Recursive Approach of H_∞ Control Problems for Singularly Perturbed Systems Under Perfect and Imperfect State Measurements

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Abstract

In this paper, we study the H_∞ control problem for singularly perturbed systems under both prefect and imperfect state measurements by using the recursive approach of Gajic *et al.*(1990). We construct a controller that guarantees a disturbance attenuation level larger than a boundary value of the reduced–order slow and fast subsystems when the singular perturbation parameter ε approaches zero. In order to obtain the controller, we must solve the generalized algebraic Riccati equations. The main results in this paper is to propose a new recursive algorithm to solve the generalized algebraic Riccati equations and to find sufficient conditions for the convergence of the proposed algorithm. Using the recursive algorithm, we show that the solution of the generalized algebraic Riccati equation converges to a positive semi–definite stabilizing solution with the rate of convergence of $O(\varepsilon^k)$ under the sufficient conditions. Furthermore, in the case of perfect state measurements, we also show that the controller achieves the performance level $\gamma + O(\varepsilon^{k+1})$. In addition, we do not assume here that A_{22} is non–singular. Therefore, our new results are applicable to both standard and nonstandard singularly perturbed systems. Finally, in order to show the effectiveness of the proposed algorithm, numerical examples are included.

1. INTRODUCTION

The H_∞ control problems for standard singularly perturbed systems have often been considered by using the singular perturbation methods, i.e., the two time scale decomposition method (Kokotović 1986). Using the singular perturbation method, one can partition a singularly perturbed system into two ε–independent subsystems. In recent years, Pan and Basar (1993, 1994) have studied the H_∞ control problem for standard singularly perturbed systems by making use of a differential game theoretic approach. In Dragan (1996), the boundary of the H_∞ norm for standard singularly perturbed systems is found. In Fridman (1996), the near–optimal H_∞ control problem of singularly perturbed systems is discussed by using a high–order accuracy controller. On the other hand, singular perturbation methods for nonstandard or implicit singularly perturbed systems have been studied by using prefeedback (Kokotović 1986, Khalil 1984, 1989). Recently, there has been interest in nonstandard singularly perturbed systems (Xu and Mizukami 1996). In Xu and Mizukami (1996), the main results of Pan and Basar (1993) is extended to the nonstandard singularly perturbed systems by using a descriptor system approach. In view of the studies above, although the H_∞ control problem for both standard and nonstandard singularly perturbed systems has been studied, the method used is the singular perturbation method. The recursive approach, which is developed by Gajic *et al.*(1990), to solve the H_∞ control problem of singularly perturbed systems has never been studied.

In this paper, we study the H_∞ control problem for singularly perturbed systems under both prefect and imperfect state measurements by using the recursive approach of Gajic *et al.*(1990). We construct a controller that guarantees a disturbance attenuation level larger than the boundary value of the H_∞ performance for the reduced–order slow and fast subsystems when the singular perturbation parameter ε approaches zero. In order to obtain the controller, we must solve the generalized algebraic Riccati equations. The main purpose here is to propose a new recursive algorithm to solve the generalized algebraic Riccati equations and to find sufficient conditions for the convergence of the recursive algorithm by using the reduced–order algebraic Riccati equations. It is important to note that the sufficient conditions derived here is independent of the parameter ε. The resulting controller is obtained by the solution to the generalized algebraic Riccati equations which may be solved using a new recursive algorithm. We prove that the solution of the generalized algebraic Riccati equation converges to a positive semi–definite stabilizing solution with the rate of convergence of $O(\varepsilon^k)$ under the sufficient conditions. Furthermore, in the perfect state measurements case we show that the proposed controller achieves the performance level $\gamma + O(\varepsilon^{k+1})$. In addition, we do not assume here that A_{22} is non–singular. Thus, our new results are applicable to both standard and nonstandard singularly perturbed systems.

This paper is organized as follows. In Section 2, we organize the existing results of the H_∞ control problems for singularly perturbed systems. The aim of Sections 3 and 4 is to propose a new recursive algorithm to solve the generalized algebraic Riccati equations and to find sufficient conditions for the convergence of the recursive algorithm. In Section 5, to show the effectiveness of the proposed algorithm, numerical examples are included. In Section 6 we includes some discussions on the results.

2 Problem Formulation

We consider a singularly perturbed linear time–invariant systems

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_{11}w + B_{21}u, \tag{1a}$$

$$\varepsilon\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_{12}w + B_{22}u, \tag{1b}$$

$$z = C_{11}x_1 + C_{12}x_2 + D_{12}u, \tag{1c}$$

$$y = C_{21}x_1 + C_{22}x_2 + D_{21}w. \tag{1d}$$

where ε is a small positive parameter, $x_1 \in \mathrm{R}^{n_1}$ and $x_2 \in \mathrm{R}^{n_2}$ are states, $y \in \mathrm{R}^{k_1}$ is the measured output, $u \in \mathrm{R}^m$ is the control input, $w \in \mathrm{R}^l$ is the disturbance, $z \in \mathrm{R}^{k_2}$ is the controlled output. The system (1) is called the nonstandard singularly perturbed systems if the matrix A_{22} is singular.

Let us introduce the partitioned matrices

$$\begin{aligned}
A &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \ A_\varepsilon = \begin{bmatrix} A_{11} & A_{12} \\ \varepsilon^{-1}A_{21} & \varepsilon^{-1}A_{22} \end{bmatrix}, \\
B_1 &= \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}, \ B_{1\varepsilon} = \begin{bmatrix} B_{11} \\ \varepsilon^{-1}B_{12} \end{bmatrix}, \\
B_2 &= \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}, \ B_{2\varepsilon} = \begin{bmatrix} B_{21} \\ \varepsilon^{-1}B_{22} \end{bmatrix}, \\
S_{\gamma\varepsilon} &= \begin{bmatrix} S_{11} & \varepsilon^{-1}S_{12} \\ \varepsilon^{-1}S_{21} & \varepsilon^{-2}S_{22} \end{bmatrix}, \ R_\gamma = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}, \\
Q &= C_1^TC_1 = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}, \ M_\varepsilon = B_{1\varepsilon}B_{1\varepsilon}^T = \begin{bmatrix} M_{11} & \varepsilon^{-1}M_{12} \\ \varepsilon^{-1}M_{12}^T & \varepsilon^{-2}M_{22} \end{bmatrix}, \\
C_1 &= \begin{bmatrix} C_{11} & C_{12} \end{bmatrix}, \ C_2 = \begin{bmatrix} C_{21} & C_{22} \end{bmatrix},
\end{aligned}$$

and define

$$\begin{aligned}
S_{ij} &= B_{2i}B_{2j}^T - \gamma^{-2}B_{1i}B_{1j}^T, \ i = 1, 2, j = 1, 2, \\
R_{ij} &= C_{2i}^TC_{2j} - \gamma^{-2}C_{1i}^TC_{1j}, \ i = 1, 2, j = 1, 2.
\end{aligned}$$

We now consider the H_∞ control problems under the following basic assumption.

Assumption 1

1. *The pair $(A_\varepsilon,\ B_{1\varepsilon})$ is stabilizable and $(C_1,\ A_\varepsilon)$ is detectable for $\varepsilon \in (0, \varepsilon^*]$ $(\varepsilon^* > 0)$.*
2. *The pair $(A_\varepsilon,\ B_{2\varepsilon})$ is stabilizable and $(C_2,\ A_\varepsilon)$ is detectable for $\varepsilon \in (0, \varepsilon^*]$ $(\varepsilon^* > 0)$.*
3. $D_{12}^T[C_1\ D_{12}] = [0\ I]$.
4. $\begin{bmatrix} B_{1\varepsilon} \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I \end{bmatrix}$.

The problem considered in this paper is the H_∞ optimal control problem for singularly perturbed systems.

Find all admissible K_ε such that $||G_\varepsilon||_\infty < \gamma$ where G_ε equals to the transfer function from w to z, that is,

$$G_\varepsilon = \left[\begin{array}{c|cc} A_\varepsilon & B_{1\varepsilon} & B_{2\varepsilon} \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right], \tag{2}$$

where the transfer matrix in terms of state–space data is denoted by

$$\left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] = \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D}.$$

The following lemma is already known (see Doyle *et al.* 1989).

