Robust Non-Fragile Controllers for Uncertain Linear Continuous-Time Systems

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Abstract— **This paper deals with a design problem of robust non-fragile controllers for linear continuous-time systems with uncertainties which are included in both the system matrix and the input one. In this paper, we deal with two classes of control gain perturbations and show that sufficient conditions for the existence of the robust non-fragile controller are given in terms of linear matrix inequalities (LMIs). Finally, illustrative examples are presented.**

I. Introduction

Robustness of control systems to uncertainties has always been the central issue in feedback control and therefore for uncertain dynamical systems, a large number of robust controller design methods have been presented (e.g. Ackermann 1980, Hagino and Komiriya 1989) Robust controller design methods can be classified roughly into "quadratic stabilizing control (e.g. Petersen and Hollot 1986)" and "$\mathcal{H}^{\infty}$ control (e.g. Doyle et. al. 1989, Khargonekar et. al. 1990)".

By the way, there have been some efforts to tackle the design problem of robust non-fragile controllers (e.g. Dorato 1998, Famularo et. al. 2000). Because controller implementation is subject to imprecision inherent in analog-digital and digital-analog conversion, finite word length, and finite resolution measuring instruments and roundoff errors in numerical computations and any useful design procedure should generate a controller which also has sufficient room for readjustment of its coefficients (Keel and Bhattacharyya 1997). For linear continuous-time systems with structured uncertainties existing in the system matrix only, a design method of a robust non-fragile state feedback controller and a have been suggested (Famularo et. al. 2000). Also, a design method of a $\mathcal{H}^{\infty}$ controller for linear systems with additive controller gain variations has been derived (Wang and Lin 2000). However, so far the design problem of robust non-fragile controllers for linear continuous-time systems with uncertainties which are included in both the system matrix and the input one has not been discussed.

From this viewpoint on the basis of the existing result for quadratic stabilization, we present a design method of a robust non-fragile controller for linear continuous-time systems with structured uncertainties existing in both the system matrix and the input one. Furthermore, we deals with a design method of robust non-fragile $\mathcal{H}^{\infty}$ controllers. In this paper, we show that sufficient conditions for the existence of the robust non-fragile controller are given in terms of linear matrix inequalities (LMIs).

This paper is organized as follows. In Sect. 2, notation and two useful lemmas which are used in this paper are shown and in Sect. 3, we introduce the classes of uncertain systems and the control gain variations under consideration. Sect. 4 contains the main results. The design method of the robust non-fragile controller is presented. Finally, illustrative examples are included to illustrate the results developed in this paper.

II. Preliminaries

In this section, we show notation and two useful lemmas which are used in this paper.

In this paper, we use the following notation. The transpose of matrix $\mathcal{A}$ and the inverse of one are denoted by $\mathcal{A}^T$ and $\mathcal{A}^{-1}$ respectively. Also $H_e\{\mathcal{A}\}$ means $\mathcal{A}+\mathcal{A}^T$ and I_n represents n-dimensional identity matrix. For real symmetric matrices $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} > \mathcal{B}$ (resp. $\mathcal{A} \geq \mathcal{B}$) means that $\mathcal{A} - \mathcal{B}$ is positive (resp. nonnegative) definite matrix. The symbol "$\stackrel{\triangle}{=}$" means equality by definition. Besides, $\mathcal{L}_2[0,\infty)$ is $\mathcal{L}_2$-space (i.e. the collection of all square integrable functions) and for a signal $f(t) \in \mathcal{L}_2[0,\infty)$, $||f(t)||_{\mathcal{L}_2}$ denotes its $\mathcal{L}_2$-norm.

Furthermore, the following two useful lemmas are used in this paper.

Lemma 1: For given constant real symmetric matrix Ξ, the following arguments are equivalent.

(i). $\Xi \stackrel{\triangle}{=} \begin{pmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{12}^T & \Xi_{22} \end{pmatrix} > 0$

(ii). $\Xi_{11} > 0$ and $\Xi_{22} - \Xi_{12}^T \Xi_{11}^{-1} \Xi_{12} > 0$

(iii). $\Xi_{22} > 0$ and $\Xi_{11} - \Xi_{12} \Xi_{22}^{-1} \Xi_{12}^T > 0$

Proof: See Boyd et. al. (1994). ■

Lemma 2: For matrices $\mathcal{G}$ and $\mathcal{H}$ which have appropriate dimensions and a positive scalar γ, the following relation holds.

$$\mathcal{G}\mathcal{H} + \mathcal{H}^T\mathcal{G}^T \leq \gamma \mathcal{G}\mathcal{G}^T + \frac{1}{\gamma}\mathcal{H}^T\mathcal{H}$$

Proof: The proof follows the same lines as Lemma 1 of Oya and Hagino (2003). ■

III. Problem Formulation

Consider the uncertain linear continuous-time system described by the following state equation.

