

Asymptotic Expansions and a New Numerical Algorithm of the Algebraic Riccati Equation for Multiparameter Singularly Perturbed Systems

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In this paper we study a continuous-time multiparameter algebraic Riccati equation (MARE) with an indefinite sign quadratic term. The existence of a unique and bounded solution of the MARE is newly established. We show that the Kleinman algorithm can be used to solve the sign indefinite MARE. The proof of the convergence and the existence of the unique solution of the Kleinman algorithm is done by using the Newton–Kantorovich theorem. Furthermore, we present new algorithms for solving the generalized multiparameter algebraic Lyapunov equation (GMALE) by means of the fixed-point algorithm. © 2002 Elsevier Science (USA)

1. INTRODUCTION

The deterministic and stochastic multimodeling controls and their filtering problems have been investigated extensively by several researchers

(see e.g., [1–6]). The multimodeling problems arise in large-scale dynamic systems. For example, these multimodel situations in practice are illustrated by the multiarea power system [1] and the passenger car model [6]. In order to obtain the optimal solution to the multimodeling problems, we must solve the multiparameter algebraic Riccati equation (MARE), which is parameterized by two small, positive, same-order parameters ε_1 and ε_2 . Various reliable approaches to the theory of the ordinary algebraic Riccati equation (ARE) have been well documented in the literature (see, e.g., [7, 8]). One of these approaches is the invariant subspace approach, which is based on the Hamiltonian matrix. However, there is no guarantee of symmetry for the solution of the ARE when the ARE is known to be ill-conditioned [7]. Note that it is very hard to solve directly the singularly perturbed ARE and the MARE due to the presence of the small parameters [5, 6, 15, 18].

A popular approach to dealing with the multiparameter singularly-perturbed systems (MSPS) is the two-time-scale design method [1]. However, it is known from [5] that $O(\|\mu\|)$ (where $\mu = (\varepsilon_1, \varepsilon_2)$) accuracy is very often not sufficient because the reduced-order controller which is based on the two-time-scale design method might not produce satisfactory results for the desired performance. More recently, the exact slow-fast decomposition method for solving the MARE has been proposed in [5, 6]. The solutions are obtained by solving the Sylvester equations of lower dimensions, which are nonsymmetric equations by means of the Newton method or the fixed point algorithm. However, the results of [5, 6] need the assumption that the sign of the quadratic term of the MARE corresponding to both the optimal control and the filtering problem is positive semidefinite and that Hamiltonian matrices for the fast subsystems have no eigenvalues in common (Assumption 5 of [6]).

In this paper, we investigate the asymptotic expansions for the MARE with an indefinite sign quadratic term and propose the iterative technique for solving such a MARE. First, we relax the condition for the existence of the solution compared with [3] in the sense that some of the assumptions for the MARE are weakened. It is worth pointing out that the existence of a unique and bounded solution of the MARE with an indefinite sign quadratic term has not been established so far in the previous literature [3]. Furthermore, note that the MSPS with either standard or nonstandard singular perturbations [4] is considered. Second, we propose a new iterative algorithm for solving the sign-indefinite MARE. The method studied here is based on the Kleinman algorithm [9]. Therefore, the algorithm achieves the quadratic convergence property. Note that the difference between the results in [9] and the present paper is that the successive approximation technique is used to prove the convergence in [9], while the approach adopted here is composed of the Newton–Kantorovich theorem [10, 11].

Thus, we do not assume here that the sign of the quadratic term for the MARE is positive semidefinite. The Newton–Kantorovich theorem also plays an important role in the proof of the existence of the unique solution. The main objective in this paper is to provide a new algorithm for solving the generalized multiparameter algebraic Lyapunov equation (GMALE). The method presented in this paper is based on the fixed-point algorithm [14]. Consequently, our proposed algorithm is extremely useful since we have to solve only an algebraic Lyapunov equation (ALE) of lower dimension. In particular, it is important to note that so far the algorithm for solving the GMALE has not been established. Finally, a numerical example is given to complement the theoretical results. The resulting algorithms are implemented for the multiparameter H_∞ optimal-control problem.

2. PROBLEM FORMULATION AND PRIMARY RESULT

We consider the MARE

$$A_{\varepsilon}^T P_{\varepsilon} + P_{\varepsilon} A_{\varepsilon} - P_{\varepsilon} S_{\varepsilon} P_{\varepsilon} + Q = 0, \quad (1)$$

where

$$\begin{aligned} P_{\varepsilon} &= \begin{bmatrix} P_{00} & \varepsilon_1 P_{10}^T & \varepsilon_2 P_{20}^T \\ \varepsilon_1 P_{10} & \varepsilon_1 P_{11} & \sqrt{\varepsilon_1 \varepsilon_2} P_{21}^T \\ \varepsilon_2 P_{20} & \sqrt{\varepsilon_1 \varepsilon_2} P_{21} & \varepsilon_2 P_{22} \end{bmatrix} \in \mathbf{R}^{N \times N}, \\ P_{00} &= P_{00}^T, \quad P_{11} = P_{11}^T, \quad P_{22} = P_{22}^T, \\ A_{\varepsilon} &= \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ \varepsilon_1^{-1} A_{10} & \varepsilon_1^{-1} A_{11} & 0 \\ \varepsilon_2^{-1} A_{20} & 0 & \varepsilon_2^{-1} A_{22} \end{bmatrix} \in \mathbf{R}^{N \times N}, \\ S_{\varepsilon} &= S_{\varepsilon}^T = \begin{bmatrix} S_{00} & \varepsilon_1^{-1} S_{01} & \varepsilon_2^{-1} S_{02} \\ \varepsilon_1^{-1} S_{01}^T & \varepsilon_1^{-2} S_{11} & 0 \\ \varepsilon_2^{-1} S_{02}^T & 0 & \varepsilon_2^{-2} S_{22} \end{bmatrix} \in \mathbf{R}^{N \times N}, \\ S_{00} &= S_{00}^T, \quad S_{11} = S_{11}^T, \quad S_{22} = S_{22}^T, \\ Q &= Q^T = \begin{bmatrix} Q_{00} & Q_{01} & Q_{02} \\ Q_{01}^T & Q_{11} & 0 \\ Q_{02}^T & 0 & Q_{22} \end{bmatrix} \in \mathbf{R}^{N \times N}, \end{aligned}$$

$$\begin{aligned}
Q_{00} &= Q_{00}^T, & Q_{11} &= Q_{11}^T, & Q_{22} &= Q_{22}^T, \\
P_{00}, A_{00}, S_{00}, Q_{00} &\in \mathbf{R}^{n_0 \times n_0}, & P_{11}, A_{11}, S_{11}, Q_{11} &\in \mathbf{R}^{n_1 \times n_1}, \\
P_{22}, A_{22}, S_{22}, Q_{22} &\in \mathbf{R}^{n_2 \times n_2}, & \varepsilon_1 > 0, \varepsilon_2 > 0, & N = n_0 + n_1 + n_2.
\end{aligned}$$

If the sign of the MARE (1) is positive semidefinite, then the equation (1) is known as a regulator ARE, appearing in the multimodeling [3]. However, we do not assume in this paper that the sign of the MARE (1) is positive semidefinite. That is, no assumption is made on the definiteness of S_{ε} . In addition, we do not assume here that A_{11} and A_{22} are nonsingular compared with [1, 3].

In order to avoid the ill-conditioning due to the large parameter ε_j^{-1} which is included in the MARE (1), we introduce the following useful lemma.

LEMMA 2.1. *The MARE (1) is equivalent to the generalized multiparameter algebraic Riccati equation (GMARE) (2a),*

$$A^T P + P^T A - P^T S P + Q = 0, \quad (2a)$$

$$P_{\varepsilon} = \Phi_{\varepsilon} P = P^T \Phi_{\varepsilon}, \quad (2b)$$

where

$$\begin{aligned}
\Phi_{\varepsilon} &= \begin{bmatrix} I_{n_0} & 0 & 0 \\ 0 & \varepsilon_1 I_{n_1} & 0 \\ 0 & 0 & \varepsilon_2 I_{n_2} \end{bmatrix}, & A &= \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & 0 \\ A_{20} & 0 & A_{22} \end{bmatrix}, \\
S &= \begin{bmatrix} S_{00} & S_{01} & S_{02} \\ S_{01}^T & S_{11} & 0 \\ S_{02}^T & 0 & S_{22} \end{bmatrix}, & P &= \begin{bmatrix} P_{00} & \varepsilon_1 P_{10}^T & \varepsilon_2 P_{20}^T \\ P_{10} & P_{11} & \frac{1}{\sqrt{\alpha}} P_{21}^T \\ P_{20} & \sqrt{\alpha} P_{21} & P_{22} \end{bmatrix}, & \alpha &= \frac{\varepsilon_1}{\varepsilon_2}.
\end{aligned}$$

Proof. First, by direct calculation we verify that $P_{\varepsilon} = \Phi_{\varepsilon} P$. Second, it is easy to verify that $A = \Phi_{\varepsilon} A_{\varepsilon}$, $S = \Phi_{\varepsilon} S_{\varepsilon} \Phi_{\varepsilon}$. Hence,

$$A^T P = A_{\varepsilon}^T \Phi_{\varepsilon} \Phi_{\varepsilon}^{-1} P_{\varepsilon} = A_{\varepsilon}^T P_{\varepsilon}.$$

By using a similar calculation, we can immediately rewrite (1) as (2a). ■

ε_1 and ε_2 are two small positive singular perturbation parameters of the same order of magnitude such that,

$$0 < k_1 \leq \alpha = \frac{\varepsilon_1}{\varepsilon_2} \leq k_2 < \infty. \quad (3)$$

It is assumed that the limit of α exists as ε_1 and ε_2 tend to zero; that is,

$$\bar{\alpha} = \lim_{\substack{\varepsilon_1 \rightarrow +0 \\ \varepsilon_2 \rightarrow +0}} \alpha.$$