Lemma 1 *Under Assumption 1, there exists an admissible controller such that* $||G_\varepsilon||_\infty < \gamma$ *iff the following three conditions hold.*

i) *The backward algebraic Riccati equation*

$$P_\varepsilon A_\varepsilon + A_\varepsilon^T P_\varepsilon - P_\varepsilon S_{\gamma\varepsilon} P_\varepsilon + Q = 0 \tag{3}$$

has the unique positive semi–definite stabilizing solution, where

$$P_\varepsilon = \begin{bmatrix} P_{11} & \varepsilon P_{21}^T \\ \varepsilon P_{21} & \varepsilon P_{22} \end{bmatrix}. \tag{4}$$

ii) *The forward algebraic Riccati equation*

$$W_\varepsilon A_\varepsilon^T + A_\varepsilon W_\varepsilon - W_\varepsilon R_\gamma W_\varepsilon + M_\varepsilon = 0 \tag{5}$$

has the unique positive semi–definite stabilizing solution, where

$$W_\varepsilon = \begin{bmatrix} W_{11} & W_{12} \\ W_{12}^T & \varepsilon^{-1} W_{22} \end{bmatrix}. \tag{6}$$

iii) $\lambda_{\max}(P_\varepsilon W_\varepsilon) < \gamma^2$,

where $\lambda_{\max}(P_\varepsilon W_\varepsilon)$ *is maximum eigenvalue of* $P_\varepsilon W_\varepsilon$.

Moreover, when these conditions hold, one such controller, i.e., the central controller with a free parameter equal to zero, is given by

$$u = -B_{2\varepsilon}^T P_\varepsilon \hat{x} \tag{7}$$

where

$$\dot{\hat{x}} = [A_\varepsilon - (B_{2\varepsilon} B_{2\varepsilon}^T - \gamma^{-2} B_{1\varepsilon} B_{1\varepsilon}^T) P_\varepsilon - Z W_\varepsilon C_2^T C_2]\hat{x} + Z W_\varepsilon C_2^T y,$$

$$Z = (I - \gamma^{-2} W_\varepsilon P_\varepsilon)^{-1}.$$

Here, $\hat{x}(t) = [\hat{x}_1^T \ \hat{x}_2^T]^T$ *is the observer state.*

3. PRELIMINARY RESULTS: STATE FEEDBACK CONTROL

3.1. The Perfect State Measurement case

In this section, we study the H_∞ control problem by using the state feedback control law for the linear time–invariant singularly perturbed system

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_{11}w + B_{21}u, \ x_1^0 = 0 \tag{8a}$$

$$\varepsilon\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_{12}w + B_{22}u, \ x_2^0 = 0 \tag{8b}$$

$$z = C_{11}x_1 + C_{12}x_2 + D_{12}u, \tag{8c}$$

$$y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{8d}$$

We discuss the H_∞ optimal control problem that the closed–loop system is internally stable and $||G_\varepsilon||_\infty < \gamma$, where

$$G_\varepsilon = \left[\begin{array}{c|cc} A_\varepsilon & B_{1\varepsilon} & B_{2\varepsilon} \\ \hline C_1 & 0 & D_{12} \\ I & 0 & 0 \end{array}\right] \tag{9}$$

by using the formula (10)

$$u = K_\varepsilon \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{10}$$

The next result was shown by Doyle *et al.* (1989).

Lemma 2 *The following are equivalent:*

i) *$A + B_{2\varepsilon}K_\varepsilon$ is stable and the transfer matrix G_ε satisfies the inequality $||G_\varepsilon||_\infty < \gamma$.*

ii) *The Riccati equation (3) has the positive semi–definite stabilizing solution.*
Moreover, one such controller is $K_\varepsilon = -B_{2\varepsilon}^T P_\varepsilon$.

To obtain the solution of the Riccati equation (3), at first we define

$$D_{1\varepsilon} = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \varepsilon I_{n_2} \end{bmatrix}.$$

In order to solve the algebraic Riccati equation (3), we introduce the following useful lemma.

Lemma 3 *The algebraic Riccati equation (3) is equivalent to the following generalized algebraic Riccati equation (11).*

$$P^T A + A^T P - P^T S_\gamma P + Q = 0, \tag{11a}$$

$$D_{1\varepsilon}^T P = P^T D_{1\varepsilon}, \tag{11b}$$

where

$$P = \begin{bmatrix} P_{11} & \varepsilon P_{21}^T \\ P_{21} & P_{22} \end{bmatrix}, \ S_\gamma = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}.$$

Proof: Firstly, from (11b), P has the following partitioned form

$$P = \begin{bmatrix} P_{11} & \varepsilon P_{21}^T \\ P_{21} & P_{22} \end{bmatrix}, \ P_{11} = P_{11}^T, \ P_{22} = P_{22}^T.$$

It is worth to note that P is not symmetric, but $P_\varepsilon = D_{1\varepsilon}^T P = P^T D_{1\varepsilon}$ is. Secondly, we can observe the following useful relationships between A_ε, $B_{1\varepsilon}$, $B_{2\varepsilon}$, $D_{1\varepsilon}$, A, B_1 and B_2.

$$A_\varepsilon = D_{1\varepsilon}^{-1} A, \ B_{1\varepsilon} = D_{1\varepsilon}^{-1} B_1, \ B_{2\varepsilon} = D_{1\varepsilon}^{-1} B_2.$$

Substituting the above relations and $P_\varepsilon = D_{1\varepsilon}^T P = P^T D_{1\varepsilon}$ into the Riccati equation (3), we obtain

$$\begin{aligned} & P_\varepsilon A_\varepsilon + A_\varepsilon^T P_\varepsilon - P_\varepsilon S_{\gamma\varepsilon} P_\varepsilon + Q = 0 \\ \Leftrightarrow \quad & P^T D_{1\varepsilon} D_{1\varepsilon}^{-1} A + A^T D_{1\varepsilon}^{-T} D_{1\varepsilon}^T P - P^T D_{1\varepsilon} D_{1\varepsilon}^{-1} S_\gamma D_{1\varepsilon}^{-T} D_{1\varepsilon}^T P + Q = 0 \\ \Leftrightarrow \quad & P^T A + A^T P - P^T S_\gamma P + Q = 0. \end{aligned}$$

Thus, to solve the algebraic Riccati equation (3) is equivalent to solving the generalized Riccati equation (11). ∎

3.2. Recursive Algorithm of the Backward Algebraic Riccati Equation

The algebraic Riccati equation (11a) can be partitioned into

$$\begin{aligned} & A_{11}^T P_{11} + P_{11}^T A_{11} + A_{21}^T P_{21} + P_{21}^T A_{21} - P_{11}^T S_{11} P_{11} - P_{21}^T S_{22} P_{21} \\ & \quad - P_{11}^T S_{12} P_{21} - P_{21}^T S_{12}^T P_{11} + Q_{11} = 0, \end{aligned} \tag{12a}$$

$$\begin{aligned} & \varepsilon P_{21} A_{11} + P_{22}^T A_{21} + A_{12}^T P_{11} + A_{22}^T P_{21} - \varepsilon P_{21} S_{11} P_{11} - \varepsilon P_{21} S_{12} P_{21} \\ & \quad - P_{22}^T S_{12}^T P_{11} - P_{22}^T S_{22} P_{21} + Q_{12}^T = 0, \end{aligned} \tag{12b}$$

$$\begin{aligned} & A_{22}^T P_{22} + P_{22}^T A_{22} + \varepsilon A_{12}^T P_{21}^T + \varepsilon P_{21} A_{12} - P_{22}^T S_{22} P_{22} \\ & \quad - \varepsilon P_{22}^T S_{12}^T P_{21}^T - \varepsilon P_{21} S_{12} P_{22} - \varepsilon^2 P_{21} S_{11} P_{21}^T + Q_{22} = 0. \end{aligned} \tag{12c}$$

For the previous equations (12), setting $\varepsilon = 0$, we obtain the following equations

$$\begin{aligned} & A_{11}^T \bar{P}_{11} + \bar{P}_{11}^T A_{11} + A_{21}^T \bar{P}_{21} + \bar{P}_{21}^T A_{21} - \bar{P}_{11}^T S_{11} \bar{P}_{11} - \bar{P}_{21}^T S_{22} \bar{P}_{21} \\ & \quad - \bar{P}_{11}^T S_{12} \bar{P}_{21} - \bar{P}_{21}^T S_{12}^T \bar{P}_{11} + Q_{11} = 0, \end{aligned} \tag{13a}$$

$$\bar{P}_{22}^T A_{21} + A_{12}^T \bar{P}_{11} + A_{22}^T \bar{P}_{21} - \bar{P}_{22}^T S_{12}^T \bar{P}_{11} - \bar{P}_{22}^T S_{22} \bar{P}_{21} + Q_{12}^T = 0, \tag{13b}$$

$$A_{22}^T \bar{P}_{22} + \bar{P}_{22}^T A_{22} - \bar{P}_{22}^T S_{22} \bar{P}_{22} + Q_{22} = 0. \tag{13c}$$

The Riccati equation (13c) will produce the unique positive definite stabilizing solution under the following conditions.