$$\frac{d}{dt}x(t) = A(t)x(t) + B(t)u(t) \tag{1}$$

where $x(t) \in \Re^n$ and $u(t) \in \Re^m$ are the vectors of the state (assumed to be available for feedback) and the control input, respectively. The matrices $A(t)$ and $B(t)$ are supposed to have appropriate dimensions and the follow-

$$
\begin{pmatrix}
\Gamma(\mathcal{S},\mathcal{W},\gamma,\delta,\mu,\nu) & \mathcal{S}\mathcal{L}^T & \mathcal{W}^T\mathcal{M}^T & \mathcal{W}^T\mathcal{N}_{\mathcal{M}}^T & 0 & \mathcal{E} & \mathcal{W}^T\mathcal{N}_{\mathcal{M}}^T \\
\mathcal{L}\mathcal{S} & -\gamma I_q & 0 & 0 & 0 & 0 & 0 \\
\mathcal{M}\mathcal{W} & 0 & -\delta I_s & 0 & 0 & 0 & 0 \\
\mathcal{N}_{\mathcal{M}}\mathcal{W} & 0 & 0 & -\mu I_m & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\Upsilon_{\mathcal{M}}(\nu) & 0 & 0 \\
\mathcal{E}^T & 0 & 0 & 0 & 0 & -\nu I_r & 0 \\
\mathcal{N}_{\mathcal{M}}\mathcal{W} & 0 & 0 & 0 & 0 & 0 & -\nu I_{l_{\mathcal{M}}}
\end{pmatrix} < 0 \tag{7}
$$

$$
\Gamma(\mathcal{S},\mathcal{W},\gamma,\delta,\mu) \stackrel{\triangle}{=} H_e\{A\mathcal{S} + B\mathcal{W}\} + \gamma\mathcal{D}\mathcal{D}^T + \delta\mathcal{E}\mathcal{E}^T + \mu\epsilon_{\mathcal{M}}B\mathcal{F}_{\mathcal{M}}\mathcal{F}_{\mathcal{M}}^T B^T \tag{8}
$$

$$
\begin{aligned}
\frac{d}{dt}\mathcal{V}(x,t) = x^T(t)\big[H_e\{\mathcal{P}(A + BK + \mathcal{D}\Delta_{\mathcal{A}}(t)\mathcal{L} + \mathcal{E}\Delta_{\mathcal{B}}(t)\mathcal{M}K + B\mathcal{F}_{\mathcal{M}}\Delta_{\mathcal{K}_{\mathcal{M}}}(t)\mathcal{N}_{\mathcal{M}}K \\
+ \mathcal{E}\Delta_{\mathcal{B}}(t)\mathcal{M}\mathcal{F}_{\mathcal{M}}\Delta_{\mathcal{K}_{\mathcal{M}}}(t)\mathcal{N}_{\mathcal{M}}K)\}\big]x(t)
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
\Phi(\mathcal{P},K) \stackrel{\triangle}{=} H_e\{\mathcal{P}(A + BK + \mathcal{D}\Delta_{\mathcal{A}}(t)\mathcal{L} + \mathcal{E}\Delta_{\mathcal{B}}(t)\mathcal{M}K + B\mathcal{F}_{\mathcal{M}}\Delta_{\mathcal{K}_{\mathcal{M}}}(t)\mathcal{N}_{\mathcal{M}}K + \mathcal{E}\Delta_{\mathcal{B}}(t)\mathcal{M}\mathcal{F}_{\mathcal{M}}\Delta_{\mathcal{K}_{\mathcal{M}}}(t)\mathcal{N}_{\mathcal{M}}K)\} \\
< 0
\end{aligned} \tag{10}
$$

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ing time-varying structure.

$$
\begin{aligned}
A(t) &= A + \mathcal{D}\Delta_{\mathcal{A}}(t)\mathcal{L} \\
B(t) &= B + \mathcal{E}\Delta_{\mathcal{B}}(t)\mathcal{M}
\end{aligned} \tag{2}
$$

In eq.(2), the matrices A and B denote the known nominal values and the matrices $\mathcal{D}, \mathcal{E}, \mathcal{L}$ and $\mathcal{M}$ represent the structure of uncertainties. The matrices $\Delta_{\mathcal{A}}(t) \in \Re^{p\times q}$ and $\Delta_{\mathcal{B}}(t) \in \Re^{r\times s}$ denote uncertainties and satisfy $\Delta_{\mathcal{A}}(t)\Delta_{\mathcal{A}}^T(t) \le I_p$ and $\Delta_{\mathcal{B}}(t)\Delta_{\mathcal{B}}^T(t) \le I_r$, respectively.

In order to consider the control gain perturbations, the actual control input implemented is assumed to be

$$
u(t) \stackrel{\triangle}{=} K(t)x(t) \tag{3}
$$

where $K(t) \in \Re^{m\times n}$ represent the control gain matrix with uncertainties. In this paper, the following two classes of the control gain matrix $K(t)$ are considered.