The GMARE (2a) can be partitioned into

$$\begin{aligned} f_1 = & A_{00}^T P_{00} + P_{00} A_{00} + A_{10}^T P_{10} + P_{10}^T A_{10} + A_{20}^T P_{20} + P_{20}^T A_{20} \\ & - P_{00} S_{00} P_{00} - P_{10}^T S_{01}^T P_{00} - P_{00} S_{01} P_{10} \\ & - P_{20}^T S_{02}^T P_{00} - P_{00} S_{02} P_{20} - P_{10}^T S_{11} P_{10} - P_{20}^T S_{22} P_{20} + Q_{00} = 0, \end{aligned} \quad (4a)$$

$$\begin{aligned} f_2 = & P_{00} A_{01} + P_{10}^T A_{11} + \varepsilon_1 A_{00}^T P_{10}^T + A_{10}^T P_{11} + \sqrt{\alpha} A_{20}^T P_{21} \\ & - \varepsilon_1 (P_{00} S_{00} P_{10}^T + P_{10}^T S_{01}^T P_{10}^T + P_{20}^T S_{02}^T P_{10}^T) - P_{00} S_{01} P_{11} - P_{10}^T S_{11} P_{11} \\ & - \sqrt{\alpha} (P_{00} S_{02} P_{21} + P_{20}^T S_{22} P_{21}) + Q_{01} = 0, \end{aligned} \quad (4b)$$

$$\begin{aligned} f_3 = & P_{00} A_{02} + P_{20}^T A_{22} + \varepsilon_2 A_{00}^T P_{20}^T + A_{20}^T P_{22} + \frac{1}{\sqrt{\alpha}} A_{10}^T P_{21}^T \\ & - \varepsilon_2 (P_{00} S_{00} P_{20}^T + P_{10}^T S_{01}^T P_{20}^T + P_{20}^T S_{02}^T P_{20}^T) - P_{00} S_{02} P_{22} - P_{20}^T S_{22} P_{22} \\ & - \frac{1}{\sqrt{\alpha}} (P_{00} S_{01} P_{21}^T + P_{10}^T S_{11} P_{21}^T) + Q_{02} = 0, \end{aligned} \quad (4c)$$

$$\begin{aligned} f_4 = & A_{11}^T P_{11} + P_{11} A_{11} + \varepsilon_1 (A_{01}^T P_{10}^T + P_{10} A_{01}) \\ & - \varepsilon_1 (\varepsilon_1 P_{10} S_{00} P_{10}^T + P_{11} S_{01}^T P_{10}^T + \sqrt{\alpha} P_{21}^T S_{02}^T P_{10}^T) \\ & - \varepsilon_1 (P_{10} S_{01} P_{11} + \sqrt{\alpha} P_{10} S_{02} P_{21}) \\ & - P_{11} S_{11} P_{11} - \alpha P_{21}^T S_{22} P_{21} + Q_{11} = 0, \end{aligned} \quad (4d)$$

$$\begin{aligned} f_5 = & \varepsilon_1 P_{10} A_{02} + \varepsilon_2 A_{01}^T P_{20}^T - \varepsilon_1 \varepsilon_2 P_{10} S_{00} P_{20}^T \\ & - \varepsilon_2 \left(P_{11} S_{01}^T P_{20}^T + \sqrt{\alpha} P_{21}^T S_{02}^T P_{20}^T \right) - \varepsilon_1 (P_{10} S_{02} P_{22} + \frac{1}{\sqrt{\alpha}} P_{10} S_{01} P_{21}^T) \\ & + \sqrt{\alpha} P_{21}^T (A_{22} - S_{22} P_{22}) + \frac{1}{\sqrt{\alpha}} (A_{11} - S_{11} P_{11})^T P_{21}^T = 0, \end{aligned} \quad (4e)$$

and

$$\begin{aligned} f_6 = & A_{22}^T P_{22} + P_{22} A_{22} + \varepsilon_2 (A_{02}^T P_{20}^T + P_{20} A_{02}) \\ & - \varepsilon_2 \left(\varepsilon_2 P_{20} S_{00} P_{20}^T + P_{22} S_{02}^T P_{20}^T + \frac{1}{\sqrt{\alpha}} P_{21} S_{01}^T P_{20}^T \right) \\ & - \varepsilon_2 \left(P_{20} S_{02} P_{22} + \frac{1}{\sqrt{\alpha}} P_{20} S_{01} P_{21}^T \right) \\ & - P_{22} S_{22} P_{22} - \frac{1}{\alpha} P_{21} S_{11} P_{21}^T + Q_{22} = 0. \end{aligned} \quad (4f)$$

By limiting solutions of the GMARE (2a) or (4) as $\varepsilon_1 \rightarrow +0$ and $\varepsilon_2 \rightarrow +0$, we obtain the equations

$$\begin{aligned} & A_{00}^T \bar{P}_{00} + \bar{P}_{00} A_{00} + A_{10}^T \bar{P}_{10} + \bar{P}_{10}^T A_{10} + A_{20}^T \bar{P}_{20} + \bar{P}_{20}^T A_{20} \\ & - \bar{P}_{00} S_{00} \bar{P}_{00} - \bar{P}_{10}^T S_{01}^T \bar{P}_{00} - \bar{P}_{00} S_{01} \bar{P}_{10} - \bar{P}_{20}^T S_{02}^T \bar{P}_{00} \\ & - \bar{P}_{00} S_{02} \bar{P}_{20} - \bar{P}_{10}^T S_{11}^T \bar{P}_{10} - \bar{P}_{20}^T S_{22} \bar{P}_{20} + Q_{00} = 0, \end{aligned} \quad (5a)$$

$$\begin{aligned} & \bar{P}_{00} A_{01} + \bar{P}_{10}^T A_{11} + A_{10}^T \bar{P}_{11} + \sqrt{\bar{\alpha}} A_{20}^T \bar{P}_{21} - \bar{P}_{00} S_{01} \bar{P}_{11} \\ & - \bar{P}_{10}^T S_{11} \bar{P}_{11} - \sqrt{\bar{\alpha}} (\bar{P}_{00} S_{02} \bar{P}_{21} + \bar{P}_{20}^T S_{22} \bar{P}_{21}) + Q_{01} = 0, \end{aligned} \quad (5b)$$

$$\begin{aligned} & \bar{P}_{00} A_{02} + \bar{P}_{20}^T A_{22} + A_{20}^T \bar{P}_{22} + \frac{1}{\sqrt{\bar{\alpha}}} A_{10}^T \bar{P}_{21}^T - \bar{P}_{00} S_{02} \bar{P}_{22} \\ & - \bar{P}_{20}^T S_{22} \bar{P}_{22} - \frac{1}{\sqrt{\bar{\alpha}}} (\bar{P}_{00} S_{01} \bar{P}_{21}^T + \bar{P}_{10}^T S_{11} \bar{P}_{21}^T) + Q_{02} = 0, \end{aligned} \quad (5c)$$

$$A_{11}^T \bar{P}_{11} + \bar{P}_{11} A_{11} - \bar{P}_{11} S_{11} \bar{P}_{11} - \bar{\alpha} \bar{P}_{21}^T S_{22} \bar{P}_{21} + Q_{11} = 0, \quad (5d)$$

$$\sqrt{\bar{\alpha}} \bar{P}_{21}^T (A_{22} - S_{22} \bar{P}_{22}) + \frac{1}{\sqrt{\bar{\alpha}}} (A_{11} - S_{11} \bar{P}_{11})^T \bar{P}_{21}^T = 0, \quad (5e)$$

and

$$A_{22}^T \bar{P}_{22} + \bar{P}_{22} A_{22} - \bar{P}_{22} S_{22} \bar{P}_{22} - \frac{1}{\bar{\alpha}} \bar{P}_{21} S_{11} \bar{P}_{21}^T + Q_{22} = 0. \quad (5f)$$

where $\bar{P}_{00}$, $\bar{P}_{10}$, $\bar{P}_{20}$, $\bar{P}_{11}$, $\bar{P}_{21}$, and $\bar{P}_{22}$ are the 0-order solutions of the GMARE (2a).

We shall set the following basic condition without loss of generality [15].

(H1) The AREs $A_{jj}^T \tilde{P}_{jj} + \tilde{P}_{jj} A_{jj} - \tilde{P}_{jj} S_{jj} \tilde{P}_{jj} + Q_{jj} = 0$, $j = 1, 2$, have positive semidefinite stabilizing solutions.

If Condition (H1) holds, there exist matrices $\tilde{P}_{jj}$, $j = 1, 2$, such that the matrices $A_{jj} - S_{jj} \tilde{P}_{jj}$, $j = 1, 2$, are stable. Therefore, we chose the solutions $\bar{P}_{jj}$, $j = 1, 2$, as $\tilde{P}_{jj}$, $j = 1, 2$. Then, the unique solution of (5e) is given by $\bar{P}_{21} = 0$ because the matrices $A_{jj} - S_{jj} \bar{P}_{jj} = A_{jj} - S_{jj} \tilde{P}_{jj}$ are stable. As a consequence, the parameter $\bar{\alpha}$ does not appear in (5) automatically; that is, it does not affect the equation (5) in the limit when ε_1 and ε_2 tend to zero. Thus the AREs (5d) and (5f) will produce the unique positive semidefinite stabilizing solution under the conditions (H1).

We now obtain the 0-order equations

$$A_s^T \bar{P}_{00} + \bar{P}_{00} A_s - \bar{P}_{00} S_s \bar{P}_{00} + Q_s = 0, \quad (6a)$$

$$\bar{P}_{j0}^T = \bar{P}_{00} N_{0j} - M_{0j}, \quad j = 1, 2, \quad (6b)$$

and

$$A_{jj}^T \bar{P}_{jj} + \bar{P}_{jj} A_{jj} - \bar{P}_{jj} S_{jj} \bar{P}_{jj} + Q_{jj} = 0, \quad j = 1, 2, \quad (6c)$$

where

$$\begin{aligned} A_s &= A_{00} + N_{01} A_{10} + N_{02} A_{20} + S_{01} M_{01}^T + S_{02} M_{02}^T \\ &\quad + N_{01} S_{11} M_{01}^T + N_{02} S_{22} M_{02}^T, \end{aligned}$$

$$\begin{aligned} S_s &= S_{00} + N_{01} S_{01}^T + S_{01} N_{01}^T + N_{02} S_{02}^T + S_{02} N_{02}^T \\ &\quad + N_{01} S_{11} N_{01}^T + N_{02} S_{22} N_{02}^T, \end{aligned}$$

$$\begin{aligned} Q_s &= Q_{00} - M_{01} A_{10} - A_{10}^T M_{01}^T - M_{02} A_{20} - A_{20}^T M_{02}^T \\ &\quad - M_{01} S_{11} M_{01}^T - M_{02} S_{22} M_{02}^T, \end{aligned}$$

$$N_{0j} = -D_{0j} D_{jj}^{-1}, \quad M_{0j} = \bar{Q}_{0j} D_{jj}^{-1}, \quad \bar{Q}_{0j} = A_{j0}^T \bar{P}_{jj} + Q_{0j},$$

$$D_{00} = A_{00} - S_{00} \bar{P}_{00} - S_{01} \bar{P}_{10} - S_{02} \bar{P}_{20}, \quad D_{0j} = A_{0j} - S_{0j} \bar{P}_{jj},$$

$$D_{j0} = A_{j0} - S_{0j}^T \bar{P}_{00} - S_{jj} \bar{P}_{j0}, \quad D_{jj} = A_{jj} - S_{jj} \bar{P}_{jj}, \quad j = 1, 2.$$

The matrices A_s , S_s , and Q_s do not depend on $\bar{P}_{jj}$, $j = 1, 2$, because their matrices can be computed using T_{pq} , $p, q = 0, 1, 2$, which is independent of $\bar{P}_{jj}$, $j = 1, 2$ ([5, 6]); that is,

$$\begin{aligned} T_s &= T_{00} - T_{01} T_{11}^{-1} T_{10} - T_{02} T_{22}^{-1} T_{20} = \begin{bmatrix} A_s & -S_s \\ -Q_s & -A_s^T \end{bmatrix}, \\ T_{00} &= \begin{bmatrix} A_{00} & -S_{00} \\ -Q_{00} & -A_{00}^T \end{bmatrix}, \quad T_{0j} = \begin{bmatrix} A_{0j} & -S_{0j} \\ -Q_{0j} & -A_{j0}^T \end{bmatrix}, \\ T_{j0} &= \begin{bmatrix} A_{j0} & -S_{0j}^T \\ -Q_{0j}^T & -A_{0j}^T \end{bmatrix}, \quad T_{jj} = \begin{bmatrix} A_{jj} & -S_{jj} \\ -Q_{jj} & -A_{jj}^T \end{bmatrix}, \quad j = 1, 2. \end{aligned}$$

Note that the Hamiltonian matrices

$$T_{jj} := \begin{bmatrix} A_{jj} & -S_{jj} \\ -Q_{jj} & -A_{jj}^T \end{bmatrix}, \quad j = 1, 2,$$

are nonsingular under the condition (H1) because of

$$\begin{aligned} T_{jj} &= \begin{bmatrix} I_{n_j} & 0 \\ \bar{P}_{jj}^T & I_{n_j} \end{bmatrix} \begin{bmatrix} D_{jj} & -S_{jj} \\ 0 & -D_{jj}^T \end{bmatrix} \begin{bmatrix} I_{n_j} & 0 \\ -\bar{P}_{jj} & I_{n_j} \end{bmatrix} \\ \iff T_{jj}^{-1} &= \begin{bmatrix} I_{n_j} & 0 \\ \bar{P}_{jj} & I_{n_j} \end{bmatrix} \begin{bmatrix} D_{jj}^{-1} & -D_{jj}^{-1} S_{jj} D_{jj}^{-T} \\ 0 & -D_{jj}^{-T} \end{bmatrix} \begin{bmatrix} I_{n_j} & 0 \\ -\bar{P}_{jj}^T & I_{n_j} \end{bmatrix}. \end{aligned}$$

The required solution of the ARE (6a) exists under the following condition [15].

