Recursive Algorithm for Mixed H_2/H_∞ Control Problem of Singularly Perturbed Systems

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Abstract

In this paper, we study the mixed H_2/H_∞ control problem for infinite horizon singularly perturbed systems. In order to solve the problem, we must solve a pair of parameterized cross–coupled algebraic Riccati equations with a small positive parameter ε. Firstly, we solve the parameterized cross–coupled algebraic Riccati equations by using a Lyapunov iteration approach. Sufficient conditions are provided such that the proposed Lyapunov iterations converge to a positive semidefinite solution. Secondly, we propose a new algorithm, which combines Lyapunov iterations and recursive techniques together, to solve the parameterized cross–coupled algebraic Riccati equations. The new algorithm ensures that the solution of the parameterized cross–coupled algebraic Riccati equations converges to a positive semidefinite solution with the rate of convergence of $O(\varepsilon^k)$. As another important feature of this paper, our method is applicable to both standard and nonstandard singularly perturbed systems.

Key Words: Singularly perturbed systems, Mixed H_2/H_∞ control problem, Generalized algebraic Riccati equation, Lyapunov iterations, Recursive algorithm

1. Introduction

The cross–coupled algebraic Riccati equations play an important role to some problems of modern control theory (see for example Starr *et al.* 1969, Abou–Kandil *et al.* 1993, Li and Gajić 1994, Limebeer *et al.* 1994, Freiling *et al.* 1996, Xu and Mizukami 1996, Xu and Mizukami 1997). In Limebeer *et al.* (1994), a state feedback mixed H_2/H_∞ control problem is formulated as a dynamic Nash game, where one performance index is used to reflect an H_∞ constraint and the other performance index reflects an H_2 optimality requirement. This problem is solved by using the established theory of nonzero-sum games and the resulting feedback controller is characterized by the solution to a pair of cross-coupled algebraic Riccati equations.

It is well known that in order to obtain the Nash equilibrium strategies, we must solve the cross–coupled algebraic Riccati equations. Li and Gajić (1994) proposed an algorithm, called the Lyapunov iterations, to solve the linear–quadratic Nash game. Freiling *et al.*(1996) found the solutions to the cross–coupled algebraic Riccati equations of the mixed H_2/H_∞ type by using the Riccati iterations. But, the convergence of the Riccati iterations was not proved.

In recent years, the recursive algorithm for various control problems of not only singularly perturbed but also weakly coupled systems have been developed in many literatures (Gajić *et al.*1990, Gajić and Shen 1993, Gajić *et al.* 1995, Mizukami and Suzumura 1993, Mukaidani *et al.* 1998). It has been shown that the recursive algorithm are very effective to solve the algebraic Riccati equations when the system matrices are functions of a small perturbation parameter ε. So far, dynamic Nash games of the weakly coupled systems were studied in Gajić *et al.* 1990 and Gajić and Shen 1993 by means of a recursive algorithm. However, the recursive algorithm for solving the cross–coupled algebraic Riccati equations with relation to the dynamic Nash games of the singularly perturbed systems has not been investigated.

In this paper, we study the mixed H_2/H_∞ control problem for infinite horizon singularly perturbed systems from a viewpoint of solving the parameterized cross–coupled algebraic Riccati equations. We first apply the Lyapunov iterations to solve the parameterized cross–coupled algebraic Riccati equations. The sufficient conditions are provided such that the proposed Lyapunov iterations converges to a positive semidefinite solution. Since the singularly perturbed systems contain a small positive perturbation parameter ε, it is difficult to solve the resulted Lyapunov equations. We then propose a new algorithm, which combines the Lyapunov iterations and the recursive techniques together, to solve the parameterized cross–coupled algebraic Riccati equations. Using the new algorithm, we will overcome the computation difficulties caused by high dimensions and numerical stiffness in the Lyapunov iteration method. The convergence of the algorithm is proved by using the successive approximations of dynamic programming. It is worth to note that the recursive approach to solve the linear–quadratic Nash games and the H_2/H_∞ control problems for singularly perturbed systems has never been studied. Also, we have not found any work concerning the Lyapunov iterations to solve the mixed H_2/H_∞ control problem. As another important feature of this paper, we do not assume that A_{22} is non-singular. Therefore, our new algorithm is applicable to both standard and nonstandard singularly perturbed systems.

This paper is organized as follows. In Section 2, the problem of the H_2/H_∞ control is formulated for the singularly perturbed systems. In Section 3, we apply the Lyapunov iterations to solve the parameterized cross–coupled algebraic Riccati equations. The sufficient conditions are proven such that the Lyapunov iterations converges to a positive semidefinite solution. In Section 4, we propose a new algorithm, which combines the Lyapunov iterations and the recursive techniques together, to solve the parameterized cross–coupled algebraic Riccati equations. In Section 5, to show the effectiveness of the proposed algorithm, numerical examples are included. Finally, in Section 6 we conclude some discussions on the results.

Notation: The notations used in this paper are fairly standard. The superscript T denotes matrix transpose. I_n denotes the $n \times n$ identity matrix. $\|\cdot\|_\infty$ denotes its H_∞ norm for a transfer matrix function. $L_2[0,\ \infty)$

stands for the space of square integrable vector functions over the interval $[0, \infty)$. $\|\cdot\|_2$ denotes its $L_2[0, \infty)$ norm for a continuous function over $[0, \infty)$. $\|\cdot\|$ denotes its Euclidean norm for a matrix. $\otimes$ denotes the Kronecker product.

2. Problem Formulation

Consider a linear time–invariant singularly perturbed system

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + D_1w(t) + B_1u(t), \ x_1(0) = 0, \tag{1a}$$

$$\varepsilon\dot{x}_2(t) = A_{21}x_1(t) + A_{22}x_2(t) + D_2w(t) + B_2u(t), \ x_2(0) = 0, \tag{1b}$$

$$z(t) = \begin{bmatrix} Cx(t) \\ Lu(t) \end{bmatrix}, \ x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{1c}$$

and a quadratic cost function

$$J(x(t), u(t)) = \int_0^\infty z^T(t)z(t)dt = \|z(t)\|_2^2, \tag{2}$$

where ε is a small positive parameter, $x_1 \in \mathrm{R}^{n_1}$ and $x_2 \in \mathrm{R}^{n_2}$ are states, $u \in \mathrm{R}^{l_1}$ is the control input, $w \in \mathrm{R}^{l_2}$ is the disturbance, $z \in \mathrm{R}^{k_2}$ is the controlled output. All matrices above are of appropriate dimensions. We suppose that $L^TL = I_{l_1}$. The system (1) is said to be in the standard form if the matrix A_{22} is nonsingular. Otherwise, it is called the nonstandard singularly perturbed systems (Kokotović *et al.* 1986).

Let us introduce the partitioned matrices

$$\begin{aligned}
A &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \ A_\varepsilon = \begin{bmatrix} A_{11} & A_{12} \\ \varepsilon^{-1}A_{21} & \varepsilon^{-1}A_{22} \end{bmatrix}, \\
B &= \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \ B_\varepsilon = \begin{bmatrix} B_1 \\ \varepsilon^{-1}B_2 \end{bmatrix}, \ D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}, \ D_\varepsilon = \begin{bmatrix} D_1 \\ \varepsilon^{-1}D_2 \end{bmatrix}, \\
S_\varepsilon &= B_\varepsilon B_\varepsilon^T = \begin{bmatrix} S_{11} & \varepsilon^{-1}S_{12} \\ \varepsilon^{-1}S_{12}^T & \varepsilon^{-2}S_{22} \end{bmatrix}, \ S = BB^T = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}, \\
U_\varepsilon &= D_\varepsilon D_\varepsilon^T = \begin{bmatrix} U_{11} & \varepsilon^{-1}U_{12} \\ \varepsilon^{-1}U_{12}^T & \varepsilon^{-2}U_{22} \end{bmatrix}, \ U = DD^T = \begin{bmatrix} U_{11} & U_{12} \\ U_{12}^T & U_{22} \end{bmatrix}, \\
Q &= C^TC = \begin{bmatrix} C_1^T \\ C_2^T \end{bmatrix} \begin{bmatrix} C_1 & C_2 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}.
\end{aligned}$$

We now consider the mixed H_2/H_∞ control problems for singularly perturbed system (1) under the following basic assumption (Gajić *et al.* 1990, Gajić and Shen 1993, Gajić *et al.* 1995).

Assumption 1 *The triplet* $(A_\varepsilon, \ B_\varepsilon, \ C)$ *and* $(A_\varepsilon, \ D_\varepsilon, \ C)$ *are stabilizable and detectable for* $\varepsilon \in (0, \varepsilon^*]$ $(\varepsilon^* > 0)$.

Assumption 2 *The triplet* $(A_{22}, \ B_2, \ C_2)$ *and* $(A_{22}, \ D_2, \ C_2)$ *are stabilizable and detectable.*

These conditions are quite natural since at least one control agent has to be able to control and observe unstable modes.

The mixed H_2/H_∞ control problem is formulated as a two–player Nash game associated with a prescribed disturbance attenuation level γ,

$$J_1(x, u, w) = \int_0^\infty \gamma^2 w^T(t)w(t)dt - J(x, u) = \gamma^2\|w(t)\|_2^2 - \|z(t)\|_2^2, \tag{3a}$$

$$J_2(x, u, w) = J(x, u) = \|z(t)\|_2^2. \tag{3b}$$

The first is used to reflect an H_∞ criterion, while the second is used for an H_2 optimality requirement. The purpose is to find a linear feedback controller $u^*(t) = K_2 x(t)$ such that

$$J_1(u^*, w^*) \le J_1(u^*, w), \tag{4a}$$

$$J_2(u^*, w^*) \le J_2(u, w^*), \tag{4b}$$

where $w^*(t) = K_1 x(t)$ represents the worst-case disturbance. When $J_1(u^*, w^*) \ge 0$, we have

$$\sup_{w \in H_w} \frac{\sqrt{J(u^*,\ w)}}{\|w(t)\|_2} \le \gamma, \tag{5}$$

a H_∞ criterion, where H_w denotes an appropriate Hilbert space. The second Nash inequality shows that $u^*(t)$ regulates the state to zero with minimum output energy when the disturbance is at its worst value $w^*(t)$. The following lemma is already known (see Limebeer *et al.* 1994).

Lemma 1 *Under Assumption 1, there exists an admissible controller such that (4) hold iff the following full–order parameterized cross–coupled algebraic Riccati equations*

$$(A_\varepsilon - S_\varepsilon Y_\varepsilon)^T X_\varepsilon + X_\varepsilon (A_\varepsilon - S_\varepsilon Y_\varepsilon) + Q + \gamma^{-2} X_\varepsilon U_\varepsilon X_\varepsilon + Y_\varepsilon S_\varepsilon Y_\varepsilon = 0, \tag{6a}$$

$$(A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon)^T Y_\varepsilon + Y_\varepsilon (A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon) + Q - Y_\varepsilon S_\varepsilon Y_\varepsilon = 0, \tag{6b}$$

have solutions $X_\varepsilon \ge 0$ *and* $Y_\varepsilon \ge 0$ *where*

$$X_\varepsilon = \begin{bmatrix} X_{11} & \varepsilon X_{21}^T \\ \varepsilon X_{21} & \varepsilon X_{22} \end{bmatrix},\quad Y_\varepsilon = \begin{bmatrix} Y_{11} & \varepsilon Y_{21}^T \\ \varepsilon Y_{21} & \varepsilon Y_{22} \end{bmatrix}.$$

Then, the strategies are given by

$$w^*(t) = K_1 x(t) = \gamma^{-2} D_\varepsilon^T X_\varepsilon x(t), \tag{7a}$$

$$u^*(t) = K_2 x(t) = -B_\varepsilon^T Y_\varepsilon x(t). \tag{7b}$$

However, it is difficult to solve the parameterized cross–coupled algebraic Riccati equations (6a) and (6b) because of the different magnitudes of their coefficients caused by the small perturbation parameter ε and high dimensions.

