New Method for Composite Optimal Control of Singularly Perturbed Systems

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Abstract. In this paper, a new method based on a generalized algebraic Riccati equation arising in descriptor systems is presented to solve the composite optimal control problem of singularly perturbed systems. Different from the existing method, the slow subsystem is viewed as a special kind of descriptor systems. A new composite optimal controller is obtained which is valid for both standard and nonstandard singularly perturbed systems. It is shown that the composite optimal control can be obtained simply by only revising the solution of the slow regulator problem. It is proven that the composite optimal control can achieve a performance which is $O(\varepsilon^2)$ close to the optimal performance. Although this result is well-known for the standard singularly perturbed systems, it is new in the nonstandard case. The equivalence between the new composite optimal controller and the existing one is also established for the standard singularly perturbed systems.

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1. Introduction

The theory of optimal control for standard singularly perturbed systems has been well-developed(cf. Kokotović, Khalil and O'Reilly 1986, and the references therein). Among many results available, a well-known result states that there exists the composite optimal control which can achieve an $O(\varepsilon^2)$ approximation of the optimal performance. Recently, there has been interest in nonstandard singularly perturbed systems (Wang *et al.* 1988, Wang and Frank 1992, Wang *et al.* 1994 and Khalil, 1989). In Wang *et al.* (1988) and Wang and Frank (1992), the linear-quadratic regulator problem for nonstandard singularly perturbed systems is studied. The sub-optimal control, where only the slow regulator problem is considered, is proven to have the property of $O(\varepsilon)$ near-optimality. The results are then extended to the near-optimal control problem of nonstandard multiparameter/multitime scale singularly perturbed systems (Wang *et al.* 1994). On the other hand, static and dynamic feedback stabilizing control of nonstandard singularly perturbed systems are investigated by Khalil (1989).

In view of the studies above, one natural question here is whether there exists the composite optimal control for nonstandard singularly perturbed systems. In this paper, based on a generalized algebraic Riccati equation arising in descriptor systems (Wang *et al.* 1993, and Xu and Mizukami 1994), we study the composite optimal control problem for singularly perturbed systems. A new composite optimal controller is obtained which is valid for both standard and nonstandard singularly perturbed systems. It is shown that the composite optimal control can be obtained by revising the solution of the slow regulator problem. Since the slow subsystem is a special kind of descriptor systems, a similar design procedure to the linear regulator of descriptor systems can be used to find the composite optimal controller. As in standard singularly perturbed systems, we prove that the composite optimal control can achieve a performance which is $O(\varepsilon^2)$ close to the optimal performance even if the system is a nonstandard singularly perturbed system. Also, we prove that the new composite optimal controller is equivalent to the existing one in the case of the standard singularly perturbed systems. Therefore, we claim that the new composite optimal controller includes the existing composite optimal controller (Kokotović, Khalil and O'Reilly 1986) as a special case.

This paper is organized as follows. In next section, we will derive the optimal feedback control for the full-order problem by using a generalized Hamilton-Jacobi equation. In Section 3, the full-order problem is decomposed into a slow regulator problem and a fast regulator problem. Their solutions are investigated. In Section 4, we will construct the new composite optimal controller and discuss its near-optimality property. The equivalence between the new composite optimal controller and the existing one is established for the standard singularly perturbed systems. Section 5 is concerned with the design procedure of the composite optimal controller. Finally, Section 6 discusses some conclusions.

2. Full-Order Regulator Problem

Consider the linear time-invariant singularly perturbed system

$$\dot{x} = A_{11}x + A_{12}z + B_1u, \;\; x(0) = x_0, \tag{1a}$$

$$\varepsilon\dot{z} = A_{21}x + A_{22}z + B_2u, \;\; z(0) = z_0, \tag{1b}$$

with a performance index

$$J = \frac{1}{2}\int_0^\infty \left(\begin{bmatrix} x \\ z \end{bmatrix}^T Q \begin{bmatrix} x \\ z \end{bmatrix} + u^T Ru\right) dt, \tag{2}$$

which has to be minimized, where

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} = \begin{bmatrix} C_1^T C_1 & C_1^T C_2 \\ C_2^T C_1 & C_2^T C_2 \end{bmatrix}, \quad R > 0, \tag{3}$$

and ε is a small positive parameter, $x(t) \in R^n$ and $z(t) \in R^m$ are states, and $u(t) \in R^r$ is the control, and all matrices are of appropriate dimensions. The system (1) is called the nonstandard singularly perturbed system if the matrix A_{22} is singular.

In order to compare the near-optimal performance with the optimal performance in Section 4, we must have an exact expression of the optimal performance. Different from the existing method (Kokotović *et al.* 1986 and Chow and Kokotović 1976), we derive the optimal feedback control for the full-order problem by using a generalized Hamilton-Jacobi equation (Xu and Mizukami 1993), and arrive at the optimal performance with the new form. Before doing that, we make a temporary assumption that the final time of the performance (2) is finite and fixed for the convenience of presentation. Let the optimal performance index for the full-order problem take the form

$$\mathcal{V}^*(E_\varepsilon y(t), t) = (1/2) y^T(t) E_\varepsilon P(t) y(t),$$

where $y(t) = [x^T(t) \;\; z^T(t)]^T$,

$$E_\varepsilon = \begin{bmatrix} I_n & 0 \\ 0 & \varepsilon I_m \end{bmatrix},$$

is a symmetric matrix with I_n, I_m denoting the $n \times n, m \times m$ identity matrices, and $P(t)$ is the $(n+m) \times (n+m)$ time-varying matrix satisfying the condition

$$E_\varepsilon P(t) = P^T(t) E_\varepsilon.$$

Then, applying the generalized Hamilton-Jacobi equation (Xu and Mizukami 1993)

$$\frac{\partial \mathcal{V}^*}{\partial t} = -\min_{u(t)} \{L(y(t), u(t), t) + \mathcal{W}^*(y(t)) f(y(t), u(t), t)\}, \tag{4a}$$

$$\{\frac{\partial \mathcal{V}^*}{\partial y}\}^T = \mathcal{W}^*(y(t)) E_\varepsilon, \tag{4b}$$

to the full-order regulator problem, where

$$L(y(t), u(t), t) = (1/2)(y^T Q y + u^T R u), \tag{5a}$$

$$f(y(t), u(t), t) = Ay + Bu, \tag{5b}$$

$$\mathcal{W}^*(y(t), t) = y^T P^T(t), \tag{5c}$$

and

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \; B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \tag{6}$$

we have

$$y^T E_\varepsilon \dot{P} y = -\min_{u(t)} \{y^T Q y + u^T R u + 2 y^T P^T (Ay + Bu)\}. \tag{7}$$

Carrying out the minimization on the right-hand side of (7) gives

$$u^*(t) = -R^{-1} B^T P(t) y(t). \tag{8}$$

Substituting (8) into (7) and utilizing the relation

$$2 y^T P^T A y = y^T (P^T A + A^T P) y, \tag{9}$$

yield

$$y^T E_\varepsilon \dot{P} y = -y^T [Q + A^T P + P^T A - P^T B R^{-1} B^T P] y. \tag{10}$$

Since the above equation holds for all $y(t)$, we arrive at a generalized differential Riccati equation

$$(i)\ E_\varepsilon \dot{P} = -Q - A^T P - P^T A + P^T B R^{-1} B^T P, \tag{11a}$$

$$(ii)\ E_\varepsilon P = P^T E_\varepsilon, \tag{11b}$$

where the boundary condition is omitted since we will consider the infinite-horizon problem from now on. Taking into account the limiting case of (11), we obtain a generalized algebraic Riccati equation as follows.

$$(i)\ A^T P + P^T A - P^T B R^{-1} B^T P + Q = 0, \tag{12a}$$

$$(ii)\ E_\varepsilon P = P^T E_\varepsilon. \tag{12b}$$

Corresponding to the parameter matrices of (12), P has the following partitioned form

$$P = \begin{bmatrix} P_{11} & \varepsilon P_{21}^T \\ P_{21} & P_{22} \end{bmatrix},\ P_{11} = P_{11}^T,\ P_{22} = P_{22}^T, \tag{13}$$

since it satisfies (12b). It is worthy to note that P is not symmetric, but $E_\varepsilon P$ is. From the derivations above, we have

Theorem 1. Suppose that there exists a small positive parameter ε^* such that, for all $\varepsilon \in (0, \varepsilon^*)$, the generalized algebraic Riccati equation (12) admits a unique solution P for which $E_\varepsilon P \geq 0$. Then,

$$u^*(t) = -R^{-1} B^T P y(t), \tag{14}$$

constitutes the optimal feedback control for the full-order regulator problem, and the optimal performance is

$$J^* = \frac{1}{2} y^T(0) E_\varepsilon P y(0). \tag{15}$$

Remark 1. The existence conditions of a unique solution P, for all $\varepsilon \in (0, \varepsilon^*)$, will be given in Section 4.

3. Decomposition of Slow and Fast Regulator Problems

Similar to the standard singularly perturbed systems, we decompose the full-order regulator problem into two subsystem regulator problems.

