New Algorithm for H_2 Guaranteed Cost Control of Singularly Perturbed Uncertain Systems and Its Application to the Manufacturing Assembly Process

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Abstract

In this paper, the H_2 guaranteed cost control problem for a singularly perturbed norm–bounded uncertain system is addressed by using the improved recursive algorithm. First we derive sufficient conditions such that full–order algebraic Riccati equation has positive definite solution. After defining the generalized algebraic Riccati equation, we propose a new recursive algorithm based on the Kleinman algorithm with the very special kind of the initial condition. The proposed algorithm is very efficient from the numerical point of view since the new recursive algorithm has property of quadratic convergence. Furthermore, we apply the new algorithm to the manufacturing assembly process and show the validity of the full–order controller proposed in this paper.

1 Introduction

Recently, new results in H_2 guaranteed cost control problem of singularly perturbed systems were obtained in [5]. These results are based on the singular perturbation methods [12]. The resulting controller is $O(\varepsilon)$–close to those of full–order controller and achieves the performance for the full–order system for small enough ε. However, for the value ε that are too small, it is usually difficult to calculate the H_2 norm of the transfer matrix function due to numerical stiffness [12].

The recursive algorithm for various control problems of the singularly perturbed systems have been developed in literatures (see [11] and [3]). It has been shown that the recursive algorithm is very effective to solve the algebraic Riccati equations when the system matrices are functions of a small perturbation parameter ε. However, when the recursive approach is applied for the control problems of the singularly perturbed systems, we note that using the zero–order solution without high–order accuracy will fail to produce the desired exact solution of the algebraic Riccati equation. In this case, the recursive algorithm converge to the approximation solution.

Motivated by the results of [11], [3] and [8], we improve the classical recursive numerical technique for the solution of algebraic Riccati equation associated with the H_2 guaranteed cost control problem of singularly perturbed systems. Our new idea is to set the initial condition to the solutions of the reduced–order algebraic Riccati equation. The improved recursive algorithm is based on the Kleinman algorithm. Therefore, while the classical recursive algorithm is the linear convergence property, the new recursive algorithm achieves the quadratic convergence property. By using the proposed recursive algorithm, we show that the solution of the algebraic Riccati equation converges very fast to the exact solution. Furthermore, we apply the new algorithm to the manufacturing assembly process. Shi *et al.* [2] considered the problem of robust disturbance attenuation with stability for a class on uncertain singular perturbed systems under the composite controller design, while we focus on the design of the H_2 guaranteed cost control problem of the uncertain singular perturbed systems based on the proposed direct approach. As a result, since we need not separate the design into slow and fast designs, it is easy to construct the high–order stabilizing controller. As another significant improvement, some assumptions made in [1, 2, 5] are not necessary in this paper. Firstly, we claim that there exist the uncertainties in all of the state and the output matrices compared with Shi *et al.*[2]. Secondly, since we do not assume the orthogonal condition, our new results are applicable to the more realistic systems in comparison with Petersen *et al.*[1]. Thirdly, since we do not assume that A_{22} is nonsingular, our new results are applicable to both standard and nonstandard singularly perturbed systems.

1.1 Problem Statement

Consider the following linear singularly perturbed uncertain system

$$\dot{x}_1(t) = (A_{11} + D_1 F E_{a1}) x_1(t)$$

$$
\begin{aligned}
&\quad +(A_{12}+D_1FE_{a2})x_2(t) \\
&\quad +G_1w(t)+(B_1+D_1FE_b)u(t), \quad (1a)\\
\varepsilon\dot{x}_2(t) &= (A_{21}+D_2FE_{a1})x_1(t) \\
&\quad +(A_{22}+D_2FE_{a2})x_2(t) \\
&\quad +G_2w(t)+(B_2+D_2FE_b)u(t), \quad (1b)\\
z(t) &= C_{11}x_1(t)+C_{12}x_2(t)+D_{12}u(t), \quad (1c)\\
F^TF &\le I_j \quad (1d)
\end{aligned}
$$

where ε is a small positive parameter, $x_1(t) \in \mathbf{R}^{n_1}$ and $x_2(t) \in \mathbf{R}^{n_2}$ are state vectors, $u(t) \in \mathbf{R}^m$ is the control input, $z(t) \in \mathbf{R}^r$ is the controlled output, $w(t) \in \mathbf{R}^q$ is the disturbance, $F \in \mathbf{R}^{k\times j}$ is a Lebesgue measurable matrix of uncertain parameters. All matrices above are of appropriate dimensions. Let us introduce the partitioned matrices