Assumption 2 *The pair $(A_{22},\ B_{22})$ is stabilizable and $(C_{12},\ A_{22})$ is observable.*

Let

$\gamma_{1f} = \inf\{\gamma > 0 |$the Riccati equation (13c) has a positive definite stabilizing solution$\}$.

Then, the matix $A_{22} - S_{22}\bar{P}_{22}$ is non–singular if Assumption 2 holds. Therefore, we obtain the following 0–order equations

$$\bar{P}_{11}^T A_P + A_P^T \bar{P}_{11} - \bar{P}_{11}^T S_P \bar{P}_{11} + Q_P = 0, \tag{14a}$$

$$\bar{P}_{21} = -N_2^T + N_1^T \bar{P}_{11}, \tag{14b}$$

$$A_{22}^T \bar{P}_{22} + \bar{P}_{22}^T A_{22} - \bar{P}_{22}^T S_{22} \bar{P}_{22} + Q_{22} = 0, \tag{14c}$$

where

$$\begin{aligned} A_P &= A_{11} + N_1 A_{21} + S_{12} N_2^T + N_1 S_{22} N_2^T, \\ S_P &= S_{11} + N_1 S_{12}^T + S_{12} N_1^T + N_1 S_{22} N_1^T, \\ Q_P &= Q_{11} - N_2 A_{21} - A_{21}^T N_2^T - N_2 S_{22} N_2^T, \\ N_2^T &= \bar{A}_{22}^{-T} \hat{Q}_{12}^T,\ N_1^T = -\bar{A}_{22}^{-T} \bar{A}_{12}^T, \\ \bar{A}_{12} &= A_{12} - S_{12}\bar{P}_{22},\ \bar{A}_{22} = A_{22} - S_{22}\bar{P}_{22}, \\ \hat{Q}_{12} &= Q_{12} + A_{21}^T \bar{P}_{22}. \end{aligned}$$

Remark 1 *Although the expressions of the matrix A_P, S_P and Q_P contain the matrix $\bar{P}_{22}$, they do not depend on it (Xu and Mizukami 1996).*

The unique positive definite stabilizing solution of (14a) exists under the following conditions.

Assumption 3

$$\operatorname{rank} \begin{bmatrix} sI_{n_1} - A_{11} & -A_{12} & B_{21} \\ -A_{21} & -A_{22} & B_{22} \end{bmatrix} = n_1 + n_2,\ \forall s \in \mathrm{C}^+, \tag{15}$$

$$\operatorname{rank} \begin{bmatrix} sI_{n_1} - A_{11}^T & -A_{21}^T & C_{11}^T \\ -A_{12}^T & -A_{22}^T & C_{12}^T \end{bmatrix} = n_1 + n_2,\ \forall s \in \mathrm{C}^+. \tag{16}$$

Remark 2 *Q_P and S_P can also be expressed as*

$$Q_P = C_P^T C_P,\ S_P = B_{20} B_{20}^T - \gamma^{-2} B_{10} B_{10}^T,$$

where

$$\begin{aligned} C_P &= C_{11} + C_{12} M_{10}^T,\ M_{10} = -\bar{A}_{210} \bar{A}_{220}^{-1}, \\ \bar{A}_{210} &= A_{21}^T + Q_{12}^T \bar{P}_{22}^{-1},\ \bar{A}_{220} = A_{22}^T + Q_{22} \bar{P}_{22}^{-1}, \\ B_{10} &= B_{11} + N_1 B_{12},\ B_{20} = B_{21} + N_1 B_{22}. \end{aligned}$$

Therefore, for every $\gamma > \gamma_{1f}$, the pair $(A_P,\ B_{20})$ is stabilizable and $(C_P,\ A_P)$ is observable if and only if Assumption 3 is satisfied (Xu and Mizukami 1996).

Associated with the Riccati equation (14c), we define the Hamiltonian matrix

$$H_{1\gamma} = \begin{bmatrix} A_{22} & -S_{22} \\ -Q_{22} & -A_{22}^T \end{bmatrix}. \tag{17}$$

Let

$\gamma_P = \max\{\gamma > 0 | \text{the Hamiltonian matrix } H_{1\gamma} \text{ is singular}\}$.

Moreover, let us define

$\gamma_{1s} = \inf\{\gamma > \gamma_P | \text{the Riccati equation (14a) has a positive definite stabilizing solution}\}$.

As the results, if Assumptions 2, 3 hold, then for every $\gamma > \bar{\gamma}_1 = \max\{\gamma_{1s}, \gamma_{1f}\}$, the Riccati equations (14a) and (14c) have the positive definite stabilizing solutions.

Now, let us introduce

$$P_{11} = \bar{P}_{11} + \varepsilon E_{11},\ P_{21} = \bar{P}_{21} + \varepsilon E_{21},\ P_{22} = \bar{P}_{22} + \varepsilon E_{22}. \tag{18}$$

The $O(\varepsilon^k)$ approximation of the error terms E_{ij} $(i,\ j = 1,\ 2)$ will result in $O(\varepsilon^{k+1})$ approximation of the required matrix P_{ij} $(i,\ j = 1,\ 2)$. That is why we are interested in finding equations of the error terms and a convenient algorithm to find their solutions. Substituting (18) into (12) and subtracting (13) from (12), we arrive at the error equations. Hence, we propose the following algorithm (19).

$$E_{11}^{T(j+1)} \bar{A}_P + \bar{A}_P^T E_{11}^{(j+1)} = -V_P^T H_{P1}^{T(j)} - H_{P1}^{(j)} V_P + V_P^T H_{P3}^{(j)} V_P + \varepsilon H_{P2}^{(j)}, \tag{19a}$$

$$E_{11}^{T(j+1)} \bar{A}_{12} + E_{21}^{T(j+1)} \bar{A}_{22} + \bar{A}_{21}^T E_{22}^{(j+1)} = H_{P1}^{(j)}, \tag{19b}$$

$$E_{22}^{T(j+1)} \bar{A}_{22} + \bar{A}_{22}^T E_{22}^{(j+1)} = H_{P3}^{(j)}, \tag{19c}$$

where

$$\begin{aligned}
H_{P1}^{(j)} &= -A_{11}^T P_{21}^{T(j)} + P_{11}^{T(j)} S_{11} P_{21}^{T(j)} + P_{21}^{T(j)} S_{12}^T P_{21}^{T(j)} + \varepsilon(E_{11}^{T(j)} S_{12} E_{22}^{(j)} + E_{21}^{T(j)} S_{22} E_{22}^{(j)}), \\
H_{P2}^{(j)} &= E_{11}^{T(j)} S_{11} E_{11}^{(j)} + E_{21}^{T(j)} S_{22} E_{21}^{(j)} + E_{11}^{T(j)} S_{12} E_{21}^{(j)} + E_{21}^{T(j)} S_{12}^T E_{11}^{(j)}, \\
H_{P3}^{(j)} &= -A_{12}^T P_{21}^{T(j)} - P_{21}^{(j)} A_{12} + \varepsilon P_{21}^{(j)} S_{11} P_{21}^{T(j)} \\
&\quad + \varepsilon E_{22}^{T(j)} S_{22} E_{22}^{(j)} + P_{21}^{(j)} S_{12} P_{22}^{(j)} + P_{22}^{T(j)} S_{12}^T P_{21}^{T(j)}, \\
P_{11}^{(j)} &= \bar{P}_{11} + \varepsilon E_{11}^{(j)},\ P_{21}^{(j)} = \bar{P}_{21} + \varepsilon E_{21}^{(j)},\ P_{22}^{(j)} = \bar{P}_{22} + \varepsilon E_{22}^{(j)},\ E_{11}^{(0)} = E_{21}^{(0)} = E_{22}^{(0)} = 0, \\
\bar{A}_{11} &= A_{11} - S_{11} \bar{P}_{11} - S_{12} \bar{P}_{21},\ \bar{A}_{21} = A_{21} - S_{12}^T \bar{P}_{11} - S_{22} \bar{P}_{21}, \\
\bar{A}_P &= \bar{A}_{11} - \bar{A}_{12} \bar{A}_{22}^{-1} \bar{A}_{21},\ V_P = \bar{A}_{22}^{-1} \bar{A}_{21}.
\end{aligned}$$

The following theorem indicates the convergence of the algorithm (19).