- the multiplicative form :

$$
K(t) \stackrel{\triangle}{=} K + \mathcal{F}_{\mathcal{M}}\Delta_{\mathcal{K}_{\mathcal{M}}}(t)\mathcal{N}_{\mathcal{M}}K \tag{4}
$$

- the additive form :

$$
K(t) \stackrel{\triangle}{=} K + \mathcal{F}_{\mathcal{A}}\Delta_{\mathcal{K}_{\mathcal{A}}}(t)\mathcal{N}_{\mathcal{A}} \tag{5}
$$

where K is the nominal control gain matrix. In eqs.(4) and (5), $\mathcal{F}_{\mathcal{M}}, \mathcal{F}_{\mathcal{A}}, \mathcal{N}_{\mathcal{M}}$ and $\mathcal{N}_{\mathcal{A}}$ are known constant matrices with appropriate dimensions and the matrices $\Delta_{\mathcal{K}_{\mathcal{M}}}(t) \in \Re^{k_{\mathcal{M}}\times l_{\mathcal{M}}}$ and $\Delta_{\mathcal{K}_{\mathcal{A}}}(t) \in \Re^{k_{\mathcal{A}}\times l_{\mathcal{A}}}$ represent the control gain variations and satisfy the relation $\Delta_{\mathcal{K}_{\mathcal{M}}}(t)\Delta_{\mathcal{K}_{\mathcal{M}}}^T(t) \le \epsilon_{\mathcal{M}} I_{k_{\mathcal{M}}}$ and $\Delta_{\mathcal{K}_{\mathcal{A}}}(t)\Delta_{\mathcal{K}_{\mathcal{A}}}^T(t) \le \epsilon_{\mathcal{A}} I_{k_{\mathcal{A}}}$ where $\epsilon_{\mathcal{M}}$ and $\epsilon_{\mathcal{A}}$ are known positive scalars.

Note that the manipulated input for the uncertain system eq.(1) is $u(t) \stackrel{\triangle}{=} Kx(t)$, because the control gain variations $\Delta_{\mathcal{K}_{\mathcal{M}}}(t) \in \Re^{k_{\mathcal{M}}\times l_{\mathcal{M}}}$ and $\Delta_{\mathcal{K}_{\mathcal{M}}}(t) \in \Re^{k_{\mathcal{A}}\times l_{\mathcal{A}}}$ cannot be handled. In this paper, we simply consider the actual control input $u(t)$ described by eqs.(3), (4) and (5) so as to design the robust stabilizing state feedback controller under control gain variations.

From eqs.(1) and (3), we get

$$
\frac{d}{dt}x(t) = (A(t) + B(t)K(t))x(t) \tag{6}
$$

From the above discussion, our control objective in this paper is to design the state feedback gain matrix K which stabilizes the closed-loop system eq.(6).

IV. Robust Non-Fragile Controllers

In this section, we show that the design method of the robust non-fragile controller based on the linear matrix inequality (LMI) framework.

Firstly, we give the following theorem for the robust stabilizing controller under multiplicative control gain perturbations of the form eq.(4).

Theorem 1: Consider the uncertain system eq.(1). There exists the state feedback gain matrix K (it is given by $K \stackrel{\triangle}{=} \mathcal{W}\mathcal{S}^{-1}$, if there exist) such that the control law eq.(3) with the multiplicative control gain perturbations of the form eq.(4) is a stabilizing control for the closed-loop system eq.(6), if there exist $\mathcal{S} > 0, \mathcal{W}, \gamma > 0, \delta > 0, \mu > 0$ and $\nu > 0$ satisfying the LMI condition eq.(7). In eq.(7), $\Upsilon_{\mathcal{M}}(\nu)$ is the matrix given by $\Upsilon_{\mathcal{M}}(\nu) \stackrel{\triangle}{=} I_s - \nu(I_s + \epsilon_{\mathcal{M}}\mathcal{M}\mathcal{F}_{\mathcal{M}}\mathcal{F}_{\mathcal{M}}^T\mathcal{M}^T)$ and $\Gamma(\mathcal{S},\mathcal{W},\gamma,\delta,\mu)$ is the matrix given by eq.(8).

Proof: Using a symmetric positive definite matrix $\mathcal{P} \in \Re^{n\times n}$, we introduce the quadratic function $\mathcal{V}(x,t) \stackrel{\triangle}{=} x^T(t)\mathcal{P}x(t)$ as a Lyapunov function candidate.

From eqs.(2), (4) and (6), the time derivative of the quadratic function $\mathcal{V}(x,t)$ along the trajectory of the closed-loop system eq.(6) can be computed as eq.(9). Therefore if there exist the state feedback gain matrix $K \in \Re^{m\times n}$ and the symmetric positive definite matrix $\mathcal{P} \in \Re^{n\times n}$ which satisfy the condition eq.(10), then the quadratic function $\mathcal{V}(x,t)$ satisfies the following relation eq.(11) and the quadratic function $\mathcal{V}(x,t)$ becomes a Lyapunov function for the closed-loop system eq.(6). Namely, the closed-loop system eq.(6) is asymptotically stable.

$$
\frac{d}{dt}\mathcal{V}(x,t) < 0 \quad \text{for} \quad \forall x(t) \neq 0 \tag{11}
$$

