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The Guaranteed Cost Control Problem of Uncertain Singularly Perturbed Systems

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In this paper we study the algebraic Riccati equation corresponding to the guaranteed cost control theory for an uncertain singularly perturbed system. The construction of the controller involves solving the full-order algebraic Riccati equation with small parameter ε. Under control-oriented assumptions, we first provide the sufficient conditions such that the full-order algebraic Riccati equation has a positive semi-definite stabilizing solution. Next we propose an iterative algorithm based on the Kleinman algorithm to solve the algebraic Riccati equation which depends on the parameter ε. Our new idea is to use the solutions of the reduced-order algebraic Riccati equations for the initial condition. By using the iterative algorithm, we can easily obtain a required solution of the algebraic Riccati equation. Moreover, using the initial conditions without ε, we show that there exists an $\tilde{\varepsilon}$ such that the proposed algorithm has quadratic convergence. Finally, in order to show the effectiveness of the proposed algorithm, numerical examples are included.

1. INTRODUCTION

In recent years, the stabilizing problem of singularly perturbed systems containing uncertain parameters has been intensively studied [4–7]. Shao

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and Sawan [4] have shown that the robust stability conditions of singularly perturbed systems are obtained by using a singular perturbation method [1]. Corless *et al.* [5] proposed a class of the nonlinear composite controllers which assure the global uniformly ultimate bounds of the trajectories of closed-loop singular perturbed systems. Garcia *et al.* [6] studied the H_2 guaranteed cost control problem for a singularly perturbed norm-bounded uncertain system by using composite control techniques. Shi *et al.* [7] also considered the problem of robust disturbance attenuation with stability for a class of uncertain singularly perturbed systems. It is found that the basic assumption in the above references, that is, that the state matrix $A_{22} + \Delta_{22}(t)$ for the fast subsystem is nonsingular, plays an important role in the study of the problem, where $\Delta_{22}(t)$ is uncertainty. However, this assumption has often been found to be too conservative in applications for the practical systems because it contains uncertainties. Moreover, in many references, in order to obtain the slow subsystem it is also assumed that A_{22} is nonsingular. Thus their results are applicable only to the standard singularly perturbed systems.

Recently, Mukaidani *et al.* [13] have studied the Riccati equation approach to reduce such conservatism for singularly perturbed systems in which structured uncertainties enter into the state matrix. However, the robust stabilizing problem of singularly perturbed systems in which structured uncertainties enter into both state and input matrices has not been investigated by using the Riccati equation approach. On the other hand, the recursive algorithm for various control problems of the singularly perturbed systems has been developed in the literature [2, 3, 14]. It has been shown that the recursive algorithm is very effective for solving the algebraic Riccati equations when the system matrices are functions of a small perturbation parameter ε. However, when the recursive approach is applied for the control problems of the singularly perturbed systems, we note that using the zero-order solution without high-order accuracy will fail to produce the desired exact solution of the algebraic Riccati equation. Moreover, since the recursive approach has the linear convergence property, the convergence rate is slow. Very recently, Mukaidani *et al.* [15] proposed a new iterative algorithm based on the Kleinman algorithm [3, 9] which is the quadratic convergence. Although the iterative algorithm has a good convergence, so far, the question remains as to whether an $\tilde{\varepsilon}$ exists such that the proposed algorithm has quadratic convergence for all $(0, \tilde{\varepsilon}) \in \varepsilon$. That is, it has never been shown that there exists an $\tilde{\varepsilon}$ such that the proposed algorithm has quadratic convergence.

In this paper, we study the algebraic Riccati equation corresponding to the guaranteed cost control problem for a singularly perturbed system. In particular, the considered problem is based on the optimal guaranteed cost control problem [10] for singularly perturbed uncertain systems. In order to obtain the stabilizing controller we must solve a full-order generalized algebraic Riccati equation which contains a small parameter $\varepsilon > 0$. First, we provide conditions for stability based on two reduced-order slow and fast algebraic Riccati equations sufficient even if the matrix A_{22} is singular. Then we show that the solution of the generalized algebraic Riccati equation has asymptotic structures. Second, we present an iterative algorithm with the special initial condition to solve the corresponding generalized algebraic Riccati equation. We improve the recursive approach of Gajic *et al*. [2, 3] and Mukaidani *et al*. [14], in the sense that a more direct iterative algorithm is given. Our new idea is to set the initial condition to the solutions of the reduced-order algebraic Riccati equation. As a result, while the classical recursive algorithm is the linear convergence property, our iterative algorithm achieves the quadratic convergence property. By using the improved recursive algorithm based on the Kleinman algorithm [3, 9], we show that the solution of the generalized algebraic Riccati equation converges very fast to a desired solution. Our main result is to prove existence of an $\tilde{\varepsilon}$ such that for all $0 < \varepsilon \leq \tilde{\varepsilon}$ the proposed iterative algorithm has quadratic convergence. Furthermore, an application based on the proposed algorithm is presented to show the validity of our proposed controller for singularly perturbed systems without uncertainty. As another important feature of this paper, our new results are applicable to both standard and nonstandard singularly perturbed systems [1, 12].

Notation. The notations used in this paper are fairly standard. The superscript T denotes matrix transpose. I_j denotes the $j \times j$ identity matrix. $\| \cdot \|$ denotes its Euclidean norm for a matrix. $\| \cdot \|_2$ denotes the 2-norm on the interval $[0, \ \infty]$. $|M|$ denotes the determinant of square matrix M. $\otimes$ denotes the Kronecker product. $E[\cdot]$ denotes the expectation.

2. PROBLEM FORMULATION

Consider the following linear singularly perturbed uncertain systems

$$\begin{aligned}\dot{x}_1(t) &= (A_{11} + D_1 F E_{a1}) x_1(t) + (A_{12} + D_1 F E_{a2}) x_2(t) \\ &\quad + (B_1 + D_1 F E_b) u(t), \qquad x_1(0) = x_1^0, \end{aligned} \tag{1a}$$

$$\begin{aligned}\varepsilon \dot{x}_2(t) &= (A_{21} + D_2 F E_{a1}) x_1(t) + (A_{22} + D_2 F E_{a2}) x_2(t) \\ &\quad + (B_2 + D_2 F E_b) u(t), \qquad x_2(0) = x_2^0, \end{aligned} \tag{1b}$$

$$J = \int_0^{\infty} z(t)^T z(t) dt = \| z(t) \|_2^2, \tag{1c}$$

$$F^T(t) F(t) \leq I_j, \tag{1d}$$

where

$$z(t) = C_1 x_1(t) + C_2 x_2(t) + D_{12} u(t),$$

ε is a small positive parameter, $x_1(t) \in \mathbf{R}^{n_1}$, $x_2(t) \in \mathbf{R}^{n_2}$, and $x(t) = [x_1^T(t) x_2^T(t)]^T \in \mathbf{R}^n (n = n_1 + n_2)$ are state vectors, $u(t) \in \mathbf{R}^m$ is the control input, and $F(t) \in \mathbf{R}^{k \times j}$ is the uncertainty matrix. Moreover, all matrices above are of appropriate dimensions. The system (1) is said to be in the standard form if the matrix A_{22} is nonsingular. Otherwise, it is called a nonstandard singularly perturbed system [1, 12].