3. The Parameterized Cross–Coupled Generalized Algebraic Riccati Equations

To obtain the solutions of the parameterized cross–coupled algebraic Riccati equations (6a) and (6b), we first define

$$\Pi_\varepsilon = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \varepsilon I_{n_2} \end{bmatrix}.$$

Then, we introduce the following useful lemma.

Lemma 2 *The parameterized cross–coupled algebraic Riccati equations (6a) and (6b) are equivalent to the following parameterized cross–coupled generalized algebraic Riccati equations (8a) and (8b) respectively.*

$$(A - SY)^T X + X^T (A - SY) + Q + \gamma^{-2} X^T U X + Y^T S Y = 0, \tag{8a}$$

$$(A + \gamma^{-2} U X)^T Y + Y^T (A + \gamma^{-2} U X) + Q - Y^T S Y = 0, \tag{8b}$$

where

$$X_\varepsilon = \Pi_\varepsilon^T X = X^T \Pi_\varepsilon,\quad Y_\varepsilon = \Pi_\varepsilon^T Y = Y^T \Pi_\varepsilon,\quad X = \begin{bmatrix} X_{11} & \varepsilon X_{21}^T \\ X_{21} & X_{22} \end{bmatrix},\quad Y = \begin{bmatrix} Y_{11} & \varepsilon Y_{21}^T \\ Y_{21} & Y_{22} \end{bmatrix}.$$

Proof: The proof is identical to the proof of Lemma 3 in Mukaidani *et al.* (1999). □

In Li and Gajić (1994), only the Lyapunov iterations for solving cross–coupled algebraic Riccati equations of Nash differential games are considered. In this paper, we give the Lyapunov iterations to solve the parameterized cross–coupled algebraic Riccati equations. An algorithm for the numerical solutions of (8) is defined as follows.

$$(A + \gamma^{-2} U X^{(n)} - S Y^{(n)})^T X^{(n+1)} + X^{(n+1)T} (A + \gamma^{-2} U X^{(n)} - S Y^{(n)}) \\ + Q - \gamma^{-2} X^{(n)T} U X^{(n)} + Y^{(n)T} S Y^{(n)} = 0, \tag{9a}$$

$$(A + \gamma^{-2} U X^{(n)} - S Y^{(n)})^T Y^{(n+1)} + Y^{(n+1)T} (A + \gamma^{-2} U X^{(n)} - S Y^{(n)}) \\ + Q + Y^{(n)T} S Y^{(n)} = 0, \tag{9b}$$

where $n = 0,\ 1,\ 2,\ 3,\ \cdots$ and initial conditions $X^{(0)},\ Y^{(0)}$ are obtained as solutions of following auxiliary generalized algebraic Riccati equations

$$A^T Y^{(0)} + Y^{(0)T} A + Q - Y^{(0)T} S Y^{(0)} = 0, \tag{10a}$$

$$(A - S Y^{(0)})^T X^{(0)} + X^{(0)T} (A - S Y^{(0)}) + Q + \gamma^{-2} X^{(0)T} U X^{(0)} + Y^{(0)T} S Y^{(0)} = 0, \tag{10b}$$

$$X^{(n)} = \begin{bmatrix} X_{11}^{(n)} & \varepsilon X_{21}^{(n)T} \\ X_{21}^{(n)} & X_{22}^{(n)} \end{bmatrix},\quad Y^{(n)} = \begin{bmatrix} Y_{11}^{(n)} & \varepsilon Y_{21}^{(n)T} \\ Y_{21}^{(n)} & Y_{22}^{(n)} \end{bmatrix}.$$

We note that the unique positive semidefinite stabilizing solution of (10a) exists under Assumptions 1 and 2 (Gajić *et al.*1990, Gajić and Shen 1993, Gajić *et al.* 1995). Concerning with the Riccati equation (10b), let us define

$$\bar{\gamma}_1 = \|\hat{E}(sI - \hat{A})^{-1}\hat{D} + \hat{H}\|_\infty, \tag{11a}$$

$$\bar{\gamma}_2 = \|\bar{E}_2(sI - \bar{A}_{22})^{-1} D_2\|_\infty, \tag{11b}$$

where

$$A - S Y^{(0)} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix},\quad Q + Y^{(0)T} S Y^{(0)} = \begin{bmatrix} \bar{E}_1 \bar{E}_1^T & \bar{E}_1 \bar{E}_2^T \\ \bar{E}_2 \bar{E}_1^T & \bar{E}_2 \bar{E}_2^T \end{bmatrix},$$

$$\hat{A} = \bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21},\quad \hat{D} = D_1 - \bar{A}_{12}\bar{A}_{22}^{-1} D_2,$$

$$\hat{E} = \bar{E}_1 - \bar{E}_2\bar{A}_{22}^{-1}\bar{A}_{21},\quad \hat{H} = -\bar{E}_2\bar{A}_{22}^{-1} D_2.$$

If Assumptions 1 and 2 hold, then for every $\gamma > \bar{\gamma} = \max\{\bar{\gamma}_1,\ \bar{\gamma}_2\}$, the Riccati equation (10b) has the positive definite stabilizing solutions since the Riccati equation (10a) has stabilizing solution (Dragan 1996, Mukaidani 1998).

The algorithm (9) is based on the Lyapunov iterations (Li and Gajić 1994, Gajić and Shen 1993, Gajić *et al.* 1995). Although the algorithm (9) is similar to as that of Li and Gajic (1994), different convergent conditions are required. In Li and Gajic (1994), the stabilizable–detectable conditions will guarantee the convergence of the Lyapunov iterations of Nash games to the positive semidefinite solutions. However, the convergence of the Lyapunov iteration in this paper depends on the value of the parameter γ. In fact, if γ is very small, Lyapunov iteration (9a) may not yield the solution of $X^{(n+1)}$. Because the last three terms of the Lyapunov iteration (9a), that is, $Q - \gamma^{-2} X^{(n)T} U X^{(n)} + Y^{(n)T} S Y^{(n)}$, is not always positive semidefinite.

In this paper, under the control–oriented assumptions and a new condition for γ, we prove that the proposed Lyapunov iterations (9) converge to the positive semidefinite solutions. The algorithm (9) has the feature given in the following theorem.

Theorem 1 *Under Assumptions 1 and 2, for a predescribed disturbance attenuation level* $\gamma > \bar{\gamma} = \max\{\bar{\gamma}_1,\ \bar{\gamma}_2\}$ *and a small parameter* $\varepsilon > 0$*, the unique positive semidefinite solutions of the parameterized cross–coupled algebraic Riccati equation (6) exist, where* $\bar{\gamma}_1,\ \bar{\gamma}_2$ *are given by (11a) and (11b) respectively. It is obtained by performing Lyapunov iterations (9a) and (9b).*

Proof: We give the proof by using a method similar to that given in the proof of Theorem2.1 in Li and Gajić (1994). The proof based on the method of successive approximations (Aganovic and Gajić 1995). Firstly, we take any stabilizable linear control law $u^{(0)}(t,\ x) = -B_\varepsilon^T Y_\varepsilon^{(0)} x(t)$ and disturbance $w^{(0)}(t,\ x) = \gamma^{-2} D_\varepsilon^T X_\varepsilon^{(0)} x(t)$ where $X_\varepsilon^{(0)}$ and $Y_\varepsilon^{(0)}$ are positive semidefinite stabilizing solutions of auxiliary generalized algebraic Riccati equations (10). Then, let us consider the following two minimization problems.

$$\dot{x}(t) = A_\varepsilon x(t) + D_\varepsilon w(t) + B_\varepsilon u^{(0)}(t) = [A_\varepsilon - S_\varepsilon Y_\varepsilon^{(0)}]x(t) + D_\varepsilon w(t), \tag{12a}$$

$$\begin{aligned} V_1(x,\ t) &= \min_{w(t)} \int_t^\infty [\gamma^2 w(\tau)^T w(\tau) - \{x(\tau)^T Q x(\tau) + u^{(0)}(\tau,\ x)^T u^{(0)}(\tau,\ x)\}] d\tau \\ &= \min_{w(t)} \int_t^\infty [\gamma^2 w(\tau)^T w(\tau) - x^T(\tau)\{Q + Y_\varepsilon^{(0)} S_\varepsilon Y_\varepsilon^{(0)}\} x(\tau)] d\tau, \end{aligned} \tag{12b}$$

$$\dot{x}(t) = A_\varepsilon x(t) + D_\varepsilon w^{(0)}(t) + B_\varepsilon u(t) = [A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon^{(0)}]x(t) + B_\varepsilon u(t), \tag{12c}$$

$$V_2(x,\ t) = \min_{u(t)} \int_t^\infty [x(\tau)^T Q x(\tau) + u(\tau)^T u(\tau)] d\tau, \tag{12d}$$

where

$$X_\varepsilon^{(0)} = \begin{bmatrix} X_{11}^{(0)} & \varepsilon X_{21}^{(0)T} \\ \varepsilon X_{21}^{(0)} & \varepsilon X_{22}^{(0)} \end{bmatrix},\quad Y_\varepsilon^{(0)} = \begin{bmatrix} Y_{11}^{(0)} & \varepsilon Y_{21}^{(0)T} \\ \varepsilon Y_{21}^{(0)} & \varepsilon Y_{22}^{(0)} \end{bmatrix}.$$

Corresponding Hamiltonians to the Nash differential games for each control agent are respectively

$$H_1(t,\ x,\ w,\ u^{(0)},\ p_1^{(0)}) = \gamma^2 w^T w - x^T Q x - u^{(0)T} u^{(0)} + p_1^{(0)T}(A_\varepsilon x + D_\varepsilon w + B_\varepsilon u^{(0)}), \tag{13a}$$

$$H_2(t,\ x,\ w^{(0)},\ u,\ p_2^{(0)}) = x^T Q x + u^T u + p_2^{(0)T}(A_\varepsilon x + D_\varepsilon w^{(0)} + B_\varepsilon u), \tag{13b}$$

where

$$\frac{\partial}{\partial x} V_i^{(0)}(x,\ t) = p_i^{(0)}(t),\ (i = 1,\ 2),\quad \dot{x}(t) = [A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon^{(0)} - S_\varepsilon Y_\varepsilon^{(0)}]x(t),$$

$$V_1^{(0)}(x,\ t) = \int_t^\infty x^T(\tau)[\gamma^{-2} X_\varepsilon^{(0)} U_\varepsilon X_\varepsilon^{(0)} - Q - Y_\varepsilon^{(0)} S_\varepsilon Y_\varepsilon^{(0)}] x(\tau) d\tau,$$

$$V_2^{(0)}(x,\ t) = \int_t^\infty x^T(\tau)[Q + Y_\varepsilon^{(0)} S_\varepsilon Y_\varepsilon^{(0)}] x(\tau) d\tau.$$

The equilibrium controls must satisfy

$$\frac{\partial H_1}{\partial w} = 0\ \Rightarrow\ w^{(1)}(t,\ x) = -\frac{1}{2}\gamma^{-2} D_\varepsilon^T p_1^{(0)}(t), \tag{14a}$$

$$\frac{\partial H_2}{\partial u} = 0\ \Rightarrow\ u^{(1)}(t,\ x) = -\frac{1}{2} B_\varepsilon^T p_2^{(0)}(t). \tag{14b}$$

