INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL
Int. J. Robust Nonlinear Control 2004; **14**:697–717 (DOI: 10.1002/rnc.918)

Numerical computation of cross-coupled algebraic Riccati equations related to H_2/H_∞ control problem for singularly perturbed systems

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SUMMARY

In this paper, we present a numerical algorithm to the cross-coupled algebraic Riccati equations(CARE) related to H_2/H_∞ control problems for singularly perturbed systems (SPS) by means of Newton's method. The resulting algorithm can be widely used to solve Nash game problems and robust control problems because the CARE is solvable even if the quadratic term has an indefinite sign. We prove that the resulting iterative algorithm has the property of the quadratic convergence. Using the solution of the CARE, we construct the high-order approximate H_2/H_∞ controller. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: singularly perturbed systems (SPS); H_2/H_∞ control problem; cross-coupled algebraic Riccati equation (CARE); Newton's method

1. INTRODUCTION

H_2/H_∞ control problems have been studied by using several approaches [1–3]. In particular, a state feedback mixed H_2/H_∞ control problem is formulated as a dynamic Nash game in Reference [3]. This problem is solved by using the established theory of non-zero-sum games [4] and the resulting feedback controller is characterized by the solution to a pair of cross-coupled algebraic Riccati equations (CARE). Various reliable approaches for solving the CARE have been studied which include the Riccati iterations [5–7] and the Lyapunov iterations [8, 9]. However, there are some very serious drawbacks in these methods. The first is that the convergence of the Riccati iterations was not proved exactly. The second is that there are no results for the convergence rate of the Lyapunov iterations and the numerical simulation shows that the convergence speed is very slow when the Lyapunov iterations are used to solve the CARE [9].

In this paper, we study the mixed H_2/H_∞ control problem for infinite horizon SPS from a viewpoint of solving the parameterized CARE. After defining the generalized cross-coupled

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Received 4 June 2002
Revised 16 May 2003
Accepted 16 October 2003

algebraic Riccati equations (GCARE), the asymptotic structure of the GCARE is established. In order to solve the parameterized CARE, Newton's method [10] is applied. Although the resulting algorithm involves the generalized linear matrix equation (GLME), it is newly proved that the proposed algorithm attains the quadratic convergence by using the Newton–Kantorovich theorem [10]. Moreover, the sufficient conditions are provided such that the proposed algorithm converges to a positive semidefinite solution. Using the new algorithm, we will improve the convergence speed compared with the previous results [6, 7, 9]. As another important feature, it is newly proved that the high-order approximate strategy has an asymptotic Nash equilibrium property compared with [11]. Furthermore, we give the ε-independent strategy such that the proposed strategy has also the asymptotic Nash equilibrium property. Finally, simulation results show that the proposed algorithm succeed in improving the convergence rate dramatically.

In the past few decades, Newton's method has been applied to the CARE without the perturbation parameter (see e.g. Reference [12]). The use of Newton's method for solving algebraic Riccati equation (ARE) of SPS originated in References [13, 14]. Nevertheless, the quadratic convergence property of the resulting algorithm for the SPS has not been studied so far. It is the first time for us to prove the quadratic convergence property of the algorithm.

Notation: The notations used in this paper are fairly standard. The superscript T denotes matrix transpose. I_n denotes the $n \times n$ identity matrix. $\|\cdot\|$ denotes its Euclidean norm for a matrix. $\det M$ denotes the determinant of the matrix M. block-diag denotes the block diagonal matrix. $\mathrm{Re}\,\lambda[M]$ denotes the real part of the eigenvalue of the matrix M. $\mathrm{vec}\,M$ denotes the column vector of the matrix M [15]. $\otimes$ denotes the Kronecker product. U_{lm} denotes a permutation matrix in the Kronecker matrix sense [15] such that $U_{lm}\,\mathrm{vec}\,M = \mathrm{vec}\,M^{\mathrm{T}}$, $M \in \mathbf{R}^{l\times m}$. $\|G(s)\|_\infty$ denotes H_∞-norm of the transfer function $G(s)$.

2. PROBLEM FORMULATION

Consider a linear time-invariant SPS

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + D_1w(t) + B_1u(t),\ \ x_1(0) = 0 \tag{1a}$$

$$\varepsilon\dot{x}_2(t) = A_{21}x_1(t) + A_{22}x_2(t) + D_2w(t) + B_2u(t),\ \ x_2(0) = 0 \tag{1b}$$

and a quadratic cost function

$$J(u, w) = \int_0^\infty z^{\mathrm{T}}(t)z(t)\,\mathrm{d}t \tag{2a}$$

$$z(t) = Cx(t) + Hu(t) = [C_1\ \ C_2]\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + Hu(t) \tag{2b}$$

where $C^{\mathrm{T}}H = 0$, $H^{\mathrm{T}}H = I_m$, ε is a small positive parameter, $x \in \mathbf{R}^N$ is the state vector with $x_1 \in \mathbf{R}^{n_1}$ and $x_2 \in \mathbf{R}^{n_2}$, $N := n_1 + n_2$, $u \in \mathbf{R}^m$ is the control input, $w \in \mathbf{R}^p$ is the disturbance and $z \in \mathbf{R}^q$ is the controlled output. All matrices above are of appropriate dimensions. System (1) is said to be in the standard form if the matrix A_{22} is non-singular. Otherwise, it is called the non-standard SPS.

Let us introduce the partitioned matrices

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_\varepsilon = \begin{bmatrix} A_{11} & A_{12} \\ \varepsilon^{-1} A_{21} & \varepsilon^{-1} A_{22} \end{bmatrix}$$

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad B_\varepsilon = \begin{bmatrix} B_1 \\ \varepsilon^{-1} B_2 \end{bmatrix}, \quad D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}, \quad D_\varepsilon = \begin{bmatrix} D_1 \\ \varepsilon^{-1} D_2 \end{bmatrix}$$

$$S_\varepsilon = B_\varepsilon B_\varepsilon^{\mathrm{T}} = \begin{bmatrix} S_{11} & \varepsilon^{-1} S_{12} \\ \varepsilon^{-1} S_{12}^{\mathrm{T}} & \varepsilon^{-2} S_{22} \end{bmatrix}, \quad S = BB^{\mathrm{T}} = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{\mathrm{T}} & S_{22} \end{bmatrix}$$

$$U_\varepsilon = D_\varepsilon D_\varepsilon^{\mathrm{T}} = \begin{bmatrix} U_{11} & \varepsilon^{-1} U_{12} \\ \varepsilon^{-1} U_{12}^{\mathrm{T}} & \varepsilon^{-2} U_{22} \end{bmatrix}, \quad U = DD^{\mathrm{T}} = \begin{bmatrix} U_{11} & U_{12} \\ U_{12}^{\mathrm{T}} & U_{22} \end{bmatrix}$$

$$Q = C^{\mathrm{T}} C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \geqslant 0$$

We now consider the mixed H_2/H_∞ control problem under the following basic conditions:

(H1) There exists a small perturbation parameter $\bar{\varepsilon} > 0$ such that the triplets $(A_\varepsilon, B_\varepsilon, C)$ and $(A_\varepsilon, D_\varepsilon, C)$ are stabilizable and detectable for all $\varepsilon \in (0, \bar{\varepsilon}]$.
(H2) The triplets (A_{22}, B_2, C_2) and (A_{22}, D_2, C_2) are stabilizable and detectable.

These conditions are quite natural since at least one control agent has to be able to control and observe unstable modes. The mixed H_2/H_∞ control problem is formulated as a two-player Nash game [3] associated with a prescribed disturbance attenuation level $\gamma > 0$,

$$J_1(u, w) := \int_0^\infty \gamma^2 w^{\mathrm{T}}(t) w(t)\, \mathrm{d}t - J(u, w) = \int_0^\infty [\gamma^2 w^{\mathrm{T}}(t) w(t) - z^{\mathrm{T}}(t) z(t)]\, \mathrm{d}t \tag{3a}$$

$$J_2(u, w) := J(u, w) = \int_0^\infty z^{\mathrm{T}}(t) z(t)\, \mathrm{d}t \tag{3b}$$

The first is used to reflect an H_∞ criterion, while the second is used for H_2 optimality requirement. The purpose of the mixed H_2/H_∞ control problem is to find a linear feedback controller $u^*(t) := K_2 x(t)$ such that

$$J_1(u^*, w^*) \leqslant J_1(u^*, w) \tag{4a}$$

$$J_2(u^*, w^*) \leqslant J_2(u, w^*) \tag{4b}$$

where $w^*(t) := K_1 x(t)$ represents the worst-case disturbance in the sense that it achieves the maximum possible energy gain from the disturbance input to the output.

When $J_1(u^*, w^*) \geqslant 0$, we have the following H_∞ criterion:

$$\sup_{w \in L_2[0,\infty)} \frac{\sqrt{J(u^*, w)}}{\|w\|} \leqslant \gamma \tag{5}$$

where the notation $\|\cdot\|$ for a vector function of time is used to denote the 2-norm on the time support $[0,\infty]$.

The second Nash inequality shows that $u^*(t)$ regulates the state to zero with minimum output energy when the disturbance is at its worst value $w^*(t)$.

For the SPS (1), the purpose of the mixed H_2/H_∞ control problem is to find a feedback control law $u^*(t)$ subject to the following conditions:

[I] For a given attenuation level $\gamma > 0$,

$$\|\mathbf{R}_{zw}\|_\infty < \gamma \tag{6}$$

where the operator $\mathbf{R}_{zw}$ maps the disturbance signal $w(t)$ to the controlled output $z(t)$ when the optimal control law $u^*(t)$ is invoked.