Slow regulator problem: Find u_s to minimize

$$J_s = \frac{1}{2}\int_0^\infty \left(\begin{bmatrix} x_s \\ z_s \end{bmatrix}^T Q \begin{bmatrix} x_s \\ z_s \end{bmatrix} + u_s^T R u_s \right) dt, \tag{16}$$

for the slow subsystem

$$E\dot{y}_s = Ay_s + Bu_s, \quad Ey_s(0) = Ey_0, \tag{17}$$

where $y_s(t) = [x_s^T(t) \;\; z_s^T(t)]^T$, $E = E_\varepsilon|_{\varepsilon=0}$, A, B are defined in (6), and Q in (3).

Remark 2. The slow subsystem (17) is formed by neglecting the fast mode, which is equivalent to letting $\varepsilon = 0$ in (1). Different from the existing method to decompose the full-order system into the slow and fast subsystems, we do not use the inverse of A_{22}, which does not exist in a nonstandard case, to eliminate z_s in (16),(17). The slow subsystem (17) is viewed as a descriptor system which may display an impulse phenomenon in the solution if A_{22} is singular. It is clear that the descriptor system method permits us to study the standard and nonstandard singularly perturbed system in a unified way.

Fast regulator problem: Find u_f to minimize

$$J_f = \frac{1}{2}\int_0^\infty (z_f^T C_2^T C_2 z_f + u_f^T R u_f) dt, \tag{18}$$

for the fast subsystem

$$\varepsilon \dot{z}_f = A_{22} z_f + B_2 u_f, \quad z_f(0) = z_0 - z_s(0), \tag{19}$$

where $z_f = z - z_s$, $u_f = u - u_s$.

The fast subsystem (19) is derived by assuming that the slow variables are constant during fast transients, that is, $\dot{z}_s = 0$ and $x_s =$ a constant.

We now consider the solution of the slow and fast regulator problems under the following assumptions.

Assumption 1. The slow subsystem (17) is stabilizable and detectable, that is, for all s with $Re[s] \geq 0$,

$$\textbf{Rank} \begin{bmatrix} sI_n - A_{11} & -A_{12} & B_1 \\ -A_{21} & -A_{22} & B_2 \end{bmatrix} = n + m, \tag{20a}$$

$$\mathbf{Rank}\begin{bmatrix} sI_n - A_{11}^T & -A_{21}^T & C_1^T \\ -A_{12}^T & -A_{22}^T & C_2^T \end{bmatrix} = n + m. \tag{20b}$$

Assumption 2. The fast subsystem (19) is stabilizable and detectable.

Let us first consider the solution of the fast regulator problem.

Proposition 1. Under Assumption 2, the fast regulator problem admits a unique optimal feedback control

$$u_f^* = -R^{-1}B_{22}P_{22f}^{+}z_f, \tag{21}$$

where P_{22f}^{+} is a unique stabilizing positive semidefinite symmetric solution of the algebraic Riccati equation

$$P_{22f}A_{22} + A_{22}^T P_{22f} - P_{22f}B_2R^{-1}B_2^T P_{22f} + Q_{22} = 0. \tag{22}$$

The slow regulator problem is similar to the regulator problem of descriptor systems except that different assumptions on the system parameters are required in two problems. In other words, for the existence of the fast regulator problem, Assumption 2 is reasonable in the study of the slow regulator problem. However, this assumption is unnecessarily strong in the study of the regulator problem for a general descriptor system. Instead of Assumption 2, we only need to assume that the system (17) is impulsively controllable and observable if it is a general descriptor system, that is,

$$\mathbf{Rank}[A_{22} \;\; B_2] = m, \quad \mathbf{Rank}[A_{22}^T \;\; C_2^T] = m. \tag{23}$$

Obviously, Assumption 2 implies the conditions (23), but not vice versa.

In the following, we will consider the solution of the slow regulator problem. Before doing that, we first introduce another generalized algebraic Riccati equation (Wang *et al.* 1993, and Xu and Mizukami 1994),

$$(i)\; A^T P_s + P_s^T A - P_s^T BR^{-1}B^T P_s + Q = 0, \tag{24a}$$

$$(ii)\; E^T P_s = P_s^T E. \tag{24b}$$

where Q is the same as that in (12). The solution P_s of (24) has a lower-triangular block form

$$P_s = \begin{bmatrix} P_{11s} & 0 \\ P_{21s} & P_{22s} \end{bmatrix}, \quad P_{11s}^T = P_{11s}, \tag{25}$$

because of (24b). It is worthy to note that P_{22s} may not be symmetric. The algebraic Riccati equation (24) can be partitioned into

$$P_{11s}A_{11} + P_{21s}^TA_{21} + A_{11}^TP_{11s} + A_{21}^TP_{21s} - P_{11s}S_{11}P_{11s}$$

$$-P_{21s}^TS_{12}^TP_{11s} - P_{11s}S_{12}P_{21s} - P_{21s}^TS_{22}P_{21s} + Q_{11} = 0, \tag{26a}$$

$$P_{22s}^TA_{21} + A_{12}^TP_{11s} + A_{22}^TP_{21s} - P_{22s}^TS_{12}^TP_{11s} - P_{22s}^TS_{22}P_{21s} + Q_{12}^T = 0, \tag{26b}$$

$$P_{22s}^TA_{22} + A_{22}^TP_{22s} - P_{22s}^TS_{22}P_{22s} + Q_{22} = 0, \tag{26c}$$

where

$$S_{11} = B_1R^{-1}B_1^T, \quad S_{12} = B_1R^{-1}B_2^T, \quad S_{22} = B_2R^{-1}B_2^T. \tag{27}$$

The equation (26c) is an algebraic Riccati equation which admits at least a stabilizing positive semidefinite symmetric solution under Assumption 2. Moreover, Assumption 2 ensures that $A_{22} - S_{22}P_{22s}$ is nonsingular (see proof of Lemma 1 below). Substituting the solution of (26c) into (26b) yields

$$P_{21s} = -N_{2s}^T + N_{1s}^TP_{11s}, \tag{28}$$

where

$$N_{2s}^T = \hat{A}_{22s}^{-T}\hat{Q}_{12s}^T, \quad N_{1s}^T = -\hat{A}_{22s}^{-T}\hat{A}_{12s}^T,$$

$$\hat{A}_{12s} = A_{12} - S_{12}P_{22s}, \ \hat{A}_{22s} = A_{22} - S_{22}P_{22s},$$

$$\hat{Q}_{12s} = Q_{12} + A_{21}^TP_{22s}.$$

Furthermore, substituting P_{21s} into (26a) and making some lengthy calculations (the detail is omitted for brevity), we get

$$P_{11s}A_s + A_s^TP_{11s} - P_{11s}S_sP_{11s} + Q_s = 0, \tag{29}$$

where

$$Q_s = Q_{11} - N_{2s}A_{21} - A_{21}^TN_{2s}^T - N_{2s}S_{22}N_{2s}^T, \tag{30a}$$

$$A_s = A_{11} + N_{1s}A_{21} + S_{12}N_{2s}^T + N_{1s}S_{22}N_{2s}^T, \tag{30b}$$

$$S_s = S_{11} + N_{1s}S_{12}^T + S_{12}N_{1s}^T + N_{1s}S_{22}N_{1s}^T. \tag{30c}$$

Lemma 1. Under Assumptions 1,2, the following results hold.

(i) The algebraic Riccati equations (29) is decoupled to the algebraic Riccati equation (26c).

(ii) There exist a $n \times r$ matrix B_s and a matrix C_s with the same dimension as C_1 such that $S_s = B_s R^{-1} B_s^T$, $Q_s = C_s^T C_s$. Moreover, the triple (A_s, B_s, C_s) is stabilizable and detectable.

Proof. see Appendix A. ■

Since the triple (A_s, B_s, C_s) is stabilizable and detectable, the algebraic Riccati equation (29) admits a unique stabilizing positive semidefinite symmetric solution, denoted by P_{11s}^+, and $A_s - S_s P_{11s}^+$ is Hurwitz.

Proposition 2. Under Assumptions 1, 2, the slow regulator problem admits a unique optimal open-loop control, which can be implemented by a class of linear feedback controls given by

$$u_s^* = -R^{-1}B^T P_s y_s, \tag{31}$$

where

$$P_s = \begin{bmatrix} P_{11s}^+ & 0 \\ P_{21s} & P_{22s} \end{bmatrix}, \tag{32}$$

is the solution of the generalized algebraic Riccati equation (24).