$$
\begin{aligned}
A_\varepsilon &= \begin{bmatrix} A_{11} & A_{12} \\ \varepsilon^{-1}A_{21} & \varepsilon^{-1}A_{22} \end{bmatrix}, \\
A &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \\
B_\varepsilon &= \begin{bmatrix} B_1 \\ \varepsilon^{-1}B_2 \end{bmatrix},\ B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \\
G_\varepsilon &= \begin{bmatrix} G_1 \\ \varepsilon^{-1}G_2 \end{bmatrix},\ G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \\
C_1 &= \begin{bmatrix} C_{11} & C_{12} \end{bmatrix}, \\
D_\varepsilon &= \begin{bmatrix} D_1 \\ \varepsilon^{-1}D_2 \end{bmatrix},\ D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}, \\
E_a &= \begin{bmatrix} E_{a1} & E_{a2} \end{bmatrix}, \\
x(t) &= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in \mathbf{R}^n,\ n = n_1+n_2.
\end{aligned}
$$

By using above relations, the system (1) can be changed as

$$
\begin{aligned}
\dot{x}(t) &= (A_\varepsilon + D_\varepsilon FE_a)x(t) + G_\varepsilon w(t) \\
&\quad +(B_\varepsilon + D_\varepsilon FE_b)u(t), \quad (2a)\\
z(t) &= C_1x(t) + D_{12}u(t). \quad (2b)
\end{aligned}
$$

For technical simplification, without loss of generality we shall make the following basic assumption.

Assumption 1 *1) The pair $(A_\varepsilon,\ B_\varepsilon)$ is stabilizable for $\varepsilon \in (0,\ \varepsilon^*]$ $(\varepsilon^* > 0)$.*
2) The pair $(A_{22},\ B_2)$ is stabilizable.
3) $D_{12}^TD_{12} > 0$.

Note that since we do not assume the orthogonal condition, that is, $C_1^TD_{12} = 0$, our new results are applicable to the more realistic systems in comparison with [1].

Our first problem is to find a state feedback $u(t) = Kx(t)$ such that the closed–loop system is asymptotically stable for all F. A way to address this problem is to use the quadratic stabilizability concept [1].

Improving the results of Petersen *et al.* [1], we can be stated in the following new theorem.

Theorem 1 *Under Assumptions 1, we associate the algebraic Riccati equation*

$$
\begin{aligned}
&[A_\varepsilon - B_\varepsilon\bar{R}E_b^TE_a]^TP_\varepsilon + P_\varepsilon[A_\varepsilon - B_\varepsilon\bar{R}E_b^TE_a] \\
&+\mu P_\varepsilon D_\varepsilon D_\varepsilon^TP_\varepsilon \\
&-\mu(P_\varepsilon B_\varepsilon + C_1^TD_{12})\bar{R}(B_\varepsilon^TP_\varepsilon + D_{12}^TC_1) \\
&+Q + \frac{1}{\mu}E_a^T[I_j - E_b\bar{R}E_b^T]E_a + C_1^TC_1 \\
&- C_1^TD_{12}\bar{R}E_b^TE_a - E_a^TE_b\bar{R}D_{12}^TC_1 = 0 \quad (3)
\end{aligned}
$$

for the matrix function

$$
P_\varepsilon = P_\varepsilon(\mu) = \begin{bmatrix} P_{11}(\varepsilon,\ \mu) & \varepsilon P_{21}(\varepsilon,\ \mu)^T \\ \varepsilon P_{21}(\varepsilon,\ \mu) & \varepsilon P_{22}(\varepsilon,\ \mu) \end{bmatrix}
$$

where μ is a positive scalar, $Q > 0$ is a positive definite symmetric matrix, $\bar{R} = (\mu D_{12}^TD_{12} + E_b^TE_b)^{-1}$. For each ε, a controller that guarantees the quadratically stable for all F : $F^TF \le I_j$ exists if and only if there exist $\mu > 0$ and (3) has a positive definite solution. If such conditions are met, a controller is determined by the formula

$$
\begin{aligned}
u(t) &= -\bar{R}[\mu(B_\varepsilon^TP_\varepsilon(\mu) + D_{12}^TC_1) \\
&\quad +E_b^TE_a]x(t). \quad (4)
\end{aligned}
$$

proof: Since the proof of Theorem 1 proceeds by similar argument of the references [1], it is omitted. □

