Proceedings of the 3rd
Asian Control Conference
July 4-7,2000,Shanghai

The High–order Approximate Controller for Mixed H_2/H_∞ Control Problem

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Abstract

In this paper, we study the cross–coupled algebraic Riccati equations corresponding to the mixed H_2/H_∞ control problem. The main results in this paper are to propose a new algorithm for solving cross–coupled algebraic Riccati equations by making use of Lyapunov iterations. The proposed algorithm is Lyapunov type algorithm and can be implemented as a synchronous parallel algorithm. Under some assumption and the condition that guarantees a disturbance attenuation level γ which is larger than the maximum of the optimal disturbance attenuation level γ^*, the Lyapunov iterations are constructed such that the proposed algorithm converges to the positive semidefinite solution of the cross–coupled algebraic Riccati equations.

1. Introduction

The cross–coupled algebraic Riccati equations play an important role to some problems of modern control theory. In Limebeer *et al.* (1994), a state feedback mixed H_2/H_∞ control problem is formulated as a dynamic Nash game.

It is well known that in order to obtain the Nash equilibrium strategies, we must solve the cross–coupled algebraic Riccati equations. Li and Gajić (1994) proposed the Lyapunov iterations to solve the linear–quadratic Nash game. However, in Li and Gajić (1994), only the Lyapunov iterations for solving cross–coupled algebraic Riccati equations of Nash game is considered. Freiling *et al.*(1996) found the solutions to the cross–coupled algebraic Riccati equations of the mixed H_2/H_∞ type by using the Riccati iterations. But, the convergence of the Riccati iterations was not proved.

In this paper, we study the cross–coupled algebraic Riccati equations with γ for the mixed H_2/H_∞ control problem. The main results in this paper are to propose a new algorithm for solving the cross–coupled algebraic Riccati equations by making use of Lyapunov iterations. The convergence of the algorithm is proved by using the successive approximations of dynamic programming. It is worth to note that the Lyapunov iterations to solve the cross–coupled algebraic Riccati equations with γ for the mixed H_2/H_∞ control problems have never been studied.

2. Problem Formulation

Consider a linear time–invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Dw(t) + Bu(t), && \text{(1a)} \\ & \quad x(0) = 0, && \text{(1b)} \\ z(t) &= \begin{bmatrix} Cx(t) \\ Lu(t) \end{bmatrix}, && \text{(1c)} \end{aligned}$$

and a quadratic cost function

$$J(x(t), u(t)) = \int_0^\infty z^T(t) z(t) dt, \qquad (2)$$

where $x \in \mathrm{R}^n$ is states, $u \in \mathrm{R}^{l_1}$ is the control input, $w \in \mathrm{R}^{l_2}$ is the disturbance, $z \in \mathrm{R}^{k_2}$ is the controlled output. All matrices above are of appropriate dimensions. We assume that $L^T L = I$.

Let us define the following matrices

$$S = BB^T,\ U = DD^T,\ Q = C^TC.$$

We now consider the mixed H_2/H_∞ control problems for the system (1) under the following basic assumption (Gajić *et al.* 1995).

Assumption 1 *The triplet* $(A,\ B,\ C)$ *and* $(A,\ D,\ C)$ *are stabilizable and detectable.*

These conditions are quite natural since at least one control agent has to be able to control and observe unstable modes.

The mixed H_2/H_∞ control problem is formulated as a two-player Nash game associated with a prescribed disturbance attenuation level γ,

$$\begin{aligned} J_1(x,u,w) &= \int_0^\infty \gamma^2 w^T(t)w(t)dt \\ &\quad - J(x,u), \quad (3a) \\ J_2(x,u,w) &= J(x,u). \quad (3b) \end{aligned}$$

The first is used to reflect an H_∞ criterion, while the second is used for an H_2 optimality requirement. The purpose is to find a linear feedback controller $u^*(t) = K_2x(t)$ such that

$$\begin{aligned} J_1(u^*,w^*) &\le J_1(u^*,w), \quad (4a) \\ J_2(u^*,w^*) &\le J_2(u,w^*), \quad (4b) \end{aligned}$$

where $w^*(t) = K_1x(t)$ represents the worst-case disturbance. When $J_1(u^*,w^*) \ge 0$, we have

$$\sup_{w\in H_w} \frac{\sqrt{J(u^*,\ w)}}{||w||} \le \gamma, \quad (5)$$

a H_∞ criterion, where H_w denotes an appropriate Hilbert space. The second Nash inequality shows that $u^*(t)$ regulates the state to zero with minimum output energy when the disturbance is at its worst value $w^*(t)$. The following lemma is already known (see Limebeer *et al.* 1994).