Proof. First, we prove that the linear feedback controls (31) are strictly feedback stabilizing and under which $x_s(t), z_s(t)$ are impulse-free and $x_s(t) \to 0$, $z_s(t) \to 0$ as $t \to \infty$ for all Ey_0. Substituting (31) into (17) yields

$$\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_s \\ \dot{z}_s \end{bmatrix} = \begin{bmatrix} \hat{A}_{11s} & \hat{A}_{12s} \\ \hat{A}_{21s} & \hat{A}_{22s} \end{bmatrix} \begin{bmatrix} x_s \\ z_s \end{bmatrix}, \tag{33}$$

where $\hat{A}_{12s}$, $\hat{A}_{22s}$ are defined under the equation (28), and

$$\hat{A}_{11s} = A_{11} - S_{11}P_{11s}^+ - S_{12}P_{21s},\ \hat{A}_{21s} = A_{21} - S_{21}P_{11s}^+ - S_{22}P_{21s}.$$

Since $\hat{A}_{22s}$ is nonsingular from the proof of Lemma 1, (33) can be further transformed to a standard state space system and an algebraic equation

$$\dot{x}_s = (\hat{A}_{11s} - \hat{A}_{12s}\hat{A}_{22s}^{-1}\hat{A}_{21s})x_s, \tag{34a}$$

$$z_s = -\hat{A}_{22s}^{-1}\hat{A}_{21s}x_s. \tag{34b}$$

Moreover, $x_s(t)$, $z_s(t)$ are impulse free for the same reason. Now, if we can prove that the system matrix of (34a) is $A_s - S_s P_{11s}^{+}$, then we have $x_s(t) \to 0$, as $t \to \infty$. That also implies $z_s(t) \to 0$ as $t \to \infty$. Substituting (28) into the system matrix of (34a) and making some manipulations yield

$$\hat{A}_{11s} - \hat{A}_{12s}\hat{A}_{22s}^{-1}\hat{A}_{21s} = A_{11} - S_{11}P_{11s}^{+} - S_{12}(N_{1s}^T P_{11s}^{+} - N_{2s}^T)$$

$$+N_{1s}[A_{21} - S_{21}P_{11s}^{+} - S_{22}(N_{1s}^T P_{11s}^{+} - N_{2s}^T)]$$

$$= A_{11} + N_{1s}A_{21} + S_{12}N_{2s}^T + N_{1s}S_{22}N_{2s}^T - (S_{11} + N_{1s}S_{12}^T + S_{12}N_{1s}^T + N_{1s}S_{22}N_{1s}^T)P_{11s}^{+}$$

$$= A_s - S_s P_{11s}^{+}. \tag{35}$$

Therefore, (31) are the strictly feedback stabilizing controls.

Secondly, from the derivations before the proposition, we have known that P_s is a solution of the generalized algebraic Riccati equation (24).

Finally, we will prove that (31) is really a optimal feedback control by using a "completion of squares" method. Utilizing the algebraic Riccati equation (24) in (16), (17), we have

$$\frac{1}{2}y_s^T(T)E^T P_s y_s(T) - \frac{1}{2}y_0^T E^T P_s y_0 = \frac{1}{2}\int_0^T \frac{d}{dt}[y_s^T E^T P_s y_s]dt$$

$$= \frac{1}{2}\int_0^T y_s^T[-A^T P_s - P_s^T A + P_s^T B R^{-1} B^T P_s - Q]y_s dt$$

$$+\frac{1}{2}\int_0^T [\dot{y}_s^T E^T P_s y_s + y_s^T P_s^T E \dot{y}_s]dt, \tag{36}$$

Substituting $E\dot{y}_s(t)$ into (36) yields

$$\frac{1}{2}y_s^T(T)E^T P_s y_s(T) - \frac{1}{2}y_0^T E^T P_s y_0$$

$$= \frac{1}{2}\int_0^T [u_s^T B^T P_s y_s + y_s^T P_s^T B u_s - y_s^T Q y_s + y_s^T P_s^T B R^{-1} B^T P_s y_s]dt. \tag{37}$$

Moving the left side of (37) to the right and adding $J_s(u_s;T)$ to the both sides of it give

$$J_s(u_s;T) = \frac{1}{2}y_0^T E^T P_s y_0 - \frac{1}{2}y_s^T(T)E^T P_s y_s(T)$$

$$+\frac{1}{2}\int_0^T \{[u_s + R^{-1}B^T P_s y_s]^T R[u_s + R^{-1}B^T P_s y_s]\}dt. \tag{38}$$

Letting $T \to \infty$ and noting $y_s(t) \to 0$ as $t \to \infty$ for all admissible u_s, we have

$$J_s(u_s;\infty) = \frac{1}{2} y_0^T E^T P_s y_0 + \frac{1}{2}\int_0^\infty \{[u_s + R^{-1}B^T P_s y_s]^T R[u_s + R^{-1}B^T P_s y_s]\} dt. \tag{39}$$

Since the first term above is independent of u_s and $E^T P_s$ is unique (since P_{11s}^+ is unique), it is obvious that u_s^* given by (31) is the optimal feedback control. ■

Remark 2. The results of Proposition 2 can also be deduced from the corresponding results of the regulator problem of descriptor systems (Wang *et al.* 1993 and Xu and Mizukami 1994). However, the proof there is not so complete. Here, we have provided a different, complete and self-contained proof.

Remark 3. Similar to descriptor systems, an important feature of the slow regulator problem is that the optimal feedback controls are not unique. This fact is easily seen by noting that any solution of the algebraic Riccati equation (26c) can be used in the feedback gain of (31). But $E^T P_s = P_s^T E \geq 0$ is unique since only $P_{11s}^+ \geq 0$, the unique stabilizing positive semidefinite symmetric solution of (29), is allowed in the feedback gain of (31).

4. Near-Optimality of Composite Optimal Control

In this section, we will construct the composite optimal control $u_c^* = u_s^* + u_f^*$ as in standard case (Kokotović *et al.* 1986). It has been known that the optimal feedback control for the slow regulator problem is not unique. However, the corresponding optimal feedback control for the fast regulator problem is unique. We will select a particular optimal feedback control of the slow regulator problem to construct the composite optimal control, that is,

$$u_s^{*+} = -R^{-1}B^T P_s^+ y_s, \tag{40}$$

where

$$P_s^+ = \begin{bmatrix} P_{11s}^+ & 0 \\ P_{21s}^+ & P_{22s}^+ \end{bmatrix}, \tag{41}$$

P_{22s}^+ is the unique stabilizing positive semidefinite symmetric solution of the algebraic Riccati equation (26c), and P_{21s}^+ is the corresponding one in (28) with $P_{22s} = P_{22s}^+$, $P_{11s} = P_{11s}^+$.

Now, comparing (26c) with (22), we readily have an important relation $P_{22s}^+ = P_{22f}^+$. As the result, we obtain

$$u_c^* = u_s^{*+} + u_f^* = -R^{-1}[B_1^T \quad B_2^T] \begin{bmatrix} P_{11s}^+ & 0 \\ P_{21s}^+ & P_{22s}^+ \end{bmatrix} \begin{bmatrix} x_s \\ z_s \end{bmatrix} - R^{-1}B_2^T P_{22f}^+ z_f$$

$$= -R^{-1}[B_1^T \quad B_2^T] \begin{bmatrix} P_{11s}^{+} & 0 \\ P_{21s}^{+} & P_{22s}^{+} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}, \tag{42}$$

where $x(t) \approx x_s(t)$ and $z(t) \approx z_s(t) + z_f(t)$.

Remark 4. Let us compare (42) with (40). Then, we can find that u_s^{*+} is different from u_c^* only in P_{22s}, P_{21s} and y_s. This fact implies that $u_c^*(t)$ can be obtained very simply by solving the slow regulator problem (the design procedure will be given in Section 5), and then revising its solution. In other words, we can select P_{22s}^{+} as the solution of (26c), and change the slow variable y_s to the original variable y in (40) to obtain the composite optimal control u_c^*.

We now apply the composite optimal control u_c^* to the full-order system (1) and compare it with the exact optimal control (14). In order to do that, we first study the existence conditions of the unique solution P of the generalized algebraic Riccati equation (12).