For such a controller, taking $K(\mu) = -\bar{R}[\mu(B_\varepsilon^TP_\varepsilon(\mu) + D_{12}^TC_1) + E_b^TE_a]$ and letting $H = C_1 + D_{12}K$, the transfer matrix from $w(t)$ to $z(t)$ is expressed by

$$
\begin{aligned}
T(s) &= H(sI_n - A_\varepsilon - D_\varepsilon F\{E_a + E_bK(\mu)\} \\
&\quad -B_\varepsilon K(\mu))^{-1}G_\varepsilon. \quad (5)
\end{aligned}
$$

Then the H_2 guaranteed cost control problem for singularly perturbed uncertain systems is given below.
Find $K(\mu) = K(\mu^*)$ and determine β as small as possible such that

$$
\|T(s)\|_2 < \beta \quad (6)
$$

where

$$
\begin{aligned}
&\|T(s)\|_2^2 = \mathrm{Trace}[G_\varepsilon^TL_o(F)G_\varepsilon], \\
&[A_\varepsilon + D_\varepsilon F(E_a + E_bK) + B_\varepsilon K]^TL_o(F) \\
&\quad +L_o(F)[A_\varepsilon + D_\varepsilon F(E_a + E_bK) + B_\varepsilon K] \\
&\quad +H^TH = 0.
\end{aligned}
$$

By using a similar technique in [5], we can easily prove the following result.

Theorem 2 *Under Assumption 1, we have*

$$
L_o(F) < P_\varepsilon. \quad (7)
$$

proof: The proof is omitted because it is similar to the references [1] and [9]. □

Consequently, the best H_2 guaranteed cost β is given by

$$\beta = \min_{\mu} \sqrt{\mathrm{Trace}[G_\varepsilon^T P_\varepsilon(\mu) G_\varepsilon]}, \tag{8a}$$

$$\mu^* = \mathrm{Arg}\min_{\mu} \sqrt{\mathrm{Trace}[G_\varepsilon^T P_\varepsilon(\mu) G_\varepsilon]}. \tag{8b}$$

Moreover, the controller is defined by $K(\mu) = K(\mu^*) = -(\mu^* D_{12}^T D_{12} + E_b^T E_b)^{-1}[\mu(B_\varepsilon^T P_\varepsilon(\mu^*) + D_{12}^T C_1) + E_b^T E_a]$.

1.2 Generalized Algebraic Riccati Equation

In order to solve the algebraic Riccati equation (3), we introduce the following useful lemma [8].

Lemma 1 *The algebraic Riccati equation (3) is equivalent to the following generalized algebraic Riccati equation (9)*

$$\begin{aligned}
&[A - B\bar{R}E_b^T E_a]^T P + P^T[A - B\bar{R}E_b^T E_a] \\
&\quad +\mu P^T DD^T P \\
&\quad -\mu(P^T B + C_1^T D_{12})\bar{R}(B^T P + D_{12}^T C_1) \\
&\quad +Q + \frac{1}{\mu}E_a^T[I_j - E_b\bar{R}E_b^T]E_a + C_1^T C_1 \\
&\quad - C_1^T D_{12}\bar{R}E_b^T E_a - E_a^T E_b \bar{R} D_{12}^T C_1 = 0,
\end{aligned} \tag{9a}$$

$$P_\varepsilon = \Pi_\varepsilon^T P = P^T \Pi_\varepsilon, \tag{9b}$$

where

$$\Pi_\varepsilon = \begin{bmatrix} I_1 & 0 \\ 0 & \varepsilon I_2 \end{bmatrix}, \; P = \begin{bmatrix} P_{11} & \varepsilon P_{21}^T \\ P_{21} & P_{22} \end{bmatrix},$$
$$P_{11} = P_{11}^T, \; P_{22} = P_{22}^T,$$
$$A = \Pi_\varepsilon A_\varepsilon, \; B = \Pi_\varepsilon B_\varepsilon, \; D = \Pi_\varepsilon D_\varepsilon.$$

Moreover, by making use of relation (9b), we can change the form of the controller (4).

$$u(t) = -\bar{R}[\mu(B^T P(\mu) + D_{12}^T C_1) + E_b^T E_a]x(t). \tag{9c}$$

proof: The proof is identical to the proof of Lemma 3 in Mukaidani *et al.* [8]. □

1.3 Solvability Condition

In this section, the linear state feedback full–order controller for singularly perturbed systems with structured uncertainties is presented.