Note that $\frac{\partial}{\partial x} V_i^{(0)}(x,\ t)$ along the system trajectory can be calculated from (15).

$$\frac{\partial}{\partial x} V_i^{(0)}(x,\ t) \cdot \frac{dx}{dt} = \frac{d}{dt} V_i^{(0)}(x,\ t),\ (i = 1,\ 2). \tag{15}$$

In fact, we obtain the following equations (16).

$$\begin{aligned} &\frac{\partial}{\partial x} V_1^{(0)}(x,\ t) \cdot [A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon^{(0)} - S_\varepsilon Y_\varepsilon^{(0)}]x(t) \\ &\qquad = -x(t)^T [\gamma^{-2} X_\varepsilon^{(0)} U_\varepsilon X_\varepsilon^{(0)} - Q - Y_\varepsilon^{(0)} S_\varepsilon Y_\varepsilon^{(0)}] x(t), \end{aligned} \tag{16a}$$

$$\frac{\partial}{\partial x} V_2^{(0)}(x,\ t) \cdot [A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon^{(0)} - S_\varepsilon Y_\varepsilon^{(0)}]x(t) = -x(t)^T [Q + Y_\varepsilon^{(0)} S_\varepsilon Y_\varepsilon^{(0)}] x(t). \tag{16b}$$

These simple partial differential equations (16) have solutions of the following form

$$V_1^{(0)}(x,\ t) = -x(t)^T X_\varepsilon^{(1)} x(t), \tag{17a}$$

$$V_2^{(0)}(x,\ t) = x(t)^T Y_\varepsilon^{(1)} x(t). \tag{17b}$$

A partial differentiation to (17) gives

$$\frac{\partial}{\partial x} V_1^{(0)}(x,\ t) = -2X_\varepsilon^{(1)} x(t) = p_1^{(0)}(t), \tag{18a}$$

$$\frac{\partial}{\partial x} V_2^{(0)}(x,\ t) = 2Y_\varepsilon^{(1)} x(t) = p_2^{(0)}(t). \tag{18b}$$

By using the following relation

$$\begin{aligned} & 2x^T(t)(A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon^{(0)} - S_\varepsilon Y_\varepsilon^{(0)})^T X_\varepsilon^{(1)} x(t) \\ = \; & x^T(t)[(A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon^{(0)} - S_\varepsilon Y_\varepsilon^{(0)})^T X_\varepsilon^{(1)} + X_\varepsilon^{(1)}(A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon^{(0)} - S_\varepsilon Y_\varepsilon^{(0)})] x(t), \end{aligned}$$

we have

$$\begin{aligned} & (A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon^{(0)} - S_\varepsilon Y_\varepsilon^{(0)})^T X_\varepsilon^{(1)} + X_\varepsilon^{(1)}(A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon^{(0)} - S_\varepsilon Y_\varepsilon^{(0)}) \\ & \quad = -(Q - \gamma^{-2} X_\varepsilon^{(0)} U_\varepsilon X_\varepsilon^{(0)} + Y_\varepsilon^{(0)} S_\varepsilon Y_\varepsilon^{(0)}), \end{aligned} \tag{19a}$$

$$\begin{aligned} & (A_\varepsilon + \gamma^{-2} U_\varepsilon X^{(0)\varepsilon} - S_\varepsilon Y_\varepsilon^{(0)})^T Y_\varepsilon^{(1)} + Y_\varepsilon^{(1)}(A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon^{(0)} - S_\varepsilon Y^{(0)})_\varepsilon \\ & \quad = -(Q + Y_\varepsilon^{(0)} S_\varepsilon Y_\varepsilon^{(0)}). \end{aligned} \tag{19b}$$

Since the Riccati equation (10b) has a positive semidefinite stabilizing solution by the bounded real lemma (Dragan 1996, Zhou 1998), $A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon^{(0)} - S_\varepsilon Y_\varepsilon^{(0)}$ is a stable matrix. Furthermore, we see that the right–hand side of equation (19b) is negative definite and $X_\varepsilon^{(1)} = X_\varepsilon^{(0)} \geq 0$ comparing equation (19a) with equation (10b). Consequently, it follow that the Lyapunov equations (19a) and (19b) have unique positive semidefinite solutions $X_\varepsilon^{(1)} \geq 0$, $Y_\varepsilon^{(1)} \geq 0$ respectively.

Thus, from (14) and (18) we get

$$w^{(1)}(t,\ x) = \gamma^{-2} D_\varepsilon^T X_\varepsilon^{(1)} x(t),\ X_\varepsilon^{(1)} \geq 0, \tag{20a}$$

$$u^{(1)}(t,\ x) = -B_\varepsilon^T Y_\varepsilon^{(1)} x(t),\ Y_\varepsilon^{(1)} \geq 0. \tag{20b}$$

On the other hand, substituting $w^{(0)}(t,\ x)$, $u^{(0)}(t,\ x)$, and (18) into Hamiltonians (13), it follows from the parameterized cross–coupled algebraic Riccati equation (19) that equality (21) holds.

$$\begin{aligned} & H_1(t,\ x,\ w^{(0)},\ u^{(0)},\ p_1^{(0)}) \\ = \; & \gamma^2 w^{(0)T} w^{(0)} - x^T Q x - u^{(0)T} u^{(0)} + p_1^{(0)T}(A_\varepsilon x + D_\varepsilon w^{(0)} + B_\varepsilon u^{(0)}) \equiv 0, \end{aligned} \tag{21a}$$

$$\begin{aligned} & H_2(t,\ x,\ w^{(0)},\ u^{(0)},\ p_2^{(0)}) \\ = \; & x^T Q x + u^{(0)T} u^{(0)} + p_2^{(0)T}(A_\varepsilon x + D_\varepsilon w^{(0)} + B_\varepsilon u^{(0)}) \equiv 0. \end{aligned} \tag{21b}$$

Thus, we obtain the inequality (22).

$$\begin{aligned} 0 \;\equiv\; & -H_1(t,\ x,\ w^{(0)},\ u^{(0)},\ p_1^{(0)}) \leq -\min_w H_1(t,\ x,\ w,\ u^{(0)},\ p_1^{(0)}) \\ = \; & -H_1(t,\ x,\ w^{(1)},\ u^{(0)},\ p_1^{(0)}), \end{aligned} \tag{22a}$$

$$\begin{aligned} 0 \;\equiv\; & -H_2(t,\ x,\ w^{(0)},\ u^{(0)},\ p_2^{(0)}) \leq -\min_u H_2(t,\ x,\ w^{(0)},\ u,\ p_2^{(0)}) \\ = \; & -H_2(t,\ x,\ w^{(0)},\ u^{(1)},\ p_2^{(0)}). \end{aligned} \tag{22b}$$

Secondly, we study a Lyapunov equations (9) for any $n \in \mathbf{N}$. Taking any stabilizable linear control law $u^{(n)}(t,\ x) = -B_\varepsilon^T Y_\varepsilon^{(n)} x(t)$ and disturbance $w^{(n)}(t,\ x) = \gamma^{-2} D_\varepsilon^T X_\varepsilon^{(n)} x(t)$, similarly to the case of $n = 0$, let us consider the following two minimization problems.

$$
\begin{aligned}
\dot{x}(t) &= A_\varepsilon x(t) + D_\varepsilon w(t) + B_\varepsilon u^{(n)}(t) = [A_\varepsilon - S_\varepsilon Y_\varepsilon^{(n)}]x(t) + D_\varepsilon w(t), && \text{(23a)}\\
V_1(x,\ t) &= \min_{w(t)} \int_t^\infty [\gamma^2 w(\tau)^T w(\tau) - \{x(\tau)^T Q x(\tau) + u^{(n)T}(\tau) u^{(n)}(\tau)\}] d\tau, && \text{(23b)}\\
\dot{x}(t) &= A_\varepsilon x(t) + D_\varepsilon w^{(n)}(t) + B_\varepsilon u(t) = [A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon^{(n)}]x(t) + B_\varepsilon u(t), && \text{(23c)}\\
V_2(\ x,\ t) &= \min_{u(t)} \int_t^\infty [x(\tau)^T Q x(\tau) + u(\tau)^T u(\tau)] d\tau, && \text{(23d)}
\end{aligned}
$$

where

$$
w^{(n)}(t,\ x) = \gamma^{-2} D_\varepsilon^T X_\varepsilon^{(n)} x(t),\ u^{(n)}(t,\ x) = -B_\varepsilon^T Y_\varepsilon^{(n)} x(t).
$$

Furthermore, $A_\varepsilon - S_\varepsilon Y_\varepsilon^{(n)}$ and $A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon^{(n)}$ are stable (Gajić *et al.* 1995, Li and Gajić 1994). According to the minimum principle, the minimization problem formulated above is equal to the problem which minimize Hamiltonian $H_1(t,\ x,\ w,\ u^{(n)},\ p_1^{(n)})$ and $H_2(t,\ x,\ w^{(n)},\ u,\ p_2^{(n)})$ in respect of the w and u respectively.

By following the similar steps in the case of $n = 0$, we get

$$
\begin{aligned}
&(A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon^{(n)} - S_\varepsilon Y_\varepsilon^{(n)})^T X_\varepsilon^{(n+1)} + X_\varepsilon^{(n+1)} (A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon^{(n)} - S_\varepsilon Y_\varepsilon^{(n)})\\
&\quad = -(Q - \gamma^{-2} X_\varepsilon^{(n)} U_\varepsilon X_\varepsilon^{(n)} + Y_\varepsilon^{(n)} S_\varepsilon Y_\varepsilon^{(n)}), && \text{(24a)}\\
&(A_\varepsilon + \gamma^{-2} U_\varepsilon X^{(n)\varepsilon} - S_\varepsilon Y_\varepsilon^{(n)})^T Y_\varepsilon^{(n+1)} + Y_\varepsilon^{(n+1)} (A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon^{(n)} - S_\varepsilon Y_\varepsilon^{(n)})\\
&\quad = -(Q + Y_\varepsilon^{(n)} S_\varepsilon Y_\varepsilon^{(n)}) && \text{(24b)}\\
&0 \equiv -H_1(t,\ x,\ w^{(n)},\ u^{(n)},\ p_1^{(n)}), && \text{(24c)}\\
&0 \equiv -H_2(t,\ x,\ w^{(n)},\ u^{(n)},\ p_2^{(n)}) && \text{(24d)}
\end{aligned}
$$

where

$$
\begin{aligned}
&\frac{\partial}{\partial x} V_i^{(n)}(x,\ t) = p_i^{(n)}(t),\ (i = 1,\ 2),\quad \dot{x}(t) = [A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon^{(n)} - S_\varepsilon Y_\varepsilon^{(n)}]x(t),\\
&V_1^{(n)}(x,\ t) = \int_t^\infty x^T(\tau)[\gamma^{-2} X_\varepsilon^{(n)} U_\varepsilon X_\varepsilon^{(n)} - Q - Y_\varepsilon^{(n)} S_\varepsilon Y_\varepsilon^{(n)}] x(\tau) d\tau = -x(t)^T X_\varepsilon^{(n+1)} x(t),\\
&V_2^{(n)}(x,\ t) = \int_t^\infty x^T(\tau)[Q + Y_\varepsilon^{(n)} S_\varepsilon Y_\varepsilon^{(n)}] x(\tau) d\tau = x(t)^T Y_\varepsilon^{(n+1)} x(t).
\end{aligned}
$$