[II] The optimal control $u^*(t)$ regulates the state $x(t)$ such that the output energy is minimized when the worst-case disturbance $w^*(t)$ is applied to the SPS (1).

The following lemma is already known [3].

Lemma 2.1

Under condition (H1), suppose that there exists $\varepsilon^*(\leqslant \bar{\varepsilon})$ such that for each $\varepsilon \in (0, \varepsilon^*]$, there exist solutions $X_\varepsilon \geqslant 0$ and $Y_\varepsilon \geqslant 0$ that satisfy

$$(A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon - S_\varepsilon Y_\varepsilon)^{\mathrm{T}} X_\varepsilon + X_\varepsilon (A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon - S_\varepsilon Y_\varepsilon) + Q - \gamma^{-2} X_\varepsilon U_\varepsilon X_\varepsilon + Y_\varepsilon S_\varepsilon Y_\varepsilon = 0 \tag{7a}$$

$$(A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon - S_\varepsilon Y_\varepsilon)^{\mathrm{T}} Y_\varepsilon + Y_\varepsilon (A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon - S_\varepsilon Y_\varepsilon) + Q + Y_\varepsilon S_\varepsilon Y_\varepsilon = 0 \tag{7b}$$

where

$$X_\varepsilon = \begin{bmatrix} X_{11} & \varepsilon X_{21}^{\mathrm{T}} \\ \varepsilon X_{21} & \varepsilon X_{22} \end{bmatrix}, \quad Y_\varepsilon = \begin{bmatrix} Y_{11} & \varepsilon Y_{21}^{\mathrm{T}} \\ \varepsilon Y_{21} & \varepsilon Y_{22} \end{bmatrix}$$

Then $\mathrm{Re}\,\lambda[A_\varepsilon - S_\varepsilon Y_\varepsilon] < 0$. Moreover, if $(A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon,\ C)$ is detectable, then $\mathrm{Re}\,\lambda[A_\varepsilon + \gamma^{-2} U_\varepsilon X_\varepsilon - S_\varepsilon Y_\varepsilon] < 0$ and the following strategies (8) result in inequalities (4b) and (6).

$$u^*(t) = -B_\varepsilon^{\mathrm{T}} Y_\varepsilon x(t) \tag{8a}$$

$$w^*(t) = \gamma^{-2} D_\varepsilon^{\mathrm{T}} X_\varepsilon x(t) \tag{8b}$$

Conversely, suppose that there exist the feedback strategies $u^*(t) = K_2 x(t)$ and $w^*(t) = K_1 x(t)$ such that inequalities (4b) and (6) hold, $\mathrm{Re}\,\lambda[A_\varepsilon + B_\varepsilon K_2] < 0$ and $(A_\varepsilon + D_\varepsilon K_1,\ C)$ is detectable. Then there exist solutions $X_\varepsilon \geqslant 0$ and $Y_\varepsilon \geqslant 0$ that satisfy (7).

Define X and Y as

$$X = \begin{bmatrix} X_{11} & \varepsilon X_{21}^{\mathrm{T}} \\ X_{21} & X_{22} \end{bmatrix}, \quad X_{11}^{\mathrm{T}} = X_{11}, \quad X_{22}^{\mathrm{T}} = X_{22} \tag{9a}$$

$$Y = \begin{bmatrix} Y_{11} & \varepsilon Y_{21}^{\mathrm{T}} \\ Y_{21} & Y_{22} \end{bmatrix}, \quad Y_{11}^{\mathrm{T}} = Y_{11}, \ Y_{22}^{\mathrm{T}} = Y_{22} \tag{9b}$$

Then, CARE (7) can be written as (10) and (11) [7].

$$(A + \gamma^{-2} UX - SY)^{\mathrm{T}} X + X^{\mathrm{T}} (A + \gamma^{-2} UX - SY) + Q - \gamma^{-2} X^{\mathrm{T}} UX + Y^{\mathrm{T}} SY = 0 \tag{10a}$$

$$X_\varepsilon = E^{\mathrm{T}} X = X^{\mathrm{T}} E \tag{10b}$$

$$(A + \gamma^{-2} UX - SY)^{\mathrm{T}} Y + Y^{\mathrm{T}}(A + \gamma^{-2} UX - SY) + Q + Y^{\mathrm{T}} SY = 0 \tag{11a}$$

$$Y_\varepsilon = E^{\mathrm{T}} Y = Y^{\mathrm{T}} E \tag{11b}$$

where

$$E = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \varepsilon I_{n_2} \end{bmatrix}$$

Our main purpose is to establish a new algorithm to solve the GCARE (10) and (11) and to prove its quadratic convergence. Moreover, the other purpose of this paper is to show that the resulting H_2/H_∞ controller via the iterative solutions achieve H_∞ criterion and H_2 near-optimality.

3. NEWTON'S METHOD

In this section, we propose a new algorithm which is based on Newton's method [10]. First we can rewrite Equations (10a) and (11a) as the following GCARE:

$$\mathscr{F}(\mathscr{P}) := \tilde{A}^{\mathrm{T}} \mathscr{P} + \mathscr{P}^{\mathrm{T}} \tilde{A} + \tilde{Q} - \mathscr{P}^{\mathrm{T}} \tilde{S} \mathscr{P} - \mathscr{J} \mathscr{P}^{\mathrm{T}} \tilde{S} \mathscr{J} \mathscr{P} - \mathscr{P}^{\mathrm{T}} \mathscr{J} \tilde{S} \mathscr{P} \mathscr{J} + \mathscr{J} \mathscr{P}^{\mathrm{T}} \tilde{G} \mathscr{P} \mathscr{J} = 0 \tag{12}$$

where

$$\mathscr{P} = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad \tilde{Q} = \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix}$$

$$\tilde{S} = \begin{bmatrix} -\gamma^{-2} U & 0 \\ 0 & S \end{bmatrix}, \quad \tilde{G} = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}, \quad \mathscr{J} = \begin{bmatrix} 0 & I_N \\ I_N & 0 \end{bmatrix}, \quad N = n_1 + n_2$$

Using Newton's method [10], we provide the following iterative algorithm. It should be noted that the Newton's method will be substituted by $\mathscr{P}^{(n+1)} = \mathscr{P}^{(n)} + \Delta\mathscr{P}^{(n)}$ and will be derived by neglecting the quadratic terms of $\Delta\mathscr{P}^{(n)}$:

$$\Phi^{(n)\mathrm{T}} \mathscr{P}^{(n+1)} + \mathscr{P}^{(n+1)\mathrm{T}} \Phi^{(n)} - \Theta^{(n)\mathrm{T}} \mathscr{P}^{(n+1)} \mathscr{J} - \mathscr{J} \mathscr{P}^{(n+1)\mathrm{T}} \Theta^{(n)} + \Xi^{(n)} = 0, \quad n = 0, 1, 2, \ldots \tag{13}$$

where

$$\mathscr{P}^{(n)} = \begin{bmatrix} X^{(n)} & 0 \\ 0 & Y^{(n)} \end{bmatrix}, \quad X^{(n)} = \begin{bmatrix} X_{11}^{(n)} & \varepsilon X_{21}^{(n)\mathrm{T}} \\ X_{21}^{(n)} & X_{22}^{(n)} \end{bmatrix}, \quad Y^{(n)} = \begin{bmatrix} Y_{11}^{(n)} & \varepsilon Y_{21}^{(n)\mathrm{T}} \\ Y_{21}^{(n)} & Y_{22}^{(n)} \end{bmatrix}$$

$$\Phi^{(n)} := \tilde{A} - \tilde{S}\mathscr{P}^{(n)} - \mathscr{J} \tilde{S} \mathscr{P}^{(n)} \mathscr{J} = \begin{bmatrix} \Phi_1^{(n)} & 0 \\ 0 & \Phi_2^{(n)} \end{bmatrix}$$

$$\Theta^{(n)} := \tilde{S} \mathscr{J} \mathscr{P}^{(n)} - \tilde{G} \mathscr{P}^{(n)} \mathscr{J} = \begin{bmatrix} 0 & \Theta_1^{(n)} \\ \Theta_2^{(n)} & 0 \end{bmatrix}$$

$$\Xi^{(n)} := \tilde{Q} + \mathscr{P}^{(n)\mathrm{T}}\tilde{S}\mathscr{P}^{(n)} + \mathscr{J}\mathscr{P}^{(n)\mathrm{T}}\tilde{S}\mathscr{J}\mathscr{P}^{(n)} + \mathscr{P}^{(n)\mathrm{T}}\mathscr{J}\tilde{S}\mathscr{P}^{(n)}\mathscr{J} - \mathscr{J}\mathscr{P}^{(n)\mathrm{T}}\tilde{G}\mathscr{P}^{(n)}\mathscr{J} = \begin{bmatrix} \Xi_1^{(n)} & 0 \\ 0 & \Xi_2^{(n)} \end{bmatrix}$$

and the initial condition $\mathscr{P}^{(0)}$ is to take the solution of the following form:

$$\mathscr{P}^{(0)} = \begin{bmatrix} X^{(0)} & 0 \\ 0 & Y^{(0)} \end{bmatrix} = \begin{bmatrix} \bar{X}_{11} & \varepsilon\bar{X}_{21}^{\mathrm{T}} & 0 & 0 \\ \bar{X}_{21} & \bar{X}_{22} & 0 & 0 \\ 0 & 0 & \bar{Y}_{11} & \varepsilon\bar{Y}_{21}^{\mathrm{T}} \\ 0 & 0 & \bar{Y}_{21} & \bar{Y}_{22} \end{bmatrix} \tag{14}$$

where

$$(A + \gamma^{-2}U\bar{X} - S\bar{Y})^{\mathrm{T}}\bar{X} + \bar{X}^{\mathrm{T}}(A + \gamma^{-2}U\bar{X} - S\bar{Y}) + Q - \gamma^{-2}\bar{X}^{\mathrm{T}}U\bar{X} + \bar{Y}^{\mathrm{T}}S\bar{Y} = 0 \tag{15a}$$

$$(A + \gamma^{-2}U\bar{X} - S\bar{Y})^{\mathrm{T}}\bar{Y} + \bar{Y}^{\mathrm{T}}(A + \gamma^{-2}U\bar{X} - S\bar{Y}) + Q + \bar{Y}^{\mathrm{T}}S\bar{Y} = 0 \tag{15b}$$

$$\bar{X} = \begin{bmatrix} \bar{X}_{11} & 0 \\ \bar{X}_{21} & \bar{X}_{22} \end{bmatrix}, \quad \bar{Y} = \begin{bmatrix} \bar{Y}_{11} & 0 \\ \bar{Y}_{21} & \bar{Y}_{22} \end{bmatrix}$$