The algebraic Riccati equation (9a) can be partitioned into

$$\begin{aligned}
&A_{11}^{\mu T} P_{11} + P_{11}^T A_{11}^\mu + A_{21}^{\mu T} P_{21} + P_{21}^T A_{21}^\mu \\
&\quad -P_{11}^T S_{11}^\mu P_{11} - P_{21}^T S_{22}^\mu P_{21} \\
&\quad -P_{11}^T S_{12}^\mu P_{21} - P_{21}^T S_{12}^{\mu T} P_{11} + Q_{11}^\mu = 0,
\end{aligned} \tag{10a}$$

$$\begin{aligned}
&\varepsilon P_{21} A_{11}^\mu + P_{22}^T A_{21}^\mu + A_{12}^{\mu T} P_{11} + A_{22}^{\mu T} P_{21} \\
&\quad -\varepsilon P_{21} S_{11}^\mu P_{11} - \varepsilon P_{21} S_{12}^\mu P_{21} \\
&\quad -P_{22}^T S_{12}^{\mu T} P_{11} - P_{22}^T S_{22}^\mu P_{21} + Q_{12}^{\mu T} = 0,
\end{aligned} \tag{10b}$$

$$\begin{aligned}
&A_{22}^{\mu T} P_{22} + P_{22}^T A_{22}^\mu + \varepsilon A_{12}^{\mu T} P_{21}^T + \varepsilon P_{21} A_{12}^\mu \\
&\quad -P_{22}^T S_{22}^\mu P_{22} - \varepsilon P_{22}^T S_{12}^{\mu T} P_{21}^T - \varepsilon P_{21} S_{12}^\mu P_{22} \\
&\quad -\varepsilon^2 P_{21} S_{11}^\mu P_{21}^T + Q_{22}^\mu = 0,
\end{aligned} \tag{10c}$$

where

$$\begin{aligned}
A^\mu &= A - B\bar{R}E_b^T E_a - \mu B\bar{R}D_{12}^T C_1 \\
&= \begin{bmatrix} A_{11}^\mu & A_{12}^\mu \\ A_{21}^\mu & A_{22}^\mu \end{bmatrix},
\end{aligned}$$

$$S^\mu = \mu(B\bar{R}B^T - DD^T) = \begin{bmatrix} S_{11}^\mu & S_{12}^\mu \\ S_{12}^{\mu T} & S_{22}^\mu \end{bmatrix},$$

$$\begin{aligned}
Q^\mu &= Q + \frac{1}{\mu}E_a^T[I_j - E_b\bar{R}E_b^T]E_a + C_1^T C_1 \\
&\quad -\mu C_1^T D_{12}\bar{R}D_{12}^T C_1 \\
&\quad -C_1^T D_{12}\bar{R}E_b^T E_a - E_a^T E_b \bar{R} D_{12}^T C_1 \\
&= \begin{bmatrix} Q_{11}^\mu & Q_{12}^\mu \\ Q_{12}^{\mu T} & Q_{22}^\mu \end{bmatrix}.
\end{aligned}$$

For the previous equations (10), setting $\varepsilon = 0$, we obtain the following equations

$$\begin{aligned}
&A_{11}^{\mu T} \bar{P}_{11} + \bar{P}_{11}^T A_{11}^\mu + A_{21}^{\mu T} \bar{P}_{21} + \bar{P}_{21}^T A_{21}^\mu \\
&\quad -\bar{P}_{11}^T S_{11}^\mu \bar{P}_{11} - \bar{P}_{21}^T S_{22}^\mu \bar{P}_{21} \\
&\quad - \bar{P}_{11}^T S_{12}^\mu \bar{P}_{21} - \bar{P}_{21}^T S_{12}^{\mu T} \bar{P}_{11} + Q_{11}^\mu = 0,
\end{aligned} \tag{11a}$$