(H2) The ARE (6a) has positive semidefinite stabilizing solution.

It should be remarked that the solution P_{ε} of (1) is a function of the multiparameters ε_1 and ε_2 . However, the solutions $\bar{P}_{00}$ and $\bar{P}_{jj}$, $j = 1, 2$, of (6a) and (6c) are independent of the multiparameters ε_1 and ε_2 , respectively. The following theorem will establish the relation between P_{ε} and the reduced-order solutions (6) (see [3]).

THEOREM 2.1. *Under the conditions (H1) and (H2), there exist small ε_1^* and ε_2^* such that, for all $\varepsilon_1 \in (0, \varepsilon_1^*)$ and $\varepsilon_2 \in (0, \varepsilon_2^*)$, the MARE (1) admits a symmetric positive semidefinite stabilizing solution P_{ε} which can be written as*

$$P_{\varepsilon} = \begin{bmatrix} \bar{P}_{00} + \mathcal{F}_{00} & \varepsilon_1(\bar{P}_{10} + \mathcal{F}_{10})^T & \varepsilon_2(\bar{P}_{20} + \mathcal{F}_{20})^T \\ \varepsilon_1(\bar{P}_{10} + \mathcal{F}_{10}) & \varepsilon_1(\bar{P}_{11} + \mathcal{F}_{11}) & \sqrt{\varepsilon_1 \varepsilon_2} \mathcal{F}_{21}^T \\ \varepsilon_2(\bar{P}_{20} + \mathcal{F}_{20}) & \sqrt{\varepsilon_1 \varepsilon_2} \mathcal{F}_{21} & \varepsilon_1(\bar{P}_{22} + \mathcal{F}_{22}) \end{bmatrix}, \quad (7)$$

where

$$\mathcal{F}_{pq} = O(\|\mu\|), \quad pq = 00, 10, 20, 11, 21, 22.$$

In order to prove Theorem 2.1, we need the following lemma [1].

LEMMA 2.2. *Consider the system*

$$\begin{aligned} \dot{x}_0(t) &= A_{00}x_0(t) + A_{01}x_1(t) + A_{02}x_2(t), & x_0(t_0) &= x_0^0, \\ \varepsilon_1 \dot{x}_1(t) &= A_{10}x_0(t) + A_{11}x_1(t) + \varepsilon_3 A_{12}x_2(t), & x_1(t_0) &= x_1^0, \end{aligned}$$

and

$$\varepsilon_2 \dot{x}_2(t) = A_{20}x_0(t) + \varepsilon_3 A_{21}x_1(t) + A_{22}x_2(t), \quad x_2(t_0) = x_2^0,$$

where $x_0 \in \mathbf{R}^{n_0}$, $x_1 \in \mathbf{R}^{n_1}$, and $x_2 \in \mathbf{R}^{n_2}$ are the state vectors. ε_3 is a small weak coupling parameter and ε_1 and ε_2 are small positive singular perturbation parameters of the same order of magnitude as (3). If A_{jj}^{-1} , $j = 1, 2$, exists, and if $A_0 \equiv A_{00} - A_{01}A_{11}^{-1}A_{10} - A_{02}A_{22}^{-1}A_{20}$, A_{jj} , $j = 1, 2$, are stable matrices, then there exist small $\hat{\varepsilon}_1$ and $\hat{\varepsilon}_2$ such that for all $\varepsilon_1 \in (0, \hat{\varepsilon}_1)$ and $\varepsilon_2 \in (0, \hat{\varepsilon}_2)$ the system is asymptotically stable.

Now, let us prove Theorem 2.1.

Proof. Since the MARE (1) is equivalent to the GMARE (2a) from Lemma 2.1, we apply the implicit function theorem [3] to (2a). To do so, it is enough to show that the corresponding Jacobian is nonsingular at $\varepsilon_1 = 0$ and $\varepsilon_2 = 0$. It can be shown, after some algebra, that the Jacobian of (2a)

in the limit is given by

$$J = \nabla \mathbf{F} = \frac{\partial \text{vec}(f_1, f_2, f_3, f_4, f_5, f_6)}{\partial \text{vec}(P_{00}, P_{10}, P_{20}, P_{11}, P_{21}, P_{22})^T} \Big|_{(\mu, \mathcal{P})=(\mu_0, \mathcal{P}_0)} \\ = \begin{bmatrix} J_{00} & J_{01} & J_{02} & 0 & 0 & 0 \\ J_{10} & J_{11} & 0 & J_{13} & J_{14} & 0 \\ J_{20} & 0 & J_{22} & 0 & J_{24} & J_{25} \\ 0 & 0 & 0 & J_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & J_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & J_{55} \end{bmatrix}, \quad (8)$$

where vec denotes an ordered stack of the columns of its matrix [12] and

$$\mu = (\varepsilon_1, \varepsilon_2), \quad \mu_0 = (0, 0),$$

$$\mathcal{P} = (P_{00}, P_{10}, P_{20}, P_{11}, P_{21}, P_{22}),$$

$$\mathcal{P}_0 = (\bar{P}_{00}, \bar{P}_{10}, \bar{P}_{20}, \bar{P}_{11}, 0, \bar{P}_{22}),$$

$$J_{00} = (I_{n_0} \otimes D_{00}^T)U_{n_0 n_0} + D_{00}^T \otimes I_{n_0},$$

$$J_{0j} = (I_{n_0} \otimes D_{j0}^T)U_{n_0 n_j} + D_{j0}^T \otimes I_{n_0},$$

$$J_{j0} = D_{0j}^T \otimes I_{n_0}, \quad J_{jj} = D_{jj}^T \otimes I_{n_0}, \quad j = 1, 2,$$

$$J_{13} = I_{n_1} \otimes D_{10}, \quad J_{14} = \sqrt{\bar{\alpha}}(I_{n_1} \otimes D_{20})U_{n_1 n_2},$$

$$J_{24} = \frac{1}{\sqrt{\bar{\alpha}}}I_{n_2} \otimes D_{10}, \quad J_{25} = I_{n_2} \otimes D_{20},$$

$$J_{33} = (I_{n_1} \otimes D_{11}^T)U_{n_1 n_1} + D_{11}^T \otimes I_{n_1},$$

$$J_{44} = \sqrt{\bar{\alpha}}D_{22}^T \otimes I_{n_1} + \frac{1}{\sqrt{\bar{\alpha}}}I_{n_2} \otimes D_{11}^T,$$

and

$$J_{55} = (I_{n_2} \otimes D_{22}^T)U_{n_2 n_2} + D_{22}^T \otimes I_{n_2},$$

where $\otimes$ denotes the Kronecker products and $U_{n_j n_j}$, $j = 0, 1, 2$, is the permutation matrix in the Kronecker matrix sense [12].

The Jacobian (8) can be expressed as

$$\det J = \det J_{33} \cdot \det J_{44} \cdot \det J_{55} \cdot \det \begin{bmatrix} J_{00} & J_{01} & J_{02} \\ J_{10} & J_{11} & 0 \\ J_{20} & 0 & J_{22} \end{bmatrix} \\ = \det J_{33} \cdot \det J_{44} \cdot \det J_{55} \cdot \det J_{11} \cdot \det J_{22}$$

$$\begin{aligned}
& \cdot \det(J_{00} - J_{01}J_{11}^{-1}J_{10} - J_{02}J_{22}^{-1}J_{20}) \\
& = \det J_{11} \cdot \det J_{22} \cdot \det J_{33} \cdot \det J_{44} \cdot \det J_{55} \\
& \quad \cdot \det[I_{n_0} \otimes D_0^T U_{n_0 n_0} + D_0^T \otimes I_{n_0}], \tag{9}
\end{aligned}$$

where $D_0 \equiv D_{00} - D_{01}D_{11}^{-1}D_{10} - D_{02}D_{22}^{-1}D_{20}$. Obviously, J_{jj} , $j = 1, \dots, 5$, are nonsingular because the matrices $D_{jj} = A_{jj} - S_{jj}\bar{P}_{jj}$, $j = 1, 2$, are nonsingular under the condition (H1). After some straightforward but tedious algebra we see that $A_s - S_s\bar{P}_{00} = D_{00} - D_{01}D_{11}^{-1}D_{10} - D_{02}D_{22}^{-1}D_{20} = D_0$. Therefore, the matrix D_0 is nonsingular if the condition (H2) holds. Thus, $\det J \neq 0$; i.e., J is nonsingular at $(\mu, \mathcal{P}) = (\mu_0, \mathcal{P}_0)$. The conclusion of the first part of Theorem 2.1 is obtained directly by using the implicit function theorem. The second part of the proof of Theorem 2.1 is performed by direct calculation. By using (7), we obtain

$$\Phi_{\varepsilon}^{-1}(A - SP) = \Phi_{\varepsilon}^{-1} \left(\begin{bmatrix} D_{00} & D_{01} & D_{02} \\ D_{10} & D_{11} & 0 \\ D_{20} & 0 & D_{22} \end{bmatrix} + O(\|\mu\|) \right).$$

We know from Lemma 2.2 that for sufficiently small $\|\mu\|$ the matrix $\Phi_{\varepsilon}^{-1}(A - SP)$ will be stable. On the other hand, since $\bar{P}_{00} \geq 0$, $\bar{P}_{11} \geq 0$ and $\bar{P}_{22} \geq 0$, P_{ε} is positive semidefinite as long as $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ by using the Schur complement [13]. Therefore, the proof of Theorem 2.1 ends. ■

3. ITERATIVE ALGORITHM

We now develop an algorithm which converges quadratically to the required solution of the MARE (1). So far, the exact decomposition method for solving the MARE with a positive semidefinite sign quadratic term has been proposed in [5, 6]. However, the result of [5, 6] needs the assumption that Hamiltonian matrices for the fast subsystems have no eigenvalues in common.