Taking the forms of X and Y for solutions (9) into account, the forms of $X^{(n)}$ and $Y^{(n)}$ are derived. The GCARE (15) are independent of the perturbation parameter ε. Nevertheless, to solve the GCARE (15) seem to be formidable due to the existence of the cross-coupled term. However, it should be noted that Newton's method can also be applied to the GCARE (15) directly. This is, in fact, quite numerically tractable because the computation is well conditioned. The different initial conditions are given later in Remark 3.3.

Newton's method has the form

$$\mathrm{vec}\,\mathscr{P}^{(n+1)} = \mathrm{vec}\,\mathscr{P}^{(n)} - [\nabla\mathscr{F}(\mathscr{P}^{(n)})]^{-1}\,\mathrm{vec}\,\mathscr{F}(\mathscr{P}^{(n)}) \tag{16}$$

We now show that algorithm (13) is equivalent to Newton's method (16). Taking the vec transformations [15] on both sides of (12) and (13), we obtain

$$\begin{aligned}\mathrm{vec}\,\mathscr{F}(\mathscr{P}^{(n)}) = {} & [(\Phi^{(n)\mathrm{T}} \otimes I_{2N})U_{2N2N} + I_{2N} \otimes \Phi^{(n)\mathrm{T}}]\,\mathrm{vec}\,\mathscr{P}^{(n)} \\ & - [(\Theta^{(n)\mathrm{T}} \otimes \mathscr{J})U_{2N2N} + \mathscr{J} \otimes \Theta^{(n)\mathrm{T}}]\,\mathrm{vec}\,\mathscr{P}^{(n)} + \mathrm{vec}\,\Xi^{(n)}\end{aligned} \tag{17a}$$

$$\begin{aligned}& [(\Phi^{(n)\mathrm{T}} \otimes I_{2N})U_{2N2N} + I_{2N} \otimes \Phi^{(n)\mathrm{T}}]\mathrm{vec}\,\mathscr{P}^{(n+1)} \\ & \quad - [(\Theta^{(n)\mathrm{T}} \otimes \mathscr{J})U_{2N2N} + \mathscr{J} \otimes \Theta^{(n)\mathrm{T}}]\mathrm{vec}\,\mathscr{P}^{(n+1)} + \mathrm{vec}\,\Xi^{(n)} = 0\end{aligned} \tag{17b}$$

Subtracting (17b) from (17a) and noting that

$$\begin{aligned}\nabla\mathscr{F}(\mathscr{P}^{(n)}) & := \left.\frac{\partial\,\mathrm{vec}\,\mathscr{F}(\mathscr{P})}{\partial(\mathrm{vec}\,\mathscr{P})^{\mathrm{T}}}\right|_{\mathscr{P}=\mathscr{P}^{(n)}} \\ & = [(\Phi^{(n)\mathrm{T}} \otimes I_{2N})U_{2N2N} + I_{2N} \otimes \Phi^{(n)\mathrm{T}}] - [(\Theta^{(n)\mathrm{T}} \otimes \mathscr{J})U_{2N2N} + \mathscr{J} \otimes \Theta^{(n)\mathrm{T}}]\end{aligned}$$

we have (16).

Before investigating the convergence property of the proposed algorithm, we study the asymptotic structure of the GCARE (12). Note that

$$\nabla\mathscr{F}(\bar{\mathscr{P}}) := \left.\frac{\partial\,\mathrm{vec}\,\mathscr{F}(\mathscr{P})}{\partial(\mathrm{vec}\,\mathscr{P})^{\mathrm{T}}}\right|_{\mathscr{P}=\bar{\mathscr{P}}} = \left.\frac{\partial\mathscr{G}(\varepsilon, \mathrm{vec}\,X_{11}, \mathrm{vec}\,X_{21}, \mathrm{vec}\,X_{22}, \mathrm{vec}\,Y_{11}, \mathrm{vec}\,Y_{21}, \mathrm{vec}\,Y_{22})}{\partial(\mathrm{vec}\,X_{11}, \mathrm{vec}\,X_{21}, \mathrm{vec}\,X_{22}, \mathrm{vec}\,Y_{11}, \mathrm{vec}\,Y_{21}, \mathrm{vec}\,Y_{22})^{\mathrm{T}}}\right|_{\varepsilon=0}$$

$$= (\tilde{A} - \tilde{S}\bar{\mathscr{P}} - \mathscr{J}\tilde{S}\bar{\mathscr{P}}\mathscr{J})^{\mathrm{T}} \otimes I_N + [I_N \otimes (\tilde{A} - \tilde{S}\bar{\mathscr{P}} - \mathscr{J}\tilde{S}\bar{\mathscr{P}}\mathscr{J})^{\mathrm{T}}]U_{2N2N}$$

$$- (\tilde{S}\mathscr{J}\bar{\mathscr{P}} - \tilde{G}\bar{\mathscr{P}}\mathscr{J})^{\mathrm{T}} \otimes \mathscr{J} - [\mathscr{J} \otimes (\tilde{S}\mathscr{J}\bar{\mathscr{P}} - \tilde{G}\bar{\mathscr{P}}\mathscr{J})]^{\mathrm{T}} U_{2N2N}$$

where

$$\bar{\mathscr{P}} := \begin{bmatrix} \bar{X} & 0 \\ 0 & \bar{Y} \end{bmatrix}, \quad \mathrm{vec}\,\mathscr{F}(\mathscr{P}) := \mathscr{G}(\varepsilon, \mathrm{vec}\,X_{11}, \mathrm{vec}\,X_{21}, \mathrm{vec}\,X_{22}, \mathrm{vec}\,Y_{11}, \mathrm{vec}\,Y_{21}, \mathrm{vec}\,Y_{22})$$

Without loss of generality the following conditions are assumed.

(H3) (i) There exist the positive semidefinite admissible solution $E_0^{\mathrm{T}} Z \geqslant 0$ and the positive semidefinite stabilizing solution $Z_{22} \geqslant 0$ such that

$$A^{\mathrm{T}} Z + Z^{\mathrm{T}} A + Q - Z^{\mathrm{T}}(S - \gamma^{-2})Z = 0$$

$$A_{22}^{\mathrm{T}} Z_{22} + Z_{22} A_{22} + Q_{22} - Z_{22}(S_{22} - \gamma^{-2} U_{22}) Z_{22} = 0$$

where

$$E_0^{\mathrm{T}} Z = Z^{\mathrm{T}} E_0, \;\; E_0 := \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0_{n_2} \end{bmatrix}, \;\; Z := \begin{bmatrix} Z_{11} & 0 \\ Z_{21} & Z_{22} \end{bmatrix}, \;\; Z_{22} = Z_{22}^{\mathrm{T}}$$

(ii) There exists the matrix $K_1 = [K_{11}\;\; K_{22}]$ such that $(E_0,\; A + DK_1,\; C)$ is impulse observable.

(iii) There exists the matrix K_{22} such that $(A_{22} + D_2 K_{22},\; C_2)$ is observable.

(iv) $(E_0,\; A,\; B)$ is impulse and finite dynamics controllable.

(v)

$$\mathrm{rank}\begin{bmatrix} A - \mathrm{j}\omega E_0 \\ C \end{bmatrix} = N, \quad \forall \omega \in \mathbf{R}$$

It should be noted that the restriction of assumption (H3) is standard because these conditions are also assumed in References [3, 16] when we consider H_∞ control problem for the descriptor systems.

Remark 3.1

Using the result of the bounded real lemma [16], if condition (H3) is met, there exist the gain matrix K_2 and the positive semidefinite admissible solution $E_0^{\mathrm{T}} W \geqslant 0$ such that

$$(A + \gamma^{-2} UW + BK_2)^{\mathrm{T}} W + W^{\mathrm{T}}(A + \gamma^{-2} UW + BK_2) + Q - \gamma^{-2} W^{\mathrm{T}} UW + K_2^{\mathrm{T}} K_2 = 0$$

where $E_0^{\mathrm{T}} W = W^{\mathrm{T}} E_0$.