Lemma 1 *Under Assumption 1, there exists an admissible controller such that (4) hold iff the following full-order cross-coupled algebraic Riccati equations*

$$\begin{aligned} &(A - SY)^TX + X(A - SY) \\ &\quad +Q + \gamma^{-2}XUX \\ &\quad + YSY \equiv F_1(X,\ Y) = 0, \quad (6a) \\ &(A + \gamma^{-2}UX)^TY + Y(A + \gamma^{-2}UX) + Q \\ &\quad - YSY \equiv F_2(X,\ Y) = 0, \quad (6b) \end{aligned}$$

have solutions $X \ge 0$ *and* $Y \ge 0$. *Then, the strategies are given by*

$$\begin{aligned} w^*(t) &= K_1x(t) = \gamma^{-2}D^TXx(t), (7a) \\ u^*(t) &= K_2x(t) = -B^TYx(t). \quad (7b) \end{aligned}$$

3. The Cross–coupled Algebraic Riccati Equations

We give the Lyapunov iterations for solving the cross–coupled algebraic Riccati equations with parameter γ. An algorithm for the numerical solutions of (6) is defined as follows.

$$\begin{aligned} &(A + \gamma^{-2}UX^{(n)} - SY^{(n)})^TX^{(n+1)} \\ &\quad +X^{(n+1)}(A + \gamma^{-2}UX^{(n)} - SY^{(n)}) \\ &\quad +Q - \gamma^{-2}X^{(n)}UX^{(n)} \\ &\quad + Y^{(n)}SY^{(n)} = 0, \quad (8a) \\ &(A + \gamma^{-2}UX^{(n)} - SY^{(n)})^TY^{(n+1)} \\ &\quad +Y^{(n+1)}(A + \gamma^{-2}UX^{(n)} - SY^{(n)}) \\ &\quad +Q + Y^{(n)}SY^{(n)} = 0, \quad (8b) \end{aligned}$$

where $n = 0,\ 1,\ 2,\ 3,\ \cdots$ and initial conditions $X^{(0)}$ and $Y^{(0)}$ are obtained as solutions of the following auxiliary algebraic Riccati equations

$$\begin{aligned} &A^TY^{(0)} + Y^{(0)}A + Q - Y^{(0)}SY^{(0)} = 0, (9a) \\ &(A - SY^{(0)})^TX^{(0)} + X^{(0)}(A - SY^{(0)}) + Q \\ &\quad + \gamma^{-2}X^{(0)}UX^{(0)} + Y^{(0)}SY^{(0)} = 0. \quad (9b) \end{aligned}$$

We note that the positive semidefinite stabilizing solution of (9a) exists under Assumptions 1 (Gajić *et al.* 1995). Concerning with the Riccati equation (9b), let us define

$$||E(sI - A + SY^{(0)})^{-1}D||_\infty = \bar{\gamma} \quad (10)$$

where $E^T E = Q + Y^{(0)} S Y^{(0)}$.

If Assumption 1 holds, then for every $\gamma > \bar{\gamma}$, the Riccati equation (9b) has the positive definite stabilizing solutions since the Riccati equation (9a) has stabilizing solution (Dragan 1996).

The algorithm (8) is based on the Lyapunov iterations (Li and Gajić 1994, Gajić *et al.* 1995). Although this algorithm has a similar form of Li and Gajić (1994), it is quite easy to show that they are different. Note that the Lyapunov iterations which were proposed in Li and Gajić (1994) do not include parameter γ. Thus, the Lyapunov iterations for the Nash games converge to the positive semidefinite solutions under the stabilizable–detectable conditions. However, the convergence of the proposed Lyapunov iterations is not clear since the Lyapunov iterations (8) include parameter γ. Under the control–oriented assumptions and a new condition for parameter γ we prove that the proposed Lyapunov iterations (8) converge to the positive semidefinite solutions. The algorithm (8) has the feature given in the following theorem.