Theorem 1 *Under the stabilizability and detectability conditions, imposed in Assumptions 1, 2 and 3, for a predescribed disturbance attenuation level $\gamma > \bar{\gamma}_1 = \max\{\gamma_{1s}, \gamma_{1f}\}$ and a small parameter $\varepsilon > 0$, the following results hold:*

i) *The proposed algorithm (19) converges to the exact solution of E with the rate of convergence of $O(\varepsilon^k)$, that is*

$$||E - E^{(k)}|| = O(\varepsilon^k), \quad (k = 1, 2, \cdots), \tag{20}$$

where

$$E = \begin{bmatrix} E_{11} & E_{21}^T \\ E_{21} & E_{22} \end{bmatrix}, \ E^{(k)} = \begin{bmatrix} E_{11}^{(k)} & E_{21}^{T(k)} \\ E_{21}^{(k)} & E_{22}^{(k)} \end{bmatrix}.$$

ii) *The solution $P_{\varepsilon}^{(k)}$ is positive semi–definite stabilizing solution of algebraic Riccati equation (3), where*

$$P_{\varepsilon}^{(k)} = \begin{bmatrix} \bar{P}_{11} + \varepsilon E_{11}^{(k)} & \varepsilon(\bar{P}_{21} + \varepsilon E_{21}^{(k)})^T \\ \varepsilon(\bar{P}_{21} + \varepsilon E_{21}^{(k)}) & \varepsilon(\bar{P}_{22} + \varepsilon E_{22}^{(k)}) \end{bmatrix}. \tag{21}$$

Proof: As a starting point we need to show the existence of a bounded solution of E in neighborhood of $\varepsilon = 0$. To prove that by the implicit function theorem (Gajic *et al.* 1990, Gajic 1986), it is enough to show that the corresponding Jacobian is non–singular at $\varepsilon = 0$. The Jacobian is given by

$$J^1|_{\varepsilon=0} \ = \ \begin{bmatrix} J_{11}^1 & 0 & 0 \\ J_{21}^1 & J_{22}^1 & J_{23}^1 \\ 0 & 0 & J_{33}^1 \end{bmatrix} \tag{22}$$

where, using the Kronecker products representation we have

$$\begin{aligned} J_{11}^1 &= I \otimes \bar{A}_P + \bar{A}_P^T \otimes I, \\ J_{22}^1 &= I \otimes \bar{A}_{22}, \\ J_{33}^1 &= I \otimes \bar{A}_{22} + \bar{A}_{22}^T \otimes I. \end{aligned}$$

When $\gamma > \bar{\gamma}_1$, the matrix $\bar{A}_{22}$ is non–singular because of Assumption 2. The matrix $A_P - S_P\bar{P}_{11}$ is non–singular if Assumption 3 holds. Therefore, we obtain

$$A_P - S_P\bar{P}_{11} = \bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21} = \bar{A}_P. \tag{23}$$

Since the matrix $\bar{A}_P$ is stable too, for a small parameter ε, the Jacobian is non–singular. As a result, we can achieve the $O(\varepsilon^k)$ approximation of E_{ij} $(i, \ j = 1, \ 2)$ by performing only k iterations using algolithm (19). The remainder of the proof is to show that $P_{\varepsilon}^{(k)}$ is positive semi–definite stabilizing solution. Firstly, from (21), we have

$$P_{\varepsilon}^{(k)} \ = \ \begin{bmatrix} \bar{P}_{11} & 0 \\ 0 & 0 \end{bmatrix} + O(\varepsilon).$$

The matrix $\bar{P}_{11}$ is positive definite if the algebraic Riccati equation (14a) has solution. Therefore, $P_{\varepsilon}^{(k)} \geq 0$. Secondly, we get

$$A_{\varepsilon} - S_{\gamma\varepsilon}P_{\varepsilon}^{(k)} = \begin{bmatrix} \bar{A}_{11} + O(\varepsilon) & \bar{A}_{12} + O(\varepsilon) \\ \varepsilon^{-1}\{\bar{A}_{21} + O(\varepsilon)\} & \varepsilon^{-1}\{\bar{A}_{22} + O(\varepsilon)\} \end{bmatrix}$$

by straightforward computations. The matrix $\bar{A}_{22}$ and $\bar{A}_P$ are non–singular since Assumptions 2 and 3 hold. If parameter ε is sufficently small, $A_{\varepsilon} - S_{\gamma\varepsilon}P_{\varepsilon}^{(k)}$ is stable. Thus, the proof of Theorem 1 is completed. ■

To the end of this section, we apply the controller $u = -B_{2\varepsilon}^T P_{\varepsilon}^{(k)}x$ to the system (1) and compare it with the exact optimal control (10).

Theorem 2 *Under the conditions given in Theorem 1, if the controller gain matrix $\bar{K}$ is designed for a prescribed disturbance attenuation level $\gamma > \bar{\gamma}_1$ and the resulted controller $u = -B_{2\varepsilon}^T P_\varepsilon^{(k)} x$ is applied to the system (1), then the following inequality will be satisfied:*

$$\begin{aligned} & ||(C_1 + D_{12}\bar{K}) \cdot (sI - A_\varepsilon - B_{2\varepsilon}\bar{K})^{-1} B_{1\varepsilon}||_\infty \\ = & ||(C_1 + D_{12}K)(sI - A_\varepsilon - B_{2\varepsilon}K)^{-1} B_{1\varepsilon}||_\infty + O(\varepsilon^{k+1}) \\ < & \gamma + O(\varepsilon^{k+1}) \end{aligned} \tag{24}$$

where $K = -B_{2\varepsilon}^T P_\varepsilon$, $\bar{K} = -B_{2\varepsilon}^T P_\varepsilon^{(k)}$.

Proof: We give the proof by using a method similar to that given in the proof of Theorem in Fridman (1996). Applying the optimal controller $u = -Kx$ to (8) yields

$$\dot{x} = \bar{A}_\varepsilon x + \varepsilon \bar{F}_\varepsilon x + B_{1\varepsilon} w, \ x^0 = 0, \tag{25a}$$

$$J = \int_0^\infty x^T(t) \bar{Q} x(t) dt, \tag{25b}$$

where

$$\begin{aligned} \bar{A}_\varepsilon &= \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \varepsilon^{-1}\bar{A}_{21} & \varepsilon^{-1}\bar{A}_{22} \end{bmatrix}, \\ \bar{F}_\varepsilon &= -\varepsilon^{-1}(B_{2\varepsilon}B_{2\varepsilon}^T P_\varepsilon - A_\varepsilon + \bar{A}_\varepsilon), \\ \bar{Q} &= Q + P_\varepsilon B_{2\varepsilon} B_{2\varepsilon}^T P_\varepsilon. \end{aligned}$$

Since $\bar{A}_{22}$ is stable, there is transformation $y = T^{-1}x$ such that $T^{-1}\bar{A}_\varepsilon T = \mathrm{diag}[A_s \ \varepsilon^{-1}A_f]$ (see Kokotović *et al.* 1986).