Theorem 2. Under Assumptions 1, 2, there exists a small positive parameter ε^* such that, for all $\varepsilon \in [0, \varepsilon^*)$, the generalized algebraic Riccati equation (12) admits a unique stabilizing solution P for which $E_\varepsilon P \geq 0$. Moreover, the solution P possesses a power series expansion at $\varepsilon = 0$, that is,

$$P = \begin{bmatrix} P_{11}^{(0)} & \varepsilon P_{21}^{(0)T} \\ P_{21}^{(0)} & P_{22}^{(0)} \end{bmatrix} + \sum_{i=1}^{\infty} \frac{\varepsilon^i}{i!} \begin{bmatrix} P_{11}^{(i)} & \varepsilon P_{21}^{(i)T} \\ P_{21}^{(i)} & P_{22}^{(i)} \end{bmatrix}. \tag{43}$$

Proof. The algebraic Riccati equation (12) can be partitioned into

$$A_{11}^T P_{11} + P_{11} A_{11} + A_{21}^T P_{21} + P_{21}^T A_{21} - P_{11} S_{11} P_{11}$$

$$-P_{21}^T S_{22} P_{21} - P_{11} S_{12} P_{21} - P_{21}^T S_{12}^T P_{11} + Q_{11} = 0, \tag{44a}$$

$$\varepsilon P_{21} A_{11} + P_{22} A_{21} + A_{12}^T P_{11} + A_{22}^T P_{21} - \varepsilon P_{21} S_{11} P_{11}$$

$$-\varepsilon P_{21} S_{12}^T P_{21} - P_{22} S_{12}^T P_{11} - P_{22} S_{22} P_{21} + Q_{12}^T = 0, \tag{44b}$$

$$A_{22}^T P_{22} + P_{22} A_{22} + \varepsilon A_{12}^T P_{21}^T + \varepsilon P_{21} A_{12} - P_{22} S_{22} P_{22}$$

$$-\varepsilon P_{22} S_{12}^T P_{21}^T - \varepsilon P_{21} S_{12} P_{22} - \varepsilon^2 P_{21} S_{11}^T P_{21}^T + Q_{22} = 0. \tag{44c}$$

Let $\varepsilon = 0$, then the zero-order equations in (44) reduce to

$$P_{11}^{(0)} A_{11} + P_{21}^{(0)T} A_{21} + A_{11}^T P_{11}^{(0)} + A_{21}^T P_{21}^{(0)} - P_{11}^{(0)} S_{11} P_{11}^{(0)}$$

$$-P_{21}^{(0)T}S_{12}^TP_{11}^{(0)} - P_{11}^{(0)}S_{12}P_{21}^{(0)} - P_{21}^{(0)T}S_{22}P_{21}^{(0)} + Q_{11} = 0, \tag{45a}$$

$$P_{22}^{(0)}A_{21} + A_{12}^TP_{11}^{(0)} + A_{22}^TP_{21}^{(0)} - P_{22}^{(0)}S_{12}^TP_{11}^{(0)} - P_{22}^{(0)}S_{22}P_{21}^{(0)} + Q_{12}^T = 0, \tag{45b}$$

$$P_{22}^{(0)}A_{22} + A_{22}^TP_{22}^{(0)} - P_{22}^{(0)}S_{22}P_{22}^{(0)} + Q_{22} = 0. \tag{45c}$$

The zero-order equations in (45) are the same as those in (26) except that $P_{22}^{(0)}$ here is required to be symmetric. Hence, (45) has a solution $P_{11}^{(0)} = P_{11s}^+$, $P_{22}^{(0)} = P_{22s}^+$ and $P_{21}^{(0)} = P_{21s}^+$ under Assumptions 1 and 2. Furthermore, applying the implicit function theorem at the point $(\varepsilon = 0, P_{11} = P_{11}^{(0)}, P_{21} = P_{21}^{(0)}, P_{22} = P_{22}^{(0)})$ to the equations (44) yields the following Jacobian matrix.

$$J_{acobi} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}_{(0, P_{11}^{(0)}, P_{21}^{(0)}, P_{22}^{(0)})}. \tag{46}$$

Using the Kronecker product representation, we have

$$J_{31} = 0, \quad J_{32} = 0, \quad J_{33} = I_m \otimes (A_{22} - S_{22}P_{22}^{(0)}) + (A_{22} - S_{22}P_{22}^{(0)})^T \otimes I_m, \tag{47}$$

from (44c), and

$$J_{22} = (A_{22} - S_{22}P_{22}^{(0)})^T \otimes I_n, \tag{48}$$

from (44b). Furthermore, from (44b), we have

$$P_{21} = -N_2^T + N_1^TP_{11} + O(\varepsilon), \tag{49}$$

where

$$N_2^T = \hat{A}_{22}^{-T}\hat{Q}_{12}^T, \quad N_1^T = -\hat{A}_{22}^{-T}\hat{A}_{12}^T,$$

$$\hat{A}_{12} = A_{12} - S_{12}P_{22}, \ \hat{A}_{22} = A_{22} - S_{22}P_{22},$$

$$\hat{Q}_{12} = Q_{12} + A_{21}^TP_{22}.$$

Substituting (49) into (44a) and calculating its derivative with respect to P_{11} at the point $(\varepsilon = 0, P_{11} = P_{11}^{(0)}, P_{21} = P_{21}^{(0)}, P_{22} = P_{22}^{(0)})$ yield

$$J_{11} = I_n \otimes [A_{11} + N_{1s}A_{21} + S_{12}N_{2s}^T + N_{1s}S_{22}N_{2s}^T$$

$$-(S_{11}+N_{1s}S_{12}^T+S_{12}N_{1s}^T+N_{1s}S_{22}N_{1s}^T)P_{11s}^+]$$

$$+[A_{11}+N_{1s}A_{21}+S_{12}N_{2s}^T+N_{1s}S_{22}N_{2s}^T$$

$$-(S_{11}+N_{1s}S_{12}^T+S_{12}N_{1s}^T+N_{1s}S_{22}N_{1s}^T)P_{11s}^+]^T\otimes I_n$$

$$=I_n\otimes(A_s-S_sP_{11s}^+)+(A_s-S_sP_{11s}^+)^T\otimes I_n, \tag{50}$$

by noting $P_{11}^{(0)}=P_{11s}^+$, $P_{22}^{(0)}=P_{22s}^+$ and $P_{21}^{(0)}=P_{21s}^+$. The exact expressions of J_{13}, J_{21} and J_{23} are not important in the analysis of the nonsingularity of Jacobian matrix J_{acobi}. Since $(A_{22}-S_{22}P_{22s}^+)$ and $(A_s-S_sP_{11s}^+)$ are all Hurwitz matrices, J_{11}, J_{22} and J_{33} are nonsingular. Therefore,

$$J_{acobi}=\begin{bmatrix} J_{11} & 0 & J_{13} \\ J_{21} & J_{22} & J_{23} \\ 0 & 0 & J_{33} \end{bmatrix}, \tag{51}$$

is nonsingular. Consequently, there exists a small positive parameter ε^* such that, for all $\varepsilon\in[0,\varepsilon^*)$, the generalized algebraic Riccati equation (12) admits a unique stabilizing solution P for which $E_\varepsilon P\geq 0$. The property $E_\varepsilon P\geq 0$ follows from $Q\geq 0$ and $R>0$. The uniqueness of $E_\varepsilon P\geq 0$ follows from the fact that $P_{11}^{(0)}=P_{11s}^+$ and $P_{22}^{(0)}=P_{22s}^+$, hence $P_{21}^{(0)}=P_{21s}^+$, are all unique. Finally, the existence of the series (43) at $\varepsilon=0$ follows also from the implicit function theorem (Dieudonné 1982). ■

Remark 5. Similar to the standard case (Kokotović *et al.* 1986), the existence conditions for the solution of the full-order regulator problem are also described in terms of stabilizability-detectability conditions on the slow and fast regulator problems, which are ε-independent (Assumptions 1 and 2). Moreover, it has been proven that Assumption 1 is equivalent to the assumption that (A_s,B_s,C_s) is stabilizable and detectable (Lemma 1), a lower-order system condition.

Now, we can compare the composite optimal control u_c^* with the exact optimal control u^* and show the $O(\varepsilon^2)$ approximation of J^*. Applying the composite optimal control u_c^* to the full-order system (1), we have

$$J^c=\frac{1}{2}y^T(0)E_\varepsilon P_c y(0), \tag{52}$$

where P_c is the solution of the generalized Lyapunov equation

$$(i)\ (A-SP_s^+)^TP_c+P_c^T(A-SP_s^+)=-P_s^{+T}SP_s^+-Q, \tag{53a}$$

$$(ii)\ E_\varepsilon P_c = P_c^T E_\varepsilon, \tag{53b}$$

with $S = BR^{-1}B^T$.

Theorem 3. Under the conditions of Theorem 2, the first two terms of the power series of J^c and J^* at $\varepsilon = 0$ are the same, that is,

$$J^c = J^* + O(\varepsilon^2), \tag{54}$$

and hence the composite optimal control (42) is an $O(\varepsilon^2)$ near-optimal solution to the full-order regulator problem (1),(2).