In this paper we develop an elegant and simple algorithm which converges globally to the positive semidefinite solution of the MARE (1). Taking into account the fact that the MARE (1) is equivalent to the GMARE (2a) from Lemma 2.1, the algorithm is given in terms of the GMALE, which has to be solved iteratively. We present the iterative algorithm based on the Kleinman algorithm [9]. Here we note that the Kleinman algorithm is based on the Newton-type algorithm. In general, the stabilizable–detectable conditions will guarantee the convergence of the Kleinman algorithm for the standard linear-quadratic regulator-type GMARE to the required solutions. However, there is no guarantee of quadratic convergence for the Kleinman algorithm (2a) because the matrix S is in general indefinite [16, 18].

In this paper we show that by using the Newton–Kantorovich theorem [10, 11] the Kleinman algorithm guarantees the quadratic convergence under the appropriate initial conditions.

We propose an algorithm for solving the GMARE (2a),

$$(A - SP^{(i)})^T P^{(i+1)} + P^{(i+1)T} (A - SP^{(i)}) + P^{(i)T} SP^{(i)} + Q = 0 \quad (10a)$$

$$P_{\mathcal{E}}^{(i)} = \Phi_{\mathcal{E}} P^{(i)} = P^{(i)T} \Phi_{\mathcal{E}}, \quad (10b)$$

$i = 0, 1, 2, 3, \dots$, with the initial condition obtained from

$$P^{(0)} = \begin{bmatrix} \bar{P}_{00} & 0 & 0 \\ \bar{P}_{10} & \bar{P}_{11} & 0 \\ \bar{P}_{20} & 0 & \bar{P}_{22} \end{bmatrix}, \quad (11)$$

where

$$P^{(i)} = \begin{bmatrix} P_{00}^{(i)} & \varepsilon_1 P_{10}^{(i)T} & \varepsilon_2 P_{20}^{(i)T} \\ P_{10}^{(i)} & P_{11}^{(i)} & \frac{1}{\sqrt{\alpha}} P_{21}^{(i)T} \\ P_{20}^{(i)} & \sqrt{\alpha} P_{21}^{(i)} & P_{22}^{(i)} \end{bmatrix},$$

$$P_{00}^{(i)} = P_{00}^{(i)T}, \quad P_{11}^{(i)} = P_{11}^{(i)T}, \quad P_{22}^{(i)} = P_{22}^{(i)T},$$

and $\bar{P}_{pq}$, $p, q = 00, 10, 20, 11, 22$, are defined by (6).

According to the Newton–Kantorovich theorem [10, 11], it is well known that if the initial condition is very close to the exact solution of the considered equation, the Newton method has the quadratic convergence property. Therefore, we choose the proposal for good choice of the initial conditions as in (11).

Although the sign of the matrix S is in general indefinite, we can prove the quadratic convergence for the resulting algorithm (10) by using the Newton–Kantorovich theorem because the initial condition is very close to the exact solution of the GMARE (2a) for sufficiently small $\|\mu\|$. This idea is derived from the fact that

$$\|P - P^{(0)}\| = \left\| \begin{bmatrix} P_{00} & \varepsilon_1 P_{10}^T & \varepsilon_2 P_{20}^T \\ P_{10} & P_{11} & \frac{1}{\sqrt{\alpha}} P_{21}^T \\ P_{20} & \sqrt{\alpha} P_{21} & P_{22} \end{bmatrix} - \begin{bmatrix} \bar{P}_{00} & 0 & 0 \\ \bar{P}_{10} & \bar{P}_{11} & 0 \\ \bar{P}_{20} & 0 & \bar{P}_{22} \end{bmatrix} \right\|$$

$$= O(\|\mu\|).$$

The algorithm (10) has the feature given in the following lemma.

LEMMA 3.1. *Under the conditions (H1) and (H2), there exist an $\bar{\varepsilon}_1$ and an $\bar{\varepsilon}_2$ such that for all $0 < \varepsilon_1 \leq \bar{\varepsilon}_1 \leq \varepsilon_1^*$ and $0 < \varepsilon_2 \leq \bar{\varepsilon}_2 \leq \varepsilon_2^*$, respectively, the iterative algorithm (10) converges to the exact solution of P^* with the rate of quadratic convergence. Then, $P_{\varepsilon}^{(i)} = \Phi_{\varepsilon} P^{(i)} = P^{(i)T} \Phi_{\varepsilon}$ is positive semidefinite. Moreover, there exists a unique solution of the GMARE (2a) with the indefinite sign quadratic term in the neighborhood of the required solution P^* . That is, the conditions*

$$\|P^{(i)} - P^*\| \leq \frac{O(\|\mu\|^{2^i})}{2^i \beta \gamma} = O(\|\mu\|^{2^i}), \quad i = 0, 1, 2, \dots, \quad (12a)$$

$$P_{\varepsilon}^{(i)} = \Phi_{\varepsilon} P^{(i)} = P^{(i)T} \Phi_{\varepsilon} \geq 0, \quad i = 1, 2, 3, \dots, \quad (12b)$$

and

$$\|P^{(0)} - P^*\| \leq \frac{1}{\beta \gamma} [1 - \sqrt{1 - 2\theta}], \quad (12c)$$

where

$$\mathcal{G}(P) = A^T P + P^T A - P^T S P + Q, \quad (13)$$

$$\gamma = 2\|S\| < \infty, \beta = \|\nabla \mathcal{G}(P^{(0)})\|^{-1}, \eta = \beta \cdot \|\mathcal{G}(P^{(0)})\|, \theta = \beta \eta \gamma,$$

$$\nabla \mathcal{G}(P) = \frac{\partial \text{vec} \mathcal{G}(P)}{\partial (\text{vec} P)^T}, \text{ and } P^* = \begin{bmatrix} P_{00}^* & \varepsilon_1 P_{10}^{*T} & \varepsilon_2 P_{20}^{*T} \\ P_{10}^* & P_{11}^* & \frac{1}{\sqrt{\alpha}} P_{21}^{*T} \\ P_{20}^* & \sqrt{\alpha} P_{21}^* & P_{22}^* \end{bmatrix},$$

are satisfied.

Proof. This proof is equivalent to the proof of existence of the unique solution for the GMARE (2a). Thus, the proof follows directly by applying the Newton–Kantorovich theorem [10, 11] for the GMARE (2a). We now verify that function $\mathcal{G}(P)$ is differentiable on a convex set $\mathcal{D}$. Using the fact that

$$\nabla \mathcal{G}(P) = (A - SP)^T \otimes I_N + I_N \otimes (A - SP)^T, \quad (14)$$

we have

$$\|\nabla \mathcal{G}(P_1) - \nabla \mathcal{G}(P_2)\| \leq \gamma \|P_1 - P_2\|, \quad (15)$$

where $\gamma = 2\|S\|$. Moreover, using the fact that

$$\nabla \mathcal{G}(P^{(0)}) = \begin{bmatrix} D_{00} & D_{01} & D_{02} \\ D_{10} & D_{11} & 0 \\ D_{20} & 0 & D_{22} \end{bmatrix}^T \otimes I_N + I_N \otimes \begin{bmatrix} D_{00} & D_{01} & D_{02} \\ D_{10} & D_{11} & 0 \\ D_{20} & 0 & D_{22} \end{bmatrix}^T, \quad (16)$$

it follows that $\nabla \mathcal{G}(P^{(0)})$ is nonsingular because D_0 and D_{jj} , $j = 1, 2$, are stable under the conditions (H1) and (H2). Therefore, there exists a β such that $\|[\nabla \mathcal{G}(P^{(0)})]^{-1}\| \equiv \beta$. On the other hand, since $\mathcal{G}(P^{(0)}) < O(\|\mu\|)$, there exists an η such that $\|[\nabla \mathcal{G}(P^{(0)})]^{-1}\| \cdot \|\mathcal{G}(P^{(0)})\| \equiv \eta = O(\|\mu\|)$. Thus, there exists a θ such that $\theta \equiv \beta\gamma\eta < 2^{-1}$ because $\eta = O(\|\mu\|)$. Using the Newton–Kantorovich theorem, the strict error estimate is given by (12a). Now, let us define

$$t^* \equiv \frac{1}{\gamma\beta}[1 - \sqrt{1 - 2\theta}] = \frac{1}{2\|S\| \cdot \|[\nabla \mathcal{G}(P^{(0)})]^{-1}\|}[1 - \sqrt{1 - 2\theta}]. \quad (17)$$

Clearly, $\mathcal{S} \equiv \{P : \|P - P^{(0)}\| \leq t^*\}$ is in the convex set $\mathcal{D}$. In the following, since $\|P^* - P^{(0)}\| = O(\|\mu\|)$ holds for small ε_1 and ε_2 , we show that P^* is the unique solution in $\mathcal{S}$.

On the other hand, using (12a), we have

$$P_{\varepsilon}^{(i)} = \begin{bmatrix} \bar{P}_{00} + O(\|\mu\|) & \varepsilon_1(\bar{P}_{10} + O(\|\mu\|))^T & \varepsilon_2(\bar{P}_{20} + O(\|\mu\|))^T \\ \varepsilon_1(\bar{P}_{10} + O(\|\mu\|)) & \varepsilon_1(\bar{P}_{11} + O(\|\mu\|)) & \sqrt{\varepsilon_1\varepsilon_2}O(\|\mu\|)^T \\ \varepsilon_2(\bar{P}_{20} + O(\|\mu\|)) & \sqrt{\varepsilon_1\varepsilon_2}O(\|\mu\|) & \varepsilon_2(\bar{P}_{22} + O(\|\mu\|)) \end{bmatrix}.$$

Since $\bar{P}_{00} \geq 0$, $\bar{P}_{11} \geq 0$, and $\bar{P}_{22} \geq 0$, $P_{\varepsilon}^{(i)}$ is positive semidefinite by using the Schur complement [13]. Therefore, the proof is completed. ■

4. MAIN RESULTS

Now, we consider a method for solving the GMALE (10a). So far, there is little argument as to the numerical method for solving the GMALE. Therefore, in order to obtain the solution of the GMALE (10a), we present a new algorithm by applying the fixed point algorithm [5, 6, 14]. Let us consider the following GMALE in general form.

$$\Lambda^T Y + Y^T \Lambda + U = 0, \quad (18)$$

where Y is the solution of the GMALE (18) and Λ and U are known matrices defined by

$$Y = \begin{bmatrix} Y_{00} & \varepsilon_1 Y_{10}^T & \varepsilon_2 Y_{20}^T \\ Y_{10} & Y_{11} & \frac{1}{\sqrt{\alpha}} Y_{21}^T \\ Y_{20} & \sqrt{\alpha} Y_{21} & Y_{22} \end{bmatrix} \in \mathbf{R}^{N \times N},$$

$$Y_{00} = Y_{00}^T, Y_{11} = Y_{11}^T, Y_{22} = Y_{22}^T,$$

$$\Lambda = \begin{bmatrix} \Lambda_{00} & \Lambda_{01} & \Lambda_{02} \\ \Lambda_{10} & \Lambda_{11} & \mathcal{E}\Lambda_{12} \\ \Lambda_{20} & \mathcal{E}\Lambda_{21} & \Lambda_{22} \end{bmatrix} \in \mathbf{R}^{N \times N},$$

$$U = U^T = \begin{bmatrix} U_{00} & U_{01} & U_{02} \\ U_{01}^T & U_{11} & \mathcal{E}U_{12} \\ U_{02}^T & \mathcal{E}U_{12}^T & U_{22} \end{bmatrix} \in \mathbf{R}^{N \times N},$$

$$U_{00} = U_{00}^T, U_{11} = U_{11}^T, U_{22} = U_{22}^T,$$

$$Y_{00}, \Lambda_{00}, U_{00} \in \mathbf{R}^{n_0 \times n_0}, Y_{11}, \Lambda_{11}, U_{11} \in \mathbf{R}^{n_1 \times n_1},$$

$$Y_{22}, \Lambda_{22}, U_{22} \in \mathbf{R}^{n_2 \times n_2},$$

$$\varepsilon_1 > 0, \varepsilon_2 > 0, \|\mu\| = \mathcal{E} = \sqrt{\varepsilon_1 \varepsilon_2}.$$

The required solution of the GMALE (18) exists under the standard condition [1].