Theorem 1 *Under Assumption 1, for a prescribed disturbance attenuation level $\gamma > \bar{\gamma}$, the positive semidefinite solution of the cross–coupled algebraic Riccati equations (6) exist. It is obtained by performing Lyapunov iterations (8a) and (8b).*

Proof: We can give the proof by using a method similar to the proof of Theorem2.1 in Li and Gajić (1994). Firstly, we take any stabilizable linear control law $u^{(0)}(t,\ x) = -B^T Y^{(0)} x(t)$ and disturbance $w^{(0)}(t,\ x) = \gamma^{-2} D^T X^{(0)} x(t)$ such that $X^{(0)}$ and $Y^{(0)}$ are positive semidefinite. Then, let us consider the following closed loop singularly perturbed system and performance criterion

$$\dot{x}(t) = [A_\varepsilon + \gamma^{-2} U X^{(0)} - S Y^{(0)}] x(t), \tag{11a}$$

$$\begin{aligned} &\bar{J}_1^{(0)}(x,\ t) \\ &= \int_t^\infty [\gamma^2 w^{(0)}(\tau,\ x)^T w^{(0)}(\tau,\ x) \\ &\qquad -\{x(\tau)^T Q x(\tau) \\ &\qquad +u^{(0)}(\tau,\ x)^T u^{(0)}(\tau,\ x)\}] d\tau \\ &= \int_t^\infty x^T(\tau) [\gamma^{-2} X^{(0)} U X^{(0)} \\ &\qquad - Q - Y^{(0)} S Y^{(0)}] x(\tau) d\tau, \end{aligned} \tag{11b}$$

$$\begin{aligned} &\bar{J}_2^{(0)}(x,\ t) \\ &= \int_t^\infty [x(\tau)^T Q x(\tau) \\ &\qquad +u^{(0)}(\tau,\ x)^T u^{(0)}(\tau,\ x)] d\tau \\ &= \int_t^\infty x^T(\tau) [Q + Y^{(0)} S Y^{(0)}] x(\tau) d\tau. \end{aligned} \tag{11c}$$

Corresponding Hamiltonians to the Nash differential games for each control agent are

$$\begin{aligned} &H_1(t,\ x,\ w,\ u^*,\ p_1^{(0)}) \\ &= \gamma^2 w^T w - x^T Q x - u^{*T} u^* \\ &\qquad + p_1^{(0)T} (Ax + Dw + Bu^*), \end{aligned} \tag{12a}$$

$$\begin{aligned} &H_2(t,\ x,\ w^*,\ u,\ p_2^{(0)}) \\ &= x^T Q x + u^T u \\ &\qquad + p_2^{(0)T} (Ax + Dw^* + Bu), \end{aligned} \tag{12b}$$

where $\frac{\partial}{\partial x} \bar{J}_i^{(0)}(x,\ t) = p_i^{(0)}(t),\ (i = 1,\ 2)$.

The equilibrium controls must satisfy

$$\begin{aligned} &\frac{\partial H_1}{\partial w} = 0 \\ &\Rightarrow w^{(1)}(t,\ x) = -\frac{1}{2} \gamma^{-2} D^T p_1^{(0)}(t), \end{aligned} \tag{13a}$$

$$\begin{aligned} &\frac{\partial H_2}{\partial u} = 0 \\ &\Rightarrow u^{(1)}(t,\ x) = -\frac{1}{2} B^T p_2^{(0)}(t). \end{aligned} \tag{13b}$$

Note that $\partial \bar{J}_i^{(0)}(x,\ t)/\partial x$ along the system trajectory can be calculated from (11) by using equality (14),

$$\frac{\partial}{\partial x} \bar{J}_i^{(0)}(x,\ t) \cdot \frac{dx}{dt} = \frac{d}{dt} \bar{J}_i^{(0)}(x,\ t), \tag{14}$$

$(i = 1,\ 2)$.