Proof. Subtracting (12) from (53) and rearranging, we obtain a generalized Lyapunov equation for $W = P_c - P$:

$$(i)\ (A - SP_s^+)^T W + W^T(A - SP_s^+) = -(P - P_s^+)^T S(P - P_s^+), \tag{55a}$$

$$(ii)\ E_\varepsilon W = W^T E_\varepsilon. \tag{55b}$$

Also from the application of the implicit function theorem to (53), P_c possesses a power series at $\varepsilon = 0$. Thus W can also be extended as follows:

$$W = \begin{bmatrix} W_{11}^{(0)} & \varepsilon W_{21}^{(0)T} \\ W_{21}^{(0)} & W_{22}^{(0)} \end{bmatrix} + \sum_{i=1}^{\infty} \frac{\varepsilon^i}{i!} \begin{bmatrix} W_{11}^{(i)} & \varepsilon W_{21}^{(i)T} \\ W_{21}^{(i)} & W_{22}^{(i)} \end{bmatrix}. \tag{56}$$

From (41), (42) and the result that $P_{11}^{(0)} = P_{11s}^+$ and $P_{22}^{(0)} = P_{22s}^+$, we have

$$(P - P_s^+)^T S(P - P_s^+) = O(\varepsilon^2), \tag{57}$$

and, since $(A_{22} - S_{22}P_{22s}^+)$ and $(A_s - S_s P_{11}^+)$ are Hurwitz matrices, the substitution of (56) into (55) yields $W_{11}^{(0)} = 0$, $W_{21}^{(0)} = 0$, $W_{22}^{(0)} = 0$, and $W_{11}^{(1)} = 0$, $W_{21}^{(1)} = 0$, $W_{22}^{(1)} = 0$. Hence, $E_\varepsilon W = O(\varepsilon^2)$, which proves (54). ∎

We have therefore provided a complete theoretic analysis of the near-optimality of the composite optimal control for both standard and nonstandard singularly perturbed systems. To the end of this section, we will show that the composite optimal controller (42) is equivalent to the existing composite optimal controller (Kokotović *et al.* 1986) in the case of the standard singularly perturbed systems. Let A_{22} of (1) be nonsingular, that is, the standard singularly perturbed system. Then, the composite optimal controller is

$$u_c^* = -R^{-1} \begin{bmatrix} B_1^T & B_2^T/\varepsilon \end{bmatrix} \begin{bmatrix} K_s & 0 \\ \varepsilon K_m^T & \varepsilon K_f \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}$$

$$= -R^{-1} \begin{bmatrix} B_1^T & B_2^T \end{bmatrix} \begin{bmatrix} K_s & 0 \\ K_m^T & K_f \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}. \tag{58}$$

In the above, K_s is the unique stabilizing positive semidefinite symmetric solution of the algebraic Riccati equation

$$0 = -K_s(A_0 - B_0 R_0^{-1} N_0^T M_0) - (A_0 - B_0 R_0^{-1} N_0^T M_0)^T K_s$$
$$+K_s B_0 R_0^{-1} B_0^T K_s - M_0^T (I - N_0 R_0^{-1} N_0^T) M_0, \tag{59}$$

where

$$R_0 = R + N_0^T N_0, \tag{60a}$$

$$A_0 = A_{11} - A_{12} A_{22}^{-1} A_{21}, \quad B_0 = -A_{12} A_{22}^{-1} B_2 + B_1, \tag{60b}$$

$$M_0 = C_1 - C_2 A_{22}^{-1} A_{21}, \quad N_0 = -C_2 A_{22}^{-1} B_2, \tag{60c}$$

K_f is the unique stabilizing positive semidefinite solution of the algebraic Riccati equation

$$0 = -K_f A_{22} - A_{22}^T K_f + K_f B_2 R^{-1} B_2^T K_f - C_2^T C_2, \tag{61}$$

and K_m is

$$K_m = [K_s(B_1 R^{-1} B_2^T K_f - A_{12}) - (A_{21}^T K_f + C_1^T C_2)](A_{22} - B_2 R^{-1} B_2^T K_f)^{-1}. \tag{62}$$

Theorem 4. Suppose that the system (1) is the standard singularly perturbed system and Assumptions 1, 2 are satisfied. Then, the following identities hold.

$$K_f = P_{22s}^+, \quad K_m^T = P_{21s}^+, \quad K_s = P_{11s}^+, \tag{63}$$

and hence the composite optimal controller (58) is the same as the composite optimal controller (42).

Proof. see Appendix B. ■

From Theorem 4, we claim that the new composite optimal controller includes the existing composite optimal controller (Kokotović, Khalil and O'Reilly 1986) as the special case.

5. Design Procedure and Example

As stated in Remark 4, the composite optimal control $u_c^*(t)$ can be obtained by revising the solution of the slow regulator problem. Therefore, the design procedure of the composite optimal controller is similar to that of the regulator problem for descriptor systems (Wang *et al.* 1993). The basic steps are as follows.

Step 1. Calculate A_s, S_s, Q_s by using

$$H_s := \begin{bmatrix} A_s & -S_s \\ -Q_s & -A_s \end{bmatrix} = T_1 - T_2 T_4^{-1} T_3, \tag{64}$$

where the matrices $T_i, i = 1, 2, 3, 4$, are defined in Appendix A.

Step 2. Find the unique positive semidefinite stabilizing solutions P_{22s}^+, P_{11s}^+ of the algebraic Riccati equations (26c), (29), respectively.

Step 3. Calculate P_{21s}^+ in (28) by using P_{22s}^+, P_{11s}^+.

Step 4. Substitute $y_s(t)$ of (40) by $y(t)$ to obtain the composite optimal controller (42).

The main part (Step 2) of the above design procedure involves solving two reduced-order algebraic Riccati equations (26c), (29). Since (26c), (29) are decoupled algebraic Riccati equations (Lemma 1), parallel computations for the solutions are possible.

Example. Consider a nonstandard singularly perturbed system

$$\begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u(t), \quad x(0) = 1, \quad z(0) = 1. \tag{65}$$

The performance index to be minimized is

$$J = \frac{1}{2} \int_0^\infty \{ \begin{bmatrix} x \\ z \end{bmatrix}^T \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + u^2 \} dt. \tag{66}$$

It is obvious that the existing method (Kokotović *et al.* 1986) to find the composite optimal control is not valid for this example. However, it is solvable by using the method of this paper. Referring the design procedure, the matrices T_i, i=1,2,3,4, are, respectively,

$$T_1 = \begin{bmatrix} 1 & -4 \\ -4 & -1 \end{bmatrix}, \; T_2 = \begin{bmatrix} 2 & -2 \\ -2 & -2 \end{bmatrix},$$

$$T_3 = \begin{bmatrix} 2 & -2 \\ -2 & -2 \end{bmatrix}, \; T_4 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Hence

$$H_s = T_1 - T_2 T_4^{-1} T_3 = \begin{bmatrix} -7 & -4 \\ -4 & 7 \end{bmatrix}.$$

Associated with the Hamiltonian matrices T_4 and H_s are two completely decoupled algebraic Riccati equations,

$$(i) \qquad 1 - p_{22s}^2 = 0,$$

$$(ii) \qquad 2 - 7p_{11s} - 2p_{11s}^2 = 0,$$

where, the small letters are used to denote scalars. From (i), $p_{22s}^+ = 1$. On the other hand, (ii) has a unique positive semidefinite solution $p_{11s}^+ = (\sqrt{65} - 7)/4 \approx 0.2656$. Therefore, $p_{21s}^+ = 4$ from (28). The composite optimal control is

$$u_c^* = -[2\ 1] \begin{bmatrix} 0.2656 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}. \tag{67}$$

Now, letting $\varepsilon = 0.1$, the optimal feedback control is

$$u^* = -[2\ 1] \begin{bmatrix} 0.7217 & 0.2472 \\ 2.4723 & 0.9158 \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}. \tag{68}$$

The values of the performance index are $J^c = 0.6845$, $J^* = 0.6539$. Hence, the loss of performance $J^c = 0.6845$ is less than 4.48% compared with $J^* = 0.6539$. When $\varepsilon = 0.01$, the optimal feedback control becomes

$$u^* = -[2\ 1] \begin{bmatrix} 0.3544 & 0.0371 \\ 3.714 & 0.997 \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}. \tag{69}$$

The values of the performance index are $J^c = 0.2216$, $J^* = 0.2193$, with the loss less than 1.08%. For $\varepsilon = 0.001$, the optimal feedback control is

$$u^* = -[2\ 1] \begin{bmatrix} 0.2755 & 0.00397 \\ 3.9676 & 0.9997 \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}. \tag{70}$$

The values of the performance index are $J^c = 0.14227$, $J^* = 0.14224$, with the loss less than 0.0211%. These computation results show a trend that $u_c^* \to u^*$ and $J^c \to J^*$ as $\varepsilon \to 0$.

6. Conclusions

In this paper, using the generalized algebraic Riccati equation arising in descriptor systems, we have studied the composite optimal regulator problem for singularly perturbed systems. The new composite optimal controller has been obtained which is valid for both standard and nonstandard singularly perturbed systems. We show that the composite optimal control can be obtained simply by a procedure of solving the slow regulator problem, and then revising its solution. Moreover, the existence conditions for the solution of the full-order regulator problem can also be described in terms of the stabilizability-detectability conditions of the slow and fast regulator problems which are ε-independent and lower-order. As in standard singularly perturbed systems, we prove that the composite optimal control can achieve a performance which is $O(\varepsilon^2)$ close to the optimal performance even if the system is a nonstandard singularly perturbed system. Finally, we prove that the new composite optimal controller is equivalent to the existing one in the case of the standard singularly perturbed systems. Therefore, we claim that the new composite optimal controller includes the existing composite optimal controller (Kokotović, Khalil and O'Reilly 1986) as a special case.

Appendix A: proof of Lemma 1.