(H3) The matrices $\Lambda_{jj}, j = 1, 2$, are nonsingular and $\Lambda_0 \equiv \Lambda_{00} - \Lambda_{01}\Lambda_{11}^{-1}\Lambda_{10} - \Lambda_{02}\Lambda_{22}^{-1}\Lambda_{20}$, $\Lambda_{jj}, j = 1, 2$, are stable.

The GMALE (18) can be partitioned into

$$\begin{aligned} & \Lambda_{00}^T Y_{00} + Y_{00} \Lambda_{00} + \Lambda_{10}^T Y_{10} + Y_{10}^T \Lambda_{10} \\ & + \Lambda_{20}^T Y_{20} + Y_{20}^T \Lambda_{20} + U_{00} = 0, \end{aligned} \quad (19a)$$

$$\begin{aligned} & Y_{00} \Lambda_{01} + Y_{10}^T \Lambda_{11} + \mathcal{E} Y_{20}^T \Lambda_{21} + \varepsilon_1 \Lambda_{00}^T Y_{10}^T + \Lambda_{10}^T Y_{11} \\ & + \sqrt{\alpha} \Lambda_{20}^T Y_{21} + U_{01} = 0, \end{aligned} \quad (19b)$$

$$\begin{aligned} & Y_{00} \Lambda_{02} + Y_{20}^T \Lambda_{22} + \mathcal{E} Y_{10}^T \Lambda_{12} + \varepsilon_2 \Lambda_{00}^T Y_{20}^T + \Lambda_{20}^T Y_{22} \\ & + \frac{1}{\sqrt{\alpha}} \Lambda_{10}^T Y_{21}^T + U_{02} = 0, \end{aligned} \quad (19c)$$

$$\begin{aligned} & \Lambda_{11}^T Y_{11} + Y_{11} \Lambda_{11} + \varepsilon_1 (\Lambda_{01}^T Y_{10}^T + Y_{10} \Lambda_{01}) \\ & + \sqrt{\alpha} \mathcal{E} (\Lambda_{21}^T Y_{21} + Y_{21}^T \Lambda_{21}) + U_{11} = 0, \end{aligned} \quad (19d)$$

$$\begin{aligned} & \varepsilon_1 Y_{10} \Lambda_{02} + \varepsilon_2 \Lambda_{01}^T Y_{20}^T + \sqrt{\alpha} Y_{21}^T \Lambda_{22} + \frac{1}{\sqrt{\alpha}} \Lambda_{11}^T Y_{21}^T \\ & + \mathcal{E} (Y_{11} \Lambda_{12} + \Lambda_{21}^T Y_{22}) + \mathcal{E} U_{12} = 0, \end{aligned} \quad (19e)$$

$$\begin{aligned} & \Lambda_{22}^T Y_{22} + Y_{22} \Lambda_{22} + \varepsilon_2 (\Lambda_{02}^T Y_{20}^T + Y_{20} \Lambda_{02}) \\ & + \frac{1}{\sqrt{\alpha}} \mathcal{E} (\Lambda_{12}^T Y_{21}^T + Y_{21} \Lambda_{12}) + U_{22} = 0. \end{aligned} \quad (19f)$$

For the equations (19) above, in the limit, as $\varepsilon_1 \rightarrow +0$ and $\varepsilon_2 \rightarrow +0$, we obtain the equations

$$\begin{aligned} \Lambda_{00}^T \bar{Y}_{00} + \bar{Y}_{00} \Lambda_{00} + \Lambda_{10}^T \bar{Y}_{10} + \bar{Y}_{10}^T \Lambda_{10} \\ + \Lambda_{20}^T \bar{Y}_{20} + \bar{Y}_{20}^T \Lambda_{20} + U_{00} = 0, \end{aligned} \quad (20a)$$

$$\bar{Y}_{00} \Lambda_{01} + \bar{Y}_{10}^T \Lambda_{11} + \Lambda_{10}^T \bar{Y}_{11} + \sqrt{\bar{\alpha}} \Lambda_{20}^T \bar{Y}_{21} + U_{01} = 0, \quad (20b)$$

$$\bar{Y}_{00} \Lambda_{02} + \bar{Y}_{20}^T \Lambda_{22} + \Lambda_{20}^T \bar{Y}_{22} + \frac{1}{\sqrt{\bar{\alpha}}} \Lambda_{10}^T \bar{Y}_{21}^T + U_{02} = 0, \quad (20c)$$

$$\Lambda_{11}^T \bar{Y}_{11} + \bar{Y}_{11} \Lambda_{11} + U_{11} = 0, \quad (20d)$$

$$\sqrt{\bar{\alpha}} \bar{Y}_{21}^T \Lambda_{22} + \frac{1}{\sqrt{\bar{\alpha}}} \Lambda_{11}^T \bar{Y}_{21}^T = 0, \quad (20e)$$

$$\Lambda_{22}^T \bar{Y}_{22} + \bar{Y}_{22} \Lambda_{22} + U_{22} = 0. \quad (20f)$$

Note that the unique solution of (20e) is given by $\bar{Y}_{21} = 0$ since the matrices Λ_{jj} , $j = 1, 2$, are nonsingular under the condition (H3). Thus the parameter $\bar{\alpha}$ does not appear in (20). Consequently, we obtain the 0-order equations

$$\begin{aligned} \Lambda_0^T \bar{Y}_{00} + \bar{Y}_{00} \Lambda_0 + U_{00} - U_{01} \Lambda_{11}^{-1} \Lambda_{10} - \Lambda_{10}^T \Lambda_{11}^{-T} U_{01}^T \\ - U_{02} \Lambda_{22}^{-1} \Lambda_{20} - \Lambda_{20}^T \Lambda_{22}^{-T} U_{02}^T \\ + \Lambda_{10}^T \Lambda_{11}^{-T} U_{11} \Lambda_{11}^{-1} \Lambda_{10} + \Lambda_{20}^T \Lambda_{22}^{-T} U_{22} \Lambda_{22}^{-1} \Lambda_{20} = 0, \end{aligned} \quad (21a)$$

$$\bar{Y}_{j0}^T = -(\bar{Y}_{00} \Lambda_{0j} + \Lambda_{j0}^T \bar{Y}_{jj} + U_{0j}) \Lambda_{jj}^{-1}, \quad j = 1, 2, \quad (21b)$$

and

$$\Lambda_{jj}^T \bar{Y}_{jj} + \bar{Y}_{jj} \Lambda_{jj} + U_{jj} = 0, \quad j = 1, 2. \quad (21c)$$

Now, let us introduce

$$Y = \begin{bmatrix} \bar{Y}_{00} + \mathcal{E} \Xi_{00} & \varepsilon_1 (\bar{Y}_{10} + \mathcal{E} \Xi_{10})^T & \varepsilon_2 (\bar{Y}_{20} + \mathcal{E} \Xi_{20})^T \\ \bar{Y}_{10} + \mathcal{E} \Xi_{10} & \bar{Y}_{11} + \mathcal{E} \Xi_{11} & \frac{\mathcal{E}}{\sqrt{\alpha}} \Xi_{21}^T \\ \bar{Y}_{20} + \mathcal{E} \Xi_{20} & \sqrt{\alpha} \mathcal{E} \Xi_{21} & \bar{Y}_{22} + \mathcal{E} \Xi_{22} \end{bmatrix}. \quad (22)$$

The approximation of the error terms Ξ_{pq} , $pq = 00, 10, 20, 11, 21, 22$, will result in an approximation of the required matrix Y_{pq} . That is why we are interested in finding equations of the error terms and a convenient algorithm to find their solutions. Substituting (22) into (19) and subtracting (20)

from (19), we arrive at the error equations.

$$\Lambda_{00}^T \Xi_{00} + \Xi_{00} \Lambda_{00} + \Lambda_{10}^T \Xi_{10} + \Xi_{10}^T \Lambda_{10} + \Lambda_{20}^T \Xi_{20} + \Xi_{20}^T \Lambda_{20} = 0, \quad (23a)$$

$$\begin{aligned} & \Xi_{00} \Lambda_{01} + \Xi_{10}^T \Lambda_{11} + \Lambda_{10}^T \Xi_{11} + \sqrt{\alpha} \Lambda_{20}^T \Xi_{21} \\ &= -\bar{Y}_{20}^T \Lambda_{21} - \frac{\varepsilon_1}{\mathcal{E}} \Lambda_{00}^T \bar{Y}_{10}^T - \varepsilon_1 \Lambda_{00}^T \Xi_{10}^T - \mathcal{E} \Xi_{20}^T \Lambda_{21}, \end{aligned} \quad (23b)$$

$$\begin{aligned} & \Xi_{00} \Lambda_{02} + \Xi_{20}^T \Lambda_{22} + \Lambda_{20}^T \Xi_{22} + \frac{1}{\sqrt{\alpha}} \Lambda_{10}^T \Xi_{21}^T \\ &= -\bar{Y}_{10}^T \Lambda_{12} - \frac{\varepsilon_2}{\mathcal{E}} \Lambda_{00}^T \bar{Y}_{20}^T - \varepsilon_2 \Lambda_{00}^T \Xi_{20}^T - \mathcal{E} \Xi_{10}^T \Lambda_{12}, \end{aligned} \quad (23c)$$

$$\begin{aligned} \Lambda_{11}^T \Xi_{11} + \Xi_{11} \Lambda_{11} &= \frac{\varepsilon_1}{\mathcal{E}} (\Lambda_{01}^T \bar{Y}_{10}^T + \bar{Y}_{10} \Lambda_{01}) - \varepsilon_1 (\Lambda_{01}^T \Xi_{10}^T + \Xi_{10} \Lambda_{01}) \\ &\quad - \mathcal{E} \sqrt{\alpha} (\Lambda_{21}^T \Xi_{21} + \Xi_{21}^T \Lambda_{21}), \end{aligned} \quad (23d)$$

$$\begin{aligned} \Lambda_{22}^T \Xi_{22} + \Xi_{22} \Lambda_{22} &= -\frac{\varepsilon_2}{\mathcal{E}} (\Lambda_{02}^T \bar{Y}_{20}^T + \bar{Y}_{20} \Lambda_{02}) - \varepsilon_2 (\Lambda_{02}^T \Xi_{20}^T + \Xi_{20} \Lambda_{02}) \\ &\quad - \frac{\mathcal{E}}{\sqrt{\alpha}} (\Lambda_{12}^T \Xi_{21}^T + \Xi_{21} \Lambda_{12}), \end{aligned} \quad (23e)$$

$$\begin{aligned} \sqrt{\alpha} \Xi_{21}^T \Lambda_{22} + \frac{1}{\sqrt{\alpha}} \Lambda_{11}^T \Xi_{21}^T &= -\frac{\varepsilon_1}{\mathcal{E}} \bar{Y}_{10} \Lambda_{02} - \frac{\varepsilon_2}{\mathcal{E}} \Lambda_{01}^T \bar{Y}_{20}^T - \bar{Y}_{11} \Lambda_{12} - \Lambda_{21}^T \bar{Y}_{22} \\ &\quad - U_{12} - \varepsilon_1 \Xi_{10} \Lambda_{02} - \varepsilon_2 \Lambda_{01}^T \Xi_{20}^T \\ &\quad - \mathcal{E} (\Lambda_{21}^T \Xi_{22} + \Xi_{11} \Lambda_{12}). \end{aligned} \quad (23f)$$