Let us introduce the partitioned matrices

$$A_\varepsilon = \begin{bmatrix} A_{11} & A_{12} \\ \varepsilon^{-1} A_{21} & \varepsilon^{-1} A_{22} \end{bmatrix}, \qquad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

$$B_\varepsilon = \begin{bmatrix} B_1 \\ \varepsilon^{-1} B_2 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix},$$

$$D_\varepsilon = \begin{bmatrix} D_1 \\ \varepsilon^{-1} D_2 \end{bmatrix}, \qquad D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}, \qquad E_a = \begin{bmatrix} E_{a1} & E_{a2} \end{bmatrix}.$$

Using the above relations, the system (1a) and (1b) can be changed to

$$\dot{x}(t) = (A_\varepsilon + D_\varepsilon F E_a) x(t) + (B_\varepsilon + D_\varepsilon F E_b) u(t), \qquad x(0) = x^0 \tag{2a}$$

$$z(t) = C x(t) + D_{12} u(t). \tag{2b}$$

Now, let us consider the guaranteed cost control problem for such singularly perturbed uncertain systems (2) using the linear state feedback controller. Through the paper, the following basic assumptions are made.

(A1) There exists a parameter $\varepsilon^* > 0$ such that the pair $(A_\varepsilon,\ B_\varepsilon)$ is stabilizable for all $\varepsilon \in (0,\ \varepsilon^*]$.

(A2) The pair $(A_{22},\ B_2)$ is stabilizable.

(A3) $C^T D_{12} = 0, \quad D_{12}^T D_{12} > 0.$

With (1) and (2) we associate the algebraic Riccati equation [10]

$$\begin{aligned} &[A_\varepsilon - B_\varepsilon \overline{R} E_b^T E_a]^T P_\varepsilon + P_\varepsilon [A_\varepsilon - B_\varepsilon \overline{R} E_b^T E_a] \\ &+ \mu P_\varepsilon D_\varepsilon D_\varepsilon^T P_\varepsilon - \mu P_\varepsilon B_\varepsilon \overline{R} B_\varepsilon^T P_\varepsilon \\ &+ \frac{1}{\mu} E_a^T [I_j - E_b \overline{R} E_b^T] E_a + Q = 0 \end{aligned} \tag{3}$$

for the matrix function

$$P_\varepsilon = P_\varepsilon(\mu) = \begin{bmatrix} P_{11}(\varepsilon,\ \mu) & \varepsilon P_{21}(\varepsilon,\ \mu)^T \\ \varepsilon P_{21}(\varepsilon,\ \mu) & \varepsilon P_{22}(\varepsilon,\ \mu) \end{bmatrix}, \tag{4}$$

where μ is positive scalar, $Q = C^T C$, $R = D_{12}^T D_{12}$, $\overline{R} = (\mu R + E_b^T E_b)^{-1}$. It is well known that a feature of quadratic stabilizability of the system (2) can be stated invoking [10].

LEMMA 2.1. *For each* $\varepsilon \in (0, \varepsilon^*]$, *if there exists* $\mu > 0$ *and* (3) *has a positive definite solution, then a controller that guarantees the quadratic stability for all* $F : F^T(t)F(t) \le I_j$ *exists. If such conditions are met, a controller is given by the formula*

$$u(t) = -\overline{R}[\mu B_\varepsilon^T P_\varepsilon(\mu) + E_b^T E_a]x(t) = Kx(t). \tag{5}$$

Furthermore, for given $\delta > 0$, *there exists a matrix* $\overline{P}_\varepsilon > 0$ *such that*

$$J \le x(0)^T \overline{P}_\varepsilon x(0), \qquad P_\varepsilon < \overline{P}_\varepsilon < P_\varepsilon + \delta I_n, \tag{6}$$

where $\overline{P}_\varepsilon$ *is said to be a cost matrix.*

Note that the bound obtained in Lemma 2.1 depends on the initial condition $x(0)$. To remove this dependence, we assume that $x(0)$ is a zero mean random variable satisfying $E[x(0)x(0)^T] = I_n$. In this case, the bound becomes

$$\overline{J} = E[J] \le \text{Trace}[\overline{P}_\varepsilon] < \text{Trace}[P_\varepsilon] + n\delta. \tag{7}$$

This result implies that since $n\delta$ is independent of μ, a control, which minimizes the cost bound, can be obtained by choosing $\mu > 0$ to minimize $\text{Trace}[P_\varepsilon]$. Then the guaranteed cost control problem for singularly perturbed uncertain systems is given below.

Find K and $0 < \beta < \infty$ such that the following conditions are satisfied. That is, the goal is to determine β as small as possible for the cost $\text{Trace}[P_\varepsilon]$,

$$\mu^* = \text{Arg}\min_{\mu} \text{Trace}[P_\varepsilon(\mu)], \tag{8a}$$

$$\beta = \min_{\mu} \text{Trace}[P_\varepsilon(\mu)]. \tag{8b}$$

If there exists a positive definite solution P_ε in the interval $(0, \hat{\mu})$, then P_ε is a convex function with respect to $\mu \in (0, \hat{\mu})$. This property ensures that a global nimimum is reachable by a one-line search algorithm [10].

2.1. *The Generalized Algebraic Riccati Equation*

In order to solve the algebraic Riccati equation (3), we introduce the following useful lemma [13, 14].

LEMMA 2.2. *The algebraic Riccati equation* (3) *is equivalent to the generalized algebraic Riccati equation*

$$\begin{aligned}[A - B\overline{R}E_b^T E_a]^T P + P[A - B\overline{R}E_b^T E_a] \\ + \mu PDD^T P - \mu PB\overline{R}B^T P \\ + \frac{1}{\mu}E_a^T[I_j - E_b\overline{R}E_b^T]E_a + Q = 0,\end{aligned} \tag{9a}$$

$$P_\varepsilon = \Pi_\varepsilon^T P = P^T \Pi_\varepsilon, \tag{9b}$$

where

$$\Pi_\varepsilon = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \varepsilon I_{n_2} \end{bmatrix}, \qquad P = \begin{bmatrix} P_{11} & \varepsilon P_{21}^T \\ P_{21} & P_{22} \end{bmatrix},$$

$$P_{11} = P_{11}^T,\ P_{22} = P_{22}^T,\ A = \Pi_\varepsilon A_\varepsilon,\ B = \Pi_\varepsilon B_\varepsilon,\ D = \Pi_\varepsilon D_\varepsilon.$$

Moreover, by making use of relation (9b), *we can change the form of the controller* (5),

$$u(t) = -\overline{R}[\mu B^T P(\mu) + E_b^T E_a]x(t). \tag{10}$$

Proof. Since the proof is identical to the proof of Lemma 3 in [14], it is omitted. ∎

2.2. *The Solvability Condition*

In this section, the linear state feedback full-order controller for singularly perturbed systems with structured uncertainties is presented.