Hence, we conclude that $0 \leqslant E_0^{\mathrm{T}} \bar{X} \leqslant E_0^{\mathrm{T}} W$ compared with (15a) as $K_2 = -B^{\mathrm{T}} \bar{Y}$. On the other hand, using condition (H3), there exists the solution $E_0^{\mathrm{T}} \bar{Y} \geqslant 0$. Finally, since the GCARE (15) has the solutions $E_0^{\mathrm{T}} \bar{X} \geqslant 0$ and $E_0^{\mathrm{T}} \bar{Y} \geqslant 0$, there exist the solutions $\bar{X}_{11} \geqslant 0$ and $\bar{Y}_{11} \geqslant 0$. Using the similar technique, it can be shown quite easily that there exist the solutions $\bar{X}_{22} \geqslant 0$ and $\bar{Y}_{22} \geqslant 0$.

The following theorem will establish the relation between $\mathscr{P}$ and reduced-order solutions of the GCARE (15).

Theorem 3.1
Assume that

$$\det \nabla \mathscr{F}(\bar{\mathscr{P}}) \neq 0 \tag{18}$$

Under conditions (H1)–(H3), the GCARE (10) and (11) admit the solutions X and Y that possess a power series expansion at $\varepsilon = 0$, respectively. That is,

$$X = \begin{bmatrix} \bar{X}_{11} & \varepsilon \bar{X}_{21}^{\mathrm{T}} \\ \bar{X}_{21} & \bar{X}_{22} \end{bmatrix} + O(\varepsilon) \tag{19a}$$

$$Y = \begin{bmatrix} \bar{Y}_{11} & \varepsilon \bar{Y}_{21}^{\mathrm{T}} \\ \bar{Y}_{21} & \bar{Y}_{22} \end{bmatrix} + O(\varepsilon) \tag{19b}$$

Proof
We apply the implicit function theorem [17] to (12). To do so, it is enough to show that the corresponding Jacobian is non-singular at $\varepsilon = 0$. Using the fact that the independent variable of the function $\mathscr{F}(\mathscr{P})$ is only ε, it can be shown, after some algebra, that the Jacobian of (12) in the limit as $\varepsilon \to +0$ is given by

$$\begin{aligned} J_{\bar{\mathscr{P}}} &= \lim_{\varepsilon \to +0} \frac{\partial \operatorname{vec} \mathscr{F}(\mathscr{P})}{\partial (\operatorname{vec} \mathscr{P})^{\mathrm{T}}} \\ &= (\tilde{A} - \tilde{S}\bar{\mathscr{P}} - \mathscr{J}\tilde{S}\bar{\mathscr{P}}\mathscr{J})^{\mathrm{T}} \otimes I_N + [I_N \otimes (\tilde{A} - \tilde{S}\bar{\mathscr{P}} - \mathscr{J}\tilde{S}\bar{\mathscr{P}}\mathscr{J})^{\mathrm{T}}]U_{2N2N} \\ &\quad - (\tilde{S}\mathscr{J}\bar{\mathscr{P}} - \tilde{G}\bar{\mathscr{P}}\mathscr{J})^{\mathrm{T}} \otimes \mathscr{J} - [\mathscr{J} \otimes (\tilde{S}\mathscr{J}\bar{\mathscr{P}} - \tilde{G}\bar{\mathscr{P}}\mathscr{J})]^{\mathrm{T}} U_{2N2N} \\ &= \nabla \mathscr{F}(\bar{\mathscr{P}}) \end{aligned} \tag{20}$$

Therefore, $\det J_{\bar{\mathscr{P}}} \neq 0$, i.e. $J_{\bar{\mathscr{P}}}$ is non-singular at $\varepsilon = 0$. The conclusion of Theorem 3.1 is obtained directly by using the implicit function theorem. See for detail e.g. References [7, 18]. □

We are concerned with good choices of the starting points which guarantee to find a required solution of the given GCARE (12). Our new idea is to set the initial conditions to matrix (14). The fundamental idea is based on the fact that $\|\mathscr{P} - \mathscr{P}^{(0)}\| = O(\varepsilon)$. Although the GCARE (12) has the general indefinite sign, we can get the required solution with the rate of quadratic convergence by using Newton's method. Moreover, using the Newton–Kantorovich theorem [10], we can prove the existence of the unique solution of the GCARE (12) in the neighbourhood of the initial guess (14). The main result of this section is as follows.

Theorem 3.2
Under conditions (H1)–(H3), the iterative algorithm (13) converges to the exact solution $\mathscr{P}^*$ of the GCARE (12) with the rate of quadratic convergence. The unique bounded solution $\mathscr{P}^*$ of

the GCARE (12) is in the neighbourhood of the matrix $\mathscr{P}^{(0)}$. That is, the following conditions are satisfied:

$$\|\mathscr{P}^{(n)} - \mathscr{P}^*\| \leqslant \frac{O(\varepsilon^{2^n})}{2^n \beta \mathscr{L}}, \quad n = 0, 1, 2, \ldots \tag{21a}$$

$$\|\mathscr{P}^{(0)} - \mathscr{P}^*\| \leqslant \frac{1}{\beta \mathscr{L}}[1 - \sqrt{1 - 2\theta}] \tag{21b}$$

where

$$\mathscr{P}^* = \begin{bmatrix} X^* & 0 \\ 0 & Y^* \end{bmatrix}, \quad \mathscr{L} := 6\|\tilde{S}\| + 2\|\tilde{G}\| < \infty, \quad \beta := \|[\nabla \mathscr{F}(\mathscr{P}^{(0)})]^{-1}\|$$

$$\eta := \|[\nabla \mathscr{F}(\mathscr{P}^{(0)})]^{-1}\| \cdot \|\mathscr{F}(\mathscr{P}^{(0)})\|, \quad \theta := \beta \eta \mathscr{L}$$

In order to prove Theorem 3.2, we need the following useful lemma which is called Newton–Kantorovich theorem [10].

Lemma 3.1
Let $\mathbf{X}$ and $\mathbf{Y}$ be Banach spaces and $\mathbf{F} : \mathbf{D} \subseteq \mathbf{X} \to \mathbf{Y}$ be Fréchet differentiable in an open convex set $\mathbf{D}_0 \subseteq \mathbf{D}$ and, for some $\mathbf{x}_0 \in \mathbf{D}_0$, $\mathbf{F}'(\mathbf{x}_0)^{-1}$ exist. Assume that $\mathbf{F}(\mathbf{x}_0) \neq 0$ without loss of generality and that

$$\|\mathbf{F}'(\mathbf{x}) - \mathbf{F}'(\mathbf{y})\| \leqslant \mathbf{L}\|\mathbf{x} - \mathbf{y}\|, \quad \mathbf{x}, \ \mathbf{y} \in \mathbf{D}_0$$

$$\hat{\beta} \geqslant \|\mathbf{F}'(\mathbf{x}_0)^{-1}\|, \quad \hat{\eta} = \|\mathbf{F}'(\mathbf{x}_0)^{-1}\mathbf{F}(\mathbf{x}_0)\|, \quad \theta = \hat{\beta}\hat{\eta}\mathbf{L} \leqslant \frac{1}{2}, \quad t^* = \frac{1 - \sqrt{1 - 2\theta}}{\hat{\beta}\mathbf{L}}$$

$$\bar{\mathbf{S}} = \{\mathbf{x} \in \mathbf{R}^n \mid \|\mathbf{x} - \mathbf{x}_0\| \leqslant t^*\} \subset \mathbf{D}_0$$

Then:
(i) The Newton iterates

$$\mathbf{x}^{k+1} = \mathbf{x}^k - \mathbf{F}'(\mathbf{x}^k)^{-1}\mathbf{F}(\mathbf{x}^k), \quad k = 0, 1, 2, \ldots$$

are well defined, $\mathbf{x}^k \in \bar{\mathbf{S}}$ and converge to a solution $\mathbf{x}^* \in \bar{\mathbf{S}}$ of $\mathbf{F}(\mathbf{x}) = 0$. Moreover, we have the error estimate

$$\|\mathbf{x}^k - \mathbf{x}^*\| \leqslant \frac{(2\theta)^{2^k}}{2^k \hat{\beta}\mathbf{L}}, \quad k = 0, 1, 2, \ldots$$

(ii) The solution $\mathbf{x}^*$ is unique in $\bar{\mathbf{S}}$.

Using Lemma 3.1, let us prove Theorem 3.2.