These equations (23) have very nice form since the unknown quantities Ξ_{pq} in the right-hand side are multiplied by the small parameters ε_1 , ε_2 , and $\mathcal{E}$. This fact suggests that a fixed-point algorithm can be efficient for their solutions. Hence, we propose the following algorithm (24).

$$\Lambda_{jj}^T \Xi_{jj}^{(i+1)} + \Xi_{jj}^{(i+1)} \Lambda_{jj} + \mathcal{G}_{jj}(i) = 0, \quad j = 1, 2, \quad (24a)$$

$$\sqrt{\alpha} \Xi_{21}^{(i+1)T} \Lambda_{22} + \frac{1}{\sqrt{\alpha}} \Lambda_{11}^T \Xi_{21}^{(i+1)T} + \mathcal{G}_{12}(i) = 0, \quad (24b)$$

$$\Lambda_{00}^T \Xi_{00}^{(i+1)} + \Xi_{00}^{(i+1)} \Lambda_{00} + \mathcal{G}_{00}(i) = 0, \quad (24c)$$

$$\Xi_{j0}^{(i+1)T} = -[\Xi_{00}^{(i+1)} \Lambda_{0j} + \mathcal{G}_{0j}(i)] \Lambda_{jj}^{-1}, \quad j = 1, 2, \quad i = 0, 1, 2, \dots, \quad (24d)$$

where

$$\begin{aligned} \mathcal{G}_{11}(i) &= \frac{\varepsilon_1}{\mathcal{E}} (\Lambda_{01}^T \bar{Y}_{10}^T + \bar{Y}_{10} \Lambda_{01}) + \varepsilon_1 (\Lambda_{01}^T \Xi_{10}^{(i)T} + \Xi_{10}^{(i)} \Lambda_{01}) \\ &\quad + \mathcal{E} \sqrt{\alpha} (\Lambda_{21}^T \Xi_{21}^{(i)} + \Xi_{21}^{(i)T} \Lambda_{21}), \end{aligned}$$

$$\begin{aligned}
\mathcal{G}_{22}(i) &= \frac{\varepsilon_2}{\mathcal{E}}(\Lambda_{02}^T \bar{Y}_{20}^T + \bar{Y}_{20} \Lambda_{02}) + \varepsilon_2(\Lambda_{02}^T \Xi_{20}^{(i)T} + \Xi_{20}^{(i)} \Lambda_{02}) \\
&\quad + \frac{\mathcal{E}}{\sqrt{\alpha}}(\Lambda_{12}^T \Xi_{21}^{(i)T} + \Xi_{21} \Lambda_{12}), \\
\mathcal{G}_{12}(i) &= \frac{\varepsilon_1}{\mathcal{E}} \bar{Y}_{10} \Lambda_{02} + \frac{\varepsilon_2}{\mathcal{E}} \Lambda_{01}^T \bar{Y}_{20}^T + \bar{Y}_{11} \Lambda_{12} + \Lambda_{21}^T \bar{Y}_{22} + U_{12} \\
&\quad + \varepsilon_1 \Xi_{10}^{(i)} \Lambda_{02} + \varepsilon_2 \Lambda_{01}^T \Xi_{20}^{(i)T} + \mathcal{E}(\Lambda_{21}^T \Xi_{22}^{(i)} + \Xi_{11}^{(i)} \Lambda_{12}), \\
\mathcal{G}_{01}(i) &= \Lambda_{10}^T \Xi_{11}^{(i+1)} + \sqrt{\alpha} \Lambda_{20}^T \Xi_{21}^{(i+1)} + \bar{Y}_{20}^T \Lambda_{21} \\
&\quad + \frac{\varepsilon_1}{\mathcal{E}} \Lambda_{00}^T \bar{Y}_{10}^T + \varepsilon_1 \Lambda_{00}^T \Xi_{10}^{(i)T} + \mathcal{E} \Xi_{20}^{(i)T} \Lambda_{21}, \\
\mathcal{G}_{02}(i) &= \Lambda_{20}^T \Xi_{22}^{(i+1)} + \frac{1}{\sqrt{\alpha}} \Lambda_{10}^T \Xi_{21}^{(i+1)T} + \bar{Y}_{10}^T \Lambda_{12} \\
&\quad + \frac{\varepsilon_2}{\mathcal{E}} \Lambda_{00}^T \bar{Y}_{20}^T + \varepsilon_2 \Lambda_{00}^T \Xi_{20}^{(i)T} + \mathcal{E} \Xi_{10}^{(i)T} \Lambda_{12}, \\
\mathcal{G}_{00}(i) &= -[\Lambda_{10}^T \Lambda_{11}^{-T} \mathcal{G}_{01}(i)^T + \mathcal{G}_{01}(i) \Lambda_{11}^{-1} \Lambda_{10} \\
&\quad + \Lambda_{20}^T \Lambda_{22}^{-T} \mathcal{G}_{02}(i)^T + \mathcal{G}_{02}(i) \Lambda_{22}^{-1} \Lambda_{20}],
\end{aligned}$$

$$\Lambda_{11}^T \Xi_{11}^{(0)} + \Xi_{11}^{(0)} \Lambda_{11} + \frac{\varepsilon_1}{\mathcal{E}}(\Lambda_{01}^T \bar{Y}_{10}^T + \bar{Y}_{10} \Lambda_{01}) = 0, \quad (25a)$$

$$\Lambda_{22}^T \Xi_{22}^{(0)} + \Xi_{22}^{(0)} \Lambda_{22} + \frac{\varepsilon_2}{\mathcal{E}}(\Lambda_{02}^T \bar{Y}_{20}^T + \bar{Y}_{20} \Lambda_{02}) = 0, \quad (25b)$$

$$\begin{aligned}
&\sqrt{\alpha} \Xi_{21}^{(0)T} \Lambda_{22} + \frac{1}{\sqrt{\alpha}} \Lambda_{11}^T \Xi_{21}^{(0)T} + \frac{\varepsilon_1}{\mathcal{E}} \bar{Y}_{10} \Lambda_{02} + \frac{\varepsilon_2}{\mathcal{E}} \Lambda_{01}^T \bar{Y}_{20}^T \\
&\quad + \bar{Y}_{11} \Lambda_{12} + \Lambda_{21}^T \bar{Y}_{22} + U_{12} = 0,
\end{aligned} \quad (25c)$$

$$\begin{aligned}
&\Lambda_{00}^T \Xi_{00}^{(0)} + \Xi_{00}^{(0)} \Lambda_{00} - \Lambda_{10}^T \Lambda_{11}^{-T} \Theta_{01}^T - \Theta_{01} \Lambda_{11}^{-1} \Lambda_{10} \\
&\quad - \Lambda_{20}^T \Lambda_{22}^{-T} \Theta_{02}^T - \Theta_{02} \Lambda_{22}^{-1} \Lambda_{20} = 0,
\end{aligned} \quad (25d)$$

$$\Xi_{10}^{(0)T} = -(\Xi_{00}^{(0)} \Lambda_{01} + \Theta_{01}) \Lambda_{11}^{-1}, \quad (25e)$$

$$\Xi_{20}^{(0)T} = -(\Xi_{00}^{(0)} \Lambda_{02} + \Theta_{02}) \Lambda_{22}^{-1}, \quad (25f)$$

$$\Theta_{01} = \Lambda_{10}^T \Xi_{11}^{(0)} + \sqrt{\alpha} \Lambda_{20}^T \Xi_{21}^{(0)} + \bar{Y}_{20}^T \Lambda_{21} + \frac{\varepsilon_1}{\mathcal{E}} \Lambda_{00}^T \bar{Y}_{10}^T,$$

and

$$\Theta_{02} = \Lambda_{20}^T \Xi_{22}^{(0)} + \frac{1}{\sqrt{\alpha}} \Lambda_{10}^T \Xi_{21}^{(0)T} + \bar{Y}_{10}^T \Lambda_{12} + \frac{\varepsilon_2}{\mathcal{E}} \Lambda_{00}^T \bar{Y}_{20}^T.$$

The following theorem indicates the convergence of the algorithm (24).

THEOREM 4.1. *The fixed-point algorithm (24) converges to the exact solution of Ξ_{pq} with the rate of convergence of $O(\|\mu\|^{i+1})$; that is,*

$$\begin{aligned} \|\Xi_{pq} - \Xi_{pq}^{(i)}\| &= O(\|\mu\|^{i+1}), \\ i &= 0, 1, 2, \dots, pq = 00, 10, 20, 11, 21, 22. \end{aligned} \quad (26)$$

Proof. The proof is done by using mathematical induction. When $i = 0$ for the equations (24), the first-order approximations Ξ_{pq} corresponding to the small parameters ε_1 , ε_2 , and $\mathcal{E}$ satisfy the equations (25). It follows from these equations that

$$\|\Xi_{pq} - \Xi_{pq}^{(0)}\| = O(\|\mu\|), \quad pq = 00, 10, 20, 11, 21, 22.$$