Using the transformation T, we obtain

$$\dot{y}_1 = A_s y_1 + \varepsilon \bar{F}_s y + B_{1s} w, \ y_1^0 = 0, \tag{26a}$$

$$\varepsilon \dot{y}_2 = A_f y_2 + \varepsilon \bar{F}_f y + B_{1f} w, \ y_2^0 = 0, \tag{26b}$$

where $[\bar{F}_s^T \ \bar{F}_f^T]^T = T^{-1}\bar{F}_\varepsilon$, $[B_{1s}^T \ B_{1f}^T]^T = T^{-1}B_{1\varepsilon}$. From (26), if ε is small enough, then we have $||y|| \le c_1||w||, c_1 > 0$. Similarly, substituting $u = -\bar{K}x$ and $f = T^{-1}x$ into system (1), we get

$$\dot{f}_1 = A_s f_1 + \varepsilon \hat{F}_s f + B_{1s} w, \ f_1^0 = 0, \tag{27a}$$

$$\varepsilon \dot{f}_2 = A_f f_2 + \varepsilon \hat{F}_f f + B_{1f} w, \ f_2^0 = 0, \tag{27b}$$

where $[\hat{F}_s^T \ \hat{F}_f^T]^T = -\varepsilon^{-1}T^{-1}(B_{2\varepsilon}B_{2\varepsilon}^T P_\varepsilon^{(k)} - A_\varepsilon + \bar{A}_\varepsilon)$. Hence, from (27), one can derive $||f|| \le c_2||w||, \ c_2 > 0$. Subtracting (27) from (26) we get the following equation (28).

$$\dot{e}_1 = A_s e_1 + \varepsilon \hat{F}_s e + O(\varepsilon^{k+1}) y, \tag{28a}$$

$$\varepsilon \dot{e}_2 = A_f e_2 + \varepsilon \hat{F}_f e + O(\varepsilon^{k+1}) y, \tag{28b}$$

where $e = y - f$. From (28), we obtain $||e|| \le c_3\varepsilon^{k+1}||y|| \le c_4\varepsilon^{k+1}||w||, \ c_3, c_4 > 0$.

Then, we note that

$$T^{-1}\hat{F}_{\varepsilon}T - T^{-1}\bar{F}_{\varepsilon}T = O(\varepsilon^k), \quad ||\hat{Q} - \bar{Q}||_2 = m_0\varepsilon^{k+1}, \ m_0 > 0,$$

where $\hat{Q} = Q + P_{\varepsilon}^{(k)}B_{2\varepsilon}B_{2\varepsilon}^T P_{\varepsilon}^{(k)}$. Applying the Schwartz inequality yields

$$\begin{aligned} |J - \hat{J}| &\leq \int_0^{\infty} [m_1|e(t)||y(t)| + m_2|e(t)||f(t)| + m_0\varepsilon^{k+1}|y(t)||f(t)|]dt \\ &\leq \bar{m}[||e||(||y|| + ||f||) + \varepsilon^{k+1}||y|| \cdot ||f||] \end{aligned} \tag{29}$$

where $\bar{m} = \max\{m_0, m_1, m_2\}$, $m_1 = ||T^T\bar{Q}T||_2$, $m_2 = ||T^T\hat{Q}T||_2$. Moreover, substituting $||y|| \leq c_1||w||$, $||f|| \leq c_2||w||$ and $||e|| \leq c_4\varepsilon^{k+1}||w||$ into (29) yields

$$|J - \hat{J}| \leq \bar{m}[c_4(c_1 + c_2) + c_1 \cdot c_2]\varepsilon^{k+1}||w||^2 \leq \bar{m}_0\varepsilon^{k+1}||w||^2. \tag{30}$$

Finally, by using condition $J \leq \gamma^2||w||^2$, we have

$$\hat{J} \leq [\gamma^2 + O(\varepsilon^{k+1})]||w||^2 = [\gamma + O(\varepsilon^{k+1})]^2||w||^2, \tag{31}$$

that is, an $O(\varepsilon^k)$ accuracy controller $u = -\bar{K}x$ achieves the performance level $\gamma + O(\varepsilon^{k+1})$. ■

4. MAIN RESULTS: OUTPUT FEEDBACK CONTROL

4.1. The Imperfect State Measurement case

In this section we now turn to the H_∞ optimal control problem by using the output feedback control law. By following the similar steps in the state feedback case, we first study the algebraic Riccati equation (5). To obtain the positive semi–definite stabilizing solution of the algebraic Riccati equation (5), we introduce the following useful lemma.

Lemma 4 *The algebraic Riccati equation (5) is equivalent to the following generalized algebraic Riccati equation (32).*

$$WA^T + AW^T - WR_\gamma W^T + M = 0, \tag{32a}$$

$$D_{2\varepsilon}W^T = WD_{2\varepsilon}^T, \tag{32b}$$

where

$$W = \begin{bmatrix} W_{11} & W_{12} \\ \varepsilon W_{12}^T & W_{22} \end{bmatrix}, \ M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}, \ D_{2\varepsilon} = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \varepsilon^{-1}I_{n_2} \end{bmatrix}.$$

Proof: The proof is omitted since it is similar to the proof of Lemma 3. ■

4.2. Recursive Algorithm of the Forward Algebraic Riccati Equation

We consider the recursive solution for the generalized forward algebraic Riccati equations (32). The equation (32a) can be partitioned into

$$\begin{aligned} &A_{11}W_{11}^T + W_{11}A_{11}^T + A_{12}W_{12}^T + W_{12}A_{12}^T \\ &\quad -W_{11}R_{11}W_{11}^T - W_{12}R_{22}W_{12}^T - W_{11}R_{12}W_{12}^T - W_{12}R_{12}^TW_{11}^T + M_{11} = 0 \end{aligned} \tag{33a}$$

$$\varepsilon A_{11}W_{12} + A_{12}W_{22}^T + W_{11}A_{21}^T + W_{12}A_{22}^T$$
$$-\varepsilon W_{11}R_{11}W_{12} - \varepsilon W_{12}R_{12}^T W_{12} - W_{11}R_{12}W_{22}^T - W_{12}R_{22}W_{22}^T + M_{12} = 0 \quad (33b)$$

$$A_{22}W_{22}^T + W_{22}A_{22}^T + \varepsilon A_{21}W_{12} + \varepsilon W_{12}^T A_{21}^T$$
$$-W_{22}R_{22}W_{22}^T - \varepsilon W_{12}^T R_{12}W_{22}^T - \varepsilon W_{22}R_{12}^T W_{12} - \varepsilon^2 W_{12}^T R_{11}W_{12} + M_{22} = 0 \quad (33c)$$

Setting $\varepsilon = 0$ in (33), we obtain the following equations (34).

$$A_{11}\bar{W}_{11}^T + \bar{W}_{11}A_{11}^T + A_{12}\bar{W}_{12}^T + \bar{W}_{12}A_{12}^T$$
$$-\bar{W}_{11}R_{11}\bar{W}_{11}^T - \bar{W}_{12}R_{22}\bar{W}_{12}^T - \bar{W}_{11}R_{12}\bar{W}_{12}^T - \bar{W}_{12}R_{12}^T\bar{W}_{11}^T + M_{11} = 0 \quad (34a)$$

$$A_{12}\bar{W}_{22}^T + \bar{W}_{11}A_{21}^T + \bar{W}_{12}A_{22}^T - \bar{W}_{11}R_{12}\bar{W}_{22}^T - \bar{W}_{12}R_{22}\bar{W}_{22}^T + M_{12} = 0 \quad (34b)$$

$$A_{22}\bar{W}_{22}^T + \bar{W}_{22}A_{22}^T - \bar{W}_{22}R_{22}\bar{W}_{22}^T + M_{22} = 0 \quad (34c)$$

The Riccati equation (34c) will produce the unique positive definite stabilizing solution under the following conditions.

Assumption 4 *The pair* $(A_{22}^T,\ C_{22}^T)$ *is stabilizable and* $(B_{12}^T,\ A_{22}^T)$ *is observable.*

Let

$\gamma_{2f} = \inf\{\gamma > 0 |$the Riccati equation (34c) has a positive definite stabilizing solution, and $\lambda_{\max}(\bar{P}_{22}\bar{W}_{22}) < \gamma^2\}$.