$$\begin{aligned}
&\bar{P}_{22}^T A_{21}^\mu + A_{12}^{\mu T} \bar{P}_{11} + A_{22}^{\mu T} \bar{P}_{21} \\
&\quad -\bar{P}_{22}^T S_{12}^{\mu T} \bar{P}_{11} - \bar{P}_{22}^T S_{22}^\mu \bar{P}_{21} + Q_{12}^{\mu T} = 0,
\end{aligned} \tag{11b}$$

$$A_{22}^{\mu T} \bar{P}_{22} + \bar{P}_{22}^T A_{22}^\mu - \bar{P}_{22}^T S_{22}^\mu \bar{P}_{22} + Q_{22}^\mu = 0, \tag{11c}$$

where $\bar{P}_{11}$, $\bar{P}_{21}$ and $\bar{P}_{22}$ are 0–order solutions of the equations (10). The Riccati equation (11c) will produce the unique positive definite stabilizing solution under the following conditions [8].

Let
$\Gamma_f := \{\mu > 0 |$the Riccati equation (11c) has a positive definite stabilizing solution$\}$,
$\mu_f := \sup\{\mu | \mu \in \Gamma_f\}$.

Then, the matrix $A_{22}^\mu - S_{22}^\mu \bar{P}_{22}$ is non–singular if we choose $0 < \mu < \mu_f$. Therefore, we obtain the following 0–order equations

$$\bar{P}_{11}^T A_0^\mu + A_0^{\mu T} \bar{P}_{11} - \bar{P}_{11}^T S_0^\mu \bar{P}_{11} + Q_0^\mu = 0, \tag{12a}$$

$$\bar{P}_{21} = -N_2^T + N_1^T \bar{P}_{11}, \tag{12b}$$

$$A_{22}^{\mu T} \bar{P}_{22} + \bar{P}_{22}^T A_{22}^\mu - \bar{P}_{22}^T S_{22}^\mu \bar{P}_{22} + Q_{22}^\mu = 0, \tag{12c}$$

where

$$\begin{aligned}
A_0^\mu &= A_{11}^\mu + N_1 A_{21}^\mu + S_{12}^\mu N_2^T + N_1 S_{22}^\mu N_2^T, \\
S_0^\mu &= S_{11}^\mu + N_1 S_{12}^{T\mu} + S_{12}^\mu N_1^T + N_1 S_{22}^\mu N_1^T, \\
Q_0^\mu &= Q_{11}^\mu - N_2 A_{21}^\mu - A_{21}^{\mu T} N_2^T - N_2 S_{22}^\mu N_2^T,
\end{aligned}$$

$$
\begin{aligned}
N_2^T &= D_4^{-T}\hat{Q}_{12}^T,\ N_1^T = -D_4^{-T}D_2^T, \\
D_1 &= A_{11}^\mu - S_{11}^\mu \bar{P}_{11} - S_{12}^\mu \bar{P}_{21}, \\
D_3 &= A_{21}^\mu - S_{12}^{\mu T}\bar{P}_{11} - S_{22}^\mu \bar{P}_{21}, \\
D_2 &= A_{12}^\mu - S_{12}^\mu \bar{P}_{22},\ D_4 = A_{22}^\mu - S_{22}^\mu \bar{P}_{22}, \\
D_0 &= D_1 - D_2 D_4^{-1} D_3,\ \hat{Q}_{12} = Q_{12} + A_{21}^{\mu T}\bar{P}_{22}.
\end{aligned}
$$

Remark 1 *Although the expressions of the matrix A_0^μ, S_0^μ and Q_0^μ contain the matrix $\bar{P}_{22}$, they do not depend on it (see, e.g., [6]). In fact, the coefficient matrices of the equation (12a) are obtained from the formula*

$$
T_0 = T_1 - T_2 T_4^{-1} T_3 = \begin{bmatrix} A_0^\mu & -S_0^\mu \\ -Q_0^\mu & -A_0^{\mu T} \end{bmatrix}, \tag{13}
$$

where

$$
T_1 = \begin{bmatrix} A_{11}^\mu & -S_{11}^\mu \\ -Q_{11}^\mu & -A_{11}^{\mu T} \end{bmatrix}, T_2 = \begin{bmatrix} A_{12}^\mu & -S_{12}^\mu \\ -Q_{12}^\mu & -A_{21}^{\mu T} \end{bmatrix},
$$
$$
T_3 = \begin{bmatrix} A_{21}^\mu & -S_{12}^{\mu T} \\ -Q_{12}^{\mu T} & -A_{12}^{\mu T} \end{bmatrix}, T_4 = \begin{bmatrix} A_{22}^\mu & -S_{22}^\mu \\ -Q_{22}^\mu & -A_{22}^{\mu T} \end{bmatrix}.
$$