When $i = k$ ($k \geq 1$), we assume that $\|\Xi_{pq} - \Xi_{pq}^{(k)}\| = O(\|\mu\|^{k+1})$. Subtracting (24) from (23), we arrive at the equations

$$\begin{aligned} & \Lambda_{00}^T(\Xi_{00} - \Xi_{00}^{(k+1)}) + (\Xi_{00} - \Xi_{00}^{(k+1)})\Lambda_{00} \\ & \quad + \Lambda_{10}^T(\Xi_{10} - \Xi_{10}^{(k+1)}) + (\Xi_{10} - \Xi_{10}^{(k+1)})^T\Lambda_{10} \\ & \quad + \Lambda_{20}^T(\Xi_{20} - \Xi_{20}^{(k+1)}) + (\Xi_{20} - \Xi_{20}^{(k+1)})^T\Lambda_{20} = 0, \\ & (\Xi_{00} - \Xi_{00}^{(k+1)})\Lambda_{01} + (\Xi_{10} - \Xi_{10}^{(k+1)})^T\Lambda_{11} \\ & \quad + \Lambda_{10}^T(\Xi_{11} - \Xi_{11}^{(k+1)}) + \sqrt{\alpha}\Lambda_{20}^T(\Xi_{21} - \Xi_{21}^{(k+1)}) \\ & \quad = -\varepsilon_1\Lambda_{00}^T(\Xi_{10} - \Xi_{10}^{(k)})^T - \mathcal{E}(\Xi_{20} - \Xi_{20}^{(k)})^T\Lambda_{21}, \\ & (\Xi_{00} - \Xi_{00}^{(k+1)})\Lambda_{02} + (\Xi_{20} - \Xi_{20}^{(k+1)})^T\Lambda_{22} \\ & \quad + \Lambda_{20}^T(\Xi_{22} - \Xi_{22}^{(k+1)}) + \frac{1}{\sqrt{\alpha}}\Lambda_{10}^T(\Xi_{21} - \Xi_{21}^{(k+1)})^T \\ & \quad = -\varepsilon_2\Lambda_{00}^T(\Xi_{20} - \Xi_{20}^{(k)})^T - \mathcal{E}(\Xi_{10} - \Xi_{10}^{(k)})^T\Lambda_{12}, \\ & \Lambda_{11}^T(\Xi_{11} - \Xi_{11}^{(k+1)}) + (\Xi_{11} - \Xi_{11}^{(k+1)})\Lambda_{11} \\ & \quad = -\varepsilon_1[\Lambda_{01}^T(\Xi_{10} - \Xi_{10}^{(k)})^T + (\Xi_{10} - \Xi_{10}^{(k)})\Lambda_{01}] \\ & \quad \quad - \mathcal{E}\sqrt{\alpha}[\Lambda_{21}^T(\Xi_{21} - \Xi_{21}^{(k)}) + (\Xi_{21} - \Xi_{21}^{(k)})^T\Lambda_{21}], \\ & \Lambda_{22}^T(\Xi_{22} - \Xi_{22}^{(k+1)}) + (\Xi_{22} - \Xi_{22}^{(k+1)})\Lambda_{22} \\ & \quad = -\varepsilon_2[\Lambda_{02}^T(\Xi_{20} - \Xi_{20}^{(k)})^T + (\Xi_{20} - \Xi_{20}^{(k)})\Lambda_{02}] \\ & \quad \quad - \frac{\mathcal{E}}{\sqrt{\alpha}}[\Lambda_{12}^T(\Xi_{21} - \Xi_{21}^{(k)})^T + (\Xi_{21} - \Xi_{21}^{(k)})\Lambda_{12}], \end{aligned}$$

$$\begin{aligned}
& \sqrt{\alpha}(\Xi_{21} - \Xi_{21}^{(k+1)})^T \Lambda_{22} + \frac{1}{\sqrt{\alpha}} \Lambda_{11}^T (\Xi_{21} - \Xi_{21}^{(k+1)})^T \\
& = -\varepsilon_1(\Xi_{10} - \Xi_{10}^{(k)}) \Lambda_{02} - \varepsilon_2 \Lambda_{01}^T (\Xi_{20} - \Xi_{20}^{(k)})^T \\
& \quad - \mathcal{E}[\Lambda_{21}^T (\Xi_{22} - \Xi_{22}^{(k)}) + (\Xi_{11} - \Xi_{11}^{(k)}) \Lambda_{12}].
\end{aligned}$$

Using the assumption $\|\Xi_{pq} - \Xi_{pq}^{(k)}\| = O(\|\mu\|^{k+1})$, we have

$$\begin{aligned}
& \Lambda_{00}^T (\Xi_{00} - \Xi_{00}^{(k+1)}) + (\Xi_{00} - \Xi_{00}^{(k+1)}) \Lambda_{00} \\
& \quad + \Lambda_{10}^T (\Xi_{10} - \Xi_{10}^{(k+1)}) + (\Xi_{10} - \Xi_{10}^{(k+1)})^T \Lambda_{10} \\
& \quad + \Lambda_{20}^T (\Xi_{20} - \Xi_{20}^{(k+1)}) + (\Xi_{20} - \Xi_{20}^{(k+1)})^T \Lambda_{20} = 0, \\
& (\Xi_{00} - \Xi_{00}^{(k+1)}) \Lambda_{01} + (\Xi_{10} - \Xi_{10}^{(k+1)})^T \Lambda_{11} + \Lambda_{10}^T (\Xi_{11} - \Xi_{11}^{(k+1)}) \\
& \quad + \sqrt{\alpha} \Lambda_{20}^T (\Xi_{21} - \Xi_{21}^{(k+1)}) = O(\|\mu\|^{k+2}), \\
& (\Xi_{00} - \Xi_{00}^{(k+1)}) \Lambda_{02} + (\Xi_{20} - \Xi_{20}^{(k+1)})^T \Lambda_{22} + \Lambda_{20}^T (\Xi_{22} - \Xi_{22}^{(k+1)}) \\
& \quad + \frac{1}{\sqrt{\alpha}} \Lambda_{10}^T (\Xi_{21} - \Xi_{21}^{(k+1)})^T = O(\|\mu\|^{k+2}), \\
& \Lambda_{11}^T (\Xi_{11} - \Xi_{11}^{(k+1)}) + (\Xi_{11} - \Xi_{11}^{(k+1)}) \Lambda_{11} = O(\|\mu\|^{k+2}), \\
& \Lambda_{22}^T (\Xi_{22} - \Xi_{22}^{(k+1)}) + (\Xi_{22} - \Xi_{22}^{(k+1)}) \Lambda_{22} = O(\|\mu\|^{k+2}), \\
& \sqrt{\alpha}(\Xi_{21} - \Xi_{21}^{(k+1)})^T \Lambda_{22} + \frac{1}{\sqrt{\alpha}} \Lambda_{11}^T (\Xi_{21} - \Xi_{21}^{(k+1)})^T = O(\|\mu\|^{k+2}).
\end{aligned}$$

After the cancellation takes place, since $\Lambda_0, \Lambda_{jj}, j = 1, 2$, are stable from the condition (H3), we get

$$\begin{aligned}
& \Lambda_0^T (\Xi_{00} - \Xi_{00}^{(k+1)}) + (\Xi_{00} - \Xi_{00}^{(k+1)}) \Lambda_0 = O(\|\mu\|^{k+2}), \\
& (\Xi_{j0} - \Xi_{j0}^{(k+1)})^T = -(\Xi_{00} - \Xi_{00}^{(k+1)}) \Lambda_{0j} \Lambda_{jj}^{-1} + O(\|\mu\|^{k+2}), \quad j = 1, 2, \\
& \Xi_{jj} - \Xi_{jj}^{(k+1)} = O(\|\mu\|^{k+2}), \quad j = 1, 2, \\
& \Xi_{21} - \Xi_{21}^{(k+1)} = O(\|\mu\|^{k+2}).
\end{aligned}$$

Therefore, we have

$$\|\Xi_{pq} - \Xi_{pq}^{(k+1)}\| = O(\|\mu\|^{k+2}), \quad pq = 00, 10, 20, 11, 21, 22.$$

Consequently, the equation (26) holds for all $i \in \mathbf{N}$. This completes the proof of Theorem 4.1 concerned with the fixed-point algorithm. ■

5. MULTIPARAMETER H_∞ OPTIMAL CONTROL PROBLEM

5.1. The Design Problem and Preliminaries

In this section, we study the H_∞ control problem by using the state feedback control law for the MSPS,

$$\begin{aligned} \dot{x}_0 &= A_{00}x_0 + A_{01}x_1 + A_{02}x_2 + B_{01}u_1 + B_{02}u_2 \\ &\quad + F_{01}w_1 + F_{02}w_2, \quad x_0^0 = 0, \end{aligned} \quad (27a)$$

$$\begin{aligned} \varepsilon_1 \dot{x}_1 &= A_{10}x_0 + A_{11}x_1 + \varepsilon_3 A_{12}x_2 \\ &\quad + B_{11}u_1 + \varepsilon_3 B_{12}u_2 + F_{11}w_1 + \varepsilon_3 F_{12}w_2, \quad x_1^0 = 0, \end{aligned} \quad (27b)$$

$$\begin{aligned} \varepsilon_2 \dot{x}_2 &= A_{20}x_0 + \varepsilon_3 A_{21}x_1 + A_{22}x_2 \\ &\quad + \varepsilon_3 B_{21}u_1 + B_{22}u_2 + \varepsilon_3 F_{21}w_1 + F_{22}w_2, \quad x_2^0 = 0, \end{aligned} \quad (27c)$$

$$z = \begin{bmatrix} C_{00} & C_{01} & 0 \\ C_{10} & 0 & C_{12} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ H_1 & 0 \\ 0 & H_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad (27d)$$

where $x_0 \in \mathbf{R}^{n_0}$, $x_1 \in \mathbf{R}^{n_1}$, and $x_2 \in \mathbf{R}^{n_2}$ are the state vector; $u_j \in \mathbf{R}^{m_j}$, $j = 1, 2$, is the control input; $w_j \in \mathbf{R}^{l_j}$, $j = 1, 2$, is the disturbance; and $z \in \mathbf{R}^n$ is the controlled output. In order to simplify derivations, without loss of generality we assume that the fast state variables are not connected among themselves; i.e., $\varepsilon_3 \equiv 0$ [3, 5, 6].

We now discuss the H_∞ optimal control problem that the closed-loop system is internally stable and $\|G_\varepsilon\|_\infty < \gamma$, where

$$\begin{aligned} G_\varepsilon &= (C + HK_\varepsilon)(sI_N - A_\varepsilon - B_\varepsilon K_\varepsilon)^{-1} F_\varepsilon, \\ B_\varepsilon &= \begin{bmatrix} B_{01} & B_{02} \\ \varepsilon_1^{-1} B_{11} & 0 \\ 0 & \varepsilon_2^{-1} B_{22} \end{bmatrix}, \quad F_\varepsilon = \begin{bmatrix} F_{01} & F_{02} \\ \varepsilon_1^{-1} F_{11} & 0 \\ 0 & \varepsilon_2^{-1} F_{22} \end{bmatrix}, \\ C &= \begin{bmatrix} C_{00} & C_{01} & 0 \\ C_{10} & 0 & C_{12} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ H_1 & 0 \\ 0 & H_2 \end{bmatrix}, \quad H^T H > 0, \end{aligned} \quad (28)$$

by using the state feedback controller (29),

$$u = K_\varepsilon \begin{bmatrix} x_0^T & x_1^T & x_2^T \end{bmatrix}^T = K_\varepsilon x. \quad (29)$$

The next result was shown by Doyle *et al.* [19].

LEMMA 5.1. *The following are equivalent:*

(i) $A_\varepsilon + B_\varepsilon K_\varepsilon$ is stable and the transfer matrix G_ε satisfies the inequality $\|G_\varepsilon\|_\infty < \gamma$.

(ii) The MARE (30) has the positive semidefinite stabilizing solution

$$\begin{aligned} A_{\varepsilon}^T X_{\varepsilon} + X_{\varepsilon} A_{\varepsilon} + \gamma^{-2} X_{\varepsilon} F_{\varepsilon} F_{\varepsilon}^T X_{\varepsilon} \\ - X_{\varepsilon} B_{\varepsilon} (H^T H)^{-1} B_{\varepsilon}^T X_{\varepsilon} + C^T C = 0. \end{aligned} \quad (30)$$

Moreover, one such optimal controller that guarantees the γ level of optimality is given by

$$u = K_{\varepsilon} x = -(H^T H)^{-1} B_{\varepsilon}^T X_{\varepsilon} x. \quad (31)$$

Note that the MARE (30) is not a convex function with respect to P_{ε} because the matrix $\gamma^{-2} F_{\varepsilon} F_{\varepsilon}^T - B_{\varepsilon} (H^T H)^{-1} B_{\varepsilon}^T$ is in general indefinite.