(i) Let us define four partitioned matrices (Wang *et al.* 1988)

$$T_1 = \begin{bmatrix} A_{11} & -S_{11} \\ -Q_{11} & -A_{11}^T \end{bmatrix}, \; T_2 = \begin{bmatrix} A_{12} & -S_{12} \\ -Q_{12} & -A_{21}^T \end{bmatrix}, \tag{71a}$$

$$T_3 = \begin{bmatrix} A_{21} & -S_{12}^T \\ -Q_{12}^T & -A_{12}^T \end{bmatrix}, \; T_4 = \begin{bmatrix} A_{22} & -S_{22} \\ -Q_{22} & -A_{22}^T \end{bmatrix}. \tag{71b}$$

Note that T_4 is a Hamiltonian matrix. Associate with T_4 is the algebraic Riccati equation (26c) or (22) which admits at least a symmetric positive semidefinite stabilizing solution P_{22s}^+ under Assumption 2. Let P_{22s} be an arbitrary solution of (26c). Then, we have

$$\begin{bmatrix} A_{22} & -S_{22} \\ -Q_{22} & -A_{22}^T \end{bmatrix} = \begin{bmatrix} I & 0 \\ P_{22s} & I \end{bmatrix} \begin{bmatrix} \hat{A}_{22s} & -S_{22} \\ 0 & -\hat{A}_{22s}^T \end{bmatrix} \begin{bmatrix} I & 0 \\ -P_{22s} & I \end{bmatrix}, \tag{72}$$

where $\hat{A}_{22s}$ is defined below (28). Since T_4 is nonsingular (Lemma 1, Wang *et al.* 1988), $\hat{A}_{22s}$

is also nonsingular. This means that T_4^{-1} can be expressed explicitly in terms of $\hat{A}_{22s}^{-1}$. Furthermore, the algebraic Riccati equation (29) corresponds to the Hamiltonian matrix, namely,

$$H_s := \begin{bmatrix} A_s & -S_s \\ -Q_s & -A_s^T \end{bmatrix}. \tag{73}$$

Therefore, it suffices the proof of (i) to show that $H_s = T_1 - T_2 T_4^{-1} T_3$. This can be done by a lengthy, but direct algebraic manipulations, which are omitted here for brevity.

(ii) From (30c), it is seen that $B_s = B_1 + N_{1s} B_2$. However, it seems difficult to find C_s from (30a). In order to do that, we introduce a dual algebraic Riccati equation of (26c), that is,

$$A_{22} K_{22s} + K_{22s} A_{22}^T - K_{22s} Q_{22} K_{22s} + S_{22} = 0, \tag{74}$$

which admits at least a symmetric positive semidefinite solution K_{22s}^+ under Assumption 2. Similar to the analysis of (72), we have

$$\begin{bmatrix} A_{22} & -S_{22} \\ -Q_{22} & -A_{22}^T \end{bmatrix} = \begin{bmatrix} I & -K_{22s} \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{A}_{22s} & 0 \\ -Q_{22} & -\tilde{A}_{22s}^T \end{bmatrix} \begin{bmatrix} I & K_{22s} \\ 0 & I \end{bmatrix}, \tag{75}$$

where $\tilde{A}_{22s} = A_{22}^T - Q_{22} K_{22s}$ is nonsingular since T_4 is nonsingular. After the calculation of $T_1 - T_2 T_4^{-1} T_3$, we arrive at another set of expressions for A_s, S_s and Q_s, that is,

$$Q_s = Q_{11} + Q_{12} M_{2s}^T + M_{2s} Q_{12}^T + M_{2s} Q_{22} M_{2s}^T, \tag{76a}$$

$$A_s = A_{11} + M_{1s} Q_{12}^T + A_{12} M_{2s}^T + M_{1s} Q_{22} M_{2s}^T, \tag{76b}$$

$$S_s = S_{11} - A_{12} M_{1s}^T - M_{1s} A_{12}^T - M_{1s} Q_{22} M_{1s}^T, \tag{76c}$$

$$M_{1s} = \tilde{S}_{12s} \tilde{A}_{22s}^{-1}, \quad M_{2s} = -\tilde{A}_{21s} \tilde{A}_{22s}^{-1}, \tag{76d}$$

$$\tilde{A}_{21s} = A_{21}^T - Q_{12} W_{22s}, \quad \tilde{S}_{12s} = S_{12} + A_{12}^T W_{22s}. \tag{76e}$$

Hence, it is easy to find $C_s = C_1 + C_2 M_{2s}^T$ from (76a).

Let us now prove the second part of (ii). Note the relation

$$\begin{bmatrix} I_n & -\hat{A}_{12s} \hat{A}_{22s}^{-1} \\ 0 & -\hat{A}_{22s}^{-1} \end{bmatrix} \begin{bmatrix} sI_n - A_{11} & -A_{12} & B_1 \\ -A_{21} & -A_{22} & B_2 \end{bmatrix}$$

$$\times \begin{bmatrix} I_n & 0 & 0 \\ -\hat{A}_{22s}^{-1}A_{21} & I_m & \hat{A}_{22s}^{-1}B_2 \\ -B_2^T P_{22s}\hat{A}_{22s}^{-1}A_{21} & B_2^T P_{22s} & I_r + B_2^T P_{22s}\hat{A}_{22s}^{-1}B_2 \end{bmatrix}$$

$$= \begin{bmatrix} sI_n - (A_{11} + N_{1s}A_{21}) & 0 & B_s \\ 0 & I_m & 0 \end{bmatrix}, \tag{77}$$

where the formulas under the equation (28) have been used in the above to simplify the expressions. Hence,

$$rank \begin{bmatrix} sI_n - A_{11} & -A_{12} & B_1 \\ -A_{21} & -A_{22} & B_2 \end{bmatrix} = n + m, \quad \text{for all} \quad s \quad \text{with} \quad Re[s] \geq 0, \tag{78}$$

if and only if $rank[sI_n - (A_{11} + N_{1s}A_{21}) \quad B_s] = n$, for all s with $Re[s] \geq 0$. In other words, the matrix pair $(A_{11} + N_{1s}A_{21}, B_s)$ is stabilizable. Since

$$A_s = A_{11} + N_{1s}A_{21} + B_s R^{-1} B_2^T N_{2s}^T,$$

and the feedback $R^{-1}B_2^T N_{2s}^T$ does not change the stabilizable property of $(A_{11} + N_1 A_{21}, B_s)$, we arrive at the conclusion that the matrix pair (A_s, B_s) is also stabilizable. Similarly, we can prove that (A_s, C_s) is detectable if and only if (20b) is satisfied. In this case, the formulas in (76) are used for the purpose. The detail is omitted for brevity. Thereby, we have finished the proof of Lemma 1. ■

Appendix B: proof of Theorem 4.

First, comparing (61) with (26c) yields $K_f = P_{22s}^+$ directly.

Second, comparing (62) with (28) and noting that $K_f = P_{22s}^+$, we have the conclusion that $K_m^T = P_{21s}^+$ if $K_s = P_{11s}^+$. Therefore, the remainder of the proof is to show that $K_s = P_{11s}^+$. In order to do that, we only need to show that the algebraic Riccati equations (29),(59) are the same equations, that is,

$$A_0 - B_0 R_0^{-1} N_0^T M_0 = A_s, \tag{79a}$$

$$B_0 R_0^{-1} B_0^T = S_s, \tag{79b}$$

$$M_0^T (I - N_0 R_0^{-1} N_0^T) M_0 = Q_s. \tag{79c}$$

Before showing these relations, let us define (pp.115, Kokotović et al. 1986)

$$H = I + R^{-1}B_2^T K_f (A_{22} - S_{22}K_f)^{-1} B_2. \tag{80}$$

Then,

$$H^{-1} = I - R^{-1}B_2^T K_f A_{22}^{-1} B_2, \tag{81}$$

and

$$\begin{aligned} R_0^{-1} &= HR^{-1}H^T \\ &= R^{-1} + R^{-1}B_2^T (A_{22} - S_{22}K_f)^{-T} K_f B_2 R^{-1} + R^{-1}B_2^T K_f (A_{22} - S_{22}K_f)^{-1} B_2 R^{-1} \\ &\quad + R^{-1}B_2^T K_f (A_{22} - S_{22}K_f)^{-1} B_2 R^{-1} B_2^T (A_{22} - S_{22}K_f)^{-T} K_f B_2 R^{-1}. \end{aligned} \tag{82}$$

Let us further introduce four useful identities.

$$A_{22}^{-1} + A_{22}^{-1} S_{22} K_f (A_{22} - S_{22}K_f)^{-1} = (A_{22} - S_{22}K_f)^{-1}, \tag{83a}$$

$$A_{22}^{-1} + (A_{22} - S_{22}K_f)^{-1} S_{22} K_f A_{22}^{-1} = (A_{22} - S_{22}K_f)^{-1}, \tag{83b}$$

$$I + S_{22}K_f (A_{22} - S_{22}K_f)^{-1} = A_{22}(A_{22} - S_{22}K_f)^{-1}, \tag{83c}$$

$$I + K_f S_{22} (A_{22} - S_{22}K_f)^{-T} = A_{22}^T (A_{22} - S_{22}K_f)^{-T}. \tag{83d}$$