Then, the matrix $A_{22}^T - R_{22}W_{22}^T$ is non–singular if Assumption 4 holds. Therefore, we obtain the following 0–order equations (35).

$$\bar{W}_{11}A_W^T + A_W\bar{W}_{11}^T - \bar{W}_{11}R_W\bar{W}_{11}^T + M_W = 0 \quad (35a)$$

$$\bar{W}_{12} = -L_2 + \bar{W}_{11}L_1 \quad (35b)$$

$$\bar{W}_{22}A_{22}^T + A_{22}\bar{W}_{22}^T - \bar{W}_{22}R_{22}\bar{W}_{22}^T + M_{22} = 0 \quad (35c)$$

where

$$\begin{aligned}
A_W &= A_{11} + A_{12}L_1^T + L_2R_{12}^T + L_2R_{22}L_1^T \\
R_W &= R_{11} + R_{12}L_1^T + L_1R_{12}^T + L_1R_{22}L_1^T \\
M_W &= M_{11} - A_{12}L_2^T - L_2A_{12}^T - L_2R_{22}L_2^T \\
L_2^T &= T_4^{-1}\hat{V}_{12},\ L_1^T = -T_4^{-1}T_2 \\
T_2 &= A_{21} - \bar{W}_{22}R_{12}^T,\ T_4 = A_{22} - \bar{W}_{22}R_{22}^T \\
\hat{V}_{12} &= \bar{W}_{22}A_{12}^T + M_{12}^T
\end{aligned}$$

Remark 3 *Although the expressions of the matrix* A_W, R_W *and* M_W *contain the matrix* $\bar{W}_{22}$, *they do not depend on it (Xu and Mizukami 1996).*

The unique positive definite stabilizing solution of (35a) exists under the following conditions.

Assumption 5

$$\mathrm{rank}\begin{bmatrix} sI_{n_1} - A_{11}^T & -A_{21}^T & C_{21}^T \\ -A_{12}^T & -A_{22}^T & C_{22}^T \end{bmatrix} = n_1 + n_2, \ \forall s \in \mathrm{C}^+, \tag{36}$$

$$\mathrm{rank}\begin{bmatrix} sI_{n_1} - A_{11} & -A_{12} & B_{11} \\ -A_{21} & -A_{22} & B_{12} \end{bmatrix} = n_1 + n_2, \ \forall s \in \mathrm{C}^+. \tag{37}$$

Remark 4 *Similar to Remark 2, M_W and R_W can also be expressed as*

$$M_W = B_W B_W^T, \ R_W = C_{20}^T C_{20} - \gamma^{-2} C_{10}^T C_{10},$$

where

$$\begin{aligned} B_W &= B_{11} + M_{20} B_{12}^T, \ M_{20} = -\tilde{A}_{120} \tilde{A}_{220}^{-1}, \\ \tilde{A}_{120} &= A_{12} + M_{12} \bar{W}_{22}^{-T}, \ \tilde{A}_{220} = A_{22} + M_{22} \bar{W}_{22}^{-T}, \\ C_{10} &= C_{11} + L_1 C_{12}, \ C_{20} = C_{21} + L_1 C_{22}. \end{aligned}$$

Therefore, for every $\gamma > \gamma_{2f}$, the pair $(A_W^T, \ C_{20}^T)$ is stabilizable and $(B_W^T, \ A_W^T)$ is observable if and only if Assumption 5 is satisfied (Xu and Mizukami 1996).

Associated with the Riccati equation (35c), we define the Hamiltonian matrix

$$H_{2\gamma} = \begin{bmatrix} A_{22}^T & -R_{22} \\ -M_{22} & -A_{22} \end{bmatrix}. \tag{38}$$

Let

$\gamma_W = \max\{\gamma > 0 |$the Hamiltonian matrix $H_{2\gamma}$ is singular$\}$.

Moreover, let us define

$\gamma_{2s} = \inf\{\gamma > \gamma_W|$ the Riccati equation (35a) has a positive definite stabilizing solution, and $\lambda_{\max}(\bar{P}_{11}\bar{W}_{11}) < \gamma^2\}$.

As the results, if Assumptions 4,5 hold, then for every $\gamma > \bar{\gamma}_2 = \max\{\gamma_{2s}, \gamma_{2f}\}$, the Riccati equation (35a) and (35c) have the positive definite stabilizing solutions.

The 0–order solution of (33) is $O(\varepsilon)$ close to the exact one. We introduce

$$W_{11} = \bar{W}_{11} + \varepsilon F_{11}, \ W_{12} = \bar{W}_{12} + \varepsilon F_{12}, \ W_{22} = \bar{W}_{22} + \varepsilon F_{22}. \tag{39}$$

The $O(\varepsilon^k)$ approximation of F_{ij} $(i, \ j = 1, \ 2)$ will produce the $O(\varepsilon^{k+1})$ approximation of the required matrix W_{ij} $(i, \ j = 1, \ 2)$. Similar to the derivations in Section 3, we also obtain the following algorithm for (33),

$$T_0 F_{11}^{T(j+1)} + F_{11}^{(j+1)} T_0^T = -Z_2 H_{W1}^{T(j)} - H_{W1}^{(j)} Z_2^T + Z_2 H_{W3}^{(j)} Z_2^T + \varepsilon H_{W2}^{(j)} \tag{40a}$$

$$F_{11}^{(j+1)} T_2^T + F_{12}^{(j+1)} T_4^T + T_3 F_{22}^{T(j+1)} = H_{W1}^{(j)} \tag{40b}$$

$$T_4 F_{22}^{T(j+1)} + F_{22}^{(j+1)} T_4^T = H_{W3}^{(j)} \tag{40c}$$

where

$$
\begin{aligned}
H_{W1}^{(j)} &= -A_{11}W_{12}^{(j)} + W_{11}^{(j)}R_{11}W_{12}^{(j)} + W_{12}^{(j)}R_{12}^TW_{12}^{(j)} + \varepsilon(F_{11}^{(j)}R_{12}F_{22}^{T(j)} + F_{12}^{(j)}R_{22}F_{22}^{T(j)}),\\
H_{W2}^{(j)} &= F_{11}^{(j)}R_{11}F_{11}^{T(j)} + F_{12}^{(j)}R_{22}F_{12}^{T(j)} + F_{11}^{(j)}R_{12}F_{12}^{T(j)} + F_{12}^{(j)}R_{12}^TF_{11}^{(j)},\\
H_{W3}^{(j)} &= -A_{21}W_{12}^{(j)} - W_{12}^{T(j)}A_{21}^T + \varepsilon W_{12}^{T(j)}R_{11}W_{12}^{(j)} + \varepsilon F_{22}^{(j)}R_{22}F_{22}^{T(j)}\\
&\quad +W_{12}^{T(j)}R_{12}W_{22}^{T(j)} + W_{22}^{(j)}R_{12}^TW_{12}^{(j)},\\
W_{11}^{(j)} &= \bar{W}_{11} + \varepsilon F_{11}^{(j)},\ W_{12}^{(j)} = \bar{W}_{12} + \varepsilon F_{12}^{(j)},\ W_{22}^{(j)} = \bar{W}_{22} + \varepsilon F_{22}^{(j)},\\
F_{11}^{(0)} &= F_{12}^{(0)} = F_{22}^{(0)} = 0,\\
T_1 &= A_{11} - \bar{W}_{12}R_{12}^T - \bar{W}_{11}R_{11}^T,\ T_3 = A_{12} - \bar{W}_{12}R_{22}^T - \bar{W}_{11}R_{12},\\
T_0 &= T_1 - T_3T_4^{-1}T_2,\ Z_2 = T_3T_4^{-1}.
\end{aligned}
$$

The following theorem indicates the convergence the algorithm (40).