Let us define
$\Gamma_s := \{0 < \mu \le \mu_f |$the Riccati equation (12a) has a positive definite stabilizing solution$\}$,
$\mu_s := \sup\{\mu | \mu \in \Gamma_s\}$.

As a result, for every $0 < \mu < \bar{\mu} = \min\{\mu_s,\ \mu_f\}$, Riccati equations (12a) and (12c) have the positive definite stabilizing solutions.

We have the following result.

Theorem 3 *Under Assumption 1, if we select a parameter $0 < \mu < \bar{\mu} = \min\{\mu_s,\ \mu_f\}$, then there exists small $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0,\ \bar{\varepsilon})$, the algebraic Riccati equation (3) admits a positive definite solution, which can be written as*

$$
P_\varepsilon = \begin{bmatrix} \bar{P}_{11} + O(\varepsilon) & \varepsilon \bar{P}_{21}^T + O(\varepsilon^2) \\ \varepsilon \bar{P}_{21} + O(\varepsilon^2) & \varepsilon \bar{P}_{22} + O(\varepsilon^2) \end{bmatrix}. \tag{14}
$$

If such condition are met, a control is given by (3). Furthermore, P_ε of the Riccati equation (3) is a positive definite stabilizing solution.

proof: By using the implicit function theorem, the theorem can be proved. The proof is omitted because it is similar to the references Dragan [4] and Mukaidani *et al.* [8]. □

Remark 2 *We can prove Theorem 3 by using a similar method to that given in the proof of Theorem 2.1 and 2.2 in [4]. Note that the proof given in [4] is made on the invertible assumption, that is, A_{22} is non-singular. However, this paper improves the proof of Theorem 2 in the sense that the invertible assumption is not needed.*

2 The New Iterative Algorithm

By the results in Garcia *et al.* [5], we have to solve the full–order algebraic Riccati equations at any risk so as to calculate Trace$[G_\varepsilon^T P_\varepsilon(\mu) G_\varepsilon]$ for every $0 < \mu < \bar{\mu}$. Furthermore, since the guaranteed cost depend on the small parameter ε, that is,

$$
\begin{aligned}
&\mathrm{Trace}[G_\varepsilon^T P_\varepsilon(\mu) G_\varepsilon] \\
= \ &\mathrm{Trace}[G_1^T P_{11}(\mu) G_1 + G_2^T P_{21}(\mu) G_1 \\
&+ G_1^T P_{21}^T(\mu) G_2 + \varepsilon^{-1} G_2^T P_{22}(\mu) G_2],
\end{aligned}
$$

we have to solve the full–order algebraic Riccati equations with high–order accuracy because of $\varepsilon^{-1}G_2^T P_{22}(\mu) G_2$ for the guaranteed cost. So far, the recursive algorithm (Gajic *et al.* [11, 3]) was very effective to solve the full–order algebraic Riccati equation with small parameter $\varepsilon > 0$. However, note that using the zero–order solution without high–order accuracy will fail to produce the desired exact solution of the algebraic Riccati equation. In this case, the recursive algorithm converge to the approximation solution.

In this paper we develop an elegant and simple algorithm which converges globally to the positive definite symmetric solution of equation (3). The algorithm is derived by the standard algebraic Riccati equations, which have to be solved iteratively. We present the new iterative algorithm based on the Kleinman algorithm [10]. Here we note that the Kleinman algorithm is based on the Newton type algorithm. In general, the stabilizable–detectable conditions will guarantee the convergence of the Kleinman algorithm for the standard linear quadratic regulator type generalized algebraic Riccati equation to the positive definite solutions. However, it is difficult to apply the Kleinman algorithm to the equation (3) presented in this paper because the matrix $S^\mu = BR^{-1}B^T - \mu DD^T$ is in general indefinite. That is, the generalized algebraic Riccati equation (9a) is not always a convex function with respect to P.