Proof
Using the fact that

$$\nabla\mathscr{F}(\mathscr{P}) = \frac{\partial\,\mathrm{vec}\,\mathscr{F}(\mathscr{P})}{\partial(\mathrm{vec}\,\mathscr{P})^{\mathrm{T}}}$$
$$= (\tilde{A} - \tilde{S}\mathscr{P} - \mathscr{J}\tilde{S}\mathscr{P}\mathscr{J})^{\mathrm{T}} \otimes I_N + [I_N \otimes (\tilde{A} - \tilde{S}\mathscr{P} - \mathscr{J}\tilde{S}\mathscr{P}\mathscr{J})^{\mathrm{T}}]U_{2N2N}$$
$$- (\tilde{S}\mathscr{J}\mathscr{P} - \tilde{G}\mathscr{P}\mathscr{J})^{\mathrm{T}} \otimes \mathscr{J} - [\mathscr{J} \otimes (\tilde{S}\mathscr{J}\mathscr{P} - \tilde{G}\mathscr{P}\mathscr{J})]^{\mathrm{T}} U_{2N2N} \tag{22}$$

we have

$$\|\nabla\mathscr{F}(\mathscr{P}_1) - \nabla\mathscr{F}(\mathscr{P}_2)\| \leqslant \mathscr{L}\|\mathscr{P}_1 - \mathscr{P}_2\| \tag{23}$$

It is obvious that $\nabla\mathscr{F}(\mathscr{P})$ is continuous at all $\mathscr{P}$. Moreover, since

$$\nabla\mathscr{F}(\mathscr{P}^{(0)}) = (\tilde{A} - \tilde{S}\mathscr{P}^{(0)} - \mathscr{J}\tilde{S}\mathscr{P}^{(0)}\mathscr{J})^{\mathrm{T}} \otimes I_N + [I_N \otimes (\tilde{A} - \tilde{S}\mathscr{P}^{(0)} - \mathscr{J}\tilde{S}\mathscr{P}^{(0)}\mathscr{J})^{\mathrm{T}}]U_{2N2N}$$
$$- (\tilde{S}\mathscr{J}\mathscr{P}^{(0)} - \tilde{G}\mathscr{P}^{(0)}\mathscr{J})^{\mathrm{T}} \otimes \mathscr{J} - [\mathscr{J} \otimes (\tilde{S}\mathscr{J}\mathscr{P}^{(0)} - \tilde{G}\mathscr{P}^{(0)}\mathscr{J})]^{\mathrm{T}} U_{2N2N} + O(\varepsilon) \tag{24}$$

it follows that $\nabla\mathscr{F}(\mathscr{P}^{(0)})$ is non-singular for sufficiently small parameter ε under condition (18). Therefore, there exists β such that $\beta = \|[\nabla\mathscr{F}(\mathscr{P}^{(0)})]^{-1}\|$. On the other hand, since $\mathscr{F}(\mathscr{P}^{(0)}) = O(\varepsilon)$ from (19), there exists η such that $\eta = \|[\nabla\mathscr{F}(\mathscr{P}^{(0)})]^{-1}\| \cdot \|\mathscr{F}(\mathscr{P}^{(0)})\| = O(\varepsilon)$. Thus, there exists θ such that $\theta = \beta\eta\mathscr{L} < 2^{-1}$ because $\eta = O(\varepsilon)$. Using Lemma 3.1, we can show that $\mathscr{P}^*$ is the unique solution in the subset $\mathscr{S} \equiv \{\mathscr{P} : \|\mathscr{P}^{(0)} - \mathscr{P}\| \leqslant t^*\}$, where

$$t^* \equiv \frac{1}{\beta\mathscr{L}}[1 - \sqrt{1 - 2\theta}] \tag{25}$$

Moreover, the error estimate is given by

$$\|\mathscr{P}^{(n)} - \mathscr{P}^*\| \leqslant \frac{(2\theta)^{2^n}}{2^n\beta\mathscr{L}}, \quad n = 1, 2, \ldots \tag{26}$$

which implies (21a) because $2\theta = O(\varepsilon)$. Furthermore, substituting $\mathscr{P}^*$ into $\mathscr{P}$ of the subset $\mathscr{S}$, we can also get (21b). Therefore, (21) holds for the small ε. □

Remark 3.2
According to Reference [19], it is well known that the solution of the GCARE (12) is not unique and several non-negative solutions exist. The Lyapunov iterations [8] guarantee that such algorithm converge to a positive semidefinite solution. Similarly, it should be noted that if the initial conditions $EX^{(0)}$ and $EY^{(0)}$ are the positive semidefinite solutions, the new algorithm which is based on the GLME (13) converges to the positive semidefinite solution.

Remark 3.3
In order to obtain the initial condition (14), we have to solve the CARE and GCARE (15) which are independent of the perturbation parameter ε. In this case, we can also apply Newton's method to (15). In fact, it is easily seen that Newton's method can be used well to solve the

CARE and GCARE (15). Then, as one of the initial condition we recommend the solutions of the following generalized algebraic Riccati equations:

$$A^{\mathrm{T}}\hat{Y} + \hat{Y}^{\mathrm{T}}A + Q - \hat{Y}^{\mathrm{T}}S\hat{Y} = 0, \quad Y^{(0)} = \hat{Y}$$

$$(A - S\hat{Y})^{\mathrm{T}}\hat{X} + \hat{X}^{\mathrm{T}}(A - S\hat{Y}) + Q + \gamma^{-2}\hat{X}^{\mathrm{T}}U\hat{X} = 0, \quad X^{(0)} = \hat{X}$$

Note that there is no guarantee of converging to the required solutions for the above initial condition. However, since $\mathscr{F}(\mathscr{P})$ is differentiable and $\nabla\mathscr{F}(\mathscr{P})$ is continuous at all $\mathscr{P}$, if the condition as $\det \nabla\mathscr{F}(\mathscr{P}^*) \neq 0$ is met, then the new algorithm has the quadratic convergence via the local convergence property for Newton's method [10].

Remark 3.4
It should be noted that the Lyapunov iterations are much simpler to implement from the computational point of view and the fast convergence of the Newton's method increases numerical complexity compared with the Lyapunov iterations.

4. HIGH-ORDER APPROXIMATE CONTROLLER

In this section, the high-order approximate H_2/H_∞ controller is given. Such a controller is obtained by using the iterative solution (13):

$$\bar{u}^{(n)}(t) = -B^{\mathrm{T}}Y^{(n)}x(t) \tag{27a}$$

$$\bar{w}^{(n)}(t) = \gamma^{-2}D^{\mathrm{T}}X^{(n)}x(t) \tag{27b}$$

Theorem 4.1
Assume that $\mathrm{Re}\,\lambda[E^{-1}(A + \gamma^{-2}UX^{(0)} - SY^{(0)})] < 0$. Under conditions (H1) and (H2), the following result holds:

$$J_i(\bar{u}^{(n)},\ \bar{w}^{(n)}) = J_i(u^*,\ w^*) + O(\varepsilon^{2^n}), \quad i = 1, 2 \tag{28}$$

where $J_i(u^*, w^*)$, $i = 1, 2$ are the equilibrium values satisfying (4).

Proof
Note that the optimal strategies (8) result in $J_2(u^*, w^*) = x(0)^{\mathrm{T}}Y_\varepsilon x(0)$. When $\bar{u}^{(n)}$ is applied to the SPS (1) under the disturbance $\bar{w}^{(n)}$, the value of the performance index (3b) is

$$J_2(\bar{u}^{(n)},\ \bar{w}^{(n)}) = x(0)^{\mathrm{T}}W_{2\varepsilon}^{(n)}x(0) \tag{29}$$

where $W_{2\varepsilon}^{(n)}$ is the positive semidefinite solution of the following algebraic Lyapunov equation (ALE):

$$(A_\varepsilon + \gamma^{-2}U_\varepsilon X_\varepsilon^{(n)} - S_\varepsilon Y_\varepsilon^{(n)})^{\mathrm{T}}W_{2\varepsilon}^{(n)} + W_{2\varepsilon}^{(n)}(A_\varepsilon + \gamma^{-2}U_\varepsilon X_\varepsilon^{(n)} - S_\varepsilon Y_\varepsilon^{(n)}) + Q + Y_\varepsilon^{(n)}S_\varepsilon Y_\varepsilon^{(n)} = 0 \tag{30}$$

where $X_\varepsilon^{(n)} = EX^{(n)}$ and $Y_\varepsilon^{(n)} = EY^{(n)}$. Subtracting (7b) from (30) we find that $V_{2\varepsilon}^{(n)} = W_{2\varepsilon}^{(n)} - Y_\varepsilon$

satisfies the following ALE:

$$(A_\varepsilon + \gamma^{-2}U_\varepsilon X_\varepsilon^{(n)} - S_\varepsilon Y_\varepsilon^{(n)})^{\mathrm{T}} V_{2\varepsilon}^{(n)} + V_{2\varepsilon}^{(n)}(A_\varepsilon + \gamma^{-2}U_\varepsilon X_\varepsilon^{(n)} - S_\varepsilon Y_\varepsilon^{(n)})$$
$$+ (Y_\varepsilon^{(n)} - Y_\varepsilon)S_\varepsilon(Y_\varepsilon^{(n)} - Y_\varepsilon)$$
$$+ \gamma^{-2} Y_\varepsilon U_\varepsilon(X_\varepsilon^{(n)} - X_\varepsilon) + \gamma^{-2}(X_\varepsilon^{(n)} - X_\varepsilon)U_\varepsilon Y_\varepsilon = 0 \quad (31)$$

By (21a), we have $X_\varepsilon^{(n)} - X_\varepsilon = O(\varepsilon^{2^n})$ and $Y_\varepsilon^{(n)} - Y_\varepsilon = O(\varepsilon^{2^n})$, so that

$$(A_\varepsilon + \gamma^{-2}U_\varepsilon X_\varepsilon^{(n)} - S_\varepsilon Y_\varepsilon^{(n)})^{\mathrm{T}} V_{2\varepsilon}^{(n)} + V_{2\varepsilon}^{(n)}(A_\varepsilon + \gamma^{-2}U_\varepsilon X_\varepsilon^{(n)} - S_\varepsilon Y_\varepsilon^{(n)}) + O(\varepsilon^{2^n}) = 0 \quad (32)$$

It is easy to verify that $V_{2\varepsilon}^{(n)} = O(\varepsilon^{2^n})$ because $A_\varepsilon + \gamma^{-2}U_\varepsilon X_\varepsilon^{(n)} - S_\varepsilon Y_\varepsilon^{(n)} = E^{-1}[A + \gamma^{-2}UX^{(0)} - SY^{(0)} + O(\varepsilon)]$ is stable by using the standard Lyapunov theorem [20] for sufficiently small ε. Consequently, the following equality holds:

$$J_2(\bar{u}^{(n)}, \quad \bar{w}^{(n)}) = J_2(u^*, w^*) + O(\varepsilon^{2^n})$$