Theorem 3 *Under the stabilizability and detectability conditions, imposed in Assumptions 1, 4 and 5, for a prescribed $\gamma > \bar{\gamma}_2 = \max\{\gamma_{2s}, \gamma_{2f}\}$ and a small parameter $\varepsilon > 0$, the followings hold:*

i) *The proposed algorithm (40) converges to the exact solution of F with the rate of convergence of $O(\varepsilon^k)$, that is*

$$
||F - F^{(k)}|| = O(\varepsilon^k), \quad (k = 1, 2, \cdots), \tag{41}
$$

where

$$
F = \begin{bmatrix} F_{11} & F_{12} \\ F_{12}^T & F_{22} \end{bmatrix},\ F^{(k)} = \begin{bmatrix} F_{11}^{(k)} & F_{12}^{(k)} \\ F_{12}^{(k)T} & F_{22}^{(k)} \end{bmatrix}.
$$

ii) *The solution $W_\varepsilon^{(k)}$ is positive semi–definite stabilizing solution of the algebraic Riccati equation (5), where*

$$
W_\varepsilon^{(k)} = \begin{bmatrix} \bar{W}_{11} + \varepsilon F_{11}^{(k)} & \bar{W}_{12} + \varepsilon F_{12}^{(k)} \\ (\bar{W}_{12} + \varepsilon F_{12}^{(k)})^T & \varepsilon^{-1}(\bar{W}_{22} + \varepsilon F_{22}^{(k)}) \end{bmatrix}. \tag{42}
$$

Proof: We consider the corresponding Jacobian matrix given by

$$
J^2|_{\varepsilon=0} = \begin{bmatrix} J_{11}^2 & 0 & 0 \\ J_{21}^2 & J_{22}^2 & J_{23}^2 \\ 0 & 0 & J_{33}^2 \end{bmatrix} \tag{43}
$$

where

$$
\begin{aligned}
J_{11}^2 &= I \otimes T_0 + T_0^T \otimes I,\\
J_{22}^2 &= I \otimes T_4,\\
J_{33}^2 &= I \otimes T_4 + T_4^T \otimes I.
\end{aligned}
$$

When $\gamma > \bar{\gamma}_2$, the matrix T_4 is non–singular because of Assumption 4. The matrix $A_W^T - R_W\bar{W}_{11}^T$ is non–singular since Assumption 5 holds. Therefore, we obtain the following equation.

$$
A_W^T - R_W\bar{W}_{11}^T = T_1^T + M_1T_3^T = T_1^T - T_2^TT_4^{-T}T_3^T = T_0^T \tag{44}
$$

The matrix T_0 is stable also. Thus, for a sufficiently small parameter ε, the Jacobian is non–singular. The remainder of the proof is to show that $W_{\varepsilon}^{(k)}$ is positive semi–definite stabilizing solution. Using (42), we have

$$W_{\varepsilon}^{(k)} = \varepsilon^{-1} \begin{bmatrix} \varepsilon(\bar{W}_{11} + \varepsilon F_{11}^{(k)}) & \varepsilon(\bar{W}_{12} + \varepsilon F_{12}^{(k)}) \\ \varepsilon(\bar{W}_{12} + \varepsilon F_{12}^{(k)})^T & \bar{W}_{22} + \varepsilon F_{22}^{(k)} \end{bmatrix} = \varepsilon^{-1}\left(\begin{bmatrix} 0 & 0 \\ 0 & \bar{W}_{22} \end{bmatrix} + O(\varepsilon)\right).$$

The matrix $\bar{W}_{22}$ is positive definite if the algebraic Riccati equation (35c) has solution. Therefore, $W_{\varepsilon}^{(k)} \geq 0$. Next, by straightforward computations, we get

$$A_{\varepsilon} - W_{\varepsilon}^{(k)} R_{\gamma}^T = \begin{bmatrix} T_1 + O(\varepsilon) & T_3 + O(\varepsilon) \\ \varepsilon^{-1}\{T_2 + O(\varepsilon)\} & \varepsilon^{-1}\{T_4 + O(\varepsilon)\} \end{bmatrix}.$$

The matrix T_4 and T_0 are non–singular since Assumptions 4 and 5 hold. Thus, for a sufficiently small parameter ε, $A_{\varepsilon} - W_{\varepsilon}^{(k)} R_{\gamma}^T$ is stable. ■

Combining the results of Theorem 1 and Theorem 3, we arrive at Theorem 4.

Theorem 4 *If Assumptions 1–5 hold, then for* $\forall\gamma > \max\{\bar{\gamma}_1,\ \bar{\gamma}_2\} = \max\{\gamma_{1s},\ \gamma_{1f},\ \gamma_{2s},\ \gamma_{2f}\}$, $\exists\varepsilon^* > 0$ *such that* $\forall\varepsilon \in [0,\ \varepsilon^*)$*, the algebraic Riccati equations (3), (5) admit a positive semi–definite stabilizing solution (21), (42) respectively. Furthermore,* $\lambda_{\max}(P_{\varepsilon}^{(k)} W_{\varepsilon}^{(k)}) < \gamma^2$

Proof: We first note that under Assumption 1–5, the algebraic Riccati equations (3), (5) admit a positve semi–definite stabilizing solution for sufficiently large values γ by applying the results of Theorem 1, 3 in this paper.

Secondly, we can easily evaluate

$$I - \gamma^{-2} W_{\varepsilon}^{(k)} P_{\varepsilon}^{(k)} = \begin{bmatrix} I - \gamma^{-2}\bar{W}_{11}\bar{P}_{11} + O(\varepsilon) & O(\varepsilon) \\ O(1) & I - \gamma^{-2}\bar{W}_{22}\bar{P}_{22} + O(\varepsilon) \end{bmatrix}. \tag{45}$$

This shows that for small enough $\varepsilon > 0$, the matrix $I - \gamma^{-2} W_{\varepsilon}^{(k)} P_{\varepsilon}^{(k)}$ can have only positive eigenvalues since $\lambda_{\max}(\bar{W}_{11}\bar{P}_{11}) < \gamma^2$, $\lambda_{\max}(\bar{W}_{22}\bar{P}_{22}) < \gamma^2$ under the definition of γ_{2f}, γ_{2s}. Thus, this proof is completed. ■

Remark 5 *If the singular perturbation parameter* $\varepsilon > 0$ *is sufficiently small, then there exist the admissible controller for any disturbance attenuation level* $\gamma > \max\{\gamma_{1s},\ \gamma_{1f},\ \gamma_{2s},\ \gamma_{2f}\}$*. Moreover, the admissible controller is given by (7).*

5. NUMERICAL EXAMPLES

In order to demonstrate the efficiency of the proposed algorithm, we have run some numerical examples for both perfect state measurements and imperfect state measurements.

5.1. Example 1

The system matrix is given by

$$A_{11} = \begin{bmatrix} 0 & 0.4 \\ 0 & 0 \end{bmatrix},\ A_{12} = \begin{bmatrix} 0 & 0 \\ 0.345 & 0 \end{bmatrix},\ A_{21} = \begin{bmatrix} 0 & -0.524 \\ 0 & 0 \end{bmatrix},\ A_{22} = \begin{bmatrix} 0 & 0.262 \\ 0 & -1 \end{bmatrix},$$

$$B_{11} = \begin{bmatrix} 1.0 \\ 0 \end{bmatrix},\ B_{12} = \begin{bmatrix} 0.2 \\ 1.2 \end{bmatrix},\ B_{21} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ B_{22} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$Q = C_1^T C_1 = \mathrm{diag}\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix},\ D_{12}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Here, $\gamma_{1f} = 0.7680$ and $\gamma_{1s} = 7.0817$. Then, for every boundaly value of $\gamma > \bar{\gamma}_1 = 7.0817$ the algebraic Riccati equation (3) has the positive semi–definite stabilizing solution. The entries show the results obtained for small parameter $\varepsilon = 0.0001$. Using the MATLAB, the values of disturbance attenuation level γ is $\gamma^* = 7.0484$. Now, we choose $\gamma = 8.0 > \bar{\gamma}_1 > \gamma^*$ to design the controller.

By using proposed recursive algorithm, we can get the following control gain matrix $\bar{K}$

$$\bar{K} = \begin{bmatrix} -2.77286 & -0.72592 & -1.03794 & -0.24411 \end{bmatrix}.$$

Applying this controller on the system, the values of H_∞ performance is $\bar{\gamma} = 7.8727$.

On the other hand, using the MATLAB, we can get the control gain matrix $\hat{K}$

$$\hat{K} = \begin{bmatrix} -2.7726 & -0.7258 & -1.0379 & -0.2441 \end{bmatrix}.$$

Thus, the values of H_∞ performance is $\hat{\gamma} = 7.8729$. As a result of simulation, we have demonstrated that proposed controller achieves the performance $\gamma + O(\varepsilon^{k+1})$.