The algebraic Riccati equation (9a) can be partitioned into

$$\begin{aligned} &A_{11}^{\mu T} P_{11} + P_{11}^T A_{11}^\mu + A_{21}^{\mu T} P_{21} + P_{21}^T A_{21}^\mu - P_{11}^T S_{11}^\mu P_{11} - P_{21}^T S_{22}^\mu P_{21} \\ &\quad - P_{11}^T S_{12}^\mu P_{21} - P_{21}^T S_{12}^{\mu T} P_{11} + Q_{11}^\mu = 0, \end{aligned} \tag{11a}$$

$$\begin{aligned} &\varepsilon P_{21} A_{11}^\mu + P_{22}^T A_{21}^\mu + A_{12}^{\mu T} P_{11} + A_{22}^{\mu T} P_{21} - \varepsilon P_{21} S_{11}^\mu P_{11} - \varepsilon P_{21} S_{12}^\mu P_{21} \\ &\quad - P_{22}^T S_{12}^{\mu T} P_{11} - P_{22}^T S_{22}^\mu P_{21} + Q_{12}^{\mu T} = 0, \end{aligned} \tag{11b}$$

$$\begin{aligned} &A_{22}^{\mu T} P_{22} + P_{22}^T A_{22}^\mu + \varepsilon A_{12}^{\mu T} P_{21}^T + \varepsilon P_{21} A_{12}^\mu - P_{22}^T S_{22}^\mu P_{22} \\ &\quad - \varepsilon P_{22}^T S_{12}^{\mu T} P_{21}^T - \varepsilon P_{21} S_{12}^\mu P_{22} - \varepsilon^2 P_{21} S_{11}^\mu P_{21}^T + Q_{22}^\mu = 0, \end{aligned} \tag{11c}$$

where

$$A^\mu = A - B\overline{R}E_b^T E_a = \begin{bmatrix} A_{11}^\mu & A_{12}^\mu \\ A_{21}^\mu & A_{22}^\mu \end{bmatrix},$$

$$S^\mu = \mu(B\overline{R}B^T - DD^T) = \begin{bmatrix} S_{11}^\mu & S_{12}^\mu \\ S_{12}^{\mu T} & S_{22}^\mu \end{bmatrix},$$

$$Q^\mu = \frac{1}{\mu} E_a^T [I_j - E_b \overline{R} E_b^T] E_a + Q = \begin{bmatrix} Q_{11}^\mu & Q_{12}^\mu \\ Q_{12}^{\mu T} & Q_{22}^\mu \end{bmatrix}.$$

Let $\overline{P}_{11}$, $\overline{P}_{21}$, and $\overline{P}_{22}$ be the limiting solutions of the above equations (11) as $\varepsilon \to +0$. Then these equations reduce to the equations

$$\begin{aligned} &A_{11}^{\mu T} \overline{P}_{11} + \overline{P}_{11}^T A_{11}^\mu + A_{21}^{\mu T} \overline{P}_{21} + \overline{P}_{21}^T A_{21}^\mu - \overline{P}_{11}^T S_{11}^\mu \overline{P}_{11} \\ &\quad - \overline{P}_{21}^T S_{22}^\mu \overline{P}_{21} - \overline{P}_{11}^T S_{12}^\mu \overline{P}_{21} - \overline{P}_{21}^T S_{12}^{\mu T} \overline{P}_{11} + Q_{11}^\mu = 0, \end{aligned} \tag{12a}$$

$$\overline{P}_{22}^T A_{21}^\mu + A_{12}^{\mu T} \overline{P}_{11} + A_{22}^{\mu T} \overline{P}_{21} - \overline{P}_{22}^T S_{12}^{\mu T} \overline{P}_{11} - \overline{P}_{22}^T S_{22}^\mu \overline{P}_{21} + Q_{12}^{\mu T} = 0, \tag{12b}$$

$$A_{22}^{\mu T} \overline{P}_{22} + \overline{P}_{22}^T A_{22}^\mu - \overline{P}_{22}^T S_{22}^\mu \overline{P}_{22} + Q_{22}^\mu = 0. \tag{12c}$$

The Riccati equation (12c) will produce the unique positive definite stabilizing solution under the following conditions [13, 14].

Let

$\Gamma_f := \{0 < \mu |$the Riccati equation (12c) has a positive definite stabilizing solution$\}$,

$\mu_f := \sup\{\mu | \mu \in \Gamma_f\}$.

Then the matrix $A_{22}^\mu - S_{22}^\mu \overline{P}_{22}$ is nonsingular if we choose $0 < \mu < \mu_f$. Therefore, we obtain the following 0-order equations

$$\overline{P}_{11}^T A_0^\mu + A_0^{\mu T} \overline{P}_{11} - \overline{P}_{11}^T S_0^\mu \overline{P}_{11} + Q_0^\mu = 0, \tag{13a}$$

$$\overline{P}_{21} = -N_2^T + N_1^T \overline{P}_{11}, \tag{13b}$$

$$A_{22}^{\mu T} \overline{P}_{22} + \overline{P}_{22}^T A_{22}^\mu - \overline{P}_{22}^T S_{22}^\mu \overline{P}_{22} + Q_{22}^\mu = 0, \tag{13c}$$

where

$$\begin{aligned}
A_0^\mu &= A_{11}^\mu + N_1 A_{21}^\mu + S_{12}^\mu N_2^T + N_1 S_{22}^\mu N_2^T, \\
S_0^\mu &= S_{11}^\mu + N_1 S_{12}^{T\mu} + S_{12}^\mu N_1^T + N_1 S_{22}^\mu N_1^T, \\
Q_0^\mu &= Q_{11}^\mu - N_2 A_{21}^\mu - A_{21}^{\mu T} N_2^T - N_2 S_{22}^\mu N_2^T, \\
N_2^T &= \overline{D}_4^{-T} \widehat{Q}_{12}^T, \qquad N_1^T = -\overline{D}_4^{-T} \overline{D}_2^T, \\
\overline{D}_1 &= A_{11}^\mu - S_{11}^\mu \overline{P}_{11} - S_{12}^\mu \overline{P}_{21}, \qquad \overline{D}_3 = A_{21}^\mu - S_{12}^{\mu T} \overline{P}_{11} - S_{22}^\mu \overline{P}_{21}, \\
\overline{D}_2 &= A_{12}^\mu - S_{12}^\mu \overline{P}_{22}, \qquad \overline{D}_4 = A_{22}^\mu - S_{22}^\mu \overline{P}_{22}, \\
\overline{D}_0 &= \overline{D}_1 - \overline{D}_2 \overline{D}_4^{-1} \overline{D}_3, \qquad \widehat{Q}_{12} = Q_{12} + A_{21}^{\mu T} \overline{P}_{22}.
\end{aligned}$$

Remark 2.1. Although the expressions of the matrix A_0^μ, S_0^μ, and Q_0^μ contain the matrix $\overline{P}_{22}$, they do not depend on it (see, e.g., [16]). In fact the coefficient matrices of Eq. (13a) are obtained from the formula

$$T_0 = T_1 - T_2 T_4^{-1} T_3 = \begin{bmatrix} A_0^\mu & -S_0^\mu \\ -Q_0^\mu & -A_0^{\mu T} \end{bmatrix}, \tag{14}$$

where

$$T_1 = \begin{bmatrix} A_{11}^\mu & -S_{11}^\mu \\ -Q_{11}^\mu & -A_{11}^{\mu T} \end{bmatrix}, T_2 = \begin{bmatrix} A_{12}^\mu & -S_{12}^\mu \\ -Q_{12}^\mu & -A_{21}^{\mu T} \end{bmatrix},$$

$$T_3 = \begin{bmatrix} A_{21}^\mu & -S_{12}^{\mu T} \\ -Q_{12}^{\mu T} & -A_{12}^{\mu T} \end{bmatrix}, \; T_4 = \begin{bmatrix} A_{22}^\mu & -S_{22}^\mu \\ -Q_{22}^\mu & -A_{22}^{\mu T} \end{bmatrix}.$$

Let us define

$\Gamma_s := \{0 < \mu |$the Riccati equation (13a) has a positive definite stabilizing solution$\}$,

$\mu_s := \sup\{\mu | \mu \in \Gamma_s\}$.

As results, for every $0 < \mu < \bar{\mu} = \min\{\mu_s, \mu_f\}$, the Riccati equations (13a) and (13c) have the positive definite stabilizing solutions.

We have the following result.