The rest of the Proof of Theorem 4.1 is omitted, since the proof of $V_{1\varepsilon}^{(n)} = W_{1\varepsilon}^{(n)} - X_\varepsilon = O(\varepsilon^{2^n})$ is performed by a similar argument. □

Theorem 4.2
Under the conditions of Theorem 4.1, the following result holds:

$$J_1(\bar{u}^{(n)}, w) = J_1(u^*, w) + O(\varepsilon^{2^n}) \quad (33a)$$

$$J_2(u, \bar{w}^{(n)}) = J_2(u, w^*) + O(\varepsilon^{2^n}) \quad (33b)$$

Proof
We first prove equality (33a). Applying the high-order approximate controller (27a) to the SPS (1) yields

$$\dot{x}(t) = \bar{A}_\varepsilon x(t) + \varepsilon \bar{F}_\varepsilon x(t) + D_\varepsilon w(t), \quad x(0) = 0 \quad (34a)$$

$$J_2(\bar{u}^{(n)}, w) = \int_0^\infty x^{\mathrm{T}}(t)\bar{Q}x(t)\,\mathrm{d}t \quad (34b)$$

where

$$\bar{A}_\varepsilon = E^{-1}(A - S\bar{Y}) = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \varepsilon^{-1}\bar{A}_{21} & \varepsilon^{-1}\bar{A}_{22} \end{bmatrix}$$

$$\bar{F}_\varepsilon = -\varepsilon^{-1}(S_\varepsilon Y_\varepsilon^{(n)} - A_\varepsilon + \bar{A}_\varepsilon), \quad \bar{Q} = Q + Y_\varepsilon^{(n)} S_\varepsilon Y_\varepsilon^{(n)}$$

Under condition (H2), $\bar{A}_{22}$ and $\bar{A}_0 := \bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}$ are stable. Hence, there exists the transformation $y(t) = T^{-1}x(t)$ such that $T^{-1}\bar{A}_\varepsilon T = \text{block-diag}[A_s \ \varepsilon^{-1}A_f]$ [21].

Using the transformation T, we obtain

$$\dot{y}_1(t) = A_s y_1(t) + \varepsilon \bar{F}_s y(t) + D_s w(t), \quad y_1(0) = 0 \quad (35a)$$

$$\varepsilon \dot{y}_2(t) = A_f y_2(t) + \varepsilon \bar{F}_f y(t) + D_f w(t), \quad y_2(0) = 0 \quad (35b)$$

where $[\bar{F}_s^{\mathrm{T}} \ \bar{F}_f^{\mathrm{T}}]^{\mathrm{T}} = T^{-1}\bar{F}_\varepsilon$ and $[D_s^{\mathrm{T}} \ D_f^{\mathrm{T}}]^{\mathrm{T}} = T^{-1}D_\varepsilon$. From (35), if ε is small enough, then we have $\|y\| \leqslant c_1\|w\|, c_1 > 0$. Similarly, substituting the optimal control $u^*(t) = -B^{\mathrm{T}}Yx(t)$ and

$f(t) = T^{-1}x(t)$ into the SPS (1), we get

$$\dot{f}_1(t) = A_s f_1(t) + \varepsilon \hat{F}_s f(t) + D_s w(t), \quad f_1(0) = 0 \tag{36a}$$

$$\varepsilon \dot{f}_2(t) = A_f f_2(t) + \varepsilon \hat{F}_f f(t) + D_f w(t), \quad f_2(0) = 0 \tag{36b}$$

where $[\hat{F}_s^{\mathrm{T}} \ \hat{F}_f^{\mathrm{T}}]^{\mathrm{T}} = T^{-1}\hat{F}_\varepsilon = -\varepsilon^{-1}T^{-1}(SY_\varepsilon - A_\varepsilon + \bar{A}_\varepsilon)$. Hence, from (36), one can derive $\|f\| \leqslant c_2\|w\|, \ c_2 > 0$. Subtracting (35) from (36) we get the following SPS (37):

$$\dot{e}_1(t) = A_s e_1(t) + \varepsilon \hat{F}_s e(t) + O(\varepsilon^{2^n})y(t) \tag{37a}$$

$$\varepsilon \dot{e}_2(t) = A_f e_2(t) + \varepsilon \hat{F}_f e(t) + O(\varepsilon^{2^n})y(t) \tag{37b}$$

where $e(t) = f(t) - y(t)$. From (37), we obtain

$$\|e\| \leqslant c_3\varepsilon^{2^n}\|y\| \leqslant c_4\varepsilon^{2^n}\|w\|, \quad c_3, c_4 > 0$$

Then, noting that $Y_\varepsilon^{(n)} - Y_\varepsilon = O(\varepsilon^{2^n})$, we get $T^{-1}\hat{F}_\varepsilon T - T^{-1}\bar{F}_\varepsilon T = O(\varepsilon^{2^n-1})$ and $\|\hat{Q} - \bar{Q}\| = m_0\varepsilon^{2^n}, m_0 > 0$, where $\hat{Q} = Q + Y_\varepsilon S_\varepsilon Y_\varepsilon$. Applying the Schwartz inequality, we obtain

$$\begin{aligned} |J_2 - \hat{J}_2| &\leqslant \int_0^\infty [m_1|e(t)||y(t)| + m_2|e(t)||f(t)| + m_0\varepsilon^{2^n}|y(t)||f(t)|]\,\mathrm{d}t \\ &\leqslant \bar{m}[\|e\|(\|y\| + \|f\|) + \varepsilon^{2^n}\|y\| \cdot \|f\|] \end{aligned} \tag{38}$$

where $\hat{J}_2 := J_2(u^*, w)$, $\bar{m} = \max\{m_0, m_1, m_2\}$, $m_1 = \|T^{\mathrm{T}}\bar{Q}T\|$ and $m_2 = \|T^{\mathrm{T}}\hat{Q}T\|$. Since $\|y\| \leqslant c_1\|w\|$, $\|f\| \leqslant c_2\|w\|$ and $\|e\| \leqslant c_4\varepsilon^{2^n}\|w\|$, we have

$$|J_2 - \hat{J}_2| \leqslant \bar{m}_0\varepsilon^{2^n}\|w\|^2 \tag{39}$$

where $\bar{m}_0 := \bar{m}[c_4(c_1 + c_2) + c_1c_2]$. Finally, by using condition $J_2 \leqslant \gamma^2\|w\|^2$, we have

$$\hat{J}_2 \leqslant [\gamma^2 + O(\varepsilon^{2^n})]\|w\|^2 \leqslant [\gamma + O(\varepsilon^{2^n})]^2\|w\|^2 \tag{40}$$

that is, an $O(\varepsilon^{2^n})$ accuracy controller $u^*(t) = -B^{\mathrm{T}}Yx(t)$ achieves the performance level $\gamma + O(\varepsilon^{2^n})$. Thus equality (33a) holds.

Secondly, we prove equality (33b). Applying the high-order approximate disturbance (27b) and the control $u(t) = K_2x(t)$ to the SPS (1) yields

$$\dot{x}(t) = (A_\varepsilon + \gamma^{-2}U_\varepsilon X^{(n)} + B_\varepsilon K_2)x(t), \quad x(0) = 0 \tag{41a}$$

$$J_2(u, \bar{w}^{(n)}) = \int_0^\infty x^{\mathrm{T}}(t)(Q + K_2^{\mathrm{T}}K_2)x(t)\,\mathrm{d}t \tag{41b}$$

On the other hand, applying the worst disturbance (8b) and the same controller $u(t) = K_2x(t)$ to the SPS (1) yields

$$\dot{x}(t) = (A_\varepsilon + \gamma^{-2}U_\varepsilon X + B_\varepsilon K_2)x(t), x(0) = 0 \tag{42a}$$

$$J_2(u, w^*) = \int_0^\infty x^{\mathrm{T}}(t)(Q + K_2^{\mathrm{T}}K_2)x(t)\,\mathrm{d}t \tag{42b}$$

If $A_\varepsilon + D_\varepsilon K_1 + B_\varepsilon K_2$ is stable, there exists the sufficient small parameter ε such that the closed-loop systems (41a) and (42a) are stable [21]. Therefore, there exist the solutions which satisfy the following ALEs [20]:

$$(A_\varepsilon + \gamma^{-2}U_\varepsilon X^{(n)} + B_\varepsilon K_2)^{\mathrm{T}}\mathscr{V}_{1\varepsilon} + \mathscr{V}_{1\varepsilon}(A_\varepsilon + \gamma^{-2}U_\varepsilon X^{(n)} + B_\varepsilon K_2) + Q + K_2^{\mathrm{T}}K_2 = 0 \tag{43a}$$

$$(A_\varepsilon + \gamma^{-2}U_\varepsilon X + B_\varepsilon K_2)\mathscr{W}_{1\varepsilon} + \mathscr{W}_{1\varepsilon}(A_\varepsilon + \gamma^{-2}U_\varepsilon X + B_\varepsilon K_2) + Q + K_2^{\mathrm{T}}K_2 = 0 \tag{43b}$$