Let us introduce the matrix $\mathcal{S} \stackrel{\triangle}{=} \mathcal{P}^{-1}$ and consider the change of variable $\mathcal{W} \stackrel{\triangle}{=} K\mathcal{S}$. Then pre- and post-

$$\Psi(\mathcal{S},\mathcal{W},\gamma,\delta,\mu) \stackrel{\triangle}{=} \varGamma(\mathcal{S},\mathcal{W},\gamma,\delta,\mu) + \frac{1}{\gamma}\mathcal{S}\mathcal{L}^T\mathcal{L}\mathcal{S} + \frac{1}{\delta}\mathcal{W}^T\mathcal{M}^T\mathcal{M}\mathcal{W} + \frac{1}{\mu}\mathcal{W}^T\mathcal{N}_{\mathcal{M}}^T\mathcal{N}_{\mathcal{M}}\mathcal{W} + \left(\mathcal{E}\Delta_{\mathcal{B}}(t) + \mathcal{W}^T\mathcal{N}_{\mathcal{M}}^T\Delta_{\mathcal{K}_{\mathcal{M}}}^T(t)\mathcal{F}_{\mathcal{M}}^T\mathcal{M}^T\right)\left(\mathcal{E}\Delta_{\mathcal{B}}(t) + \mathcal{W}^T\mathcal{N}_{\mathcal{M}}^T\Delta_{\mathcal{K}_{\mathcal{M}}}^T(t)\mathcal{F}_{\mathcal{M}}^T\mathcal{M}^T\right)^T < 0 \tag{12}$$

$$\begin{pmatrix} \varGamma(\mathcal{S},\mathcal{W},\gamma,\delta,\mu) & \mathcal{S}\mathcal{L}^T & \mathcal{W}^T\mathcal{M}^T & \mathcal{W}^T\mathcal{N}_{\mathcal{M}}^T & 0 \\ \mathcal{L}\mathcal{S} & -\gamma I_q & 0 & 0 & 0 \\ \mathcal{M}\mathcal{W} & 0 & -\delta I_s & 0 & 0 \\ \mathcal{W}\mathcal{N}_{\mathcal{M}} & 0 & 0 & -\mu I_m & 0 \\ 0 & 0 & 0 & 0 & -I_s \end{pmatrix} + H_e\left\{\begin{pmatrix} \mathcal{E}\Delta_{\mathcal{B}}(t) & \mathcal{W}^T\mathcal{N}_{\mathcal{M}}^T \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 & I_s \\ 0 & 0 & 0 & 0 & \Delta_{\mathcal{K}_{\mathcal{M}}}^T(t)\mathcal{F}_{\mathcal{M}}^T\mathcal{M}^T \end{pmatrix}\right\} < 0 \tag{13}$$

$$\begin{pmatrix} \varGamma(\mathcal{S},\mathcal{W},\gamma,\delta,\mu) + \frac{1}{\nu}\mathcal{E}\mathcal{E}^T + \frac{1}{\nu}\mathcal{W}^T\mathcal{N}_{\mathcal{M}}^T\mathcal{N}_{\mathcal{M}}\mathcal{W} & \mathcal{S}\mathcal{L}^T & \mathcal{W}^T\mathcal{M}^T & \mathcal{W}^T\mathcal{N}_{\mathcal{M}}^T & 0 \\ \mathcal{L}\mathcal{S} & -\gamma I_q & 0 & 0 & 0 \\ \mathcal{M}\mathcal{W} & 0 & -\delta I_s & 0 & 0 \\ \mathcal{W}\mathcal{N}_{\mathcal{M}} & 0 & 0 & -\mu I_m & 0 \\ 0 & 0 & 0 & 0 & -\varUpsilon_{\mathcal{M}}(\nu) \end{pmatrix} < 0 \tag{14}$$

$$\begin{pmatrix} \varGamma(\mathcal{S},\mathcal{W},\gamma,\delta,\mu) & \mathcal{S}\mathcal{L}^T & \mathcal{W}^T\mathcal{M}^T & \mathcal{S}\mathcal{N}_{\mathcal{A}}^T & 0 & \mathcal{E} & \mathcal{S}\mathcal{N}_{\mathcal{A}}^T \\ \mathcal{L}\mathcal{S} & -\gamma I_q & 0 & 0 & 0 & 0 & 0 \\ \mathcal{M}\mathcal{W} & 0 & -\delta I_s & 0 & 0 & 0 & 0 \\ \mathcal{N}_{\mathcal{A}}\mathcal{S} & 0 & 0 & -\mu I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\varUpsilon_{\mathcal{A}}(\nu) & 0 & 0 \\ \mathcal{E}^T & 0 & 0 & 0 & 0 & -\nu I_r & 0 \\ \mathcal{N}_{\mathcal{A}}\mathcal{S} & 0 & 0 & 0 & 0 & 0 & -\nu I_{l_{\mathcal{A}}} \end{pmatrix} < 0 \tag{15}$$

multiplying eq.(10) by $\mathcal{S} \in \Re^{n\times n}$ and using **Lemma 2**, we get the inequality eq.(12). Furthermore applying **Lemma 1** to the inequality eq.(12) and simple algebraic manipulation gives the matrix inequality eq.(13). Also by using **Lemma 2**, the condition eq.(14) can be obtained. Therefore by applying **Lemma 1** to the matrix inequality eq.(14), it is easy to verify that the condition eq.(14) is equivalent to the LMI condition eq.(7) for $\mathcal{S},\mathcal{W},\gamma,\delta,\mu$ and ν.