THEOREM 2.1. *Under the assumptions* $(A1)$–$(A3)$, *if we select a parameter* $0 < \mu < \bar{\mu} = \min\{\mu_s, \ \mu_f\}$, *then there exists small* $0 < \bar{\varepsilon} \leq \varepsilon^*$ *such that for all* $\varepsilon \in (0, \ \bar{\varepsilon})$, *the generalized algebraic Riccati equation* (9*a*) *admits a positive definite solution*, *which can be written as*

$$P = \begin{bmatrix} \overline{P}_{11} + \varepsilon E_{11}(\varepsilon) & \varepsilon(\overline{P}_{21} + \varepsilon E_{21}(\varepsilon))^T \\ \overline{P}_{21} + \varepsilon E_{21}(\varepsilon) & \overline{P}_{22} + \varepsilon E_{22}(\varepsilon) \end{bmatrix}. \tag{15}$$

If such conditions are met, *a control is given by* (10). *Furthermore*, $P_\varepsilon = \Pi_\varepsilon^T P$ *of the Riccati equation* (3) *is a positive definite stabilizing solution*.

Proof. By using the implicit function theorem, the theorem can be proved. The proof is omitted since it is similar to Refs. [8, 13, 14]. ∎

Remark 2.2. We can prove Theorem 2.1 by using a similar method given in the proof of Theorems 2.1 and 2.2 in [8]. Note that the proof given in [8] is made on the invertible assumption, that is, A_{22} is nonsingular. However, this paper improves the proof of Dragan [8] in the sense that the invertible assumption is not needed.

3. PRELIMINARY RESULTS

By the results in [10], we have to solve by all means the full-order algebraic Riccati equations for every $0 < \mu < \bar{\mu}$. So far, the recursive algorithm [2, 3, 14] was very effective for solving the full-order algebraic Riccati equation with small parameter $\varepsilon > 0$. However, note that using the zero-order solution without high-order accuracy will fail to produce the desired exact solution of the algebraic Riccati equation. In this case, the recursive algorithm converges to the approximation solution. Moreover, since the recursive approach has the linear convergence property, the convergence rate is not efficient.

In this paper we develop an elegant and simple algorithm which converges globally to the positive definite symmetric solution of Eq. (3). The algorithm is given in terms of the standard algebraic Riccati equations, which have to be solved iteratively. We present the iterative algorithm based on the Kleinman algorithm [3, 9]. Here we note that the Kleinman algorithm is based on the Newton type algorithm. In general, the stabilizable-detectable conditions will guarantee the convergence of the Kleinman algorithm for the standard linear-quadratic regulator type algebraic Riccati equation to the positive definite solutions. However, it is difficult to apply the Kleinman algorithm to Eq. (3) presented in this paper because the matrix $S_\varepsilon^\mu = B_\varepsilon R^{-1} B_\varepsilon^T - \mu D_\varepsilon D_\varepsilon^T$ is in general indefinite. That is, the generalized algebraic Riccati equation (9a) is not always a convex function with respect to P because the matrix $S^\mu = BR^{-1}B^T - \mu DD^T$ is also in general indefinite.

We propose the following algorithm for solving the generalized algebraic Riccati equation (9a)

$$\begin{aligned}(A^\mu - S^\mu P^{(i)})^T P^{(i+1)} + P^{(i+1)T}(A^\mu - S^\mu P^{(i)})\\ +P^{(i)T} S^\mu P^{(i)} + Q^\mu = 0 \qquad (i = 0, 1, \ldots),\end{aligned} \tag{16a}$$

with the initial condition obtained from

$$P^{(0)} = \begin{bmatrix} \overline{P}_{11} & 0 \\ \overline{P}_{21} & \overline{P}_{22} \end{bmatrix}, \tag{16b}$$

where

$$P^{(i)} = \begin{bmatrix} P_{11}^{(i)} & \varepsilon P_{21}^{(i)T} \\ P_{21}^{(i)} & P_{22}^{(i)} \end{bmatrix} \quad (i = 1, 2, \cdots),$$

$\overline{P}_{11}$, $\overline{P}_{21}$, $\overline{P}_{22}$ are defined by (13). Note that even if the initial condition looks the same as in [15], it is quite different.

The Kleinman algorithm is well known and widely used to find a solution of the algebraic Riccati equation, and its local convergence properties are well understood. The convergence of the proposed algorithm is concerned with the good choice of the initial condition which guarantees finding the required solution of a given generalized algebraic Riccati equation. Our new idea is to set the initial condition $P^{(0)}$ to Eq. (16b). The fundamental idea is based on $\|P - P^{(0)}\| = O(\varepsilon)$ from Theorem 2.1. Although the matrix S^μ is in general indefinite, we prove convergence to the required solution by using the Kleinman algorithm (16). The algorithm (16) has the feature given in the following theorem.

THEOREM 3.1. *Under the assumptions* (A1)–(A3), *if we select a parameter* $0 < \mu < \bar{\mu} = \min\{\mu_s,\ \mu_f\}$, *then the new iterative algorithm* (16) *converges to the exact solution of* P^* *with the rate of quadratic convergence such that*

$P_{\varepsilon}^{(i)} = \Pi_{\varepsilon}^{T} P^{(i)} = P^{(i)T}\Pi_{\varepsilon}$ *is positive definite and* $A_{\varepsilon}^{\mu} - S_{\varepsilon}^{\mu} P_{\varepsilon}^{(i)}$ *is stable for all* $i (i = 1, 2, 3, \ldots)$. *That is,*

$$\lim_{i\to\infty} \frac{\|P^{(i+1)} - P^*\|}{\|P^{(i)} - P^*\|} = 0, \tag{17a}$$

$$\|P^{(i+1)} - P^*\| \le \mathcal{M}\|P^{(i)} - P^*\|^2, \qquad 0 < \mathcal{M} < \infty$$

$$\Leftrightarrow \|P^{(i)} - P^*\| = O(\varepsilon^{2^i}), \tag{17b}$$

$$\|P^{(i)}\| \le c < \infty, \tag{17c}$$

$$P_{\varepsilon}^{(i)} = \Pi_{\varepsilon}^{T} P^{(i)} = \begin{bmatrix} \overline{P}_{11} + O(\varepsilon) & \varepsilon\overline{P}_{21}^{T} + O(\varepsilon^2) \\ \varepsilon\overline{P}_{21} + O(\varepsilon^2) & \varepsilon\overline{P}_{22} + O(\varepsilon^2) \end{bmatrix} > 0, \tag{17d}$$

$$\mathrm{Re}\lambda[A_{\varepsilon}^{\mu} - S_{\varepsilon}^{\mu} P_{\varepsilon}^{(i)}] < 0, \tag{17e}$$

where

$$P = P^* = \begin{bmatrix} P_{11} & \varepsilon P_{21}^{T} \\ P_{21} & P_{22} \end{bmatrix}, P^{(i)} = \begin{bmatrix} P_{11}^{(i)} & \varepsilon P_{21}^{(i)T} \\ P_{21}^{(i)} & P_{22}^{(i)} \end{bmatrix}.$$

Moreover, let $P_{11}^{(\infty)}$, $P_{21}^{(\infty)}$, *and* $P_{22}^{(\infty)}$ *be the limit point of the iterative algorithm* (16). *As results, we have*

$$A^{\mu T} P^{(\infty)} + P^{(\infty)T} A^{\mu} - P^{(\infty)T} S^{\mu} P^{(\infty)} + Q^{\mu} = 0, \tag{18}$$

where

$$P^* = P^{(\infty)} = \begin{bmatrix} P_{11}^{(\infty)} & \varepsilon P_{21}^{(\infty)T} \\ P_{21}^{(\infty)} & P_{22}^{(\infty)} \end{bmatrix} = \begin{bmatrix} \overline{P}_{11} + O(\varepsilon) & \varepsilon\overline{P}_{21}^{T} + O(\varepsilon^2) \\ \overline{P}_{21} + O(\varepsilon) & \overline{P}_{22} + O(\varepsilon) \end{bmatrix}.$$

Thus, by using the linear state feedback full-order controller

$$u_{exa}(t) = -\overline{R}[\mu B^{T} P^{(\infty)} + E_b^{T} E_a] x(t), \tag{19}$$

the uncertain linear singularly perturbed system (1) *is quadratically stable.*

In order to prove Theorem 3.1, we need the following result on properties of Newton's method [17].