Thus, we have

$$J_2(u, \bar{w}^{(n)}) = x^{\mathrm{T}}(0)\mathscr{V}_{1\varepsilon}x(0) \tag{44a}$$

$$J_2(u, w^*) = x^{\mathrm{T}}(0)\mathscr{W}_{1\varepsilon}x(0) \tag{44b}$$

Using $X^{(n)} = X + O(\varepsilon^{2^n})$, we can change the form of (43b) into

$$(A_\varepsilon + \gamma^{-2}U_\varepsilon X^{(n)} + B_\varepsilon K_2)^{\mathrm{T}}\mathscr{W}_{1\varepsilon} + \mathscr{W}_{1\varepsilon}(A_\varepsilon + \gamma^{-2}U_\varepsilon X^{(n)} + B_\varepsilon K_2) + Q + K_2^{\mathrm{T}}K_2 + O(\varepsilon^{2^n}) = 0 \tag{45}$$

Subtracting (45) from (43a), we get

$$\begin{aligned}(A_\varepsilon + \gamma^{-2}U_\varepsilon X^{(n)} + B_\varepsilon K_2)^{\mathrm{T}}(\mathscr{V}_{1\varepsilon} - \mathscr{W}_{1\varepsilon})&\\ + (\mathscr{V}_{1\varepsilon} - \mathscr{W}_{1\varepsilon})(A_\varepsilon + \gamma^{-2}U_\varepsilon X^{(n)} + B_\varepsilon K_2) + O(\varepsilon^{2^n}) = 0&\end{aligned} \tag{46}$$

Then, (46) yields $\mathscr{V}_{1\varepsilon} - \mathscr{W}_{1\varepsilon} = O(\varepsilon^{2^n})$, which implies (33b) because

$$J_2(u, \bar{w}^{(n)}) - J_2(u, w^*) = x^{\mathrm{T}}(0)(\mathscr{V}_{1\varepsilon} - \mathscr{W}_{1\varepsilon})x(0) \tag{47}$$

This is the desired result. □

Finally we give the main result in this section.

Theorem 4.3

Under the conditions of Theorem 4.1, the following result holds:

$$J_1(\bar{u}^{(n)}, \bar{w}^{(n)}) \leqslant J_1(\bar{u}^{(n)}, w) + O(\varepsilon^{2^n}) \tag{48a}$$

$$J_2(\bar{u}^{(n)}, \bar{w}^{(n)}) \leqslant J_2(u, \bar{w}^{(n)}) + O(\varepsilon^{2^n}) \tag{48b}$$

Proof

Using (4a), Theorems 4.1 and 4.2, we have

$$\begin{aligned}J_1(\bar{u}^{(n)}, \bar{w}^{(n)}) &= J_1(\bar{u}^{(n)}, w) + J_1(\bar{u}^{(n)}, \bar{w}^{(n)}) - J_1(u^*, w^*) + J_1(u^*, w^*) - J_1(\bar{u}^{(n)}, w)\\ &\leqslant J_1(\bar{u}^{(n)}, w) + J_1(\bar{u}^{(n)}, \bar{w}^{(n)}) - J_1(u^*, w^*) + J_1(u^*, w) - J_1(\bar{u}^{(n)}, w)\\ &\leqslant J_1(\bar{u}^{(n)}, w) + O(\varepsilon^{2^n})\end{aligned} \tag{49}$$

which proves (48a). The other case is similar. □

Consequently, when ε is known, we can get the high-order $O(\varepsilon^{2^n})$ approximate strategy which achieves $O(\varepsilon^{2^n})$ approximation for the equilibrium values of the cost functionals.

In the rest of this section, we will present an important implication. If the perturbation parameter ε is unknown, then the following corollary is easily seen in view of Theorem 4.3.

Corollary 4.1

Under the conditions of Theorem 4.1, the following result holds:

$$J_1(\tilde{u}, \tilde{w}) \leqslant J_1(\tilde{u}, w) + O(\varepsilon) \tag{50a}$$

$$J_2(\tilde{u}, \tilde{w}) \leqslant J_2(u, \tilde{w}) + O(\varepsilon) \tag{50b}$$

where

$$\tilde{u}(t) = -B^{\mathrm{T}} \bar{Y} x(t) \tag{51a}$$

$$\tilde{w}(t) = \gamma^{-2} D^{\mathrm{T}} \bar{X} x(t) \tag{51b}$$

Proof

Since the proof can be carried out via a similar technique used in the proof of Theorems 4.1, 4.2 and 4.3 and setting $\varepsilon = 0$ and $n = 0$, it is omitted. □

5. NUMERICAL EXAMPLE

In order to demonstrate the efficiency of our proposed algorithm, we have run two numerical examples.

5.1. Example 1

Let us consider the following SPS:

$$\begin{bmatrix} \dot{x}_1 \\ \varepsilon \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u \tag{52}$$

with the performance index

$$J_1(u, w) = \int_0^\infty \gamma^2 w^{\mathrm{T}} w \, \mathrm{d}t - J_2(u, w) \tag{53a}$$

$$J_2(u, w) = \int_0^\infty (x_1^2 + x_2^2 + u^2) \, \mathrm{d}t \tag{53b}$$

The numerical results are obtained for small parameter $\varepsilon = 10^{-2}$. Since $\det A_{22} = 0$, the system is non-standard SPS. Now, we choose as $\gamma = 5.0$ to design the controller. We give the initial condition (14) and the solution of the GCARE (12), respectively.

$$\mathscr{P}^{(0)} = \begin{bmatrix} X^{(0)} & 0 \\ 0 & Y^{(0)} \end{bmatrix}, \quad \mathscr{P}^{(3)} = \begin{bmatrix} X^{(3)} & 0 \\ 0 & Y^{(3)} \end{bmatrix}$$

$$X^{(0)} = \begin{bmatrix} 1.1548e+00 & 8.4467e-03 \\ 8.4467e-01 & 5.0253e-01 \end{bmatrix}, \quad Y^{(0)} = \begin{bmatrix} 1.1880e+00 & 8.7860e-03 \\ 8.7860e-01 & 5.0505e-01 \end{bmatrix}$$

$$X^{(3)} = \begin{bmatrix} 1.1645e+00 & 8.4503e-03 \\ 8.4503e-01 & 5.0684e-01 \end{bmatrix}, \quad Y^{(3)} = \begin{bmatrix} 1.1984e+00 & 8.7929e-03 \\ 8.7929e-01 & 5.0965e-01 \end{bmatrix}$$

Table I. Errors per iteration.

i	$\|\mathscr{F}(\mathscr{P}^{(n)})\|$
0	$1.8379e-02$
1	$8.4512e-05$
2	$1.7208e-09$
3	$1.3323e-15$

Table II. Number of iterations such that $\|\mathscr{F}(\mathscr{P}^{(n)})\| < e-12$.

ε	Lyapunov iterations	Newton's method
10^{-1}	11	4
10^{-2}	11	3
10^{-3}	11	2
10^{-4}	11	2
10^{-5}	11	2
10^{-6}	11	1

Table I shows the results of the errors $\|\mathscr{F}(\mathscr{P}^{(n)})\|$ per iterations. We find that the solutions of the GCARE (12) converge to the exact solution with accuracy of $\|\mathscr{F}(\mathscr{P}^{(n)})\| < e-12$ after three iterations. Moreover, it is interesting to see that the result of Table I shows that algorithm (13) has quadratic convergence. Table II shows the results of the iterations in order to converge to the required solution with the same accuracy of $\|\mathscr{F}(\mathscr{P}^{(n)})\| < e-12$ for the Lyapunov iterations algorithm [8] versus the proposed algorithm. It can be seen that the convergence rate of the resulting algorithm is stable for all ε since the initial conditions $\mathscr{P}^{(0)}$ is quite good. On the other hand, the Lyapunov iterations converge very slowly.

5.2. Example 2

The system matrices are given as follows:

$$A_{11} = \begin{bmatrix} 0 & 0 & 4.5 & 0 & 1 \\ 0 & 0 & 0 & 4.5 & -1 \\ 0 & 0 & -0.05 & 0 & -0.1 \\ 0 & 0 & 0 & -0.05 & 0.1 \\ 0 & 0 & 32.7 & -32.7 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.4 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -0.05 & 0.05 & 0 & 0 \\ 0 & -0.1 & 0 & 0 \\ 0 & 0 & -0.05 & 0.05 \\ 0 & 0 & 0 & -0.1 \end{bmatrix}$$

$$D_1 = B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & 0 \\ 0.05 & 0 \\ 0 & 0 \\ 0 & 0.05 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ 0.1 & 0 \\ 0 & 0 \\ 0 & 0.1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Although H_∞ control problem has been considered in Reference [22], H_2/H_∞ control has not been investigated. Taking the optimality for the cost function into account, H_2/H_∞ control is applied to the SPS (1). The numerical results are obtained for small parameter $\varepsilon = 10^{-3}$. It is found that there exists the solution of H_∞ control problem for all $\gamma \in \{\gamma | 0.501673 < \gamma\}$ via MATLAB. Now, we choose as $\gamma = 1.0$ to solve the GCARE (12). We give the initial condition (14) and solutions of the GCARE (12) as the convergence solution $\mathscr{P}^{(4)}$, respectively.