In fact, by substituting (11a), (11b) and (11c) into (14) we have

$$\frac{\partial}{\partial x} \bar{J}_1^{(0)}(x,\ t) \cdot [A$$

$$+\gamma^{-2}UX^{(0)} - SY^{(0)}]x(t)$$
$$= -x(t)^T[\gamma^{-2}X^{(0)}UX^{(0)}$$
$$- Q - Y^{(0)}SY^{(0)}]x(t), \quad (15a)$$
$$\frac{\partial}{\partial x}\bar{J}_2^{(0)}(x,\ t)\cdot[A$$
$$+\gamma^{-2}UX^{(0)} - SY^{(0)}]x(t)$$
$$= -x(t)^T[Q + Y^{(0)}SY^{(0)}]x(t). \quad (15b)$$

These simple partial differential equations (15) have solutions of the following form

$$\bar{J}_1^{(0)}(x,\ t) = -x(t)^T X^{(1)}x(t), \quad (16a)$$
$$\bar{J}_2^{(0)}(x,\ t) = x(t)^T Y^{(1)}x(t). \quad (16b)$$

A partial differentiation to (16) gives

$$\frac{\partial}{\partial x}\bar{J}_1^{(0)}(x,\ t) = -2X^{(1)}x(t) = p_1^{(0)}(t), \quad (17a)$$
$$\frac{\partial}{\partial x}\bar{J}_2^{(0)}(x,\ t) = 2Y^{(1)}x(t) = p_2^{(0)}(t). \quad (17b)$$

Thus, we have

$$(A+\gamma^{-2}UX^{(0)} - SY^{(0)})^T X^{(1)}$$
$$+X^{(1)}(A+\gamma^{-2}UX^{(0)} - SY^{(0)})$$
$$= -(Q - \gamma^{-2}X^{(0)}UX^{(0)}$$
$$+ Y^{(0)}SY^{(0)}), \quad (18a)$$
$$(A+\gamma^{-2}UX^{(0)} - SY^{(0)})^T Y^{(1)}$$
$$+Y^{(1)}(A+\gamma^{-2}UX^{(0)} - SY^{(0)})$$
$$= -(Q + Y^{(0)}SY^{(0)}). \quad (18b)$$

Since the Riccati equation (9b) has a positive semidefinite stabilizing solution by the bounded real lemma (Dragan), $A + \gamma^{-2}UX^{(0)} - SY^{(0)}$ is a stable matrix. Furthermore, we see that the right–hand side of equation (18b) is negative definite and $X^{(1)} = X^{(0)} \geq 0$ by comparing equation (18a) with equation (9b). Consequently, it follows that the Lyapunov equations (18a) and (18b) have unique positive semidefinite solutions $X^{(1)} \geq 0$ and $Y^{(1)} \geq 0$ respectively.

Thus, from (13) and (17) we get

$$w^{(1)}(t,\ x) = \gamma^{-2}D^T X^{(1)}x(t), \quad (19a)$$
$$(X^{(1)} \geq 0),$$
$$u^{(1)}(t,\ x) = -B^T Y^{(1)}x(t), \quad (19b)$$
$$(Y^{(1)} \geq 0).$$

Continuing the same procedure, we get the sequences of solutions and the control agents $X^{(2)}$, $Y^{(2)}$, $w^{(2)}(t,\ x)$, $u^{(2)}(t,\ x)$, $X^{(3)}$, $Y^{(3)}$, $w^{(3)}(t,\ x)$, $u^{(3)}(t,\ x)$, $\cdots$. That is, $\{X^{(n+1)}\}$, $\{Y^{(n+1)}\}$, $w^{(n+1)}(t,\ x)$ and $u^{(n+1)}(t,\ x)$ are obtained as follows.

$$(A+\gamma^{-2}UX^{(n)} - SY^{(n)})^T X^{(n+1)}$$
$$+X^{(n+1)T}(A+\gamma^{-2}UX^{(n)} - SY^{(n)})$$
$$= -(Q - \gamma^{-2}X^{(n)T}UX^{(n)}$$
$$+ Y^{(n)T}SY^{(n)}), \quad (20a)$$
$$(A+\gamma^{-2}UX^{(n)} - SY^{(n)})^T Y^{(n+1)}$$
$$+Y^{(n+1)T}(A+\gamma^{-2}UX^{(n)} - SY^{(n)})$$
$$= -(Q + Y^{(n)T}SY^{(n)}), \quad (20b)$$
$$w^{(n+1)}(t,\ x) = \gamma^{-2}D^T X^{(n+1)}x(t), \quad (20c)$$
$$(X^{(n+1)} \geq 0),$$
$$u^{(n+1)}(t,\ x) = -B^T Y^{(n+1)}x(t), \quad (20d)$$
$$(Y^{(n+1)} \geq 0).$$