It follows that the result of the theorem is true. Thus the proof of **Theorem 1** is completed. ∎

Theorem 1 provides a sufficient condition for the existence of the robust controller under multiplicative control gain perturbations of the form eq.(4). Next, we show the theorem for the robust controller under additive control gain perturbations of the form eq.(5).

Theorem 2: Consider the uncertain system eq.(1). There exists the state feedback gain matrix K (it is given by $K \stackrel{\triangle}{=} \mathcal{W}\mathcal{S}^{-1}$, if there exist) such that the control law eq.(3) with the additive control gain perturbations of the form eq.(5) is a stabilizing control for the closed-loop system eq.(6), if there exist $\mathcal{S} > 0, \mathcal{W}, \gamma > 0, \delta > 0$, $\mu > 0$ and $\nu > 0$ satisfying the LMI condition eq.(15). In eq.(15), $\varUpsilon_{\mathcal{A}}(\nu)$ is the matrix given by $\varUpsilon_{\mathcal{A}}(\nu) \stackrel{\triangle}{=} I_s - \nu(I_s + \epsilon_{\mathcal{A}}\mathcal{M}\mathcal{F}_{\mathcal{A}}\mathcal{F}_{\mathcal{A}}^T\mathcal{M}^T)$ and $\varGamma(\mathcal{S},\mathcal{W},\gamma,\delta,\mu)$ is the matrix given by eq.(8).

Proof: The result of the **Theorem 2** is derived in a similar way as for **Theorem 1**. ∎

Remark 1: In this paper, we present a LMI-based design method of robust non-fragile stabilizing controllers for linear continuous-time systems with uncertainties which are included in both system matrix and input one. Therefore, the proposed design method can be implemented for more practical uncertain systems. Besides, the proposed design method can be easily extended to robust non-fragile $\mathcal{H}^{\infty}$ controllers (see Appendix).

V. Illustrative Examples

In order to demonstrate the efficiency of the proposed robust non-fragile stabilizing controller, we have run a simple example. In this example, we consider the robust non-fragile controller under the multiplicative control gain perturbations of the form eq.(4). Also, the simulation results are shown for the proposed robust non-fragile stabilizing controller and the conventional quadratic stabilizing controller designed without thinking of control gain perturbations. The control problem considered here are not necessary practical. However, the simulation results stated below illustrate the distinct feature of the proposed robust non-fragile controller.

Consider the uncertain linear continuous-time system eq.(16). In this example, we assume that $k_{\mathcal{M}} = l_{\mathcal{M}} = 1$ (i.e. $\Delta_{\mathcal{K}_{\mathcal{M}}}(t) \in \Re^{1\times 1}, \mathcal{F}_{\mathcal{M}} \in \Re^{1\times 1}$ and $\mathcal{N}_{\mathcal{M}} \in \Re^{1\times 1}$) and

$$\frac{d}{dt}x(t) = \begin{pmatrix} -1.0 & 1.0 \\ 0.0 & 5.0 \end{pmatrix} x(t) + \begin{pmatrix} 1.00 \\ 1.25 \end{pmatrix} \Delta_{\mathcal{A}}(t) \begin{pmatrix} 0.25 & 0.05 \end{pmatrix} x(t) + \begin{pmatrix} 0.0 \\ 2.0 \end{pmatrix} u(t) + \begin{pmatrix} 0.050 \\ 0.175 \end{pmatrix} \Delta_{\mathcal{B}}(t) u(t) \tag{16}$$

- Case 1) : $\Delta_{\mathcal{A}}(t) = \sin(5.00\pi t), \quad \Delta_{\mathcal{B}}(t) = -\cos(2.00\pi t), \quad \Delta_{\mathcal{K}_{\mathcal{M}}}(t) = \sqrt{\epsilon_{\mathcal{M}}}\sin(10.00\pi t)$ (20)

- Case 2) :

$$\begin{array}{llll} \Delta_{\mathcal{A}}(t) = 1.0, & \Delta_{\mathcal{B}}(t) = -1.0, & \Delta_{\mathcal{K}}(t) = \sqrt{\epsilon_{\mathcal{M}}} & \text{for} \;\; 0 \le t \le 1.0 \\ \Delta_{\mathcal{A}}(t) = -1.0, & \Delta_{\mathcal{B}}(t) = 1.0, & \Delta_{\mathcal{K}}(t) = \sqrt{\epsilon_{\mathcal{M}}} & \text{for} \;\; 1.0 < t \le 2.0 \\ \Delta_{\mathcal{A}}(t) = -1.0, & \Delta_{\mathcal{B}}(t) = 1.0, & \Delta_{\mathcal{K}}(t) = -\sqrt{\epsilon_{\mathcal{M}}} & \text{for} \;\; t > 2.0 \end{array} \tag{21}$$