5.2. Example 2

Consider the system

$$\begin{bmatrix} x_1 \\ \varepsilon \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} w + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \tag{46a}$$

$$z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \tag{46b}$$

$$y = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \end{bmatrix} w. \tag{46c}$$

Since the matrix $A_{22} = 0$, the system (46) is nonstandard singularly perturbed systems. The four basic performance levels for the system (46) are

$$\gamma_{1f} = 3.0, \quad \gamma_{1s} = \gamma_{2f} = \sqrt{10}, \quad \gamma_{2s} = 3.17720 \tag{47}$$

Thus, for every boundaly value $\gamma > \max\{\bar{\gamma}_1, \ \bar{\gamma}_2\} = \max\{\gamma_{1f}, \ \gamma_{1s}, \ \gamma_{2f}, \ \gamma_{2s}\} = 3.17720$, the algebraic Riccati equation (3), (5) have the positive semi–definite stabilizing solution. The numerical results are obtained for small parameter $\varepsilon = 10^{-8}$. Using the MATLAB, the values of disturbance attenuation level γ is $\gamma^* = 3.1686$ for $\varepsilon = 10^{-8}$. Now, we choose $\gamma = 3.5 > \max\{\bar{\gamma}_1, \ \bar{\gamma}_2\} > \gamma^*$ and solve the algebraic Riccati equation (3), (5) by using proposed recursive algorithm. First, in case of the algebraic Riccati equation (3), the proposed algorithm (19) has produced the positive semi–definite stabilizing solution after 3 iterations. The result of simulation is shown in Table 1, where

$$\bar{P} = \begin{bmatrix} P_{11} & \varepsilon P_{21}^T \\ \varepsilon P_{21} & \varepsilon P_{22} \end{bmatrix}. \tag{48}$$

Table 1. Value of P when $\varepsilon = 10^{-8}$, $\gamma = 3.5$.

j	P_{11}	P_{21}	P_{22}
1	4.2772536	16.021542	1.9414507
2	4.2772543	16.021543	1.9414511
3	4.2772543	16.021543	1.9414511

Secondly, in case of the algebraic Riccati equation (5), the result of simulation is shown in Table 2, where

$$\bar{W} \;=\; \begin{bmatrix} W_{11} & W_{12} \\ W_{12}^T & \varepsilon^{-1}W_{22} \end{bmatrix}. \tag{49}$$

Table 2. Value of W when $\varepsilon = 10^{-8}$, $\gamma = 3.5$.

j	W_{11}	W_{12}	W_{22}
1	0.15376483	1.6835733	3.1304952
2	0.15376484	1.6835732	3.1304951
3	0.15376484	1.6835732	3.1304951

Hence

$$I - \gamma^{-2}\bar{W}\bar{P} = \begin{bmatrix} 0.94631 & -4.67928 \times 10^{-9} \\ -4.68216 & 0.50386 \end{bmatrix}.$$

This shows that the matrix $I - \gamma^{-2}\bar{W}\bar{P}$ have only positive eigenvalues.

In order to verify the exactitude of the solution, we calculate the remainder when substitute $\tilde{P}$ and $\tilde{W}$ into the generalized algebraic Riccati equation (11a) and (32a) respectively.

$$\tilde{P}^T A + A^T\tilde{P} - \tilde{P}^T S_\gamma \tilde{P} + Q = 10^{-7} \times \begin{bmatrix} 7.882 & 1.894 \\ 1.894 & 0.471 \end{bmatrix}$$

$$\tilde{W}A^T + A\tilde{W}^T - \tilde{W}R_\gamma\tilde{W}^T + M = 10^{-7} \times \begin{bmatrix} 0.756 & 1.200 \\ 1.200 & 1.113 \end{bmatrix}$$

where

$$\tilde{P} = \begin{bmatrix} P_{11} & \varepsilon P_{21}^T \\ P_{21} & P_{22} \end{bmatrix}, \quad \tilde{W} = \begin{bmatrix} W_{11} & W_{12} \\ \varepsilon W_{12}^T & W_{22} \end{bmatrix}.$$

Therefore, the numerical example illustrates the effectiveness of the proposed algorithm since the solutions $\tilde{P}$ and $\tilde{W}$ converge to the exact solutions P and W which were defined by (11a) and (32a). Indeed, we can obtain the solution of the algebraic Riccati equations (3) and (5) even though A_{22} is singular.

Furthermore, applying the controller by substituting (48), (49) into (7) to the system (46), the values of H_∞ performance is $\bar{\gamma} = 3.3736$. We find that when we choose a γ larger than the maximum of γ_{1f}, γ_{1s}, γ_{2f} and γ_{2s}, the obtained admissible controller by using recursive algorithm achieves the desired performance bound for a sufficiently small value $\varepsilon > 0$.

6. CONCLUSION

In this paper, we have considered the H_∞ control problem for singularly perturbed systems under both prefect and imperfect state measurements. We have provided a controller that guarantees a disturbance attenuation level larger than the boundary value of the H_∞ performance for the reduced–order slow and fast subsystems. Such a controller is designed by using recursive algorithm. The main contribution of the paper is to propose a new recursive algorithm to solve the generalized algebraic Riccati equations and to find sufficient conditions for the convergence of the recursive algorithm. In case of imperfect state measurements, if we choose any $\gamma > \max\{\bar{\gamma}_1, \ \bar{\gamma}_2\} > \max\{\gamma_{1s}, \ \gamma_{1f}, \ \gamma_{2s}, \ \gamma_{2f}\}$ as sufficient conditions, then there exist the admissible controller that attain the disturbance attenuation level γ. This time, recursive solutions of generalized algebraic Riccati equations converge to a positive semi–definite stabilizing solutions with the rate of convergence of $O(\varepsilon^k)$. In addition, our new results are applicable to both standard and nonstandard singularly perturbed systems. On the other hand, in case of the H_∞ control problem for prefect state measurements, we proposed a controller with an accuracy $O(\varepsilon^k)$ which is different from the accuracy controller given in Fridman (1996). That is, the structure of the controller proposed in this paper is simpler than the structures previously proposed in Fridman (1996). This simpler structure is achieved by using the recursive algorithm. We have also shown that an $O(\varepsilon^k)$ accuracy controller achieves the performance $\gamma + O(\varepsilon^{k+1})$.

References

Doyle, J. C., Glover, K., Khargonekar, P. P., and Francis, B. A., 1989, State Space Solution to Standard H_2, and H_∞ Control Problems. *IEEE Transaction on Automatic Control*, **34**, 831–847.

Dragan, V., 1996, H_∞–norms and Disturbance Attenuation for Systems with Fast Transients. *IEEE Transaction on Automatic Control*, **41**, 747–750.

Fridman, E., 1996, Near–Optimal H_∞– Control of Linear Singularly Perturbed Systems. *IEEE Transaction on Automatic Control*, **41**, 236–240.

Gajic, Z., Petkovski, D., and Shen, X., 1990, Singularly Perturbed and Weakly Coupled Linear System–a Recursive Approach: Lecture Notes in Control and Information Sciences, 140, (Berlin: Springer–Verlag).

Gajic, Z., 1986, Numerical fixed–point solution for near–optimum regulators of linear quadratic gaussian control problems for singular perturbed systems. *International Journal of Control*, **43**, 373–387.

Khalil, H. K., 1984, Time Scale Decomposition of Linear Implicit Singularly Perturbed Systems. *IEEE Transaction on Automatic Control*, **29**, 1054–1056.

Khalil, H. K., 1989, Feedback Control of Nonstandard Singularly Perturbed Systems. *IEEE Transaction on Automatic Control*, **34**, 1052–1060.

Kokotović, P., Khalil, H. K., and O'Reilly, J., 1986, Singular Perturbation Methods in Control: Analysis and Design (New York: Academic Press).

Mukaidani, H., Xu, H., and Mizukami, K., 1997, The Recursive Algorithm of H_∞ Control Problems for Nonstandard Singularly Perturbed Systems. *Proceedings of the 2th Asian Control Conference*, Korea, 22–25 July, pp.591–594.

Pan, Z., and Basar, T., 1993, H_∞–Optimal Control for Singularly Perturbed Systems–Part I. Perfect state measurements. *Automatica*, **29**, 401–423.

Pan, Z., and Basar, T., 1994, H_∞–Optimal Control for Singularly Perturbed Systems–Part II. Imperfect state measurements. *IEEE Transaction on Automatic Control*, **39**, 230–299.

Xu, H., and Mizukami, K., 1996, Nonstandard extension of $H_\infty-$optimal control for singularly perturbed systems. *Proceedings of the 7th International Symposium on Dynamic Games and Applications*, Japan, 16–18 December, pp.931–948.