Rearranging the Lyapunov equation (20a), we get

$$(A - SY^{(n)})^T X^{(n+1)} + X^{(n+1)}(A - SY^{(n)})$$
$$= -[Q -$$
$$\gamma^{-2}(X^{(n+1)} - X^{(n)})U(X^{(n+1)} - X^{(n)})$$
$$+\gamma^{-2}X^{(n+1)}UX^{(n+1)} + Y^{(n)}SY^{(n)}].$$

Note that since $A - SY^{(n)}$ is stable matrix for all $n \in \mathrm{N}$ (Li and Gajić 1994) and there exists $n_0 \in \mathrm{N}$ such that right–hand side of above equation is negative definite for all $n \in \mathrm{N}$ with $n_0 \leq n$, the Lyapunov equation (20a) has a unique positive semidefinite solution $X^{(n+1)} \geq 0$ for all $n \in \mathrm{N}$ with $n_0 \leq n$. On the other hand, the Lyapunov equation (20b) has a unique positive semidefinite solution $Y^{(n+1)} \geq 0$ for all $n \in \mathrm{N}$ since the right–hand side of equation (20b) is negative definite.

It is easy to show that these sequences converge because the corresponding Hamiltonians consisting of the matrices $\{X^{(n)}\}$ and $\{Y^{(n)}\}$, $(n = 0,\ 1,\ 2,\ \cdots)$ tend to zero as $n \to \infty$. In addition, let us define the limits $\{X^{(\infty)}\}$ and $\{Y^{(\infty)}\}$ for the corresponding sequences, we have

$$(A+\gamma^{-2}UX^{(\infty)} - SY^{(\infty)})^T X^{(\infty)}$$

$$+X^{(\infty)}(A+\gamma^{-2}UX^{(\infty)}-SY^{(\infty)})$$
$$+Q-\gamma^{-2}X^{(\infty)}UX^{(\infty)}$$
$$+Y^{(\infty)}SY^{(\infty)}=0, \qquad (21a)$$
$$(A+\gamma^{-2}UX^{(\infty)}-SY^{(\infty)})^TY^{(\infty)}$$
$$+Y^{(\infty)}(A+\gamma^{-2}UX^{(\infty)}-SY^{(\infty)})$$
$$+Q+Y^{(\infty)}SY^{(\infty)}=0, \qquad (21b)$$

that is, $\{X^{(\infty)}\}$ and $\{Y^{(\infty)}\}$ satisfy the cross-coupled algebraic Riccati equations (6) so that they represent the sought solutions of these equations. Thus, the proof of Theorem 1 is completed. □

Given the system (1) and any desired H_∞-norm bound constraint $\gamma > \bar{\gamma}$, we define the following high-order approximate controller (22) corresponding to the mixed H_2/H_∞ control problem.

$$w_{app}(t) = \gamma^{-2}D^TX^{(n)}x(t), \qquad (22a)$$
$$u_{app}(t) = -B^TY^{(n)}x(t). \qquad (22b)$$

In view of Theorem 1, we can derive the following theorem.

Theorem 2 *Under Assumption 1, for a sufficient large $n \in \mathbb{N}$, the high-order approximate controller (22) is a near-optimal solution to the mixed H_2/H_∞ control problem.*

Proof: The proof is omitted since it is clear that $X^{(n)} \to X$ and $Y^{(n)} \to Y$ as $n \to \infty$. □