We propose the following algorithm for solving the generalized algebraic Riccati equation (3)

$$
\begin{aligned}
&(A^\mu - S^\mu P^{(i)})^T P^{(i+1)} + P^{(i+1)T}(A^\mu - S^\mu P^{(i)}) \\
&+ P^{(i)T} S^\mu P^{(i)} + Q^\mu = 0,
\end{aligned} \tag{15a}
$$

with the initial condition obtained from

$$
P^{(0)} = \begin{bmatrix} \bar{P}_{11} & 0 \\ \bar{P}_{21} & \bar{P}_{22} \end{bmatrix}, \tag{15b}
$$

where $\bar{P}_{11}$, $\bar{P}_{21}$, $\bar{P}_{22}$ are defined by (12).

Kleinman algorithm is well known, and widely is used to find a solution of algebraic Riccati equation, and its local convergence properties are well understood. We propose a good choice of the initial condition which guarantee to find required solution of a given generalized algebraic Riccati equation. Our new idea is to set the initial condition $P^{(0)}$ to the relation (15b). The fundamental idea is based on $\|P - P^{(0)}\| = O(\varepsilon)$ from Theorem 3. Although the matrix S^μ is in general indefinite, we prove to converge to the required solution by using the Kleinman algorithm. The algorithm (15) has the feature given in the following theorem.

Theorem 4 *Under Assumptions 1, if we select a parameter $0 < \mu < \bar{\mu} = \min\{\mu_s, \ \mu_f\}$, then the new iterative algorithm (15) converges to the exact solution of P^* with the rate of quadratic convergence such that $P_\varepsilon^{(i)} = \Pi_\varepsilon^T P^{(i)} = P^{(i)T}\Pi_\varepsilon$ is positive definite. That is,*

$$\lim_{i\to\infty} \frac{\|P^{(i+1)} - P^*\|}{\|P^{(i)} - P^*\|} = 0, \tag{16a}$$

$$\|P^{(i+1)} - P^*\| \le \mathcal{M}\|P^{(i)} - P^*\|^2,$$
$$0 < \mathcal{M} < \infty \Leftrightarrow \|P^{(i)} - P^*\| = O(\varepsilon^{2^i}). \tag{16b}$$

Moreover, let $P_{11}^{(\infty)}$, $P_{21}^{(\infty)}$ and $P_{22}^{(\infty)}$ be the limit points of the iterative algorithm (15). As the results, we have

$$A^{\mu T}P^{(\infty)} + P^{(\infty)T}A^\mu - P^{(\infty)T}S^\mu P^{(\infty)} + Q^\mu = 0 \tag{17}$$

where

$$P^* = P^{(\infty)} = \begin{bmatrix} P_{11}^{(\infty)} & \varepsilon P_{21}^{T(\infty)} \\ P_{21}^{(\infty)} & P_{22}^{(\infty)} \end{bmatrix}.$$

Thus, by using the linear state feedback full–order controller

$$u_{exa}(t) = -\bar{R}[\mu(B^T P^{(\infty)} + D_{12}^T C_1) + E_b^T E_a]x(t), \tag{18}$$

the uncertain singularly perturbed system (1) is quadratically stable.

proof: The proof is omitted because it is similar to that of [9]. □

As a result of the application of the idea of Kleinman algorithm, we have managed to replace the computation of the generalized Riccati equation (3) which contains the small parameter ε by a sequence of algebraic Lyapunov equations (15a).

3 Numerical Example

In order to demonstrate the efficiency of the proposed algorithm, we consider a fourth order real world example, that is, manufacturing assembly process [2]. The system matrix is given by (See Shi *et al.* [2])

$$A_{11} = \begin{bmatrix} 0 & 1 & -1 \\ -2 & -0.2 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \ A_{12} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} 0 & -5.2 \times 10^{-2} & 0 \end{bmatrix}, \ A_{22} = \begin{bmatrix} -2 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.2 & 0 \end{bmatrix}, \ G_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \ G_2 = \begin{bmatrix} 0 & 0 & 0.1 \end{bmatrix},$$

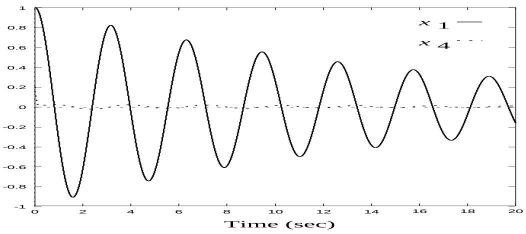

Fig.1 Response of the open loop system without any controller.