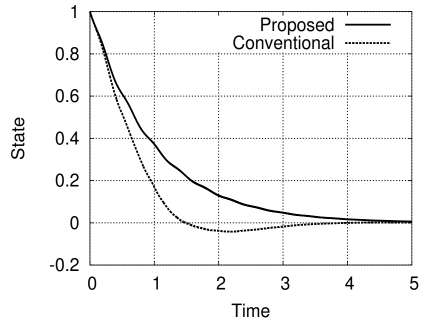

Fig. 1. Transient time-response of the state variable $x_1(t)$: Case 1)

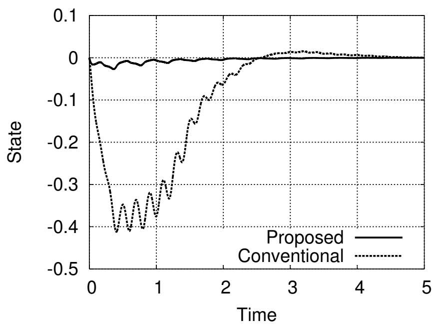

Fig. 2. Transient time-response of the state variable $x_2(t)$: Case 1)

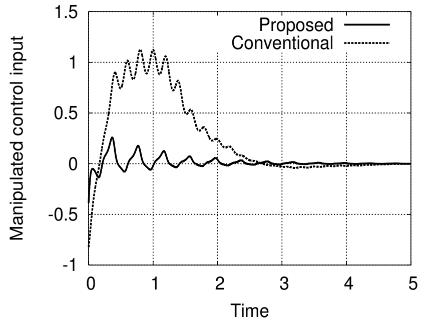

Fig. 3. Time histories of the manipulated input $u(t) \stackrel{\triangle}{=} Kx(t)$: Case 1)

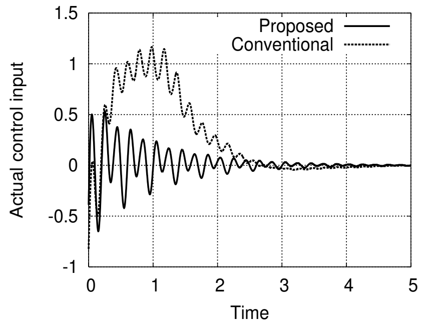

Fig. 4. Time histories of the actual control input $u(t) \stackrel{\triangle}{=} K(t)x(t)$: Case 1)

$\mathcal{F}_{\mathcal{M}}$ and $\mathcal{N}_{\mathcal{M}}$ in eq.(5) and $\epsilon_{\mathcal{M}}$ are given as $\mathcal{F}_{\mathcal{M}} = \mathcal{N}_{\mathcal{M}} = 1.0$ and $\epsilon_{\mathcal{M}} = 0.35$, respectively. Namely from eqs.(3) and (4), the actual control input $u(t)$ can be written as

$$u(t) \stackrel{\triangle}{=} (1.0 + \Delta_{\mathcal{K}_{\mathcal{M}}}(t))Kx(t) \tag{17}$$

By applying **Theorem 1** and solving the LMI condition eq.(7), we obtain the following state feedback gain matrix.

$$K = \begin{pmatrix} -3.88076 \times 10^{-1} & -19.57276 \end{pmatrix} \tag{18}$$

On the other hand, the feedback gain matrix for the conventional quadratic stabilizing control, denoted by $K_{\mathcal{C}}$, has been derived as

$$K_{\mathcal{C}} = \begin{pmatrix} -8.23760 \times 10^{-1} & -3.36386 \end{pmatrix} \tag{19}$$

In this example, we assume that the initial value for the uncertain system eq.(16) is selected as $x(0) = \begin{pmatrix} 1.0 & 0.0 \end{pmatrix}^T$ and the uncertain parameters $\Delta_{\mathcal{A}}(t)$ and $\Delta_{\mathcal{B}}(t)$ and the control gain perturbations $\Delta_{\mathcal{K}_{\mathcal{M}}}(t)$ are set as Case 1) and Case 2) (see eqs.(20) and (21)). The results of the simulation of this example are depicted in Figs. 1–8. In these figures, Proposed represents the transient time-response, the manipulated control input and the actual control input generated by the proposed robust stabilizing controller. Furthermore, Conventional shows time histories of the state, the manipulated control input and the actual control input for the conventional quadratic stabilizing controller designed without thinking of control gain perturbations.

From Figs. 1–4, we find that both the proposed robust stabilizing controller (Proposed in figures) and the conventional quadratic stabilizing controller (Conventional in figures) stabilize the uncertain linear system eq.(16) under the control gain perturbation eq.(20).