For the sequel, from the equality (24), it follows that

$$
\begin{aligned}
0 &\equiv -H_1(t,\ x,\ w^{(n)},\ u^{(n)},\ p_1^{(n)}) \le -\min_w H_1(t,\ x,\ w,\ u^{(n)},\ p_1^{(n)})\\
&= -H_1(t,\ x,\ w^{(n+1)},\ u^{(n)},\ p_1^{(n)}), && \text{(25a)}\\
0 &\equiv -H_2(t,\ x,\ w^{(n)},\ u^{(n)},\ p_2^{(n)}) \le -\min_u H_2(t,\ x,\ w^{(n)},\ u,\ p_2^{(n)})\\
&= -H_2(t,\ x,\ w^{(n)},\ u^{(n+1)},\ p_2^{(n)}). && \text{(25b)}
\end{aligned}
$$

On the other hand, by using the monotonicity result of the successive approximations and the minimization technique in the negative gradient direction (Li and Gajić 1994), we get a monotonically decrease sequences

$$
\begin{aligned}
V_1^{(n)}(x,\ t) &\ge V_1^{(n+1)}(x,\ t), && \text{(26a)}\\
V_2^{(n)}(x,\ t) &\ge V_2^{(n+1)}(x,\ t), && \text{(26b)}
\end{aligned}
$$

where $0 \ge V_1^{(n)}(x,\ t) \ge \bar{V}_1$, $V_2^{(n)}(x,\ t) \ge 0$. We note that there exists lower bound $\bar{V}_1$ (Limebeer *et al.* 1994, Lemma 2.2). Thus, these sequences (26) are convergent. Note that the sequences $p_i^{(n)}(t),\ (i = 1,\ 2),\ w^{(n)}(t, x)$

and $u^{(n)}(t, x)$ are also convergent, since $\dfrac{\partial}{\partial x} V_i^{(n)}(x,\ t) = p_i^{(n)}(t),\ (i = 1,\ 2),\ w^{(n)}(t, x) = -\dfrac{1}{2}\gamma^{-2} D_\varepsilon^T p_1^{(n-1)}(t)$, $u^{(n)}(t, x) = -\dfrac{1}{2} B_\varepsilon^T p_2^{(n-1)}(t)$. Consequently, from the method of successive approximations (Aganovic and Gajić 1995), we have the convergence form (27) when n is very large.

$$
\begin{aligned}
&-\min_w H_1(t,\ x,\ w,\ u^{(n)},\ p_1^{(n)}) = -H_1(t,\ x,\ w^{(n+1)},\ u^{(n)},\ p_1^{(n)}) \\
\to &- H_1(t,\ x,\ w^{(n+1)},\ u^{(n+1)},\ p_1^{(n+1)}) \equiv 0 \qquad (27a) \\
&-\min_u H_2(t,\ x,\ w^{(n)},\ u,\ p_2^{(n)}) = -H_2(t,\ x,\ w^{(n)},\ u^{(n+1)},\ p_2^{(n)}) \\
\to &-H_2(t,\ x,\ w^{(n+1)},\ u^{(n+1)},\ p_2^{(n+1)}) \equiv 0 \qquad (27b)
\end{aligned}
$$

It is easy to show that sequences $\{X_\varepsilon^{(n)}\}$, $\{Y_\varepsilon^{(n)}\}$, $(n = 0,\ 1,\ 2,\ \cdots)$ are convergent since the matrices corresponding Hamiltonians to tend to zero, that is, equation (27) holds for $n \to \infty$. In addition, let $\{X_\varepsilon^{(\infty)}\}$ and $\{Y_\varepsilon^{(\infty)}\}$ be the limit points of the corresponding sequences, we have

$$
\begin{aligned}
&(A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon^{(\infty)} - S_\varepsilon Y_\varepsilon^{(\infty)})^T X_\varepsilon^{(\infty)} + X_\varepsilon^{(\infty)} (A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon^{(\infty)} - S_\varepsilon Y_\varepsilon^{(\infty)}) \\
&\quad + Q - \gamma^{-2} X_\varepsilon^{(\infty)} U_\varepsilon X_\varepsilon^{(\infty)} + Y_\varepsilon^{(\infty)} S_\varepsilon Y_\varepsilon^{(\infty)} = 0, \qquad (28a) \\
&(A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon^{(\infty)} - S_\varepsilon Y_\varepsilon^{(\infty)})^T Y_\varepsilon^{(\infty)} + Y_\varepsilon^{(\infty)} (A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon^{(\infty)} - S_\varepsilon Y_\varepsilon^{(\infty)}) \\
&\quad + Q + Y_\varepsilon^{(\infty)} S_\varepsilon Y_\varepsilon^{(\infty)} = 0, \qquad (28b)
\end{aligned}
$$

that is, $\{X_\varepsilon^{(\infty)}\}$, $\{Y_\varepsilon^{(\infty)}\}$ satisfy the parameterized cross–coupled algebraic Riccati equations (6) so that they represent the sought solutions of these equations.

Thirdly, we prove that $X_\varepsilon^{(n+1)}$, $Y_\varepsilon^{(n+1)}$ are positive semidefinite and $A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon^{(n)} - S_\varepsilon Y_\varepsilon^{(n)} \equiv \mathcal{A}(n)$ is stable. The first stage is to prove that $\mathcal{A}(n)$ is stable. The proof is done by using mathematical induction. When $n = 0$, $\mathcal{A}(0)$ is stable by taking into account bounded real lemma for the equation (10b). Next $n = i$, we assume that $\mathcal{A}(i)$ is stable. Substituting $n = i$ into (23), the minimization problem (23) produce a stabilizing control given by

$$
w^{(i+1)}(t,\ x) = \gamma^{-2} D_\varepsilon^T X_\varepsilon^{(i+1)} x(t),\ u^{(i+1)}(t,\ x) = -B_\varepsilon^T Y_\varepsilon^{(i+1)} x(t).
$$

It is obvious from the method of successive approximations (Aganovic and Gajić 1995) that $\mathcal{A}(i+1)$ is stable since it is the stable matrix of the closed–loop system, i.e. $\dot{x}(t) = [A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon^{(i+1)} - S_\varepsilon Y_\varepsilon^{(i+1)}] x(t) = \mathcal{A}(i+1) x(t)$. Thus, $\mathcal{A}(n)$ is stable for all $n \in \mathbf{N}$. Furthermore, $\mathcal{A}(\infty) = A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon^{(\infty)} - S_\varepsilon Y_\varepsilon^{(\infty)}$ is also stable because the sequences $\{X_\varepsilon^{(n)}\}$, $\{Y_\varepsilon^{(n)}\}$ are convergence when $n \to \infty$.

The next stage is to prove that $X_\varepsilon^{(n+1)}$ and $Y_\varepsilon^{(n+1)}$ are positive semidefinite. Rearranging the Lyapunov equation (24a), we get

$$
\begin{aligned}
&(A_\varepsilon - S_\varepsilon Y_\varepsilon^{(n)})^T X_\varepsilon^{(n+1)} + X_\varepsilon^{(n+1)} (A_\varepsilon - S_\varepsilon Y_\varepsilon^{(n)}) \\
&\quad = -[Q - \gamma^{-2} (X_\varepsilon^{(n+1)} - X_\varepsilon^{(n)}) U_\varepsilon (X_\varepsilon^{(n+1)} - X_\varepsilon^{(n)}) + \gamma^{-2} X_\varepsilon^{(n+1)} U_\varepsilon X_\varepsilon^{(n+1)} + Y_\varepsilon^{(n)} S_\varepsilon Y_\varepsilon^{(n)}].
\end{aligned}
$$

Note that the Lyapunov equation (24a) has a unique positive semidefinite solution $X_\varepsilon^{(n+1)} \geq 0$ for all $n \in \mathbf{N}$ with $n_0 \leq n$ since $A_\varepsilon - S_\varepsilon Y_\varepsilon^{(n)}$ is stable matrix for all $n \in \mathbf{N}$ (Li and Gajić 1994). Here $A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon^{(n)} - S_\varepsilon Y_\varepsilon^{(n)}$ is stable and there exist $n_0 \in \mathbf{N}$ such that right–hand side of above equation is negative definite for all $n \in \mathbf{N}$ with $n_0 \leq n$. On the other hand, the Lyapunov equation (24b) has a unique positive semidefinite solution $Y_\varepsilon^{(n+1)} \geq 0$ for all $n \in \mathbf{N}$ since the right–hand side of equation (24b) is negative definite and $\mathcal{A}(n)$ is stable.

Finally, rearranging (24a) and (24b), we have (9a) and (9b) respectively since

$$
\begin{aligned}
&X_\varepsilon^{(n)} = \Pi_\varepsilon^T X^{(n)} = X^{(n)T} \Pi_\varepsilon,\ \ Y_\varepsilon^{(n)} = \Pi_\varepsilon^T Y^{(n)} = Y^{(n)T} \Pi_\varepsilon, \\
&X_\varepsilon^{(n+1)} = \Pi_\varepsilon^T X^{(n+1)} = X^{(n+1)T} \Pi_\varepsilon,\ \ Y_\varepsilon^{(n+1)} = \Pi_\varepsilon^T Y^{(n+1)} = Y^{(n+1)T} \Pi_\varepsilon, \\
&A_\varepsilon = \Pi_\varepsilon^{-1} A,\ B_\varepsilon = \Pi_\varepsilon^{-1} B,\ D_\varepsilon = \Pi_\varepsilon^{-1} D.
\end{aligned}
$$

Thus, the proof of Theorem 1 is completed. □

Remark 1 *Subtracting (24a) from (24b) we have*

$$(A_\varepsilon+\gamma^{-2}U_\varepsilon X_\varepsilon^{(n)}-S_\varepsilon Y_\varepsilon^{(n)})^T(Y_\varepsilon^{(n+1)}-X_\varepsilon^{(n+1)}) \\ +(Y_\varepsilon^{(n+1)}-X_\varepsilon^{(n+1)})(A_\varepsilon+\gamma^{-2}U_\varepsilon X_\varepsilon^{(n)}-S_\varepsilon Y_\varepsilon^{(n)})+\gamma^{-2}X_\varepsilon^{(n)}U_\varepsilon X_\varepsilon^{(n)}=0.$$

Since $A_\varepsilon+\gamma^{-2}U_\varepsilon X_\varepsilon^{(n)}-S_\varepsilon Y_\varepsilon^{(n)}$ *is stable for all* $n\in\mathbf{N}$*, it is easy to establish that*

$$Y_\varepsilon^{(n)}\geq X_\varepsilon^{(n)},\quad n=1,\ 2,\ 3,\cdots.$$

For each $0<\gamma^*<\gamma\leq\bar{\gamma}$, one natural question here is whether the Lyapunov iterations can be apply to the parameterized cross–coupled algebraic Riccati equations, where γ^* is an infimum of (5). We show that a stronger result than the one stated in Theorem 1 can be similarly obtained by choosing the initial value matrices $X_\varepsilon^{(0)}$ and $Y_\varepsilon^{(0)}$ such that $A_\varepsilon+\gamma^{-2}U_\varepsilon X_\varepsilon^{(0)}-S_\varepsilon Y_\varepsilon^{(0)}$ is stable. In this case, we have the following theorem.