LEMMA 3.1. *Assume that* $F : R^n \to R^n$ *is differentiable at each point of an open neighborhood of a solution* $\mathbf{x}^*$ *of* $F\mathbf{x} = 0$, *that* F' *is continuous at* $\mathbf{x}^*$, *and that* $F'(\mathbf{x}^*)$ *is nonsingular. Then* $\mathbf{x}^*$ *is a point of attraction of the iterations*

$$\mathbf{x}^{k+1} = \mathbf{x}^{k} - F'(\mathbf{x}^{k})^{-1} F\mathbf{x}^{k}, \qquad k = 0, 1, \ldots, \tag{20}$$

where $F'(\mathbf{x}^k)$ denotes the Jacobian matrix

$$F'(\mathbf{x}) = \begin{bmatrix} \partial_1 f_1(\mathbf{x}) & \cdots & \partial_n f_1(\mathbf{x}) \\ \vdots & & \vdots \\ \partial_1 f_n(\mathbf{x}) & \cdots & \partial_n f_n(\mathbf{x}) \end{bmatrix},$$

where we have used $\partial_i f_j(\mathbf{x})$ to denote the partial derivative of f_j with respect to the ith variable evaluated at $\mathbf{x}$, *and where*

$$\lim_{k\to\infty} \frac{\|\mathbf{x}^{k+1} - \mathbf{x}^*\|}{\|\mathbf{x}^k - \mathbf{x}^*\|} = 0. \tag{21}$$

Moreover, if

$$\|F'(\mathbf{x}) - F'(\mathbf{x}^*)\| \le L\|\mathbf{x} - \mathbf{x}^*\|, \qquad L > 0 \tag{22}$$

for all $\mathbf{x}$ *in some open neighborhood of* $\mathbf{x}^*$, *then there is a constant $\alpha < +\infty$ such that*

$$\|\mathbf{x}^{k+1} - \mathbf{x}^*\| \le \mathcal{N}\|\mathbf{x}^k - \mathbf{x}^*\|^2 \tag{23}$$

for all $k \ge k_0$ where k_0 depends on the initial condition of $\mathbf{x}^0$.

Proof. We begin by rearranging the generalized Lyapunov equation (16a) as

$$\begin{aligned}(A_\varepsilon^\mu - S_\varepsilon^\mu P_\varepsilon^{(i)})^T P_\varepsilon^{(i+1)} + P_\varepsilon^{(i+1)T}(A_\varepsilon^\mu - S_\varepsilon^\mu P_\varepsilon^{(i)}) \\ + P_\varepsilon^{(i)T} S_\varepsilon^\mu P_\varepsilon^{(i)} + Q^\mu = 0,\end{aligned} \tag{24}$$

where $S_\varepsilon^\mu = \Pi_\varepsilon^{-1} S^\mu \Pi_\varepsilon^{-T}$.

We first prove that $P_\varepsilon^{(i)}$ is the positive definite symmetric matrix and $A_\varepsilon^\mu - S_\varepsilon^\mu P_\varepsilon^{(i)} \equiv \mathcal{A}_\varepsilon(i)$ is stable for all i. The proof is done by using mathematical induction. When $i = 0$, since $\overline{P}_{11} > 0$ and $\overline{P}_{22} > 0$, $P_\varepsilon^{(0)}$ is positive definite as long as $\varepsilon > 0$. On the other hand, $\mathcal{A}_\varepsilon(0)$ is given by

$$\mathcal{A}_\varepsilon(0) = \Pi_\varepsilon^{-1}(A^\mu - S^\mu P^{(0)}) = \Pi_\varepsilon^{-1}\begin{bmatrix} \overline{D}_1 & \overline{D}_2 \\ \overline{D}_3 & \overline{D}_4 \end{bmatrix}.$$

It is easy to show that $\overline{D}_4 = A_{22}^\mu - S_{22}^\mu \overline{P}_{22}$ and $\overline{D}_0 = \overline{D}_1 - \overline{D}_2 \overline{D}_4^{-1} \overline{D}_3 = A_0^\mu - S_0^\mu \overline{P}_{11}$ are stable since $\overline{P}_{22}$ and $\overline{P}_{11}$ are the stabilizing solutions corresponding to the Riccati equations (13c) and (13a), respectively. Therefore, when the parameter ε is very small, $\mathcal{A}_\varepsilon(0)$ is stable from Corollary 3.1 in [1]. Using the standard properties of Lyapunov equations [18], there exists the solution of Eq. (24) because $\mathcal{A}_\varepsilon(0)$ is stable. Furthermore, by using the implicit function theorem for Eq. (24) in which $i = 0$, we observe that

$$P_\varepsilon^{(1)} = \begin{bmatrix} \overline{P}_{11} + O(\varepsilon) & \varepsilon \overline{P}_{21}^T + O(\varepsilon^2) \\ \varepsilon \overline{P}_{21} + O(\varepsilon^2) & \varepsilon \overline{P}_{22} + O(\varepsilon^2) \end{bmatrix}.$$

Since $\overline{P}_{11} > 0$ and $\overline{P}_{22} > 0$, $P_\varepsilon^{(1)}$ is also the positive definite matrix. Then

$$\mathscr{A}_\varepsilon(1) = \Pi_\varepsilon^{-1}(A^\mu - S^\mu P^{(1)}) = \Pi_\varepsilon^{-1}\begin{bmatrix} \overline{D}_1 + O(\varepsilon) & \overline{D}_2 + O(\varepsilon) \\ \overline{D}_3 + O(\varepsilon) & \overline{D}_4 + O(\varepsilon) \end{bmatrix}.$$

Thus, $\mathscr{A}_\varepsilon(1)$ is also stable from Corollary 3.1 in [1].

When $i = k$ $(k \geq 2)$, we assume that $P_\varepsilon^{(k)}$ is the positive definite matrix and $\mathscr{A}_\varepsilon(k)$ is stable. By following similar steps in the case of $k = 1$, we get the positive definite solution $P_\varepsilon^{(k+1)}$ since $\mathscr{A}_\varepsilon(k)$ is stable. Using the implicit function theorem for Eq. (24) in which $i = k$, we have

$$P_\varepsilon^{(k+1)} = \begin{bmatrix} \overline{P}_{11} + O(\varepsilon) & \varepsilon\overline{P}_{21}^T + O(\varepsilon^2) \\ \varepsilon\overline{P}_{21} + O(\varepsilon^2) & \varepsilon\overline{P}_{22} + O(\varepsilon^2) \end{bmatrix} > 0.$$

Then,

$$\mathscr{A}_\varepsilon(k+1) = \Pi_\varepsilon^{-1}(A^\mu - S^\mu P^{(k+1)}) = \Pi_\varepsilon^{-1}\begin{bmatrix} \overline{D}_1 + O(\varepsilon) & \overline{D}_2 + O(\varepsilon) \\ \overline{D}_3 + O(\varepsilon) & \overline{D}_4 + O(\varepsilon) \end{bmatrix}$$

is stable. Thus, $P_\varepsilon^{(i)}$ is the positive definite matrix and $\mathscr{A}_\varepsilon(i)$ is stable for all $i \in \mathbb{N}$.