4. Numerical Example

In order to demonstrate the efficiency of the proposed algorithm, we have run a simple numerical example (Freiling *et al.* 1996). Matrices A, D, B and Q are given by

$$A = \begin{bmatrix} A_{11} & O_{4\times 2} \\ O_{1\times 3} & A_{22} \end{bmatrix},$$
$$A_{22} = \begin{bmatrix} 0 & -100.0 \end{bmatrix},$$

$$A_{11}$$
$$= \begin{bmatrix} -0.016896 & -0.066099 & -28.891 \\ -0.24501 & -0.015623 & -0.14625 \\ 0.53061 & -0.034715 & -0.50025 \\ 0 & 1.0 & 0 \end{bmatrix},$$
$$D = \begin{bmatrix} 0.19474 & 0 \\ 0.08988 & 0 \\ 0.21103 & 0 \\ 0 & -0.08 \\ 0.8 & 0 \end{bmatrix},$$
$$B = \begin{bmatrix} 0.24342 \\ 0.11235 \\ 0.26379 \\ 0 \\ 1 \end{bmatrix},$$
$$Q = \mathrm{diag}\left(\, 0, \ 0, \ 0, \ 0.36, \ 100.0 \,\right).$$

Here, $\bar{\gamma} = 10.9316$ from (9). Then, for every boundary value of $\gamma > \bar{\gamma} = 10.9316$ the cross-couple algebraic Riccati equations (6) have positive semi-definite solutions. Now, we choose $\gamma = 12.0 > \bar{\gamma} = 10.9316$ to design the controller.

It can be seen that the solutions of the cross-coupled algebraic Riccati equations (6) converge to the following solutions after 30 Lyapunov iterations. We give the solutions of the cross-couple algebraic Riccati equations (6) at the bottom of the next page.

In order to verify the exactitude of the solution, we calculate the remainder by substituting $X^{(30)}$ and $Y^{(30)}$ into the cross-coupled algebraic Riccati equations (6a) and (6b) respectively.

$$\|F_1(X^{(30)},\ Y^{(30)})\| = 2.622e-09$$
$$\|F_2(X^{(30)},\ Y^{(30)})\| = 8.037e-09$$

where the errors $F_1(X,\ Y)$ and $F_2(X,\ Y)$ are defined as (6). Therefore, the numerical example illustrates the effectiveness of the proposed algorithm since the solutions $X^{(n)}$ and $Y^{(n)}$ converge to the exact solutions X and Y which are defined by (6a) and (6b).

5. Conclusions

By using the Lyapunov iteration, a new algorithm have been obtained in order to solve the cross–coupled algebraic Riccati equations for the mixed H_2/H_∞ control problem. Since these equations are generalized from the Nash games type cross–coupled algebraic Riccati equations, there are no direct conditions ensuring the existence of solutions and the proof of convergence. However, we have shown that the solution of the generalized cross–coupled algebraic Riccati equations with parameter γ converge to a positive semi–definite solution under the new sufficient condition for the parameter γ and give the convergence proof. That is, if we choose any $\gamma > \bar{\gamma}$ under the new sufficient condition, then there exists the mixed H_2/H_∞ controller, which attains the disturbance attenuation level γ and minimizes the performance index (2).

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$$X^{(30)} = \begin{bmatrix} 4.4656875e-03 & -2.5810736e-01 & -1.1741421e-01 & -7.2154140e-02 & 1.7223564e-04 \\ -2.5810736e-01 & 1.5869379e+01 & 7.2412482 & 3.7292626 & -1.0623733e-02 \\ -1.1741421e-01 & 7.2412482 & 3.3101126 & 1.6725734 & -4.8526282e-03 \\ -7.2154140e-02 & 3.7292626 & 1.6725734 & 1.6494192 & -2.4615843e-03 \\ 1.7223564e-04 & -1.0623733e-02 & -4.8526282e-03 & -2.4615843e-03 & 4.9927131e-01 \end{bmatrix}$$

$$Y^{(30)} = \begin{bmatrix} 5.6276701e-03 & -3.3203697e-01 & -1.5121685e-01 & -8.7081504e-02 & 1.0353647e-04 \\ -3.3203697e-01 & 2.0599731e+01 & 9.4047342 & 4.6575894 & -6.5030550e-03 \\ -1.5121685e-01 & 9.4047342 & 4.2996266 & 2.0965135 & -2.9723399e-03 \\ -8.7081504e-02 & 4.6575894 & 2.0965135 & 1.8584421 & -1.3917130e-03 \\ 1.0353647e-04 & -6.5030550e-03 & -2.9723399e-03 & -1.3917130e-03 & 4.9977260e-01 \end{bmatrix}$$