Theorem 2 *Under Assumptions 1 and 2, for a predescribed disturbance attenuation level* $\gamma^*<\gamma\leq\bar{\gamma}$ *and a small parameter* $\varepsilon>0$*, if solutions of the parameterized cross–coupled algebraic Riccati equations (6) exist, then the Lyapunov iterations (9a) and (9b) converge to the unique positive semidefinite solutions of the parameterized cross–coupled algebraic Riccati equations (6). Here we choose the initial value of the positive semidefinite symmetric matrices* $X_\varepsilon^{(0)}$ *and* $Y_\varepsilon^{(0)}$ *such that* $A_\varepsilon+\gamma^{-2}U_\varepsilon X_\varepsilon^{(0)}-S_\varepsilon Y_\varepsilon^{(0)}$ *is stable.*

Proof: Since $A_\varepsilon+\gamma^{-2}U_\varepsilon X_\varepsilon^{(0)}-S_\varepsilon Y_\varepsilon^{(0)}$ is stable matrix, the Riccati equation (19a), (19b) have unique positive semidefinite solution $X_\varepsilon^{(1)}\geq0$ and $Y_\varepsilon^{(1)}\geq0$ respectively. By repeating the similar steps to the proof of Theorem 1, the theorem can be proved. □

4. The Recursive Reduced–Order Algorithm for mixed H_2/H_∞ Control Problem of Singularly Perturbed Systems

In this section, we will derive the recursive reduced–order algorithm for solving the mixed H_2/H_∞ control problem of singularly perturbed systems. In order to obtain the solutions for the parameterized cross–coupled generalized algebraic Riccati equations (6), we introduce the notation

$$A+\gamma^{-2}UX^{(n)}-SY^{(n)}=\bar{A}^{(n)}=\begin{bmatrix}\bar{A}_{11}^{(n)} & \bar{A}_{12}^{(n)}\\ \bar{A}_{21}^{(n)} & \bar{A}_{22}^{(n)}\end{bmatrix}, \tag{29a}$$

$$Q-\gamma^{-2}X^{(n)T}UX^{(n)}+Y^{(n)T}SY^{(n)}=\bar{Q}^{(n)}=\begin{bmatrix}\bar{Q}_{11}^{(n)} & \bar{Q}_{12}^{(n)}\\ \bar{Q}_{12}^{(n)T} & \bar{Q}_{22}^{(n)}\end{bmatrix}, \tag{29b}$$

$$Q+Y^{(n)T}SY^{(n)}=\hat{Q}^{(n)}=\begin{bmatrix}\hat{Q}_{11}^{(n)} & \hat{Q}_{12}^{(n)}\\ \hat{Q}_{12}^{(n)T} & \hat{Q}_{22}^{(n)}\end{bmatrix}. \tag{29c}$$

The following set of equations can be produced by substituting (29) into (9).

$$\bar{A}^{(n)T}X^{(n+1)}+X^{(n+1)T}\bar{A}^{(n)}+\bar{Q}^{(n)}=0, \tag{30a}$$

$$\bar{A}^{(n)T}Y^{(n+1)}+Y^{(n+1)T}\bar{A}^{(n)}+\hat{Q}^{(n)}=0. \tag{30b}$$

The algorithm (30) is the Lyapunov iteration. The proof for the convergence has been given in Theorem 1. Thus, we can obtain the solution of the parameterized cross–coupled algebraic Riccati equations by performing Lyapunov iteration (30) directly. However, the Lyapunov iteration (30) involves the small positive parameter ε when we consider the singularly perturbed system. To remedy this, we propose a new combined algorithm to find the solution to the Lyapunov iteration (30) or the generalized Lyapunov equation (30). The main point is to separate the solution of the generalized Lyapunov equation (30) into two terms, that is, the 0–order term and the error terms. The 0–order terms are obtained by reduced–order ε–independent Lyapunov iteration and the error terms are obtained by a reduced–order recursive technique. By combining the Lyapunov iteration and the recursive technique, we can overcome the computation difficulties caused by the small parameter ε and the high dimension.

In order to use the combined algorithm above to solve the problem, we have to strictly establish the convergent conditions related to the recursive algorithm. If it is less conservative than the conditions of convergence for the Lyapunov iterations, then the truncation error of the algorithm (30) increases by performing iteration because the solutions $X^{(n+1)}$ and $Y^{(n+1)}$ are not exact. Also, we note that ε has to be very small. The number of iterations required for desired accuracy depends on the parameter ε. If ε is not very small, then error term of the solutions $X^{(n+1)}$ and $Y^{(n+1)}$ are not bounded. The recursive algorithm diverges in a finite iteration. Therefore, the Lyapunov iterations can not yield the required solutions. However, if the oriented above conditions are satisfied, then proposed new algorithm converges in a finite number of steps.

In the next section we derive the recursive algorithms for the generalized algebraic Lyapunov equations (30).

4.1 The Recursive Algorithm for $X^{(n+1)}$

Here we study the generalized algebraic Lyapunov equation (30a). The generalized algebraic Lyapunov equation (30a) can be partitioned into

$$\bar{A}_{11}^{(n)T}X_{11}^{(n+1)}+X_{11}^{(n+1)T}\bar{A}_{11}^{(n)}+\bar{A}_{21}^{(n)T}X_{21}^{(n+1)}+X_{21}^{(n+1)T}\bar{A}_{21}^{(n)}+\bar{Q}_{11}^{(n)}=0, \tag{31a}$$
$$\varepsilon X_{21}^{(n+1)}\bar{A}_{11}^{(n)}+X_{22}^{(n+1)T}\bar{A}_{21}^{(n)}+\bar{A}_{12}^{(n)T}X_{11}^{(n+1)}+\bar{A}_{22}^{(n)T}X_{21}^{(n+1)}+\bar{Q}_{12}^{(n)T}=0, \tag{31b}$$
$$\bar{A}_{22}^{(n)T}X_{22}^{(n+1)}+X_{22}^{(n+1)T}\bar{A}_{22}^{(n)}+\varepsilon\bar{A}_{12}^{(n)T}X_{21}^{(n+1)T}+\varepsilon X_{21}^{(n+1)}\bar{A}_{12}^{(n)}+\bar{Q}_{22}^{(n)}=0. \tag{31c}$$

Let us define the following $O(\varepsilon)$ perturbation of $X_{11}^{(n+1)}$, $X_{21}^{(n+1)}$ and $X_{22}^{(n+1)}$ for (31)

$$\bar{A}_{11}^{(n)T}\bar{X}_{11}^{(n+1)}+\bar{X}_{11}^{(n+1)T}\bar{A}_{11}^{(n)}+\bar{A}_{21}^{(n)T}\bar{X}_{21}^{(n+1)}+\bar{X}_{21}^{(n+1)T}\bar{A}_{21}^{(n)}+\bar{Q}_{11}^{(n)}=0, \tag{32a}$$
$$\bar{X}_{22}^{(n+1)T}\bar{A}_{21}^{(n)}+\bar{A}_{12}^{(n)T}\bar{X}_{11}^{(n+1)}+\bar{A}_{22}^{(n)T}\bar{X}_{21}^{(n+1)}+\bar{Q}_{12}^{(n)T}=0, \tag{32b}$$
$$\bar{A}_{22}^{(n)T}\bar{X}_{22}^{(n+1)}+\bar{X}_{22}^{(n+1)T}\bar{A}_{22}^{(n)}+\bar{Q}_{22}^{(n)}=0. \tag{32c}$$

Note that we do not set $\varepsilon=0$ in $\bar{A}_{ij}^{(n)}$ $(i,\ j=1,\ 2)$ and $\bar{Q}_{ij}^{(n)}$ $(i,\ j=1,\ 2)$ (Gajić *et al.* 1990). In the rest of section, we assume that all matrices are function of ε. However, the explicit dependence on ε of the problem matrices is omitted in order to simplify notation (Gajić *et al.* 1990).

If Assumption 2 holds, then the matrix $\bar{A}_{22}^{(n)}$ is non–singular. Therefore, we obtain the following 0–order equations

$$\bar{A}_{0}^{(n)T}\bar{X}_{11}^{(n+1)}+\bar{X}_{11}^{(n+1)T}\bar{A}_{0}^{(n)}+\bar{Q}_{0}^{(n)}=0, \tag{33a}$$
$$\bar{X}_{21}^{(n+1)}=-[\bar{A}_{22}^{(n)}]^{-T}\cdot(\bar{X}_{22}^{(n+1)T}\bar{A}_{21}^{(n)}+\bar{A}_{12}^{(n)T}\bar{X}_{11}^{(n+1)}+\bar{Q}_{12}^{(n)T}), \tag{33b}$$
$$\bar{A}_{22}^{(n)T}\bar{X}_{22}^{(n+1)}+\bar{X}_{22}^{(n+1)T}\bar{A}_{22}^{(n)}+\bar{Q}_{22}^{(n)}=0, \tag{33c}$$

where

$$\bar{A}_{0}^{(n)}=\bar{A}_{11}^{(n)}-\bar{A}_{12}^{(n)}[\bar{A}_{22}^{(n)}]^{-1}\bar{A}_{21}^{(n)},$$
$$\bar{Q}_{0}^{(n)}=\bar{Q}_{11}^{(n)}-\bar{Q}_{12}^{(n)}[\bar{A}_{22}^{(n)}]^{-1}\bar{A}_{21}^{(n)}-\bar{A}_{21}^{(n)T}[\bar{A}_{22}^{(n)}]^{-T}\bar{Q}_{12}^{(n)T}+\bar{A}_{21}^{(n)T}[\bar{A}_{22}^{(n)}]^{-T}\bar{Q}_{22}^{(n)}[\bar{A}_{22}^{(n)}]^{-1}\bar{A}_{21}^{(n)}.$$

The 0–order solutions $\bar{X}_{11}^{(n+1)}$, $\bar{X}_{21}^{(n+1)}$ and $\bar{X}_{22}^{(n+1)}$ are $O(\varepsilon)$ close to the exact one. Defining the error term $E_{ij}^{(n+1)}$ $(ij=11,\ 21,\ 22)$ with respect to $X_{ij}^{(n+1)}$ $(ij=11,\ 21,\ 22)$ the exact solutions can be produced in the following form

$$X_{11}^{(n+1)} = \bar{X}_{11}^{(n+1)} + \varepsilon E_{11}^{(n+1)},\ X_{21}^{(n+1)} = \bar{X}_{21}^{(n+1)} + \varepsilon E_{21}^{(n+1)},\ X_{22}^{(n+1)} = \bar{X}_{22}^{(n+1)} + \varepsilon E_{22}^{(n+1)}. \tag{34}$$

Substituting (34) into (31) and subtracting (32) from (31), we arrive at the error equation (35).

$$E_{11}^{(n+1)T}\bar{A}_0^{(n)} + \bar{A}_0^{(n)T}E_{11}^{(n+1)} = -\bar{A}_{21}^{(n)T}[\bar{A}_{22}^{(n)}]^{-T}\bar{H}^{(n+1)}[\bar{A}_{22}^{(n)}]^{-1}\bar{A}_{21}^{(n)}$$
$$+\bar{A}_{11}^{(n)T}X_{21}^{(n+1)T}[\bar{A}_{22}^{(n)}]^{-1}\bar{A}_{21}^{(n)} + \bar{A}_{21}^{(n)T}[\bar{A}_{22}^{(n)}]^{-T}X_{21}^{(n+1)}\bar{A}_{11}^{(n)}, \tag{35a}$$

$$E_{21}^{(n+1)} = -[\bar{A}_{22}^{(n)}]^{-T}\cdot[\bar{A}_{12}^{(n)T}E_{11}^{(n+1)} + E_{22}^{(n+1)T}\bar{A}_{21}^{(n)} + X_{21}^{(n+1)}\bar{A}_{11}^{(n)}], \tag{35b}$$

$$E_{22}^{(n+1)T}\bar{A}_{22}^{(n)} + \bar{A}_{22}^{(n)T}E_{22}^{(n+1)} + \bar{H}^{(n+1)} = 0, \tag{35c}$$

where

$$\bar{H}^{(n+1)} = (\bar{X}_{21}^{(n+1)} + \varepsilon E_{21}^{(n+1)})\bar{A}_{12}^{(n)} + \bar{A}_{12}^{(n)T}(\bar{X}_{21}^{(n+1)} + \varepsilon E_{21}^{(n+1)})^T.$$

