 0 & 0 \end{bmatrix} + O(\|\mu\|), \quad (20a)$$

$$Y = \begin{bmatrix} \bar{Y}_{00} & 0 & 0 \\ \bar{Y}_{10} & 0 & 0 \\ \bar{Y}_{20} & 0 & \bar{Y}_{22} \end{bmatrix} + O(\|\mu\|). \quad (20b)$$

Proof: We apply the implicit function theorem [5] to the GCMARE (7). To do so, it is enough to show that the corresponding Jacobian is nonsingular at $\|\mu\| = 0$. It can be shown, after some algebra, that the Jacobian of GCMARE (7) in the limit as $\mu \to \mu_0$ is given by (21).

Note that

$$J_{00} = I_{n_0} \otimes D_{00}^T + D_{00}^T \otimes I_{n_0},$$
$$J_{01} = I_{n_0} \otimes D_{x10}^T + (D_{x10}^T \otimes I_{n_0})U_{n_1n_0},$$
$$J_{02} = I_{n_0} \otimes D_{y20}^T + (D_{y20}^T \otimes I_{n_0})U_{n_2n_0},$$
$$J_{10} = D_{x01}^T \otimes I_{n_0},\ J_{20} = D_{y02}^T \otimes I_{n_0},$$
$$J_{11} = (D_{x11}^T \otimes I_{n_0})U_{n_1n_0},\ J_{22} = (D_{y22}^T \otimes I_{n_0})U_{n_2n_0},$$
$$J_{13} = I_{n_1} \otimes D_{x10}^T,\ J_{14} = \sqrt{\bar{\alpha}}(I_{n_1} \otimes D_{y20}^T),$$
$$J_{24} = \sqrt{\bar{\alpha}}^{-1}(I_{n_2} \otimes D_{x10}^T)U_{n_2n_1},\ J_{25} = I_{n_2} \otimes D_{y20}^T,$$
$$J_{33} = I_{n_1} \otimes D_{x11}^T + D_{x11}^T \otimes I_{n_1},$$
$$J_{44} = \sqrt{\bar{\alpha}}(D_{y22}^T \otimes I_{n_1})U_{n_2n_1} + \sqrt{\bar{\alpha}}^{-1}(I_{n_2} \otimes D_{x11}^T)U_{n_2n_1},$$
$$J_{55} = I_{n_2} \otimes D_{y22}^T + D_{y22}^T \otimes I_{n_2},$$
$$J_{06} = -I_{n_0} \otimes E_{x00}^T - E_{x00}^T \otimes I_{n_0},$$
$$J_{08} = -I_{n_0} \otimes E_{x20}^T - (E_{x20}^T \otimes I_{n_0})U_{n_2n_0},$$
$$J_{110} = -\sqrt{\bar{\alpha}}(I_{n_1} \otimes E_{x20}),\ J_{211} = -I_{n_2} \otimes E_{x20},$$
$$J_{60} = -I_{n_0} \otimes E_{y00}^T - E_{y00}^T \otimes I_{n_0},$$
$$J_{61} = -I_{n_0} \otimes E_{y10}^T - (E_{y10}^T \otimes I_{n_0})U_{n_1n_0},$$
$$J_{73} = -I_{n_1} \otimes E_{y10}^T,\ J_{84} = -\sqrt{\bar{\alpha}}^{-1}(I_{n_2} \otimes E_{y10}^T)U_{n_2n_1},$$
$$D_{00} = A_{00} - S_{001}\bar{X}_{00} - S_{002}\bar{Y}_{00} - S_{011}\bar{X}_{10} - S_{022}\bar{Y}_{20},$$
$$D_{x10} = A_{10} - S_{011}^T\bar{X}_{00} - S_{111}\bar{X}_{10},$$
$$D_{y20} = A_{20} - S_{022}^T\bar{Y}_{00} - S_{222}\bar{Y}_{20},$$
$$E_{x00} = S_{002}\bar{X}_{00} + S_{022}\bar{X}_{20},\ E_{x20} = S_{022}^T\bar{X}_{00} + S_{222}\bar{X}_{20},$$

$$E_{y00} = S_{001}\bar{Y}_{00} + S_{011}\bar{Y}_{10},\ E_{y10} = S_{011}^T\bar{Y}_{00} + S_{111}\bar{Y}_{10}.$$

The Jacobian (21) can be expressed as

$$\begin{aligned} &\det J \\ &= \left[\prod_{j=1}^{5}(\det J_{jj})^2\right] \\ &\quad \cdot\det\begin{bmatrix} J_s & J_{06} - J_{08}J_{22}^{-1}J_{20} \\ J_{60} - J_{61}J_{11}^{-1}J_{10} & J_s \end{bmatrix}. \end{aligned}$$

where $J_s = J_{00} - J_{01}J_{11}^{-1}J_{10} - J_{02}J_{22}^{-1}J_{20}$.

After some straightforward but tedious algebra, it is easy to show that the following relations hold.

$$J_s = \hat{A}_s^T \otimes I_{n_0} + I_{n_0} \otimes \hat{A}_s, \tag{22a}$$

$$\begin{aligned} &J_{06} - J_{08}J_{22}^{-1}J_{20} \\ &= -[(S_{s_1}\bar{Y}_{00})^T \otimes I_{n_0} + I_{n_0} \otimes (S_{s_1}\bar{Y}_{00})^T], \end{aligned} \tag{22b}$$

$$\begin{aligned} &J_{60} - J_{61}J_{11}^{-1}J_{10} \\ &= -[(S_{s_s}\bar{X}_{00})^T \otimes I_{n_0} + I_{n_0} \otimes (S_{s_2}\bar{X}_{00})^T], \end{aligned} \tag{22c}$$

where

$$\begin{aligned} \hat{A}_s &= D_{00} - D_{x01}D_{x11}^{-1}D_{x10} - D_{y02}D_{y22}^{-1}D_{y20}, \\ S_{s1}\bar{Y}_{00} &= E_{y00} - D_{x01}D_{x11}^{-1}E_{y10}, \\ S_{s2}\bar{X}_{00} &= E_{x00} - D_{y02}D_{y22}^{-1}E_{x20}. \end{aligned}$$

Hense, we have $\det J = \left[\prod_{j=1}^{5}(\det J_{jj})^2\right] \cdot \det\Gamma$. Obviously, $J_{jj},\ j = 1, \cdots, 5$ are nonsingular because the matrices $D_{x11} = A_{11} - S_{111}\bar{X}_{11}$ and $D_{y22} = A_{22} - S_{222}\bar{Y}_{22}$ are stable under Assumption 2. Moreover, the nonsingularlity assumption of the matrix Γ is made. Thus, $\det J \neq 0$, i.e., J is nonsingular at $(\mu,\ \mathbf{P}) = (\mu_0,\ \mathbf{P}_0)$. The conclusion of Theorem 1 is obtained directly by using the implicit function theorem. ■

IV. Conclusions

The linear quadratic Nash games for infinite horizon MSPS have been considered. The main contribution is that the boundedness and the asymptotic expansions for the solutions of the GCMARE have been newly investigated.

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