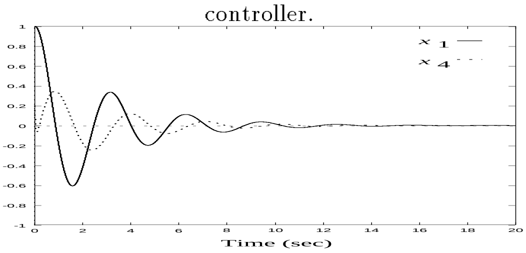

Fig.2 Response of the closed–loop system with the proposed controller.

$$C_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \ C_{12} = D_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \end{bmatrix}, \ D_2 = \begin{bmatrix} 0.2 \end{bmatrix}, \ E_b = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

$$E_{a1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \ E_{a2} = \begin{bmatrix} 0 \end{bmatrix}, \ Q = 0.05 I_4.$$

The reader is referred to [2] for a discussion of the modeling aspects. The numerical results are obtained for small parameter $\varepsilon = 0.1$. For every boundary value

$0 < \mu < \bar{\mu} = \min\{\mu_f, \ \mu_s\} = 2.5097$,

the algebraic Riccati equations (12a) and (12c) have the positive definite stabilizing solution. The best bound $\beta^* = 1.6486$ is obtained for $\mu^* = 0.1455$. Now, we choose $0 < \mu = 0.1455 < \bar{\mu}$ to design the controller. We give the solutions of the algebraic Riccati equations (3) at the top of the next page. We also give the full–order controller based on this value of $\mu^* = 0.1455$. We find that the solution of the algebraic Riccati equation (3) converges to the exact solution with accuracy of $\|F(P^{(i)})\| < 10^{-12}$ after 2 recursive iterations. Note that the result of Table 1 shows that the proposed algorithm (15a) is quadratic convergence because the initial condition (15b) is sufficiently close to the exact solution.

Table 1.

i	$\|F(P_\varepsilon^{(i)})\|$
0	6.66944×10^{-3}
1	9.73263×10^{-7}
2	4.70403×10^{-14}

In this example, the proposed controller (9c) will be employed as the manufacturing system under bounded uncer-

$$P_\varepsilon^{(3)} = \begin{bmatrix} 3.442463 & 1.515721 \times 10^{-1} & -2.964068 \times 10^{-1} & -1.871900 \times 10^{-2} \\ 1.515721 \times 10^{-1} & 1.436862 & -4.372471 \times 10^{-1} & 8.547088 \times 10^{-2} \\ -2.964068 \times 10^{-1} & -4.372471 \times 10^{-1} & 1.264558 & -2.712478 \times 10^{-2} \\ -1.871900 \times 10^{-2} & 8.547088 \times 10^{-2} & -2.712478 \times 10^{-2} & 1.653024 \times 10^{-2} \end{bmatrix}.$$

$$K = \begin{bmatrix} -5.216888 \times 10^{-1} & -2.102910 \times 10^{-2} & -1.261458 & 5.460343 \times 10^{-1} \\ -3.937801 \times 10^{-1} & -9.631873 \times 10^{-1} & 2.627014 \times 10^{-1} & -6.246930 \times 10^{-1} \end{bmatrix}.$$

tain assembly goods. The results of the simulation of this example are depicted in Figures 1 and 2. The initial state is set as $x(0) = \begin{bmatrix} 1 & 0 & 0 & 0.2 \end{bmatrix}^T$. It is shown from Fig. 2 that the closed–loop system is asymptotically stable.

4 Conclusions

In this paper, the H_2 guaranteed cost control problem for singularly perturbed systems with uncertainties has been investigated based on the iterative numerical technique. We presented a new recursive algorithm under the special initial condition. Comparing with [11] and [3], since the proposed algorithm is quadratic convergence, the required solution can be easily obtained up to an arbitrary order of accuracy, that is, $O(\varepsilon^{2^i})$ where i is a iteration number. Another important feature, our new results are applicable to the more realistic systems in comparison with [1, 2, 5].

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