These equations have very nice form since the unknown quantity E_{21} in equations for E_{11} and E_{22} are multiplied by a small parameter ε. This fact suggests that a fixed point algorithm can be efficient for their solutions. Therefore we proposed the following recursive algorithm for solving (35)

$$E_{11(k+1)}^{(n+1)T}\bar{A}_0^{(n)} + \bar{A}_0^{(n)T}E_{11(k+1)}^{(n+1)} = -\bar{A}_{21}^{(n)T}[\bar{A}_{22}^{(n)}]^{-T}\bar{H}_{(k)}^{(n+1)}[\bar{A}_{22}^{(n)}]^{-1}\bar{A}_{21}^{(n)}$$
$$+\bar{A}_{11}^{(n)T}X_{21(k)}^{(n+1)T}[\bar{A}_{22}^{(n)}]^{-1}\bar{A}_{21}^{(n)} + \bar{A}_{21}^{(n)T}[\bar{A}_{22}^{(n)}]^{-T}X_{21(k)}^{(n+1)}\bar{A}_{11}^{(n)}, \tag{36a}$$

$$E_{21(k+1)}^{(n+1)} = -[\bar{A}_{22}^{(n)}]^{-T}\cdot[\bar{A}_{12}^{(n)T}E_{11(k+1)}^{(n+1)} + E_{22(k+1)}^{(n+1)}\bar{A}_{21}^{(n)} + X_{21(k)}^{(n+1)}\bar{A}_{11}^{(n)}], \tag{36b}$$

$$E_{22(k+1)}^{(n+1)T}\bar{A}_{22}^{(n)} + \bar{A}_{12}^{(n)T}E_{22(k+1)}^{(n+1)} + \bar{H}_{(k)}^{(n+1)} = 0, \tag{36c}$$

$$(k = 0,\ 1,\ 2,\ 3,\cdots),$$

where

$$\bar{H}_{(k)}^{(n+1)} = (\bar{X}_{21}^{(n+1)} + \varepsilon E_{21(k)}^{(n+1)})\bar{A}_{12}^{(n)} + \bar{A}_{12}^{(n)T}(\bar{X}_{21}^{(n+1)} + \varepsilon E_{21(k)}^{(n+1)})^T,\ E_{21(0)}^{(n+1)} = 0$$
$$X_{ij(k)}^{(n+1)} = \bar{X}_{ij}^{(n+1)} + \varepsilon E_{ij(k)}^{(n+1)},\ (ij = 11, 21, 22).$$

The following theorem indicates the convergence of the recursive algorithm (36).

Theorem 3 *Under the conditions stated in Assumptions 1 and 2, there exists $\bar{\varepsilon} > 0$ such that, for every $\varepsilon \in (0,\ \bar{\varepsilon}]$, Lyapunov equation (35) admits a positive semidefinite solution $E^{(n+1)}$. Moreover, the following results hold:*

$$\|E^{(n+1)} - E_{(k)}^{(n+1)}\| = O(\varepsilon^k),\quad (k = 1,\ 2,\ 3,\cdots) \tag{37}$$

or equivalently

$$\|E^{(n+1)} - E_{(k+1)}^{(n+1)}\| = O(\varepsilon)\|E^{(n+1)} - E_{(k)}^{(n+1)}\|,\quad (k = 1,\ 2,\ 3,\cdots) \tag{38}$$

where

$$\|E_{(k)}^{(n+1)}(\varepsilon)\| \le c_e < \infty,\quad \forall\varepsilon \in (0,\ \bar{\varepsilon}],$$
$$E_{(k)}^{(n+1)}(\varepsilon) = E_{(k)}^{(n+1)},\ E^{(n+1)}(\varepsilon) = E^{(n+1)},$$
$$E_{(k)}^{(n+1)} = \begin{bmatrix} E_{11(k)}^{(n+1)} & E_{21(k)}^{(n+1)T} \\ E_{21(k)}^{(n+1)} & E_{22(k)}^{(n+1)} \end{bmatrix},\quad E^{(n+1)} = \begin{bmatrix} E_{11}^{(n+1)} & E_{21}^{(n+1)T} \\ E_{21}^{(n+1)} & E_{22}^{(n+1)} \end{bmatrix}.$$

Proof: As a starting point we need to show the existence of a bounded solution of $E^{(n+1)}$ in neighborhood of $\varepsilon = 0$. To prove that by the implicit function theorem (Gajić *et al.* 1990, Gajić 1986), it is enough to show that the corresponding Jacobian is non–singular at $\varepsilon = 0$. The Jacobian is given by

$$J|_{\varepsilon=0} = \begin{bmatrix} J_{11} & 0 & 0 \\ J_{21} & J_{22} & J_{23} \\ 0 & 0 & J_{33} \end{bmatrix} \tag{39}$$

where, using the Kronecker products representation we have

$$\begin{aligned}
J_{11} &= I_{n_1} \otimes \bar{A}_0^{(n)} + \bar{A}_0^{(n)T} \otimes I_{n_1},\ J_{22} = I_{n_1} \otimes \bar{A}_{22}^{(n)},\ J_{33} = I_{n_2} \otimes \bar{A}_{22}^{(n)} + \bar{A}_{22}^{(n)T} \otimes I_{n_2}, \\
J_{m1} &= \frac{\partial L_m}{\partial E_{11}^{(n+1)}}|_{\varepsilon=0},\ J_{m2} = \frac{\partial L_m}{\partial E_{21}^{(n+1)}}|_{\varepsilon=0},\ J_{m3} = \frac{\partial L_m}{\partial E_{22}^{(n+1)}}|_{\varepsilon=0},\ \ (m = 1, 2, 3), \\
L_1 &= E_{11}^{(n+1)T} \bar{A}_0^{(n)} + \bar{A}_0^{(n)T} E_{11}^{(n+1)} + \bar{A}_{21}^{(n)T} [\bar{A}_{22}^{(n)}]^{-T} \bar{H}^{(n+1)} [\bar{A}_{22}^{(n)}]^{-1} \bar{A}_{21}^{(n)} \\
&\quad - \bar{A}_{11}^{(n)T} X_{21}^{(n+1)T} [\bar{A}_{22}^{(n)}]^{-1} \bar{A}_{21}^{(n)} - \bar{A}_{21}^{(n)T} [\bar{A}_{22}^{(n)}]^{-T} X_{21}^{(n+1)} \bar{A}_{11}^{(n)} \\
L_2 &= \bar{A}_{22}^{(n)T} E_{21}^{(n+1)} + \bar{A}_{12}^{(n)T} E_{11}^{(n+1)} + E_{22}^{(n+1)} \bar{A}_{21}^{(n)} + X_{21}^{(n+1)} \bar{A}_{11}^{(n)} \\
L_3 &= E_{22}^{(n+1)T} \bar{A}_{22}^{(n)} + \bar{A}_{12}^{(n)T} E_{22}^{(n+1)} + \bar{H}^{(n+1)}
\end{aligned}$$

The matrix $\bar{A}_{22}^{(n)}$ is non–singular for all $n \in \mathbf{N}$ because of Assumption 2. On the other hand, for all $n \in \mathbf{N}$ the matrix $\bar{A}_0^{(n)}$ is also stable because of Assumption 1 (Gajić *et al.* 1990, Gajić 1986) for a small parameter ε. Thus, the Jacobian (39) is non–singular. As a result, we can achieve the $O(\varepsilon^k)$ approximation of $E_{ij}^{(n+1)}$ $(ij = 11,\ 21,\ 22)$ by performing only $k \in \mathbf{N}$ iterations using algorithm (36). Similarly, existence of $\bar{\varepsilon}$ is shown directly by again using the implicit function theorem.

The second step in the proof of the given theorem is to give an estimate of the rate of convergence. Setting $k = 0$ for equations (36) and subtracting (36) from (35), we obtain the following equations

$$(E_{11}^{(n+1)} - E_{11(1)}^{(n+1)}) \bar{A}_0^{(n)} + \bar{A}_0^{(n)T} (E_{11}^{(n+1)} - E_{11(1)}^{(n+1)}) = \varepsilon \mathcal{F}_1 (E_{21}^{(n+1)} - E_{21(0)}^{(n+1)}) \tag{40a}$$

$$(E_{21}^{(n+1)} - E_{21(1)}^{(n+1)})^T \bar{A}_{22}^{(n)} = \varepsilon \mathcal{F}_2 (E_{21}^{(n+1)} - E_{21(0)}^{(n+1)}) \tag{40b}$$

$$(E_{22}^{(n+1)} - E_{22(1)}^{(n+1)}) \bar{A}_{22}^{(n)} + \bar{A}_{22}^{(n)T} (E_{22}^{(n+1)} - E_{22(1)}^{(n+1)}) = \varepsilon \mathcal{F}_3 (E_{21}^{(n+1)} - E_{21(0)}^{(n+1)}) \tag{40c}$$

where, $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ are appropriate implicit functions with ε and $E_{21}^{(n+1)} - E_{21(0)}^{(n+1)}$. By making use of stability of $\bar{A}_{22}^{(n)}$ and $\bar{A}_0^{(n)}$ for all $n \in \mathbf{N}$ and existence of the bounded solutions of (35), we have

$$\|E_{11}^{(n+1)} - E_{11(1)}^{(n+1)}\| = O(\varepsilon),\ \|E_{21}^{(n+1)} - E_{21(1)}^{(n+1)}\| = O(\varepsilon),\ \|E_{22}^{(n+1)} - E_{22(1)}^{(n+1)}\| = O(\varepsilon). \tag{41}$$

Continuing the same procedure it can easily be shown that

$$\|E_{11}^{(n+1)} - E_{11(k_0)}^{(n+1)}\| = O(\varepsilon^{k_0}),\ \|E_{21}^{(n+1)} - E_{21(k_0)}^{(n+1)}\| = O(\varepsilon^{k_0}),\ \|E_{22}^{(n+1)} - E_{22(k_0)}^{(n+1)}\| = O(\varepsilon^{k_0}). \tag{42}$$

Thus $\|E^{(n+1)} - E_{(k_0)}^{(n+1)}\| = O(\varepsilon^{k_0}) \Leftrightarrow \|E^{(n+1)} - E_{(k)}^{(n+1)}\| = O(\varepsilon^k)$.