Then,

$$\begin{aligned} N_0^T M_0 &= -B_2^T A_{22}^{-T} C_2^T (C_1 - C_2 A_{22}^{-1} A_{21}) \\ &= -B_2^T A_{22}^{-T} C_2^T C_1 + B_2^T A_{22}^{-T} [K_f S_{22} K_f - A_{22}^T K_f - K_f A_{22}] A_{22}^{-1} A_{21} \\ &= -B_2^T A_{22}^{-T} C_2^T C_1 - B_2^T A_{22}^{-T} K_f S_{22} K_f A_{22}^{-1} A_{21} - B_2^T K_f A_{22}^{-1} A_{21} - B_2^T A_{22}^{-T} K_f A_{21}. \end{aligned} \tag{84}$$

Combining (82) and (84), and utilizing the identities (83a,b) to simplify the corresponding expressions give

$$\begin{aligned} R_0^{-1} N_0^T M_0 &= -R^{-1}B_2^T (A_{22} - S_{22}K_f)^{-T} C_2^T C_1 - R^{-1}B_2^T (A_{22} - S_{22}K_f)^{-T} K_f A_{21} \\ &\quad - R^{-1}B_2^T K_f A_{22}^{-1} A_{21} - R^{-1}B_2^T K_f (A_{22} - S_{22}K_f)^{-1} S_{22} (A_{22} - S_{22}K_f)^{-T} C_2^T C_1 \\ &\quad - R^{-1}B_2^T K_f (A_{22} - S_{22}K_f)^{-1} S_{22} (A_{22} - S_{22}K_f)^{-T} K_f A_{21} \end{aligned}$$

$$-R^{-1}B_2^TK_f(A_{22}-S_{22}K_f)^{-1}S_{22}K_fA_{22}^{-1}A_{21}. \tag{85}$$

Hence,

$$A_0-B_0R_0^{-1}N_0^TM_0=A_{11}-A_{12}A_{22}^{-1}A_{21}-[B_1-A_{12}A_{22}^{-1}B_2]\times$$

$$[-R^{-1}B_2^T(A_{22}-S_{22}K_f)^{-T}C_2^TC_1-R^{-1}B_2^T(A_{22}-S_{22}K_f)^{-T}K_fA_{21}$$

$$-R^{-1}B_2^TK_fA_{22}^{-1}A_{21}-R^{-1}B_2^TK_f(A_{22}-S_{22}K_f)^{-1}S_{22}(A_{22}-S_{22}K_f)^{-T}C_2^TC_1$$

$$-R^{-1}B_2^TK_f(A_{22}-S_{22}K_f)^{-1}S_{22}(A_{22}-S_{22}K_f)^{-T}K_fA_{21}$$

$$-R^{-1}B_2^TK_f(A_{22}-S_{22}K_f)^{-1}S_{22}K_fA_{22}^{-1}A_{21}]$$

$$=A_{11}-A_{12}(A_{22}-S_{22}K_f)^{-1}A_{21}+S_{12}K_f(A_{22}-S_{22}K_f)^{-1}A_{21}$$

$$+S_{12}(A_{22}-S_{22}K_f)^{-T}C_2^TC_1+S_{12}(A_{22}-S_{22}K_f)^{-T}K_fA_{21}$$

$$+S_{12}K_f(A_{22}-S_{22}K_f)^{-1}S_{22}(A_{22}-S_{22}K_f)^{-T}C_2^TC_1$$

$$+S_{12}K_f(A_{22}-S_{22}K_f)^{-1}S_{22}(A_{22}-S_{22}K_f)^{-T}K_fA_{21}$$

$$-A_{12}(A_{22}-S_{22}K_f)^{-1}S_{22}(A_{22}-S_{22}K_f)^{-T}C_2^TC_1$$

$$-A_{12}(A_{22}-S_{22}K_f)^{-1}S_{22}(A_{22}-S_{22}K_f)^{-T}K_fA_{21}, \tag{86}$$

where, the identities (83a,b) have also been used to simplify the expressions. After expanding A_s of (30b), we arrive at the conclusion that A_s is the same as (86), which proves (79a). Now, considering (79b), we have

$$B_0H=(B_1-A_{12}A_{22}^{-1}B_2)[I+R^{-1}B_2^TK_f(A_{22}-S_{22}K_f)^{-1}B_2]$$

$$=B_1+S_{12}K_f(A_{22}-S_{22}K_f)^{-1}B_2-A_{12}[A_{22}^{-1}+A_{22}^{-1}S_{22}K_f(A_{22}-S_{22}K_f)^{-1}]B_2$$

$$=B_1+S_{12}K_f(A_{22}-S_{22}K_f)^{-1}B_2-A_{12}(A_{22}-S_{22}K_f)^{-1}B_2$$

$$=B_1-(A_{12}-S_{12}K_f)(A_{22}-S_{22}K_f)^{-1}B_2$$

$$=B_1+N_{1s}B_2. \tag{87}$$

Therefore,

$$
\begin{aligned}
B_0R_0^{-1}B_0^T &= B_0HR^{-1}H^TB_0^T \\
&= (B_1+N_{1s}B_2)R^{-1}(B_1+N_{1s}B_2)^T \\
&= B_sR^{-1}B_s^T = S_s,
\end{aligned} \tag{88}
$$

which proves (79b). Finally,

$$
\begin{aligned}
&-M_0^TN_0R_0^{-1}N_0^TM_0 = \\
&[-C_1^TC_2A_{22}^{-1}B_2 - A_{21}^TK_fA_{22}^{-1}B_2 - A_{21}^TA_{22}^{-T}K_fB_2 + A_{21}^TA_{22}^{-T}K_fS_{22}K_fA_{22}^{-1}B_2] \\
&\times[R^{-1}B_2^T(A_{22}-S_{22}K_f)^{-T}C_2^TC_1 + R^{-1}B_2^T(A_{22}-S_{22}K_f)^{-T}K_fA_{21} \\
&+R^{-1}B_2^TK_fA_{22}^{-1}A_{21} + R^{-1}B_2^TK_f(A_{22}-S_{22}K_f)^{-1}S_{22}(A_{22}-S_{22}K_f)^{-T}C_2^TC_1 \\
&+R^{-1}B_2^TK_f(A_{22}-S_{22}K_f)^{-1}S_{22}(A_{22}-S_{22}K_f)^{-T}K_fA_{21} \\
&+R^{-1}B_2^TK_f(A_{22}-S_{22}K_f)^{-1}S_{22}K_fA_{22}^{-1}A_{21}] \\
&= -C_1^TC_2A_{22}^{-1}S_{22}(A_{22}-S_{22}K_f)^{-T}C_2^TC_1 - C_1^TC_2A_{22}^{-1}S_{22}(A_{22}-S_{22}K_f)^{-T}K_fA_{21} \\
&-C_1^TC_2A_{22}^{-1}S_{22}K_f(A_{22}-S_{22}K_f)^{-1}S_{22}(A_{22}-S_{22}K_f)^{-T}C_2^TC_1 \\
&-C_1^TC_2A_{22}^{-1}S_{22}K_f(A_{22}-S_{22}K_f)^{-1}S_{22}(A_{22}-S_{22}K_f)^{-T}K_fA_{21} \\
&-C_1^TC_2A_{22}^{-1}S_{22}K_fA_{22}^{-1}A_{21} - C_1^TC_2A_{22}^{-1}S_{22}K_f(A_{22}-S_{22}K_f)^{-1}S_{22}K_fA_{22}^{-1}A_{21} \\
&-A_{21}^TK_fA_{22}^{-1}S_{22}(A_{22}-S_{22}K_f)^{-T}C_2^TC_1 - A_{21}^TK_fA_{22}^{-1}S_{22}(A_{22}-S_{22}K_f)^{-T}K_fA_{21} \\
&-A_{21}^TK_fA_{22}^{-1}S_{22}K_fA_{22}^{-1}A_{21} \\
&-A_{21}^TK_fA_{22}^{-1}S_{22}K_f(A_{22}-S_{22}K_f)^{-1}S_{22}(A_{22}-S_{22}K_f)^{-T}C_2^TC_1 \\
&-A_{21}^TK_fA_{22}^{-1}S_{22}K_f(A_{22}-S_{22}K_f)^{-1}S_{22}(A_{22}-S_{22}K_f)^{-T}K_fA_{21} \\
&-A_{21}^TK_fA_{22}^{-1}S_{22}K_f(A_{22}-S_{22}K_f)^{-1}S_{22}K_fA_{22}^{-1}A_{21} \\
&-A_{21}^TA_{22}^{-T}K_fS_{22}(A_{22}-S_{22}K_f)^{-T}C_2^TC_1 - A_{21}^TA_{22}^{-T}K_fS_{22}(A_{22}-S_{22}K_f)^{-T}K_fA_{21}
\end{aligned}
$$