$$X^{(0)} = \begin{bmatrix} \bar{X}_{11} & \varepsilon \bar{X}_{21}^{\mathrm{T}} \\ \bar{X}_{21} & \bar{X}_{22} \end{bmatrix}, \quad Y^{(0)} = \begin{bmatrix} \bar{Y}_{11} & \varepsilon \bar{Y}_{21}^{\mathrm{T}} \\ \bar{Y}_{21} & \bar{Y}_{22} \end{bmatrix}$$

$$\bar{X}_{11} = \begin{bmatrix} 3.7133e+00 & 3.6038e-01 & 3.2010e+01 & 6.4624e-14 & 2.4552e-01 \\ 3.6038e-01 & 3.7133e+00 & 2.0421e-13 & 3.2010e+01 & -2.4552e-01 \\ 3.2010e+01 & 1.3319e-13 & 6.5090e+02 & -2.0457e+02 & 4.8400e+00 \\ -2.8560e-13 & 3.2010e+01 & -2.0457e+02 & 6.5090e+02 & -4.8400e+00 \\ 2.4552e-01 & -2.4552e-01 & 4.8400e+00 & -4.8400e+00 & 1.7478e+00 \end{bmatrix}$$

$$\bar{X}_{21} = \begin{bmatrix} 5.2916e+01 & -9.0222e-14 & 1.0465e+03 & -3.3809e+02 & 7.9970e+00 \\ 1.4558e+01 & 8.3316e-14 & 2.5005e+02 & -9.2913e+01 & 2.1958e+00 \\ 9.1116e-14 & 5.2916e+01 & -3.3809e+02 & 1.0465e+03 & -7.9970e+00 \\ -1.2361e-13 & 1.4558e+01 & -9.2913e+01 & 2.5005e+02 & -2.1958e+00 \end{bmatrix}$$

$$\bar{X}_{22} = \begin{bmatrix} 3.6935e+01 & 7.9373e+00 & -5.8814e-14 & 8.9092e-15 \\ 7.9373e+00 & 1.6947e+01 & -5.4794e-14 & 2.5610e-14 \\ -3.6404e-14 & 2.7806e-14 & 3.6935e+01 & 7.9373e+00 \\ -2.6960e-14 & -8.2431e-14 & 7.9373e+00 & 1.6947e+01 \end{bmatrix}$$

$$\bar{Y}_{11} = \begin{bmatrix} 3.8150e+00 & 3.8110e-01 & 3.2877e+01 & 2.1161e-13 & 2.4431e-01 \\ 3.8110e-01 & 3.8150e+00 & 1.6920e-13 & 3.2877e+01 & -2.4431e-01 \\ 3.2877e+01 & 1.8630e-13 & 6.6051e+02 & -2.0697e+02 & 4.8362e+00 \\ -1.7764e-15 & 3.2877e+01 & -2.0697e+02 & 6.6051e+02 & -4.8362e+00 \\ 2.4431e-01 & -2.4431e-01 & 4.8362e+00 & -4.8362e+00 & 1.7547e+00 \end{bmatrix}$$

$$\bar{Y}_{21} = \begin{bmatrix} 5.5739e+01 & 1.4397e-13 & 1.0832e+03 & -3.5113e+02 & 8.2091e+00 \\ 1.7378e+01 & 8.6035e-14 & 2.9194e+02 & -1.0976e+02 & 2.5717e+00 \\ 2.2926e-13 & 5.5739e+01 & -3.5113e+02 & 1.0832e+03 & -8.2091e+00 \\ -3.6731e-14 & 1.7378e+01 & -1.0976e+02 & 2.9194e+02 & -2.5717e+00 \end{bmatrix}$$

$$\bar{Y}_{22} = \begin{bmatrix} 3.7496e+01 & 9.0012e+00 & -5.1047e-14 & -1.7491e-14 \\ 9.0012e+00 & 1.9007e+01 & -2.4476e-14 & -1.3656e-14 \\ -6.7641e-14 & -3.0107e-14 & 3.7496e+01 & 9.0012e+00 \\ -2.8620e-14 & -8.7900e-15 & 9.0012e+00 & 1.9007e+01 \end{bmatrix}$$

$$X^{(4)} = \begin{bmatrix} X_{11} & \varepsilon X_{21}^{\mathrm{T}} \\ X_{21} & X_{22} \end{bmatrix}, \quad Y^{(4)} = \begin{bmatrix} Y_{11} & \varepsilon Y_{21}^{\mathrm{T}} \\ Y_{21} & Y_{22} \end{bmatrix}$$

$$X_{11} = \begin{bmatrix} 3.7318e+00 & 3.5834e-01 & 3.2414e+01 & -1.3395e-01 & 2.4760e-01 \\ 3.5834e-01 & 3.7318e+00 & -1.3395e-01 & 3.2414e+01 & -2.4760e-01 \\ 3.2414e+01 & -1.3395e-01 & 6.6025e+02 & -2.1004e+02 & 4.9057e+00 \\ -1.3395e-01 & 3.2414e+01 & -2.1004e+02 & 6.6025e+02 & -4.9057e+00 \\ 2.4760e-01 & -2.4760e-01 & 4.9057e+00 & -4.9057e+00 & 1.7472e+00 \end{bmatrix}$$

$$X_{21}=\begin{bmatrix} 5.3086e+01 & -5.6524e-02 & 1.0556e+03 & -3.4322e+02 & 6.6619e+00 \\ 1.4558e+01 & -3.5609e-14 & 2.5196e+02 & -9.3934e+01 & 1.7101e+00 \\ -5.6524e-02 & 5.3086e+01 & -3.4322e+02 & 1.0556e+03 & -6.6619e+00 \\ -1.0252e-14 & 1.4558e+01 & -9.3934e+01 & 2.5196e+02 & -1.7101e+00 \end{bmatrix}$$

$$X_{22}=\begin{bmatrix} 3.8681e+01 & 8.4009e+00 & -5.6223e-01 & -1.5430e-01 \\ 8.4009e+00 & 1.7076e+01 & -1.5430e-01 & -4.2769e-02 \\ -5.6223e-01 & -1.5430e-01 & 3.8681e+01 & 8.4009e+00 \\ -1.5430e-01 & -4.2769e-02 & 8.4009e+00 & 1.7076e+01 \end{bmatrix}$$

$$Y_{11}=\begin{bmatrix} 3.8344e+00 & 3.7844e-01 & 3.3299e+01 & -1.4166e-01 & 2.4625e-01 \\ 3.7844e-01 & 3.8344e+00 & -1.4166e-01 & 3.3299e+01 & -2.4625e-01 \\ 3.3299e+01 & -1.4166e-01 & 6.7015e+02 & -2.1260e+02 & 4.8999e+00 \\ -1.4166e-01 & 3.3299e+01 & -2.1260e+02 & 6.7015e+02 & -4.8999e+00 \\ 2.4625e-01 & -2.4625e-01 & 4.8999e+00 & -4.8999e+00 & 1.7536e+00 \end{bmatrix}$$

$$Y_{21}=\begin{bmatrix} 5.5945e+01 & -6.8681e-02 & 1.0931e+03 & -3.5668e+02 & 6.8048e+00 \\ 1.7378e+01 & -2.8948e-14 & 2.9421e+02 & -1.1096e+02 & 1.9823e+00 \\ -6.8681e-02 & 5.5945e+01 & -3.5668e+02 & 1.0931e+03 & -6.8048e+00 \\ 1.3661e-14 & 1.7378e+01 & -1.1096e+02 & 2.9421e+02 & -1.9823e+00 \end{bmatrix}$$

$$Y_{22}=\begin{bmatrix} 3.9351e+01 & 9.5556e+00 & -5.9900e-01 & -1.8561e-01 \\ 9.5556e+00 & 1.9175e+01 & -1.8561e-01 & -5.6336e-02 \\ -5.9900e-01 & -1.8561e-01 & 3.9351e+01 & 9.5556e+00 \\ -1.8561e-01 & -5.6336e-02 & 9.5556e+00 & 1.9175e+01 \end{bmatrix}$$

We find that the solution of the GCARE (12) converges to the exact solution with accuracy of $\|\mathscr{F}(\mathscr{P}^{(n)})\| < e-10$ after four iterations. In order to verify the accuracy of the numerical solution, we calculate the remainder per iteration by substituting $\mathscr{P}^{(n)}$ into the GCARE (12). In Table III, we present results for the error $\|\mathscr{F}(\mathscr{P}^{(n)})\|$. It can be seen that the initial guess (14) for

Table III. Errors per iteration.

i	$\|\mathscr{F}(\mathscr{P}^{(n)})\|$
0	$9.1844e-01$
1	$5.3443e-02$
2	$4.5728e-04$
3	$1.7249e-10$
4	$5.6420e-12$

Table IV. Number of iterations such that $\|\mathscr{F}(\mathscr{P}^{(n)})\| < e - 10$.

ε	Lyapunov iterations	Newton's method
10^{-2}	21	5
10^{-3}	18	4
10^{-4}	16	3
10^{-5}	15	2
10^{-6}	14	2
10^{-7}	12	2
10^{-8}	10	1

algorithm (13) is quite good and the proposed algorithm has the quadratic convergence property. Table IV shows the results of iterations for both the Lyapunov iterations and the proposed algorithm. This table shows that the convergence speed of the Lyapunov iterations is very slow. Therefore, the simulation results have shown that the proposed algorithm succeeded in improving the convergence rate dramatically.

6. CONCLUSION

We have proposed a new algorithm to solve the GCARE associated with the mixed H_2/H_∞ control problem for infinite horizon SPS. It is very important to note that the resulting algorithm is quite different from the existing methods [5–9], since the proposed algorithm is based on Newton's method. Consequently, it has been newly proved that the resulting algorithm has the quadratic convergence via the Newton–Kantorovich theorem. Although the proposed Newton's method increases numerical complexity and is not easy to implement from the computational point of view compared with Lyapunov iterations [8, 9], we have succeeded in improving the convergence rate dramatically. Furthermore, it has also been shown that the resulting H_2/H_∞ controller via the iterative solutions achieved the properties of H_∞ criterion and H_2 near-optimality.

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