On the other hand, for the boundedness of $E_{(k)}^{(n+1)}$ on ε, we show that $\|E_{(k)}^{(n+1)}\| \le \|E^{(n+1)}\| + O(\varepsilon^k) \le c_e < \infty$ since $\|E^{(n+1)}\|$ is bounded by using the implicit function theorem (Gajić 1986). This completes the proof of Theorem 3. □

4.2 The Recursive Algorithm for $Y^{(n+1)}$

In this section we now turn to the recursive algorithm for $Y^{(n+1)}$. By following the similar steps in the matrix $X^{(n+1)}$ case, the algebraic Riccati equation (30b) can be partitioned into

$$\bar{A}_0^{(n)T} \bar{Y}_{11}^{(n+1)} + \bar{Y}_{11}^{(n+1)T} \bar{A}_0^{(n)} + \hat{Q}_0^{(n)} = 0, \tag{43a}$$

$$\bar{Y}_{21}^{(n+1)} = -[\bar{A}_{22}^{(n)}]^{-T} (\bar{Y}_{22}^{(n+1)T} \bar{A}_{21}^{(n)} + \bar{A}_{12}^{(n)T} \bar{Y}_{11}^{(n+1)} + \hat{Q}_{12}^{(n)T}), \tag{43b}$$

$$\bar{A}_{22}^{(n)T} \bar{Y}_{22}^{(n+1)} + \bar{Y}_{22}^{(n+1)T} \bar{A}_{22}^{(n)} + \hat{Q}_{22}^{(n)} = 0, \tag{43c}$$

Defining the error term $F_{ij}^{(n+1)}$ $(ij = 11,\ 21,\ 22)$ with respect to $Y_{ij}^{(n+1)}$ $(ij = 11,\ 21,\ 22)$ the exact solutions can be produced in the following form

$$Y_{11}^{(n+1)} = \bar{Y}_{11}^{(n+1)} + \varepsilon F_{11}^{(n+1)},\ Y_{21}^{(n+1)} = \bar{Y}_{21}^{(n+1)} + \varepsilon F_{21}^{(n+1)},\ Y_{22}^{(n+1)} = \bar{Y}_{22}^{(n+1)} + \varepsilon F_{22}^{(n+1)}. \tag{44}$$

In this case, matrices $\bar{Y}_{11}^{(n+1)}$, $\bar{Y}_{21}^{(n+1)}$, $\bar{Y}_{22}^{(n+1)}$, $F_{11}^{(n+1)}$, $F_{21}^{(n+1)}$ and $F_{22}^{(n+1)}$ satisfy coupled algebraic Lyapunov equations (45) and (46) respectively.

$$\bar{A}_0^{(n)T}\bar{Y}_{11}^{(n+1)} + \bar{Y}_{11}^{(n+1)T}\bar{A}_0^{(n)} + \hat{Q}_0^{(n)} = 0, \tag{45a}$$

$$\bar{Y}_{21}^{(n+1)} = -[\bar{A}_{22}^{(n)}]^{-T}(\bar{Y}_{22}^{(n+1)T}\bar{A}_{21}^{(n)} + \bar{A}_{12}^{(n)T}\bar{Y}_{11}^{(n+1)} + \hat{Q}_{12}^{(n)T}), \tag{45b}$$

$$\bar{A}_{22}^{(n)T}\bar{Y}_{22}^{(n+1)} + \bar{Y}_{22}^{(n+1)T}\bar{A}_{22}^{(n)} + \hat{Q}_{22}^{(n)} = 0, \tag{45c}$$

$$\begin{aligned} F_{11}^{(n+1)T}\bar{A}_0^{(n)} + \bar{A}_0^{(n)T}F_{11}^{(n+1)} &= -\bar{A}_{21}^{(n)T}[\bar{A}_{22}^{(n)}]^{-T}\hat{H}^{(n)}[\bar{A}_{22}^{(n)}]^{-1}\bar{A}_{21}^{(n)} \\ &\quad + \bar{A}_{11}^{(n)T}(\bar{Y}_{21}^{(n+1)} + \varepsilon F_{21}^{(n+1)})^T[\bar{A}_{22}^{(n)}]^{-1}\bar{A}_{21}^{(n)} + \bar{A}_{21}^{(n)T}[\bar{A}_{22}^{(n)}]^{-T}(\bar{Y}_{21}^{(n+1)} + \varepsilon F_{21}^{(n+1)})\bar{A}_{11}^{(n)}, \end{aligned} \tag{46a}$$

$$F_{21}^{(n+1)} = -[\bar{A}_{22}^{(n)}]^{-T}\cdot[\bar{A}_{12}^{(n)T}F_{11}^{(n+1)} + F_{22}^{(n+1)}\bar{A}_{21}^{(n)} + (\bar{Y}_{21}^{(n+1)} + \varepsilon F_{21}^{(n+1)})\bar{A}_{11}^{(n)}], \tag{46b}$$

$$F_{22}^{(n+1)T}\bar{A}_{22}^{(n)} + \bar{A}_{22}^{(n)T}F_{22}^{(n+1)} + \hat{H}^{(n+1)} = 0, \tag{46c}$$

where

$$\begin{aligned} \hat{Q}_0^{(n)} &= \hat{Q}_{11}^{(n)} - \hat{Q}_{12}^{(n)}[\bar{A}_{22}^{(n)}]^{-1}\bar{A}_{21}^{(n)} - \bar{A}_{21}^{(n)T}[\bar{A}_{22}^{(n)}]^{-T}\hat{Q}_{12}^{(n)T} + \bar{A}_{21}^{(n)T}[\bar{A}_{22}^{(n)}]^{-T}\hat{Q}_{22}^{(n)}[\bar{A}_{22}^{(n)}]^{-1}\bar{A}_{21}^{(n)}, \\ \hat{H}^{(n+1)} &= (\bar{Y}_{21}^{(n+1)} + \varepsilon F_{21}^{(n+1)})\bar{A}_{12}^{(n)} + \bar{A}_{12}^{(n)T}(\bar{Y}_{21}^{(n+1)} + \varepsilon F_{21}^{(n+1)})^T. \end{aligned}$$

Similar to the derivations in Section 4.1, we also obtain the following algorithm for solving (46)

$$\begin{aligned} F_{11(k+1)}^{(n+1)T}\bar{A}_0^{(n)} + \bar{A}_0^{(n)T}F_{11(k+1)}^{(n+1)} &= -\bar{A}_{21}^{(n)T}[\bar{A}_{22}^{(n)}]^{-T}\hat{H}_{(k)}^{(n+1)}[\bar{A}_{22}^{(n)}]^{-1}\bar{A}_{21}^{(n)} \\ &\quad + \bar{A}_{11}^{(n)T}Y_{21(k)}^{(n+1)T}[\bar{A}_{22}^{(n)}]^{-1}\bar{A}_{21}^{(n)} + \bar{A}_{21}^{(n)T}[\bar{A}_{22}^{(n)}]^{-T}Y_{21(k)}^{(n+1)}\bar{A}_{11}^{(n)}, \end{aligned} \tag{47a}$$

$$F_{21(k+1)}^{(n+1)} = -[\bar{A}_{22}^{(n)}]^{-T}\cdot[\bar{A}_{12}^{(n)T}F_{11(k+1)}^{(n+1)} + F_{22(k+1)}^{(n+1)}\bar{A}_{21}^{(n)} + Y_{21(k)}^{(n+1)}\bar{A}_{11}^{(n)}], \tag{47b}$$

$$F_{22(k+1)}^{(n+1)T}\bar{A}_{22}^{(n)} + \bar{A}_{22}^{(n)T}F_{22(k+1)}^{(n+1)} + \hat{H}_{(k)}^{(n+1)} = 0, \tag{47c}$$

$$(k = 0,\ 1,\ 2,\ 3, \cdots)$$

where

$$\begin{aligned} \hat{H}_{(k)}^{(n+1)} &= (\bar{Y}_{21}^{(n+1)} + \varepsilon F_{21(k)}^{(n+1)})\bar{A}_{12}^{(n)} + \bar{A}_{12}^{(n)T}(\bar{Y}_{21}^{(n+1)} + \varepsilon F_{21(k)}^{(n+1)})^T,\ F_{21(0)}^{(n+1)} = 0, \\ Y_{ij(k)}^{(n+1)} &= \bar{Y}_{ij}^{(n+1)} + \varepsilon F_{ij(k)}^{(n+1)},\ (ij = 11, 21, 22). \end{aligned}$$

The following theorem indicates the convergence of the algorithm (47).

Theorem 4 *Under the conditions stated in Assumptions 1 and 2, there exists $\hat{\varepsilon} > 0$ such that, for every $\varepsilon \in (0,\ \hat{\varepsilon}]$, Lyapunov equation (46) admits a positive semidefinite solution $F^{(n+1)}$. Moreover, the following results hold:*

$$\|F^{(n+1)} - F_{(k)}^{(n+1)}\| = O(\varepsilon^k),\quad (k = 1,\ 2,\ 3, \cdots) \tag{48}$$

or equivalently

$$\|F^{(n+1)} - F_{(k+1)}^{(n+1)}\| = O(\varepsilon)\|F^{(n+1)} - F_{(k)}^{(n+1)}\|,\quad (k = 1,\ 2,\ 3, \cdots) \tag{49}$$

where

$$\|F_{(k)}^{(n+1)}(\varepsilon)\| \le c_f < \infty, \quad \forall \varepsilon \in (0,\ \hat{\varepsilon}],$$
$$F_{(k)}^{(n+1)}(\varepsilon) = F_{(k)}^{(n+1)},\ F^{(n+1)}(\varepsilon) = F^{(n+1)},$$
$$F_{(k)}^{(n+1)} = \begin{bmatrix} F_{11(k)}^{(n+1)} & F_{21}^{(n+1)T} \\ F_{21(k)}^{(n+1)} & F_{22(k)}^{(n+1)} \end{bmatrix}, \quad F^{(n+1)} = \begin{bmatrix} F_{11}^{(n+1)} & F_{21}^{(n+1)T} \\ F_{21}^{(n+1)} & F_{22}^{(n+1)} \end{bmatrix}.$$

Proof: The proof is omitted since it is similar to the proof of Theorem 4. □

Note that if ε is very small, then the truncation error in the algorithms (36) and (47) is a little bit too small since the equations (37) and (48) hold. Therefore, for the Lyapunov equation (9), affection of the truncation error in the algorithms (36) and (47) is very small.

An algorithm which solves the parameterized cross–coupled algebraic Riccati equation (6) with small positive parameter ε is as follows.

Step 1. Calculate $\bar{\gamma}_1$ and $\bar{\gamma}_2$ by using the norm conditions (11).

Step 2. Choose γ such that $\gamma > \bar{\gamma} = \max\{\bar{\gamma}_1,\ \bar{\gamma}_2\}$ and solve the algebraic Riccati equations (10) by using the recursive technique proposed by Mukaidani *et al.* (1999). Starting with an initial matrices of $X^{(0)}$ and $Y^{(0)}$.

Step 3 Calculate $\bar{A}^n$, $\bar{Q}^n$ and $\hat{Q}^n$ by using the relation (29).

Step 4. Compute the solutions $X^{(n+1)}$, $Y^{(n+1)}$ of the parameterized cross–coupled algebraic Riccati equation (9) by using the recursive algorithms (36) and (47).

Step 5. If $\min\{\|F_1(X_\varepsilon^{(n)},\ Y_\varepsilon^{(n)})\|,\ \|F_2(X_\varepsilon^{(n)},\ Y_\varepsilon^{(n)})\|\} < O(\varepsilon^N)$ for a given integer $N > 0$, go to Step 6. Otherwise, increment $n \to n+1$ and go to Step 3. Here $F_1(\cdot)$, $F_2(\cdot)$ are given by (51) after.

Step 6. Calculate $w^*(t) = \gamma^{-2} D_\varepsilon^T X_\varepsilon x(t)$, $u^*(t) = K_2 x(t) = -B_\varepsilon^T Y_\varepsilon x(t)$.

5. Numerical Example

In order to demonstrate the efficiency of the proposed algorithm, we have run a simple numerical example. Matrices A_ε, D_ε and B_ε are chosen randomly. These matrices are given by

$$A_{11} = \begin{bmatrix} 0 & 0.4 \\ 0 & 0 \end{bmatrix},\ A_{12} = \begin{bmatrix} 0 & 0 \\ 0.345 & 0 \end{bmatrix},$$
$$A_{21} = \begin{bmatrix} 0 & -0.524 \\ 0 & 0 \end{bmatrix},\ A_{22} = \begin{bmatrix} 0 & 0.262 \\ 0 & -1 \end{bmatrix},$$
$$D_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ D_2 = \begin{bmatrix} 0.2 \\ 1.2 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and a quadratic cost function

$$J(x(t), u(t)) = \int_0^\infty [x(t)^T Q x(t) + u(t)^T u(t)] dt, \tag{50}$$

where $Q = \text{diag}\begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}$.