On the other hand, we see from Figs. 5–8 that though Conventional cannot stabilize the uncertain system eq.(16) under the control gain perturbation eq.(21), Proposed stabilizes it. Namely this result shows that al-

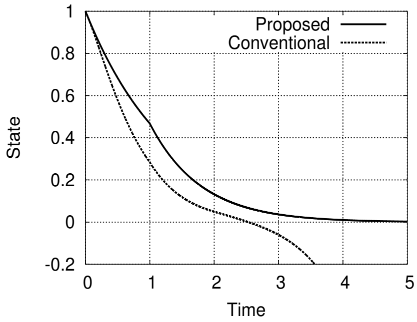

Fig. 5. Transient time-response of the state variable $x_1(t)$: Case 2)

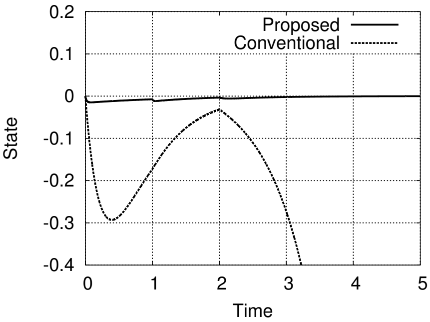

Fig. 6. Transient time-response of the state variable $x_2(t)$: Case 2)

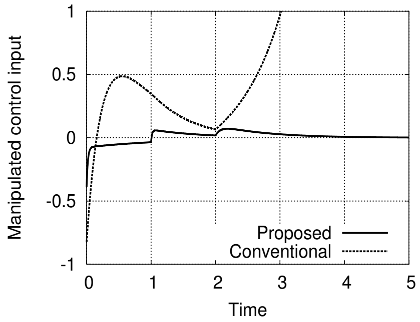

Fig. 7. Time histories of the manipulated input $u(t) \stackrel{\triangle}{=} Kx(t)$: Case 2)

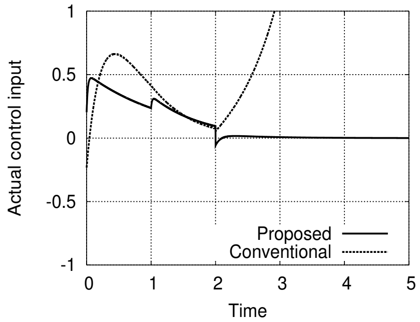

Fig. 8. Time histories of the actual control input $u(t) \stackrel{\triangle}{=} K(t)x(t)$: Case 2)

though, the conventional quadratic stabilizing controller designed without thinking of control gain perturbations is fragile under the control gain perturbation, the proposed robust controller is not fragile.

Therefore the effectiveness of the proposed robust non-fragile stabilizing controller is shown.

VI. Conclusions

In this paper, a LMI-based design method of a robust non-fragile stabilizing controller for linear continuous-time systems with structured uncertainties which are included in both the system matrix and the input one under multiplicative or additive control gain variations has been presented. Furthremore, simple examples are given for illustration of the proposed controller design, and the simulation result has shown that the closed-loop system is well stabilized in spite of plant uncertainties and control gain variations.

We have shown that the proposed robust non-fragile controller can be easily obtained by solving LMI conditions. Therefore, the proposed robust controller for linear systems with structured uncertainties and control gain perturbations can be easily obtained by using software such as MATLAB's LMI Control Toolbox and Scilab's LMITOOL. Moreover, the proposed design method can be easily extended to robust non-fragile $\mathcal{H}^\infty$ controllers.

The future research subjects are the extension of proposed controller to uncertain large-scale systems, uncertain discrete-time systems and so on.

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$$
\begin{pmatrix}
\Gamma(\mathcal{S},\mathcal{W},\gamma,\delta,\mu) & E & \mathcal{S}C^T+\mathcal{W}^TD^T & \mathcal{S}\mathcal{L}^T & \mathcal{W}^T\mathcal{M}^T & \mathcal{W}^T\mathcal{N}_{\mathcal{M}}^T & \mathcal{W}^T\mathcal{N}_{\mathcal{M}}^T & 0 & \mathcal{E} & \mathcal{W}^T\mathcal{N}_{\mathcal{M}}^T \\
E^T & -(\gamma^*)^2I_w & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C\mathcal{S}+D\mathcal{W} & 0 & -\Pi(\nu) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathcal{L}\mathcal{S} & 0 & 0 & -\gamma I_q & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathcal{M}\mathcal{W} & 0 & 0 & 0 & -\delta I_s & 0 & 0 & 0 & 0 & 0 \\
\mathcal{N}_{\mathcal{M}}\mathcal{W} & 0 & 0 & 0 & 0 & -\mu I_m & 0 & 0 & 0 & 0 \\
\mathcal{N}_{\mathcal{M}}\mathcal{W} & 0 & 0 & 0 & 0 & 0 & -\nu I_m & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\Upsilon_{\mathcal{M}}(\xi) & 0 & 0 \\
\mathcal{E}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\xi I_r & 0 \\
\mathcal{N}_{\mathcal{M}}\mathcal{W} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\xi I_{l_{\mathcal{M}}}
\end{pmatrix}
< 0 \tag{A.3}
$$

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Appendix

A. Extention to Robust Non-Fragile $\mathcal{H}^\infty$ Controllers

In this appendix, we extend the proposed design method of robust non-fragile stabilizing controllers to robust non-fragile $\mathcal{H}^\infty$ controllers.