Next we show that $\|P^{(i)}\| \leq c < \infty$. Note that

$$\Pi_\varepsilon^T P = P_\varepsilon = \begin{bmatrix} \overline{P}_{11} + O(\varepsilon) & \varepsilon\overline{P}_{21}^T + O(\varepsilon^2) \\ \varepsilon\overline{P}_{21} + O(\varepsilon^2) & \varepsilon\overline{P}_{22} + O(\varepsilon^2) \end{bmatrix},$$

$$\Pi_\varepsilon^T P^{(i)} = P_\varepsilon^{(i)} = \begin{bmatrix} \overline{P}_{11} + O(\varepsilon) & \varepsilon\overline{P}_{21}^T + O(\varepsilon^2) \\ \varepsilon\overline{P}_{21} + O(\varepsilon^2) & \varepsilon\overline{P}_{22} + O(\varepsilon^2) \end{bmatrix}, \qquad (i = 1, 2, 3, \ldots).$$

It now follows that

$$\|P_\varepsilon - P_\varepsilon^{(i)}\| = O(\varepsilon) \Leftrightarrow \|P - P^{(i)}\| = O(\varepsilon).$$

As a consequence, we can show that $\|P^{(i)}\| \leq \|P\| + O(\varepsilon) \leq c < \infty$ since $\|P\|$ is bounded from the proof of Theorem 2.1.

Finally, in the rest of the proof we show that under the assumptions (A1)–(A3), the algorithm (16) converges to the desired solution of (3). Let us define

$$\begin{aligned} F(P) &= (A - B\overline{R}E_b^T E_a)^T P + P^T(A - B\overline{R}E_b^T E_a) \\ &\quad + \mu P^T DD^T P - \mu P^T B\overline{R}B^T P + \frac{1}{\mu}E_a^T(I_j - E_b\overline{R}E_b^T)E_a + Q \\ &= A^{\mu T}P + P^T A^\mu - P^T S^\mu P + Q^\mu. \end{aligned}$$

Under the assumptions (A1)–(A3), the generalized algebraic Riccati equation (9a) (i.e., $F(P) = 0$) has the solution P^*. Taking the partial derivative of the function $F(P)$ with respect to P yields

$$\begin{aligned}\nabla F(P)|_{P=P^*} &= \left.\frac{\partial F(P)}{\partial P}\right|_{P=P^*}\\ &= (A^{\mu} - S^{\mu}P^*) \otimes I_n + I_n \otimes (A^{\mu} - S^{\mu}P^*).\end{aligned}$$

It is obvious that $\nabla F(P)$ is continuous at $P = P^*$, and that $\nabla F(P^*)$ is nonsingular since $A^{\mu} - S^{\mu}P^*$ is nonsingular via nonsingularity of $\overline{D}_4$ and $\overline{D}_0$. Therefore, P^* is the point of the attraction for the iterations (16a) since $P^{(0)}$ is sufficiently close to P^*. Moreover, it is immediately obtained from the above equation that

$$\|\nabla F(P) - \nabla F(P^*)\| \leq \mathscr{L}\|P - P^*\|, \qquad 0 < \mathscr{L} < \infty.$$

Taking into account the fact that $\|P^{(i)} - P^*\| = O(\varepsilon)$ and both $\overline{D}_4$ and $\overline{D}_0$ are stable for all $i \in \mathbb{N}$, there exists a constant $\mathscr{M} < \infty$ such that $\|P^{(i+1)} - P^*\| \leq \mathscr{M}\|P^{(i)} - P^*\|^2$. This is equivalent to the equation

$$\|P^{(i)} - P^*\| \leq \mathscr{M}^{2^i-1}\|P^{(0)} - P^*\|^{2^i} = O(\varepsilon^{2^i}).$$

Consequently, we found that the sequence $\{P_{\varepsilon}^{(i)}\} = \{\Pi_{\varepsilon}P^{(i)}\}$ $(i = 1, 2, 3, \cdots)$ has quadratic convergence by using Lemma 3.1 even if the matrix S_{ε}^{μ} is in general indefinite. In addition, let $P_{\varepsilon}^{(\infty)} = \Pi_{\varepsilon}P^{(\infty)}$ be the limit point of the corresponding sequence (16a). Then, from (16a) we also found that the solution $P_{\varepsilon}^{(\infty)}$ satisfies the algebraic Riccati equation (3) so that it represents the sought solution of this equation.

It remains to establish the controller (19). This can be done in a similar way to [13]. The proof of Theorem 3.1 is completed. ∎

As a result of the application of the idea of the Kleinman algorithm, we have managed to replace the computation of the generalized Riccati equation (9a) which contains the small parameter ε by a sequence of the generalized algebraic Lyapunov equations (16a).

4. MAIN RESULTS

Mukaidani *et al.* [15] proposed the new iterative algorithm based on the Kleinman algorithm [3, 9] which has quadratic convergence. However, so far, the question remains as to whether an $\tilde{\varepsilon}$ exists such that proposed algorithm has quadratic convergence. In this section, we give a positive answer to this question. Our main result is to show the existence of an $\tilde{\varepsilon}$ such that for all $0 < \varepsilon \leq \tilde{\varepsilon}$ the proposed iterative algorithm has quadratic convergence.

We now present the following theorem.

THEOREM 4.1. *Under the assumptions* (A1)–(A3), *there exists an* $\tilde{\varepsilon}$ *such that for all* $0 < \varepsilon \le \tilde{\varepsilon} \le \bar{\varepsilon} \le \varepsilon^*$ *the proposed iterative algorithm* (16) *has quadratic convergence, where* $\bar{\varepsilon}$ *and* ε^* *are given by Theorem* 2.1.

In order to prove Theorem 4.1, we need the following lemma [17].

LEMMA 4.1. *Assume that* $F : R^n \to R^n$ *is differentiable on a convex set* $\mathscr{D}$ *and that*

$$\|F'(\mathbf{x}) - F'(\mathbf{y})\| \le \gamma \|\mathbf{x} - \mathbf{y}\|$$

for all $\mathbf{x},\ \mathbf{y} \in \mathscr{D}$. *Suppose that there is a* $\mathbf{x}^0 \in \mathscr{D}$ *such that*

$$\|F'(\mathbf{x}^0)^{-1}\| \le \beta, \qquad \|F'(\mathbf{x}^0)^{-1} F\mathbf{x}^0\| \le \eta$$

and $\alpha \equiv \beta\gamma\eta < 2^{-1}$. *Assume that*

$$S \equiv \left\{ \mathbf{x} : \|\mathbf{x} - \mathbf{x}^0\| \le t^*,\ t^* = \frac{1}{\beta\gamma}\left[1 - \sqrt{1 - 2\alpha}\right] \right\} \subset \mathscr{D}.$$

Then the Newton iterations (20) *are well defined and converge to a solution* $\mathbf{x}^*$ *of* $F\mathbf{x} = 0$.