5.2. The Solvability Condition

The H_{∞} control problem for the MSPS defined in (27) will be solved using the algorithm (10). In that respect, we set

$$X_{\varepsilon} \Rightarrow P_{\varepsilon}, B_{\varepsilon} (H^T H)^{-1} B_{\varepsilon}^T - \gamma^{-2} F_{\varepsilon} F_{\varepsilon}^T \Rightarrow S_{\varepsilon}, C^T C \Rightarrow Q, \quad (32)$$

where $\Rightarrow$ stands for the replacement.

The AREs (6c) will produce the unique positive semidefinite stabilizing solution under the condition (H1) if γ is large enough. Therefore, let us define the sets as in [17, 18];

$\Gamma_{jf} := \{\gamma > 0 \mid \text{the pair of AREs (6c) have the positive semidefinite stabilizing solutions}\},$

$$\gamma_{jf} := \inf\{\gamma \mid \gamma \in \Gamma_{jf}\}.$$

Moreover, let us define the set as

$\Gamma_{1s} := \{\gamma > 0 \mid \text{the ARE (6a) has a positive semidefinite stabilizing solution}\},$

$$\gamma_{1s} := \inf\{\gamma \mid \gamma \in \Gamma_{1s}\}.$$

As a result, for every $\gamma > \bar{\gamma} = \max\{\gamma_{1s}, \gamma_{jf}\}$, the MARE (30) has the positive semidefinite stabilizing solutions if $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are small enough. Then, we have the following result.

COROLLARY 5.1. *If we select a parameter $\gamma > \bar{\gamma} = \max\{\gamma_{1s}, \gamma_{jf}\}$, then there exist small $\tilde{\varepsilon}_1$ and $\tilde{\varepsilon}_2$ such that for all $\varepsilon_1 \in (0, \tilde{\varepsilon}_1)$ and $\varepsilon_2 \in (0, \tilde{\varepsilon}_2)$ the MARE (30) admits a solution such that P_{ε} is the symmetric positive semidefinite stabilizing solution, which can be written as (7).*

Proof. Since the proof is similar to that for Theorem 2.1, it is omitted. ■

Remark 5.1. Note that a condition such as $\gamma > \bar{\gamma} = \max\{\gamma_{1s}, \gamma_{jf}\}$ corresponding to the parameter γ is equivalent to the conditions that the AREs (6c) have the positive semidefinite stabilizing solutions under the conditions (H1) and (H2).

5.3. Numerical Example

In the rest of this section, in order to demonstrate the efficiency of our proposed algorithm we run a numerical example. The system matrix is given as a modification of Appendix A in [1].

$$\begin{aligned}
 A_{00} &= \begin{bmatrix} 0 & 0 & 4.5 & 0 & 1 \\ 0 & 0 & 0 & 4.5 & -1 \\ 0 & 0 & -0.05 & 0 & -0.1 \\ 0 & 0 & 0 & -0.05 & 0.1 \\ 0 & 0 & 32.7 & -32.7 & 0 \end{bmatrix}, \\
 A_{01} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{02} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.1 & 0 \\ 0 & 0 \end{bmatrix}, \\
 A_{10} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.4 & 0 & 0 \end{bmatrix}, \quad A_{20} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.4 & 0 \end{bmatrix}, \\
 A_{11} &= A_{22} = \begin{bmatrix} -0.05 & 0.05 \\ 0 & -0.1 \end{bmatrix}, \quad F_{11} = F_{22} = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}, \\
 F_{01} &= F_{02} = B_{01} = B_{02} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_{11} = B_{22} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix},
 \end{aligned}$$

$$C^T C = \text{diag}(1, 1, 1, 1, 1, 0.5, 0.5, 0.5, 0.5), \quad H^T H = \text{diag}(20, 20).$$

First, the numerical results are obtained for a small parameter $\varepsilon_1 = \varepsilon_2 = 10^{-3}$. The simulation results for the different parameter ε_j will be discussed later. Note that we cannot apply the technique proposed in [5, 6] to the MARE (30) since the Hamiltonian matrices T_{jj} , $j = 1, 2$, have eigenvalues in common. The two basic quantities for the system are $\gamma_{jf} = 9.7590 \times 10^{-2}$ and $\gamma_{1s} = 4.4721 \times 10^{-1}$. Thus, for every boundary value $\gamma > \bar{\gamma} = \max\{\gamma_{1s}, \gamma_{jf}\} = 4.472 \times 10^{-1}$, the AREs (6c) and (6a) have positive semidefinite stabilizing solutions. On the other hand, by using MATLAB, the minimum value $\hat{\gamma}$ such that there exists a feedback controller is $\hat{\gamma} = 4.472 \times 10^{-1}$.

Now, we choose $\gamma = 1.0 (> \bar{\gamma})$ to solve the MARE (30). We give a solution of the MARE (30).

$$P_{\mathcal{E}} = \begin{bmatrix} P_{00} & \varepsilon_1 P_{10}^T & \varepsilon_2 P_{20}^T \\ \varepsilon_1 P_{10} & \varepsilon_1 P_{11} & \sqrt{\varepsilon_1 \varepsilon_2} P_{21}^T \\ \varepsilon_2 P_{20} & \sqrt{\varepsilon_1 \varepsilon_2} P_{21} & \varepsilon_2 P_{22} \end{bmatrix}$$

$$\begin{aligned}
P_{00} &= \begin{bmatrix} 6.0730e+000 & 8.4607e-001 & 5.1386e+001 \\ 8.4607e-001 & 6.0730e+000 & -1.3695e-001 \\ 5.1386e+001 & -1.3695e-001 & 6.9744e+002 \\ -1.3695e-001 & 5.1386e+001 & -2.3924e+002 \\ 2.5846e-001 & -2.5846e-001 & 5.0568e+000 \\ & -1.3695e-001 & 2.5846e-001 \\ & 5.1386e+001 & -2.5846e-001 \\ & -2.3924e+002 & 5.0568e+000 \\ & 6.9744e+002 & -5.0568e+000 \\ & -5.0568e+000 & 1.3473e+000 \end{bmatrix} \\
\varepsilon_1 P_{10} &= \begin{bmatrix} 1.0158e-001 & -9.0348e-005 & 1.3678e+000 \\ 5.0000e-002 & 7.2948e-015 & 6.6053e-001 \\ & -4.7187e-001 & 8.3654e-003 \\ & -2.3187e-001 & 3.7279e-003 \end{bmatrix} \\
\varepsilon_2 P_{20} &= \begin{bmatrix} -9.0348e-005 & 1.0158e-001 & -4.7187e-001 \\ 6.0176e-015 & 5.0000e-002 & -2.3187e-001 \\ & 1.3678e+000 & -8.3654e-003 \\ & 6.6053e-001 & -3.7279e-003 \end{bmatrix} \\
\varepsilon_1 P_{11} &= \begin{bmatrix} 7.6993e-003 & 2.9751e-003 \\ 2.9751e-003 & 3.9561e-003 \end{bmatrix} \\
\varepsilon_2 P_{22} &= \begin{bmatrix} 7.6993e-003 & 2.9751e-003 \\ 2.9751e-003 & 3.9561e-003 \end{bmatrix} \\
\sqrt{\varepsilon_1 \varepsilon_2} P_{21} &= \begin{bmatrix} -9.3283e-004 & -4.5889e-004 \\ -4.5889e-004 & -2.2587e-004 \end{bmatrix}
\end{aligned}$$

We find that the solution of the MARE (30) converges to the exact solution with an accuracy of $\|\mathcal{G}(P_{\varepsilon}^{(i)})\| < 10^{-10}$ after three iterations. In order to verify the exactitude of the solution, we calculate the remainder per iteration by substituting $P_{\varepsilon}^{(i)}$ into the MARE (30). In Table I we present

TABLE I
Errors per Iteration

i	$\ \mathcal{G}(P_{\varepsilon}^{(i)})\ $
0	$3.2505e-010$
1	$1.0193e-002$
2	$5.0362e-005$
3	$4.2618e-012$

TABLE II
Error $\|\mathcal{G}(P_{\varepsilon})\|$

$\varepsilon_1 = \varepsilon_2$	Revised Kleinman Algorithm	MATLAB
10^{-2}	$8.8142e - 011$	$3.3465e - 010$
10^{-3}	$5.9038e - 012$	$1.7864e - 009$
10^{-4}	$3.4592e - 011$	$2.2509e - 008$
10^{-5}	$4.1606e - 012$	$1.3073e - 005$
10^{-6}	$8.7978e - 012$	$5.2618e - 004$
10^{-7}	$6.1600e - 012$	$1.4103e - 003$
10^{-8}	$1.5099e - 011$	$3.0732e - 002$

results for the error $\|\mathcal{G}(P_{\varepsilon}^{(i)})\|$. It can be seen that the initial guess (11) for the algorithm (10) is quite good.

In order to verify the exactness of the solution, when we substitute the obtained reference solution $P_{\varepsilon}^{\text{sch}}$ into the MARE (30) by using the function are of MATLAB, the remainder is $\|\mathcal{G}(P_{\varepsilon}^{\text{sch}})\| = 1.7864e - 009$. For the different values of ε_1 and ε_2 , the remainders of the algorithm (10) versus MATLAB are given in Table II.

From Table II, it should be noted that, although the dimensionality of the MARE (30) is small, when the parameter ε_j is quite small the loss of accuracy corresponding to the error $\|\mathcal{G}(P_{\varepsilon})\|$ for MATLAB is obvious for this numerical example. On the other hand, the resulting algorithm which combines the Kleinman algorithm (10) and the fixed point algorithm (24) computes the solution to full accuracy for all ε_j . Hence, the resulting algorithm of this paper is very useful at least in this example. In Table III, we give the resulting CPU times when we ran the new method versus MATLAB. From Table III, it can be seen that although the iterative algorithm (10) takes a lot of CPU time in the case of a not very small value of the singular perturbation parameter, the new algorithm converge to the exact solution.

TABLE III
CPU Time [s]

$\varepsilon_1 = \varepsilon_2$	Revised Kleinman Algorithm	MATLAB
10^{-2}	$5.44e - 001$	$2.80e - 002$
10^{-3}	$1.32e - 001$	$2.70e - 002$
10^{-4}	$8.00e - 002$	$2.60e - 002$
10^{-5}	$8.00e - 002$	$2.70e - 002$
10^{-6}	$4.10e - 002$	$2.60e - 002$
10^{-7}	$4.30e - 002$	$2.70e - 002$
10^{-8}	$2.50e - 002$	$2.70e - 002$

6. CONCLUSION

In this paper, we have investigated the MARE with an indefinite quadratic term in general associated with the MSPS. We have shown that there exists a unique and bounded solution for the MARE. Furthermore, we have presented the iterative method for solving the sign-indefinite GMARE. Finally, based on the fixed-point algorithm, we have presented the new numerical methods for solving the GMALE appearing in the Kleinman algorithm. It should be noted that so far an algorithm for solving the GMALE with multiple parameters has not been established.

The algorithms for solving the GMARE and GMALE were applied to a wide class of control law synthesis involving a solution of the MARE, such as the robust stabilizing control problem and the guaranteed cost control problem.

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