Consider the uncertain linear dynamical system described by the following state equation.

$$
\begin{aligned}
\frac{d}{dt}x(t) &= A(t)x(t) + B(t)u(t) + Ew(t) \\
z(t) &= Cx(t) + Du(t)
\end{aligned} \tag{A.1}
$$

where $x(t) \in \Re^n$ and $u(t) \in \Re^m$ are the vectors of the state (assumed to be available for feedback) and the control input, respectively. In eq.(A.1), $z(t) \in \Re^z$ and $w(t) \in \Re^w$ are the controlled output and the disturbance input which is assumed to be square integrable, i.e. $w(t) \in \mathcal{L}_2[0,\infty)$, respectively and the matrices $A(t)$ and $B(t)$ are given by eq.(2). Furthermore, the matrices C, D and E have appropriate dimensions.

In this appendix, we consider the design problem of a robust state feedback $\mathcal{H}^\infty$ control with multiplicative control gain variations eq.(4) only, because the design method for the uncertain system with additive control gain perturbations (5) can be easily obtained by similar way for the multiplicative case.

From eqs.(3) and (A.1), the closed-loop system with uncertainties and control gain variations is given by

$$
\frac{d}{dt}x(t) = (A(t) + B(t)K(t))x(t) + Ew(t) \tag{A.2}
$$

For the problem of robust non-fragile state feedback $\mathcal{H}^\infty$ control with control gain perturbations, namely, the controller of the form eq.(4) with the feedback gain matrix K to be designed, the following theorem gives a LMI-based design method.

Theorem A.1: Consider the uncertain linear system eq.(A.1) with the multiplicative control gain variations eq.(4). There exists the state feedback gain matrix K (it is given by $K \stackrel{\triangle}{=} \mathcal{W}\mathcal{S}^{-1}$, if there exist) such that the closed-loop system eq.(A.2) is internally stable and the relation $||z(t)||_{\mathcal{L}_2} < \gamma^*||w(t)||_{\mathcal{L}_2}$ holds, if for a constant scalar $\gamma^* > 0$, there exist the matrices $\mathcal{S} > 0$ and $\mathcal{W}$ and the scalars $\gamma > 0, \delta > 0, \mu > 0, \nu > 0$ and $\xi > 0$ satisfying the LMI eq.(A.3). In eq.(A.3), the matrix $\Gamma(\mathcal{S},\mathcal{W},\gamma,\delta,\mu)$ is given by eq.(8) and $\Pi(\nu)$ and $\Upsilon_{\mathcal{M}}(\xi)$ are the matrices expressed as eqs.(A.5) and (A.6), respectively.

$$
\Pi(\nu) \stackrel{\triangle}{=} I_z - \nu\epsilon_{\mathcal{M}}D\mathcal{F}_{\mathcal{M}}\mathcal{F}_{\mathcal{M}}^TD^T \tag{A.5}
$$

$$
\Upsilon_{\mathcal{M}}(\xi) \stackrel{\triangle}{=} I_s - \xi(I_s + \epsilon_{\mathcal{M}}\mathcal{M}\mathcal{F}_{\mathcal{M}}\mathcal{F}_{\mathcal{M}}^T\mathcal{M}^T) \tag{A.6}
$$

Proof: We consider the quadratic function $\mathcal{V}(x,t) \stackrel{\triangle}{=} x^T(t)\mathcal{P}x(t)$. By evaluating the time derivative of the quadratic function $\mathcal{V}(x,t)$ along the trajectory of the uncertain closed-loop system eq.(A.2) we consider the following Hamiltonian.

$$
\mathcal{H}(z,w) \stackrel{\triangle}{=} \frac{d}{dt}\mathcal{V}(x,t) + z^T(t)z(t) - (\gamma^*)^2w^T(t)w(t) \tag{A.7}
$$

Introducing two matrices $\mathcal{S} \stackrel{\triangle}{=} \mathcal{P}^{-1}$ and $\mathcal{W} \stackrel{\triangle}{=} K\mathcal{S}$ and using the similar procedure for proof of **Theorem 1**, we easily see that if there exist the matrices $\mathcal{S} > 0$ and $\mathcal{W}$ and the scalars $\gamma > 0, \delta > 0, \mu > 0, \nu > 0$ and $\xi > 0$ satisfying the LMI condition eq.(A.3), then the closed-loop system eq.(A.2) with uncertainties and control gain variations is internally stable and the following relation holds.

$$
\mathcal{H}(z,w) < 0 \tag{A.8}
$$

By integrating both sides of the inequality eq.(A.8) from 0 to ∞ with $x(0) = 0$, we easily see from $\mathcal{V}(x,0) = 0$ that

$$
\int_0^\infty z^T(t)z(t)dt - (\gamma^*)^2\int_0^\infty w^T(t)w(t)dt + \mathcal{V}(x,\infty) < 0 \tag{A.9}
$$

The relation eq.(A.9) means

$$
||z(t)||_{\mathcal{L}_2} < \gamma^*||w(t)||_{\mathcal{L}_2} \tag{A.10}
$$

From the above discussion, if there exist the matrices $\mathcal{S} > 0, \mathcal{W}, \gamma > 0, \delta > 0, \mu > 0$ and $\nu > 0$ satisfying the LMI condition eq.(A.3) then the closed-loop system eq.(A.2) with uncertainties and control gain variations eq.(4) is internally stable and the relation eq.(A.10) holds.

It follows that the result of the theorem is true. Therefore the proof of **Theorem A.1** is completed. ∎