$$-A_{21}^T A_{22}^{-T} K_f S_{22} K_f (A_{22} - S_{22}K_f)^{-1} S_{22} (A_{22} - S_{22}K_f)^{-T} C_2^T C_1$$

$$-A_{21}^T A_{22}^{-T} K_f S_{22} K_f (A_{22} - S_{22}K_f)^{-1} S_{22} (A_{22} - S_{22}K_f)^{-T} K_f A_{21}$$

$$-A_{21}^T A_{22}^{-T} K_f S_{22} K_f A_{22}^{-1} A_{21} - A_{21}^T A_{22}^{-T} K_f S_{22} K_f (A_{22} - S_{22}K_f)^{-1} S_{22} K_f A_{22}^{-1} A_{21}$$

$$+A_{21}^T A_{22}^{-T} K_f S_{22} K_f A_{22}^{-1} S_{22} (A_{22} - S_{22}K_f)^{-T} C_2^T C_1$$

$$+A_{21}^T A_{22}^{-T} K_f S_{22} K_f A_{22}^{-1} S_{22} (A_{22} - S_{22}K_f)^{-T} K_f A_{21}$$

$$+A_{21}^T A_{22}^{-T} K_f S_{22} K_f A_{22}^{-1} S_{22} (A_{22} - S_{22}K_f)^{-T} K_f A_{21}$$

$$+A_{21}^T A_{22}^{-T} K_f S_{22} K_f A_{22}^{-1} S_{22} K_f (A_{22} - S_{22}K_f)^{-1} S_{22} (A_{22} - S_{22}K_f)^{-T} C_2^T C_1$$

$$+A_{21}^T A_{22}^{-T} K_f S_{22} K_f A_{22}^{-1} S_{22} K_f (A_{22} - S_{22}K_f)^{-1} S_{22} (A_{22} - S_{22}K_f)^{-T} K_f A_{21}$$

$$+A_{21}^T A_{22}^{-T} K_f S_{22} K_f A_{22}^{-1} S_{22} K_f (A_{22} - S_{22}K_f)^{-1} S_{22} K_f A_{22}^{-1} A_{21}. \tag{89}$$

Using the identities of (83), (89) reduces to

$$-M_0^T N_0 R_0^{-1} N_0^T M_0 =$$

$$-C_1^T C_2 (A_{22} - S_{22}K_f)^{-1} S_{22} (A_{22} - S_{22}K_f)^{-T} C_2^T C_1$$

$$-C_1^T C_2 (A_{22} - S_{22}K_f)^{-1} S_{22} (A_{22} - S_{22}K_f)^{-T} K_f A_{21}$$

$$-C_1^T C_2 (A_{22} - S_{22}K_f)^{-1} S_{22} K_f A_{22}^{-1} A_{21}$$

$$-A_{21}^T K_f (A_{22} - S_{22}K_f)^{-1} S_{22} (A_{22} - S_{22}K_f)^{-T} C_2^T C_1$$

$$-A_{21}^T K_f (A_{22} - S_{22}K_f)^{-1} S_{22} (A_{22} - S_{22}K_f)^{-T} K_f A_{21}$$

$$-A_{21}^T K_f (A_{22} - S_{22}K_f)^{-1} S_{22} K_f A_{22}^{-1} A_{21}$$

$$-A_{21}^T A_{22}^{-T} K_f A_{22} (A_{22} - S_{22}K_f)^{-1} S_{22} (A_{22} - S_{22}K_f)^{-T} C_2^T C_1$$

$$-A_{21}^T A_{22}^{-T} K_f A_{22} (A_{22} - S_{22}K_f)^{-1} S_{22} (A_{22} - S_{22}K_f)^{-T} K_f A_{21}$$

$$-A_{21}^T A_{22}^{-T} K_f A_{22} (A_{22} - S_{22}K_f)^{-1} S_{22} K_f A_{22}^{-1} A_{21}$$

$$+A_{21}^T A_{22}^{-T} K_f S_{22} K_f (A_{22} - S_{22}K_f)^{-1} S_{22} (A_{22} - S_{22}K_f)^{-T} C_2^T C_1$$

$$+A_{21}^TA_{22}^{-T}K_fS_{22}K_f(A_{22}-S_{22}K_f)^{-1}S_{22}(A_{22}-S_{22}K_f)^{-T}K_fA_{21}$$

$$+A_{21}^TA_{22}^{-T}K_fS_{22}K_f(A_{22}-S_{22}K_f)^{-1}S_{22}K_fS_{22}K_fA_{22}^{-1}A_{21}$$

$$=-C_1^TC_2(A_{22}-S_{22}K_f)^{-1}S_{22}(A_{22}-S_{22}K_f)^{-T}C_2^TC_1$$

$$-C_1^TC_2(A_{22}-S_{22}K_f)^{-1}S_{22}(A_{22}-S_{22}K_f)^{-T}K_fA_{21}$$

$$-C_1^TC_2(A_{22}-S_{22}K_f)^{-1}S_{22}K_fA_{22}^{-1}A_{21}$$

$$-A_{21}^TK_f(A_{22}-S_{22}K_f)^{-1}S_{22}(A_{22}-S_{22}K_f)^{-T}C_2^TC_1$$

$$-A_{21}^TK_f(A_{22}-S_{22}K_f)^{-1}S_{22}(A_{22}-S_{22}K_f)^{-T}K_fA_{21}$$

$$-A_{21}^TK_f(A_{22}-S_{22}K_f)^{-1}S_{22}K_fA_{22}^{-1}A_{21}$$

$$-A_{21}^TA_{22}^{-T}K_fS_{22}(A_{22}-S_{22}K_f)^{-T}C_2^TC_1$$

$$-A_{21}^TA_{22}^{-T}K_fS_{22}(A_{22}-S_{22}K_f)^{-T}K_fA_{21}$$

$$-A_{21}^TA_{22}^{-T}K_fS_{22}K_fA_{22}^{-1}A_{21}. \tag{90}$$

Therefore,

$$M_0^TM_0-M_0^TN_0R_0^{-1}N_0^TM_0=$$

$$C_1^TC_1-A_{21}^TA_{22}^{-T}C_2^TC_1-C_1^TC_2A_{22}^{-1}A_{21}$$

$$+A_{21}^TA_{22}^{-T}[-K_fA_{22}-A_{22}^TK_f+K_fS_{22}K_f]A_{22}^{-1}A_{21}$$

$$-C_1^TC_2(A_{22}-S_{22}K_f)^{-1}S_{22}(A_{22}-S_{22}K_f)^{-T}C_2^TC_1$$

$$-C_1^TC_2(A_{22}-S_{22}K_f)^{-1}S_{22}(A_{22}-S_{22}K_f)^{-T}K_fA_{21}$$

$$-C_1^TC_2(A_{22}-S_{22}K_f)^{-1}S_{22}K_fA_{22}^{-1}A_{21}$$

$$-A_{21}^TK_f(A_{22}-S_{22}K_f)^{-1}S_{22}(A_{22}-S_{22}K_f)^{-T}C_2^TC_1$$

$$-A_{21}^TK_f(A_{22}-S_{22}K_f)^{-1}S_{22}(A_{22}-S_{22}K_f)^{-T}K_fA_{21}$$

$$-A_{21}^TK_f(A_{22}-S_{22}K_f)^{-1}S_{22}K_fA_{22}^{-1}A_{21}$$

$$-A_{21}^T A_{22}^{-T} K_f S_{22}(A_{22} - S_{22}K_f)^{-T} C_2^T C_1$$

$$-A_{21}^T A_{22}^{-T} K_f S_{22}(A_{22} - S_{22}K_f)^{-T} K_f A_{21}$$

$$-A_{21}^T A_{22}^{-T} K_f S_{22} K_f A_{22}^{-1} A_{21}$$

$$= C_1^T C_1 - A_{21}^T (A_{22} - S_{22}K_f)^{-T} C_2^T C_1 - C_1^T C_2 (A_{22} - S_{22}K_f)^{-1} A_{21}$$

$$-C_1^T C_2 (A_{22} - S_{22}K_f)^{-1} S_{22} (A_{22} - S_{22}K_f)^{-T} C_2^T C_1$$

$$-C_1^T C_2 (A_{22} - S_{22}K_f)^{-1} S_{22} (A_{22} - S_{22}K_f)^{-T} K_f A_{21}$$

$$-A_{21}^T K_f (A_{22} - S_{22}K_f)^{-1} S_{22} (A_{22} - S_{22}K_f)^{-T} C_2^T C_1$$

$$-A_{21}^T K_f (A_{22} - S_{22}K_f)^{-1} S_{22} (A_{22} - S_{22}K_f)^{-T} K_f A_{21}$$

$$-A_{21}^T K_f (A_{22} - S_{22}K_f)^{-1} A_{21} - A_{21}^T (A_{22} - S_{22}K_f)^{-T} K_f A_{21}. \tag{91}$$

On the other hand, expanding Q_s of (30a) and noting $K_f = P_{22s}^+$ lead to the conclusion that Q_s is the same as (91), which proves (79c). In consequence, we have $K_s = P_{11s}^+$, hence, $K_m^T = P_{21s}^+$. The proof of Theorem 4 is completed. ■

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