Proof. Using

$$\nabla F(P) = (A^\mu - S^\mu P) \otimes I_n + I_n \otimes (A^\mu - S^\mu P),$$

we have

$$\|\nabla F(P_1) - \nabla F(P_2)\| \le \bar{\gamma} \|P_1 - P_2\|,$$

where $\bar{\gamma} = 2\|S^\mu\|$ $(= \mathscr{L})$. Moreover, using the fact that

$$\nabla F(P^{(0)}) = \begin{bmatrix} \overline{D}_1 & \overline{D}_2 \\ \overline{D}_3 & \overline{D}_4 \end{bmatrix} \otimes I_n + I_n \otimes \begin{bmatrix} \overline{D}_1 & \overline{D}_2 \\ \overline{D}_3 & \overline{D}_4 \end{bmatrix},$$

it follows that $\nabla F(P^{(0)})$ is nonsingular because $\overline{D}_4$ and $\overline{D}_0 = \overline{D}_1 - \overline{D}_2 \overline{D}_4^{-1} \overline{D}_3$ are stable. Therefore, there exists $\bar{\beta}$ such that $\|[\nabla F(P^{(0)})]^{-1}\| \equiv \bar{\beta}$. Moreover, since $F(P^{(0)}) = 0$, there exists $\bar{\eta} > 0$ independent of ε such that $0 < \bar{\beta}\bar{\gamma}\bar{\eta} < 2^{-1}$. Let us define $\bar{\alpha} \equiv \bar{\beta}\bar{\gamma}\bar{\eta}$ and using Lemma 4.1, a t^* is given by

$$t^* \equiv \frac{1}{\bar{\beta}\bar{\gamma}}\left[1 - \sqrt{1 - 2\bar{\alpha}}\right] = \frac{1}{2\|S^\mu\| \cdot \|[\nabla F(P^{(0)})]^{-1}\|}\left[1 - \sqrt{1 - 2\bar{\alpha}}\right].$$

On the other hand, from Theorem 2.1 and the initial condition (16b), we have

$$
\begin{aligned}
&\|P^* - P^{(0)}\| \\
&= \left\| \varepsilon \begin{bmatrix} E_{11}(0) & \overline{P}_{21}^T \\ E_{21}(0) & E_{22}(0) \end{bmatrix} + \sum_{k=1}^{\infty} \frac{\varepsilon^{k+1}}{k!} \begin{bmatrix} E_{11}^{(k)}(0) & E_{21}^{(k-1)}(0)^T \\ E_{21}^{(k)}(0) & E_{22}^{(k)}(0) \end{bmatrix} \right\| \\
&\le \varepsilon \left\| \begin{bmatrix} E_{11}(0) & \overline{P}_{21}^T \\ E_{21}(0) & E_{22}(0) \end{bmatrix} \right\| + \sum_{k=1}^{\infty} \frac{\varepsilon^{k+1}}{k!} \left\| \begin{bmatrix} E_{11}^{(k)}(0) & E_{21}^{(k-1)}(0)^T \\ E_{21}^{(k)}(0) & E_{22}^{(k)}(0) \end{bmatrix} \right\| \\
&= \sum_{k=0}^{\infty} a_k \varepsilon^{k+1} = g(\varepsilon), \qquad (a_k > 0),
\end{aligned}
$$

where

$$
\begin{aligned}
E_{ij}^{(k)}(0) &= \frac{d^k}{d\varepsilon^k} E_{ij}(\varepsilon)|_{\varepsilon=0}, \qquad (ij = 11,\ 21,\ 22), \\
a_0 &= \left\| \begin{bmatrix} E_{11}(0) & \overline{P}_{21}^T \\ E_{21}(0) & E_{22}(0) \end{bmatrix} \right\| \\
a_k &= \frac{1}{k!} \left\| \begin{bmatrix} E_{11}^{(k)}(0) & E_{21}^{(k-1)}(0)^T \\ E_{21}^{(k)}(0) & E_{22}^{(k)}(0) \end{bmatrix} \right\|, \quad (k \ge 1).
\end{aligned}
$$

Therefore, since t^* is independent of ε and $g(0) = 0$, it is clear that there exists $\tilde{\varepsilon}$ such that the following inequality is satisfied:

$$
\sum_{k=0}^{\infty} a_k \varepsilon^{k+1} \le t^* \Leftrightarrow g(\varepsilon) \le t^*.
$$

This completes the proof of Theorem 4.1. ∎

5. APPLICATION

In this section, we will present an important application. If there is no uncertainty in the state and input matrix, that is, $E_a \equiv 0$, $E_b \equiv 0$, and $D_\varepsilon \equiv 0$, then the following corollary is easily seen in view of Theorem 3.1.

COROLLARY 5.1. *Under the conditions of Theorem* 3.1, *the approximate controller* $u_m(t) = -R^{-1}[B_1^T \ \ B_2^T]X^{(i)}x(t) = -K_m x(t)$ *attains the suboptimal performance*, *that is*,

$$
J^m = J^* + O(\varepsilon^{2^{i+1}}), \tag{25}
$$

where $X^{(i)}$ *defined as* $E_a \equiv 0$, $E_b \equiv 0$, $D_\varepsilon \equiv 0$ *for* (16) *and* $J^* = x(0)^T X_\varepsilon x(0)$, $J^m = x(0)^T X_{m\varepsilon} x(0)$,

$$X_\varepsilon A_\varepsilon + A_\varepsilon^T X_\varepsilon - X_\varepsilon B_\varepsilon R^{-1} B_\varepsilon^T X_\varepsilon + Q = 0, \tag{26a}$$

$$X_{m\varepsilon}(A_\varepsilon - B_\varepsilon K_m) + (A_\varepsilon - B_\varepsilon K_m)^T X_{m\varepsilon} + Q + K_m^T R K_m = 0. \tag{26b}$$

Remark 5.1. The solution X_ε of the Riccati equation (26a) corresponds to the solution P_ε of the Riccati equation (3) when $E_a \equiv 0$, $E_b \equiv 0$, $D_\varepsilon \equiv 0$.

Proof. It can be carried out via a similar technique used in [11]. Subtracting (26a) from (26b), we obtain the Lyapunov equation for $W_\varepsilon = X_{m\varepsilon} - X_\varepsilon$

$$\begin{aligned}(A_\varepsilon - B_\varepsilon R^{-1} B_\varepsilon^T X_\varepsilon^{(i)})^T W_\varepsilon + W_\varepsilon (A_\varepsilon - B_\varepsilon R^{-1} B_\varepsilon^T X_\varepsilon^{(i)}) \\ + (X_\varepsilon - X_\varepsilon^{(i)}) B_\varepsilon R^{-1} B_\varepsilon^T (X_\varepsilon - X_\varepsilon^{(i)}) = 0.\end{aligned} \tag{27}$$

Also applying the implicit function theorem [2] to (27), $W_\varepsilon^{(i)}$ possesses a power series at $\varepsilon = 0$. Thus W_ε can also be extended as

$$W_\varepsilon = \begin{bmatrix} W_{11}^{(0)} & \varepsilon W_{21}^{(0)T} \\ \varepsilon W_{21}^{(0)} & \varepsilon W_{22}^{(0)} \end{bmatrix} + \sum_{k=1}^{\infty} \frac{\varepsilon^k}{k!} \begin{bmatrix} W_{11}^{(k)} & \varepsilon W_{21}^{(k)T} \\ \varepsilon W_{21}^{(k)} & \varepsilon W_{22}^{(k)} \end{bmatrix}. \tag{28}$$

From (17b) and the result that $\|X_\varepsilon - X_\varepsilon^{(i)}\| = O(\varepsilon^{2^i})$, we have

$$(X_\varepsilon - X_\varepsilon^{(i)}) B_\varepsilon R^{-1} B_\varepsilon^T (X_\varepsilon - X_\varepsilon^{(i)}) = O(\varepsilon^{2^{i+1}}), \tag{29}$$

and, since $A_\varepsilon - B_\varepsilon R^{-1} B_\varepsilon^T X_\varepsilon^{(i)}$ are Hurwitz matrices from (17e), substituting (29) into (27) yields $W_{11}^{(k)} = 0$, $W_{21}^{(k)} = 0$, $W_{22}^{(k)} = 0$ ($k = 0, 1, 2, \ldots 2^{i+1} - 1$). Hence, $W_\varepsilon = O(\varepsilon^{2^{i+1}})$, which proves (25). ∎

6. AN ILLUSTRATIVE EXAMPLE

In order to demonstrate the efficiency of the proposed algorithm, we have run a simple example.