Since $\det A_{22} = 0$, the system (1) is a nonstandard singularly perturbed system.

Here, $\bar{\gamma}_1 = 10.9316$ and $\bar{\gamma}_2 = 2.9058$ from (11). Then, for every boundary value of $\gamma > \bar{\gamma} = \max\{\bar{\gamma}_1,\ \bar{\gamma}_2\} = 10.9316$ the parameterized cross–couple algebraic Riccati equations (6) have positive semi–definite solutions.

On the other hand, the minimum values of disturbance attenuation level γ such that there exists H_2/H_∞ controller is $\gamma^* = 9.1980$. The entries show the results obtained for small parameter $\varepsilon = 0.0001$. Now, we choose $\gamma = 12.0 > \bar{\gamma} = 10.9316$ to design the controller.

Firstly, we give the following solutions of the algebraic Riccati equations (10):

$$X^{(0)} = \begin{bmatrix} X_{11}^{(0)} & \varepsilon X_{21}^{(0)T} \\ X_{21}^{(0)} & X_{22}^{(0)} \end{bmatrix}$$

$$= \begin{bmatrix} 8.6262480 & 3.8257410 & 6.8423004 \times 10^{-4} & 1.4811918 \times 10^{-4} \\ 3.8257410 & 7.7141072 & 3.0497997 \times 10^{-4} & 2.2865074 \times 10^{-5} \\ 6.8423004 & 3.0497997 & 4.7775023 & 1.0165454 \\ 1.4811918 & 2.2865074 \times 10^{-1} & 1.0165454 & 2.3869035 \times 10^{-1} \end{bmatrix},$$

$$Y^{(0)} = \begin{bmatrix} Y_{11}^{(0)} & \varepsilon Y_{21}^{(0)T} \\ Y_{21}^{(0)} & Y_{22}^{(0)} \end{bmatrix}$$

$$= \begin{bmatrix} 6.2844935 & 2.8987718 & 4.7119089 \times 10^{-4} & 1.0000000 \times 10^{-4} \\ 2.8987718 & 7.2863698 & 2.2108915 \times 10^{-4} & 4.4757142 \times 10^{-6} \\ 4.7119089 & 2.2108915 & 4.7122624 & 1.0000763 \\ 1.0000000 & 4.4757142 \times 10^{-2} & 1.0000763 & 2.3452014 \times 10^{-1} \end{bmatrix}.$$

Secondly, it can be seen that the solutions of the parameterized cross–coupled algebraic Riccati equations (6) converge to the following solutions with accuracy of $O(10^{-8})$ after 23 Lyapunov iterations.

$$X_\varepsilon^{(23)} = \begin{bmatrix} X_{11}^{(23)} & \varepsilon X_{21}^{(23)T} \\ \varepsilon X_{21}^{(23)} & \varepsilon X_{22}^{(23)} \end{bmatrix}$$

$$= \begin{bmatrix} 8.3126876 & 3.6708548 & 6.5605827 \times 10^{-4} & 1.4199535 \times 10^{-4} \\ 3.6708548 & 7.6380956 & 2.9112022 \times 10^{-4} & 1.9850687 \times 10^{-5} \\ 6.5605827 \times 10^{-4} & 2.9112022 \times 10^{-4} & 4.7774746 \times 10^{-4} & 1.0165392 \times 10^{-4} \\ 1.4199535 \times 10^{-4} & 1.9850687 \times 10^{-5} & 1.0165392 \times 10^{-4} & 2.3868896 \times 10^{-5} \end{bmatrix},$$

$$Y_\varepsilon^{(23)} = \begin{bmatrix} Y_{11}^{(23)} & \varepsilon Y_{21}^{(23)T} \\ \varepsilon Y_{21}^{(23)} & \varepsilon Y_{22}^{(23)} \end{bmatrix}$$

$$= \begin{bmatrix} 9.8895917 & 4.2091443 & 8.0863985 \times 10^{-4} & 1.7735270 \times 10^{-4} \\ 4.2091443 & 7.8666697 & 3.4322839 \times 10^{-4} & 3.1503598 \times 10^{-5} \\ 8.0863985 \times 10^{-4} & 3.4322839 \times 10^{-4} & 4.8425400 \times 10^{-4} & 1.0329538 \times 10^{-4} \\ 1.7735270 \times 10^{-4} & 3.1503598 \times 10^{-5} & 1.0329538 \times 10^{-4} & 2.4284243 \times 10^{-5} \end{bmatrix}.$$

In order to verify the exactitude of the solution, we calculate the remainder when substitute $X_\varepsilon^{(23)}$ and $Y_\varepsilon^{(23)}$ into the parameterized cross-coupled algebraic Riccati equations (6a) and (6b) respectively.

$$\|F_1(X_\varepsilon^{(23)},\ Y_\varepsilon^{(23)})\| = 2.914 \times 10^{-9}, \quad \|F_2(X_\varepsilon^{(23)},\ Y_\varepsilon^{(23)})\| = 2.911 \times 10^{-10}$$

where the errors $F_1(X_\varepsilon,\ Y_\varepsilon)$ and $F_2(X_\varepsilon,\ Y_\varepsilon)$ are defined as follows

$$(A_\varepsilon + \gamma^{-2}U_\varepsilon X_\varepsilon - S_\varepsilon Y_\varepsilon)^T X_\varepsilon + X_\varepsilon(A_\varepsilon + \gamma^{-2}U_\varepsilon X_\varepsilon - S_\varepsilon Y_\varepsilon)$$
$$+Q - \gamma^{-2}X_\varepsilon U_\varepsilon X_\varepsilon + Y_\varepsilon S_\varepsilon Y_\varepsilon \equiv F_1(X_\varepsilon,\ Y_\varepsilon), \tag{51a}$$
$$(A_\varepsilon + \gamma^{-2}U_\varepsilon X_\varepsilon - S_\varepsilon Y_\varepsilon)^T Y_\varepsilon + Y_\varepsilon(A_\varepsilon + \gamma^{-2}U_\varepsilon X_\varepsilon - S_\varepsilon Y_\varepsilon)$$
$$+Q + Y_\varepsilon S_\varepsilon Y_\varepsilon \equiv F_2(X_\varepsilon,\ Y_\varepsilon). \tag{51b}$$

Therefore, the numerical example illustrates the effectiveness of the proposed algorithm since the solutions $X_\varepsilon^{(n)}$ and $Y_\varepsilon^{(n)}$ converge to the exact solutions X_ε and Y_ε which are defined by (6a) and (6b). Indeed, we can obtain the solution of the parameterized cross–coupled algebraic Riccati equations (6a) and (6b) even though A_{22} is singular.

6. Conclusions

We have developed an algorithm for solving the parameterized cross–coupled algebraic Riccati equations with a small positive parameter ε for mixed H_2/H_∞ control problem. So far, there were no condition ensuring the existence of solutions of the parameterized cross–coupled algebraic Riccati equations and proof of convergence. Furthermore, the recursive algorithm for solving the cross–coupled algebraic Riccati equations with relation to the dynamic Nash games of the singularly perturbed systems had not been investigated. In this paper, we first derived the new sufficient condition for γ such that the solution of the parameterized cross–coupled algebraic Riccati equations converges to a positive semi–definite solution. We have also given the convergence proof for the new Lyapunov iteration algorithm. That is, if we choose any $\gamma > \max\{\bar{\gamma}_1,\ \bar{\gamma}_2\}$ as a new sufficient condition, then there exist the mixed H_2/H_∞ controller to attain the disturbance attenuation level γ and minimize the performance index. Next we proposed the new algorithm to solve the parameterized cross–coupled algebraic Riccati equations with a small positive parameter ε by combining the Lyapunov iteration and the recursive technique. By using the new algorithm, we overcame the computation difficulties caused by high dimensions and numerical stiffness in the Lyapunov iteration method. In this case, the Lyapunov iteration and the recursive algorithm converge to positive semi–definite solutions with the rate of convergence of $O(\varepsilon^k)$. In addition, our new results are applicable to both standard and nonstandard singularly perturbed systems and include the existing methods (Li and Gajić 1994, Gajić and Shen 1993) as a special case.

References

Aganovic, Z., and Gajić, Z., 1995, Successive approximation procedure for steady–state optimal control of bilinear systems. *Journal of Optimization Theory and Applications*, **84**, 273–291.

Abou–Kandil, H., Freiling, G., and Jank, G., 1993, Necessary conditions for constant solutions of coupled Riccati equations in Nash games. *System and Control letters*, **21**, 295–306.

Dragan, V., 1996, H_∞–norms and disturbance attenuation for systems with fast transients. *IEEE Transaction on Automatic Control*, **41**, 747–750.

Freiling, F., Jank, G., and Abou–Kandil, H., 1996, On global existence of solutions to coupled matrix Riccati equations in closed–loop Nash games. *IEEE Transactions on Automatic Control*, **41**, 264–269.

Gajić, Z., Petkovski, D., and Shen, X., 1990, *Singularly Perturbed and Weakly Coupled Linear System-a Recursive Approach*, Lecture Notes in Control and Information Sciences Vol.140 (Berlin: Springer–Verlag).

Gajić, Z., and Shen, X., 1993, *Parallel Algorithms for Optimal Control of Large Scale Linear Systems* (London: Springer–Verlag).

Gajić, Z., and Qureshi. M. T. J., 1995, *Lyapunov Matrix Equation in System Stability and Control* (New York: Academic Press).

Kokotović, P. V., Khalil, H. K., and O'Reilly, J., 1986, *Singular Perturbation Methods in Control: Analysis and Design* (New York: Academic Press).

Limebeer, D. J. N., Anderson, B. D. O., and Hendel, B., 1994, A Nash game approach to mixed H_2/H_∞ control. *IEEE Transactions on Automatic Control*, **39**, 69–82.

Li, T., and Gajić, Z., 1994, Lyapunov iteration for solving coupled algebraic Lyapunov equations of Nash differential games and algebraic Riccati equations of zero-sum games. *Proceedings of the 6th International Symposium on Dynamic Games and Applications*, St-Jovite Canada, December 1994, pp.332–351.

Mizukami, K., and Suzumura, F., 1993, Closed–loop Stackelberg strategies for singularly perturbed systems: the recursive approach. *International Journal of Systems Science*, **24**, 887–900.

Mukaidani, H., Xu, H. and Mizukami, K., 1999, Recursive approach of H_∞ control problems for singularly perturbed systems under perfect and imperfect state measurements. *International Journal of Systems Science*, **30**, 467–477.

Starr, A.W., and Ho, Y.C., 1969, Nonzero–sum differential games. *Journal of Optimization Theory and Applications*, **3**, 184–206.

Xu, H., and Mizukami, K., 1996, Nonstandard extension of H_∞–optimal control for singularly perturbed systems. *Proceedings of the 7th International Symposium on Dynamic Games and Applications*, Shounan Japan, 16–18 December 1996, pp.931–948.

Xu, H., and Mizukami, K., 1997, Infinite-horizon differential games of singularly perturbed systems: a unified approach. *Automatica*, **33**, 273–276.

Zhou, K, 1998, *Essentials of Robust Control* (New Jersey: Prentice–Hall, Inc.).