Consider the system (1) with

$$A_{11} = \begin{bmatrix} 0 & 0.4 \\ 0 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 0 \\ 0.345 & 0 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} 0 & -0.524 \\ 0 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0 & 0.262 \\ 0 & -1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1.0 \\ 0.0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.2 \\ 1.5 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.1 \end{bmatrix},$$

$$E_{a1} = \begin{bmatrix} 1.0 & 0.1 \\ 0 & 0.2192 \end{bmatrix}, \quad E_{a2} = \begin{bmatrix} 1.0 & 0 \\ 0 & 1.2031 \end{bmatrix}, \quad E_b = \begin{bmatrix} 1.0 \\ 0.5 \end{bmatrix},$$

$$Q = \begin{bmatrix} 1.0 & 0.5 & 0 & 1 \\ 0.5 & 1.25 & 1 & 2.5 \\ 0 & 1 & 1 & 2 \\ 1 & 2.5 & 2 & 5 \end{bmatrix}, \quad R = 1.$$

The numerical results are obtained for small parameters $\varepsilon = 10^{-4}$. Since $|A_{22}| = 0$, the system is a nonstandard singularly perturbed system. Furthermore, different from [6], the input matrix B_ε in (2) also contains the norm bounded uncertain. Thus, it is obvious that the existing method to find the composite stabilizing controller in [6] is not valid for this example. However, it is solvable by using the method of this paper.

For the system (1), two basic quantities are $\mu_f = 3.20494$, $\mu_s = 3.19766$, respectively. Therefore, for every boundary value $0 < \mu < \bar{\mu} = \min\{\mu_s, \ \mu_f\} = 3.19766$ the algebraic Riccati equation (3) has the positive definite stabilizing solution.

Using the one-line search algorithm, the best bound $\beta^* = 11.66974$ is obtained for $\mu^* = 2.5614$. The corresponding solution of the algebraic Riccati equation (3) is

$$P_\varepsilon^{(2)} = \begin{bmatrix} 3.3845999129 & 3.7019510648 & -9.4132041359 \times 10^{-5} & -1.4085605316 \times 10^{-4} \\ 3.7019510648 & 8.2846828247 & 1.3225605870 \times 10^{-4} & -9.1165808033 \times 10^{-5} \\ -9.4132041359 \times 10^{-5} & 1.3225605870 \times 10^{-4} & 3.2722523985 \times 10^{-4} & 4.5974263273 \times 10^{-5} \\ -1.4085605316 \times 10^{-4} & -9.1165808033 \times 10^{-5} & 4.5974263273 \times 10^{-5} & 1.2701719477 \times 10^{-4} \end{bmatrix}.$$

We find that the solutions of the algebraic Riccati equation (3) converge to the exact solution with accuracy of $\|F(P^{(i)})\| < 10^{-12}$ after 2 recursive iterations. In order to verify the exactitude of the solution, we calculate the remainder per iteration by substituting $P^{(i)}$ into the algebraic Riccati equation (9a). In Table I we present results for the error $\|F(P^{(i)})\|$. Note that $\|f(P_\varepsilon^{(i)})\| = \|F(P^{(i)})\|$, where $f(P_\varepsilon)$ is the left hand side of (3).

Note that the results of Table I show that the proposed algorithm (16a) has quadratic convergence because the initial condition (16b) is sufficiently

TABLE I
Errors per Iteration

i	$\|F(P^{(i)})\|$
0	1.29782×10^{-14}
1	1.79269×10^{-7}
2	1.27407×10^{-14}

close to the exact solution. On the other hand, the classical recursive algorithm [2, 3, 14] converges to the exact solution with accuracy of $\|P^{(i)}\| < 10^{-12}$ after 3 iterations. At this point, the numerical example illustrates the effectiveness of the proposed algorithm in comparison with [2, 3, 14] since the solution $P_\varepsilon^{(i)} = \Pi_\varepsilon P^{(i)}$ converges to the exact solution P_ε which is defined by (3). Furthermore, from Table II, it can be seen that the classical recursive algorithm converges very slowly for the ε which is not small. Therefore, this algorithm demands many more iterations than the proposed algorithm (16) in order to achieve high accuracy. It should be noted that the solution obtained by using the classical recursive algorithm is based on the zero-order solutions with the high-order accuracy, that is, 10^{-14}. Thus, this algorithm takes a lot of time since we have to obtain the zero-order solutions with the high-order accuracy. On the other hand, the proposed algorithm of this paper converges to the exact solution although the zero-order solutions are low-order accuracy. In fact, when we set the initial condition to the solutions

$$\overline{P}_{11} = \begin{bmatrix} 3.3843879280 & 3.7011936967 \\ 3.7011936967 & 8.2819271606 \end{bmatrix},$$

$$\overline{P}_{21} = \begin{bmatrix} -9.4096721841 \times 10^{-1} & 1.3220118597 \\ -1.4084795643 & -9.1133857289 \times 10^{-1} \end{bmatrix},$$

$$\overline{P}_{22} = \begin{bmatrix} 3.2709011976 & 4.5968491468 \times 10^{-1} \\ 4.5968491468 \times 10^{-1} & 1.2701069737 \end{bmatrix},$$

the classical recursive algorithm does not converge to the exact solution with accuracy of 10^{-12}, while the proposed algorithm converges to the exact solution with accuracy of 10^{-12} after 2 iterations.

For different values of ε, the number of iterations of the proposed algorithm versus the recursive algorithm is given in Table II with accuracy of $\|F(P^{(i)})\| < 10^{-12}$, where μ^* is defined by (8a) for each ε.

Finally, we choose $\mu = 2.5614$ and design the full-order controller based on this value of μ,

$$K = \begin{bmatrix} -0.99052 & -1.80160 & -1.16557 & -1.49993 \end{bmatrix}, \tag{30}$$

where $\varepsilon = 10^{-4}$.

TABLE II
Number of Iterations Such That $\|F(P^{(i)})\| < 10^{-12}$

ε	μ^*	Recursive algorithm	Proposed algorithm
10^{-2}	2.4451	11	3
10^{-3}	2.5504	5	3
10^{-4}	2.5614	3	2
10^{-5}	2.5625	2	2
10^{-6}	2.5626	2	2

7. CONCLUSIONS

In this paper, the guaranteed cost control problem for singularly perturbed systems has been investigated based on the iterative numerical technique. We presented the iterative algorithm under the special initial condition. Comparing with [2, 3], since the proposed algorithm has quadratic convergence, the required solution can be easily obtained up to an arbitrary order of accuracy, that is, $O(\varepsilon^{2^i})$, where i is a iteration number. Moreover, we have shown the existence of an $\tilde{\varepsilon}$ such that for all $0 < \varepsilon \leq \tilde{\varepsilon}$ the proposed algorithm has quadratic convergence.

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